QFT as a set of ODEs

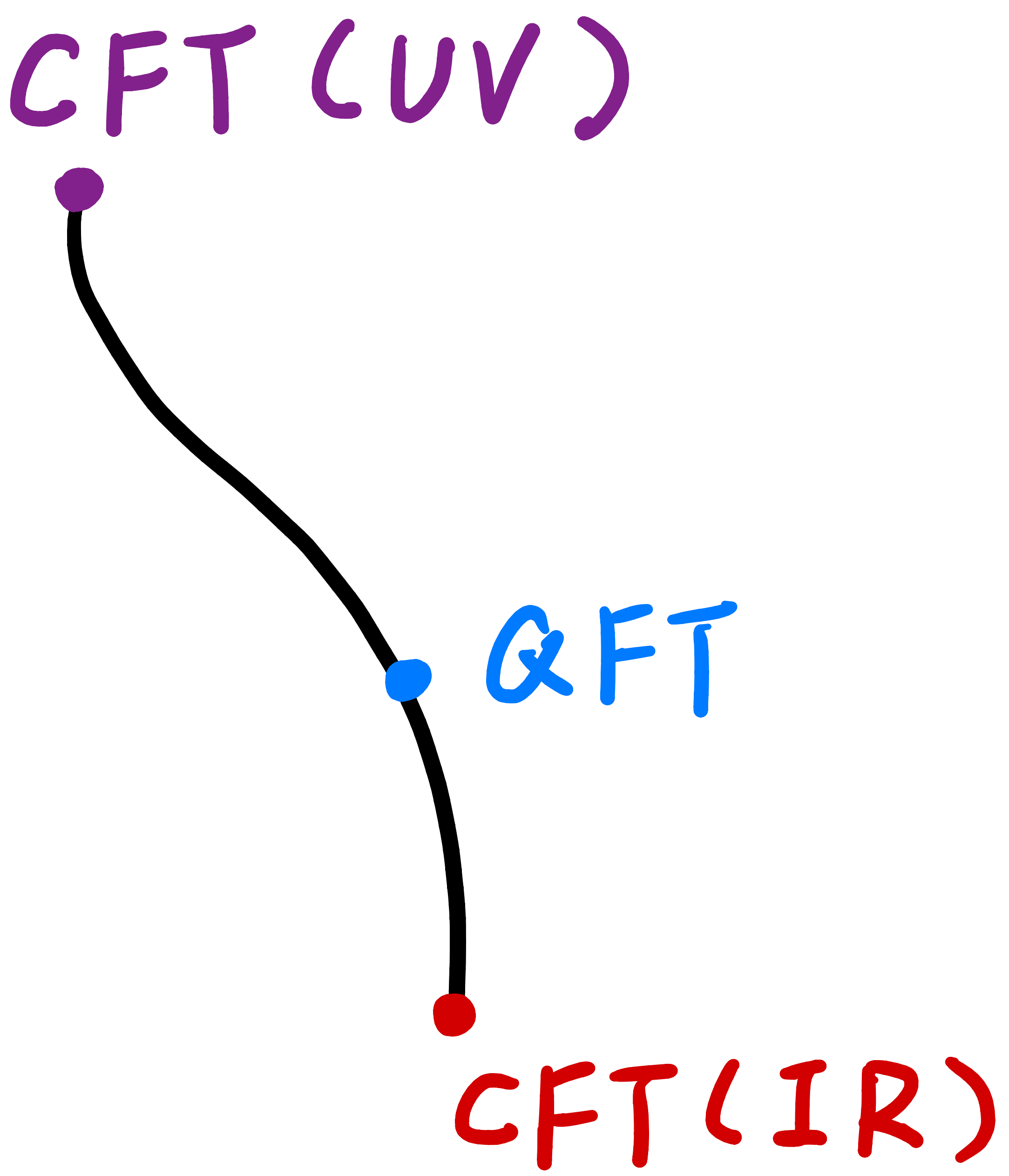
Jiaxin Qiao (EPFL -> Kavli IPMU)
2025.10.06 @YITP workshop

$$\frac{dX}{dt} = F(X)$$

Based on joint work with M.Loparco, G.Mathys, J.Penedones, and X.Zhao (to appear)

QFT in AdS

AdS is a good "box" for QFTs [Callan, Wilczek]



$$S_{QFT} = S_{CFT} + \lambda \int d^D x \Phi_\Delta(x), \quad [\lambda] = [E^{D-\Delta}]$$

For perturbation theory, the dimensionless parameter is

$$\hat{\lambda} = \lambda \times (R_{\text{AdS}})^{D-\Delta}$$

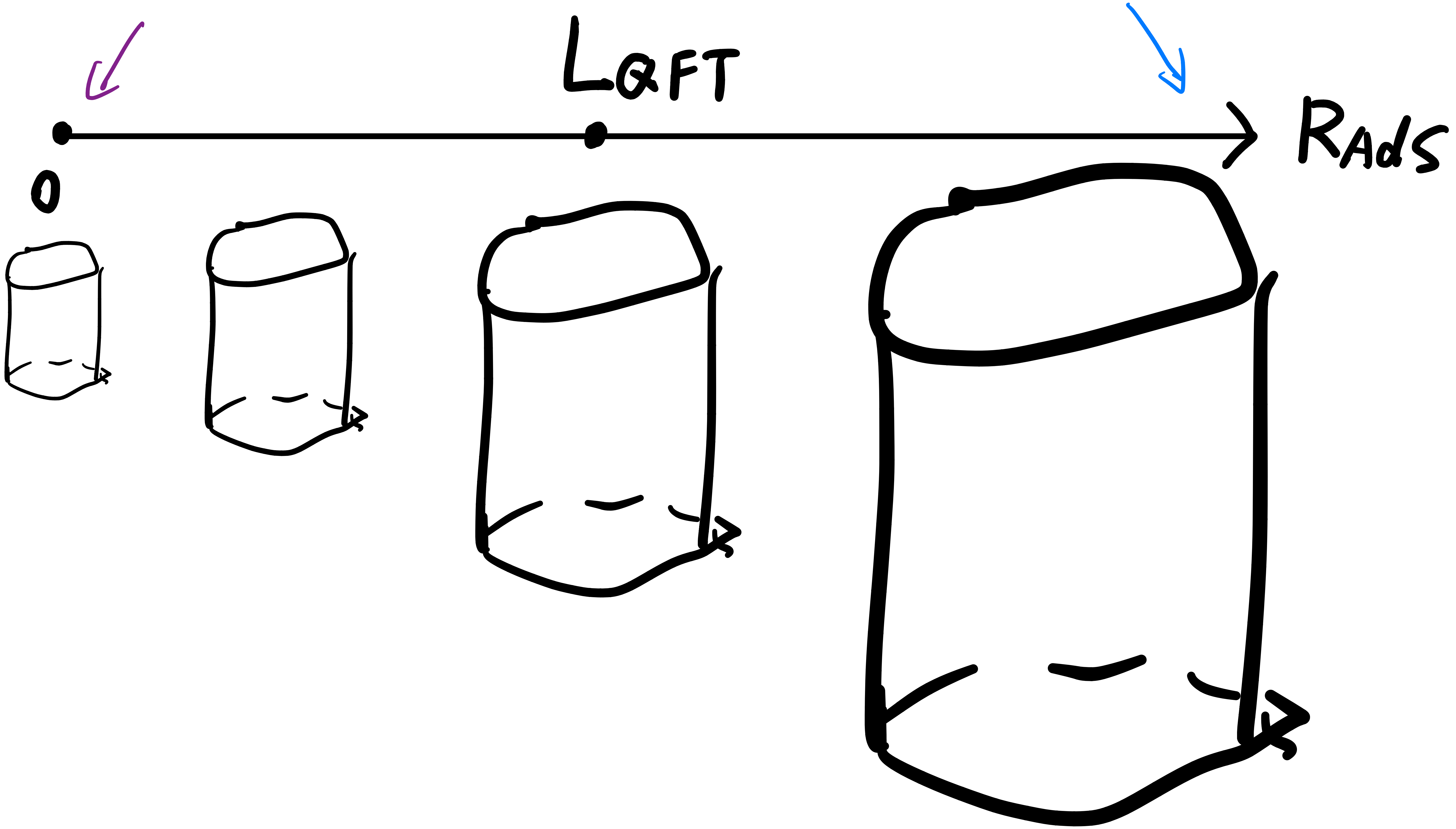
QFT in AdS

$$\hat{\lambda} = \lambda \times (R_{\text{AdS}})^{D-\Delta}$$

$$(\Delta < D)$$

CFT + small perturbation

QFT in flat space



QFT in AdS

$$\hat{\lambda} = \lambda \times (R_{\text{AdS}})^{D-\Delta}$$

- This process is equivalent to setting AdS radius = 1, and change the coupling.
- No dynamical gravity.
- Conformal boundary condition: d-dimensional boundary CFT (nonlocal)

$$\{(\Delta_i, J_i)\}, \quad \{C_{ijk}\}$$

- Boundary operator expansion (BOE)

$$\Phi(z,x) = \sum_i b_i^{\Phi} \left[z^{\Delta_i} \mathcal{O}_i + \text{descendants} \right]$$

QFT in AdS

$$\hat{\lambda} = \lambda \times (R_{\text{AdS}})^{D-\Delta}$$

- This process is equivalent to setting AdS radius = 1, and change the coupling.
- No dynamical gravity.
- Conformal boundary condition: d-dimensional boundary CFT (nonlocal)

$$\{(\Delta_i, J_i)\}, \quad \{C_{ijk}\}\ , \quad \{b_i^{\Phi}\}$$

- Boundary operator expansion (BOE)

$$\Phi(z,x) = \sum_i b_i^{\Phi} \left[z^{\Delta_i} \mathcal{O}_i + \text{descendants} \right]$$

$$\{\Delta_i(\lambda)\}, \quad \{C_{ijk}(\lambda)\} \quad \{b_i^\Phi(\lambda)\}$$

How do these data flow from $\lambda = 0$ to $\lambda = \infty$?

$$\{\Delta_i(\lambda)\}, \quad \{C_{ijk}(\lambda)\} \quad \{b_i^{\Phi}(\lambda)\}$$

How do these data flow from $\lambda = 0$ to $\lambda = \infty$?

Wilson already told us the answer, in the context of RG flow.

RG: consider infinitesimal change of the scale.

$$\mu \to \mu + \delta\mu \Rightarrow \frac{dg}{d\log\mu} = \beta(g)$$

$$\{\Delta_i(\lambda)\}, \quad \{C_{ijk}(\lambda)\} \quad \{b_i^{\Phi}(\lambda)\}$$

How do these data flow from $\lambda = 0$ to $\lambda = \infty$?

Here: consider infinitesimal change of the coupling.

$$S_{\mathrm{QFT}}(\lambda) \to S_{\mathrm{QFT}}(\lambda) + \delta\lambda \int_{\mathrm{AdS}} \Phi$$

1st order perturbation theory

$$\frac{dX}{d\lambda} = F(X), \quad X = (\Delta_1, \Delta_2, \ldots, C_{123}, \ldots, b_1^{\Phi}, \ldots)$$

Similar idea was explored for QFT in flat space [Holland, Hollands, Wald,...]

and exactly marginal deformation of 1D CFT [Behan]

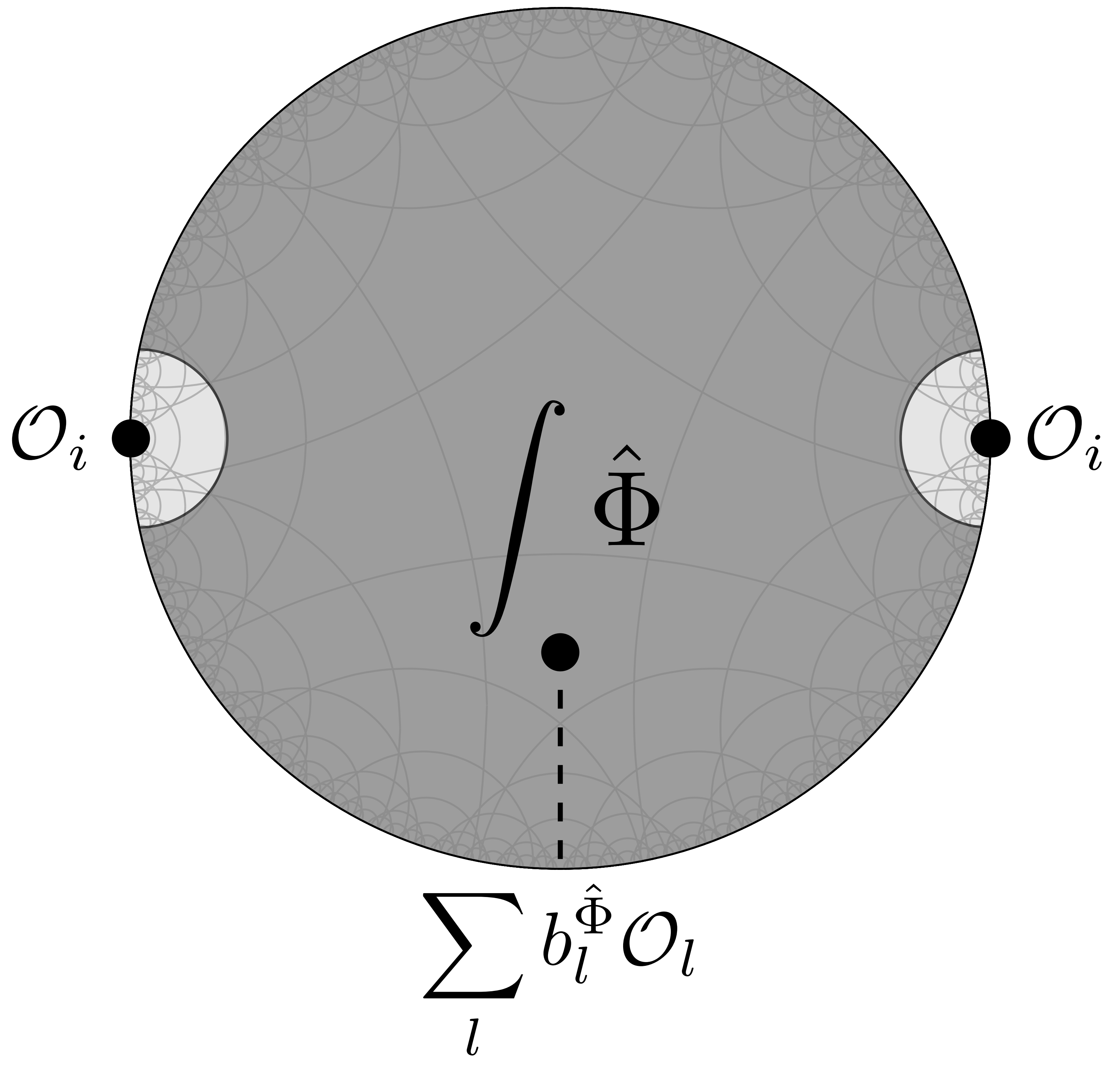
From now on, I will focus on AdS_2 (hyperbolic disk)

$$ds^2 = \frac{d\tau^2 + dz^2}{z^2} \quad (z > 0)\,.$$

$$\Delta_i(\lambda)$$

Measured in $$\langle \mathcal{O}_i(\tau_1)\mathcal{O}_i(\tau_2)\rangle = \frac{1}{|\tau_1-\tau_2|^{2\Delta_i}}$$

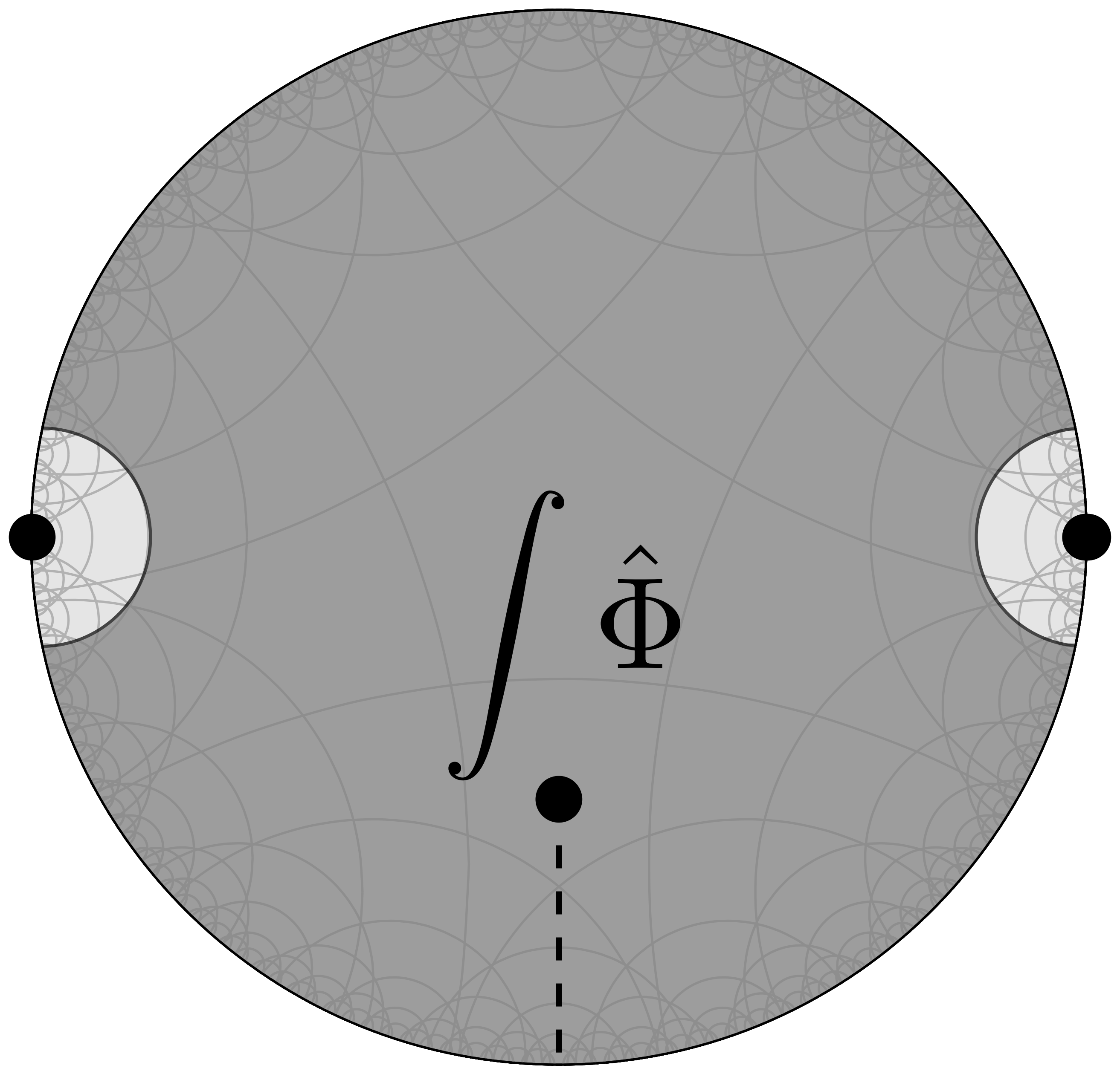
$$\frac{d\Delta_i}{d\lambda} =$$



$\Delta_i(\lambda)$

Measured in $\langle \mathcal{O}_i(\tau_1)\mathcal{O}_i(\tau_2)\rangle = \frac{1}{|\tau_1 - \tau_2|^{2\Delta_i}}$

$$\frac{d\Delta_i}{d\lambda} = \mathcal{O}_i$$



$\mathcal{O}_i$

$$\int_{\mathrm{AdS}} dx \langle \hat{\Phi}(x)\mathcal{O}_i\mathcal{O}_i\rangle$$

$$\sum_l b_l^{\hat{\Phi}} \mathcal{O}_l$$

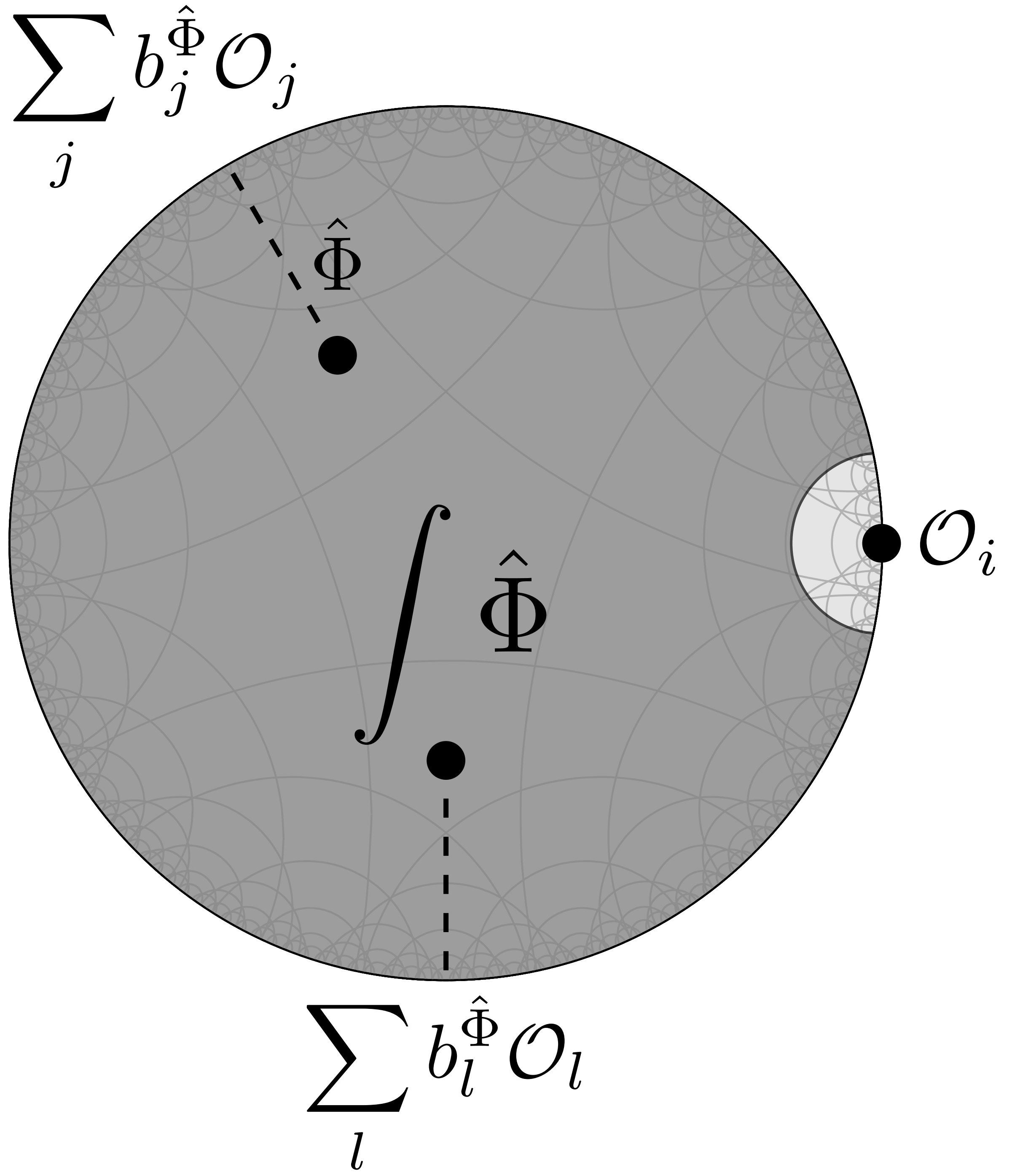
$$= \sum_l b_l^{\hat{\Phi}}\, C_{iil}\, \mathcal{I}_{\Delta_i}(\Delta_l)$$

actually only depend on Δ_l

$$b_i^{\hat{\Phi}}(\lambda)$$

Measured in $\langle \hat{\Phi}(\tau_1, z_1)\mathcal{O}_i(\tau_2)\rangle = b_i^{\hat{\Phi}} \left(\frac{z_1}{(\tau_1 - \tau_2)^2 + z_1^2} \right)^{\Delta_i}$

$$\frac{db_i^{\hat{\Phi}}}{d\lambda} =$$



$$= \sum_{l,j} b_l^{\hat{\Phi}} \, b_j^{\hat{\Phi}} \underbrace{\frac{C_{ilj} + C_{ijl}}{2}} \mathcal{J}_{\Delta_i}(\Delta_l, \Delta_j)$$

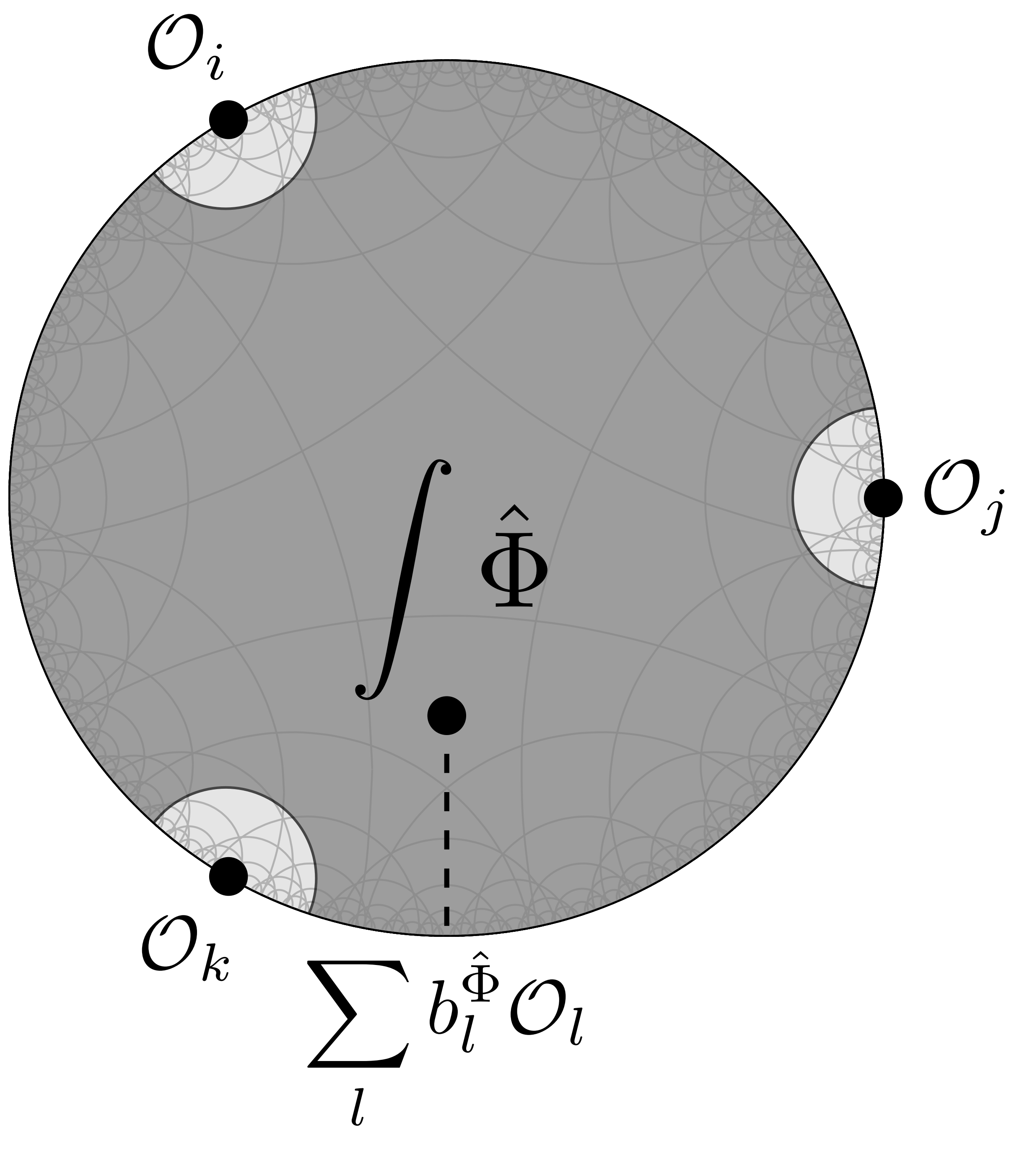
D=2 is special

C_{ijk}

Measured in $\langle \mathcal{O}_i(\tau_1)\mathcal{O}_j(\tau_2)\mathcal{O}_k(\tau_3)\rangle = \dfrac{C_{ijk}}{|\tau_{12}|^{\Delta_{ijk}}|\tau_{23}|^{\Delta_{jki}}|\tau_{23}|^{\Delta_{kij}}}$

$(\tau_1 < \tau_2 < \tau_3)$

$\dfrac{dC_{ijk}}{d\lambda} =$



$$= \sum_{l,m} b_l^{\hat{\Phi}}\, C_{jkm}\Big(C_{ilm}\mathcal{K}_{\Delta_i,\Delta_j,\Delta_k}(\Delta_l,\Delta_m) + C_{iml}\mathcal{K}_{\Delta_i,\Delta_k,\Delta_j}(\Delta_l,\Delta_m)\Big)$$

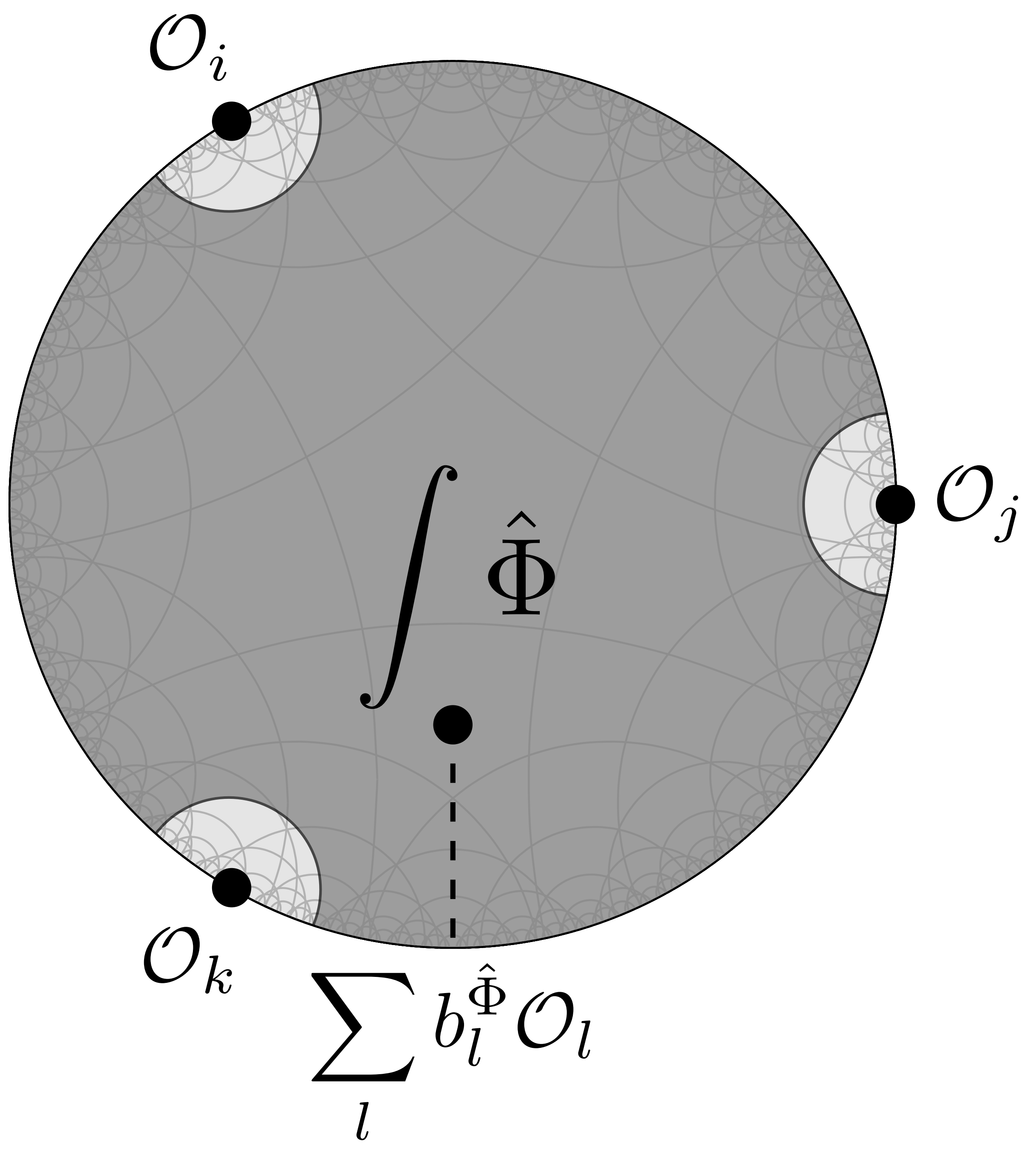
$$+ \text{ cyclic permutations of } ijk$$

C_{ijk}

Measured in $\langle \mathcal{O}_i(\tau_1)\mathcal{O}_j(\tau_2)\mathcal{O}_k(\tau_3)\rangle = \dfrac{C_{ijk}}{|\tau_{12}|^{\Delta_{ijk}}|\tau_{23}|^{\Delta_{jki}}|\tau_{23}|^{\Delta_{kij}}}$

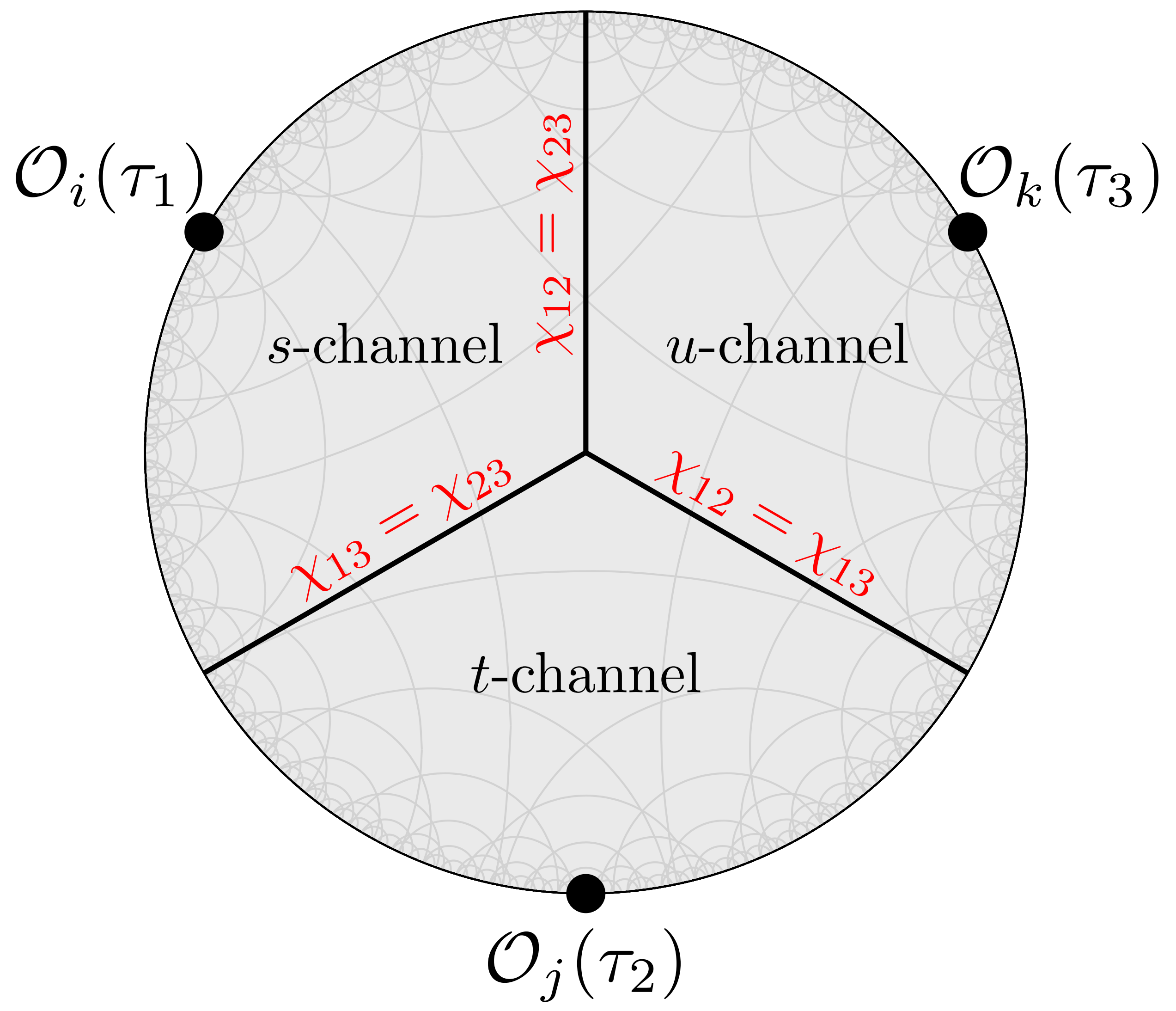
$(\tau_1 < \tau_2 < \tau_3)$

$\dfrac{dC_{ijk}}{d\lambda} =$



$$= \sum_{l,m} b_l^{\hat{\Phi}}\, C_{jkm}\Big(C_{ilm}\mathcal{K}_{\Delta_i,\Delta_j,\Delta_k}(\Delta_l,\Delta_m) + C_{iml}\mathcal{K}_{\Delta_i,\Delta_k,\Delta_j}(\Delta_l,\Delta_m)\Big)$$

$$+ \text{ cyclic permutations of } ijk$$



So we get a set of ODEs

$$\frac{d\Delta_i}{d\lambda} = \sum_l b_l^{\hat{\Phi}} C_{iil} \mathcal{I}(\Delta_l)$$

$$\frac{db_i^{\hat{\Phi}}}{d\lambda} = \sum_{j,l} b_l^{\hat{\Phi}} b_j^{\hat{\Phi}} \frac{C_{lji} + C_{lij}}{2} \mathcal{J}_{\Delta_i}(\Delta_l, \Delta_j)$$

$$\frac{dC_{ijk}}{d\lambda} = \sum_{m,l} \Big(b_l^{\hat{\Phi}} C_{lim} C_{mjk} \mathcal{K}_{\Delta_i \Delta_j \Delta_k}(\Delta_l, \Delta_m) + b_l^{\hat{\Phi}} C_{ilm} C_{mjk} \mathcal{K}_{\Delta_i \Delta_k \Delta_j}(\Delta_l, \Delta_m) \Big)$$

$$+ \quad (ijk) \to (jki) \quad + \quad (ijk) \to (kij)$$

Play with it!

It doesn't work, even in the free theory.

$$S = \int_{\mathrm{AdS}_2} \left[(\partial\phi)^2 + \lambda\phi^2 \right], \quad \text{mass deformation}$$

Play with $$\frac{d\Delta_i}{d\lambda} = \sum_l b_l^{\hat{\Phi}} C_{iil} \mathcal{I}(\Delta_l)$$

$$b_l^{\hat{\Phi}} \sim \Delta_l^{\Delta_{\hat{\Phi}}-3/4}, \quad C_{iil} \sim 2^{-\Delta_l} \Delta_l^{2\Delta_i - 3/4}, \quad \mathcal{I}(\Delta_l) \sim \frac{2^{\Delta_l}}{\Delta_l}, \qquad (\Delta_l \to \infty).$$

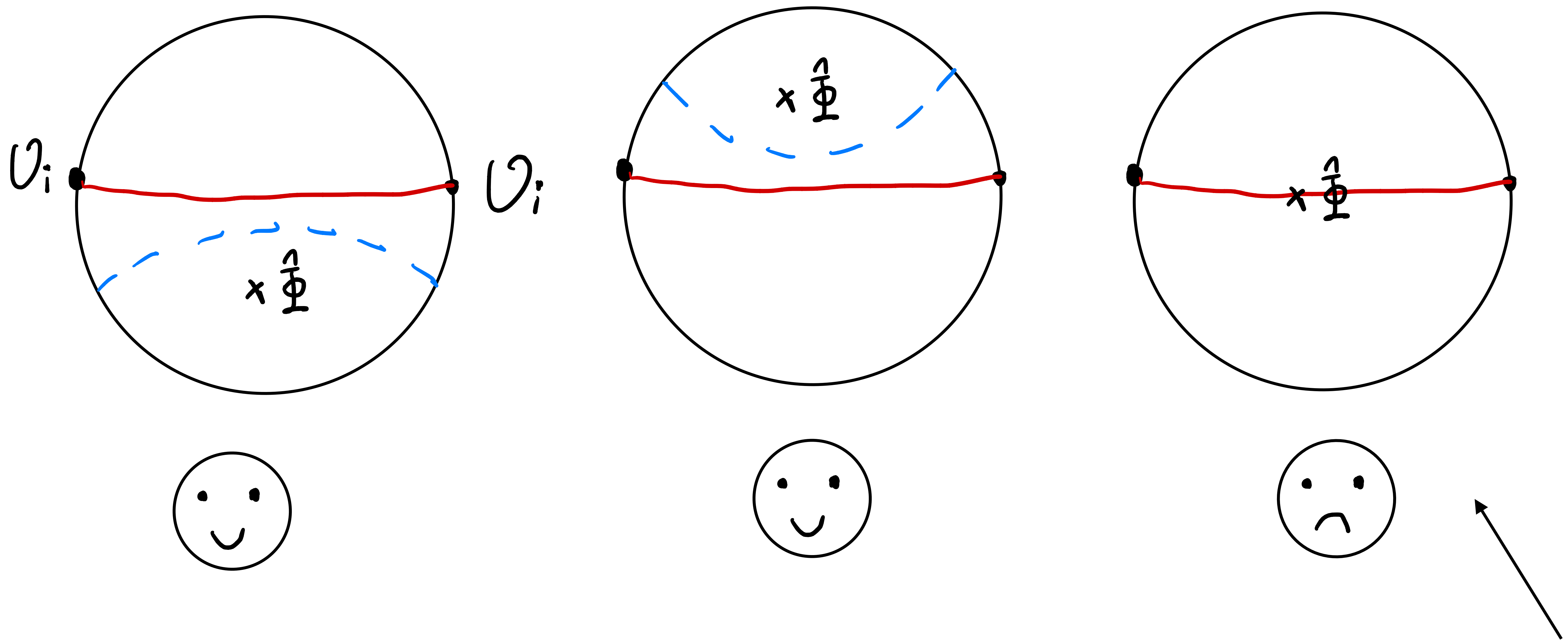
Check absolute convergence

$$\sum_{\Delta_l \leqslant \Delta_{max}} \left| b_l^{\hat{\Phi}} C_{iil} \mathcal{I}(\Delta_l) \right| \sim \int^{\Delta_{max}} d\Delta_l \, \Delta_l^{2\Delta_i + \Delta_{\hat{\Phi}} - 5/2}$$

What's wrong?

OPE convergence

- The OPE for bulk-boundary-boundary correlator does not converge everywhere.



OPE diverges in power law

What's wrong?

OPE convergence

- The OPE for bulk-boundary-boundary correlator does not converge everywhere.

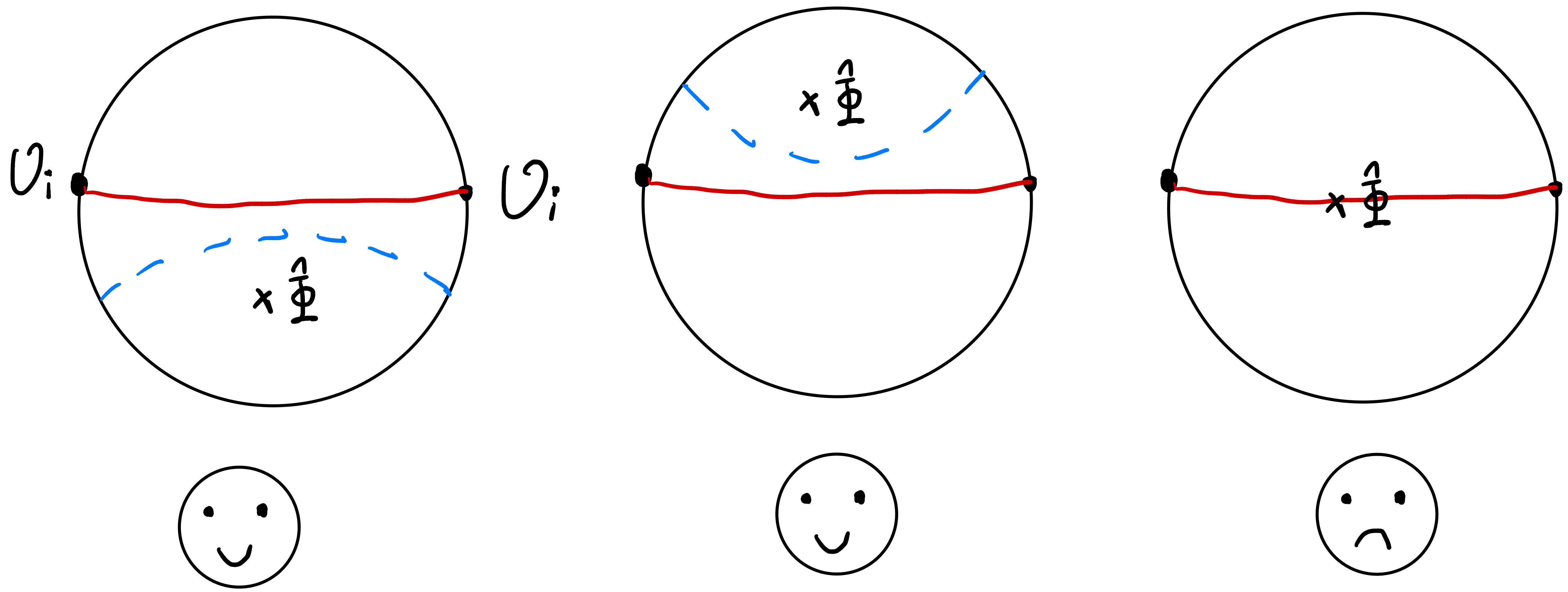
$$\sum_{\Delta_l} \int_{\mathrm{AdS}} \neq \int_{\mathrm{AdS}} \sum_{\Delta_l}$$

- Nevertheless, the full correlator is still finite at the "singular point".
- The problem is just a bad choice of the basis.

What's wrong?

OPE convergence

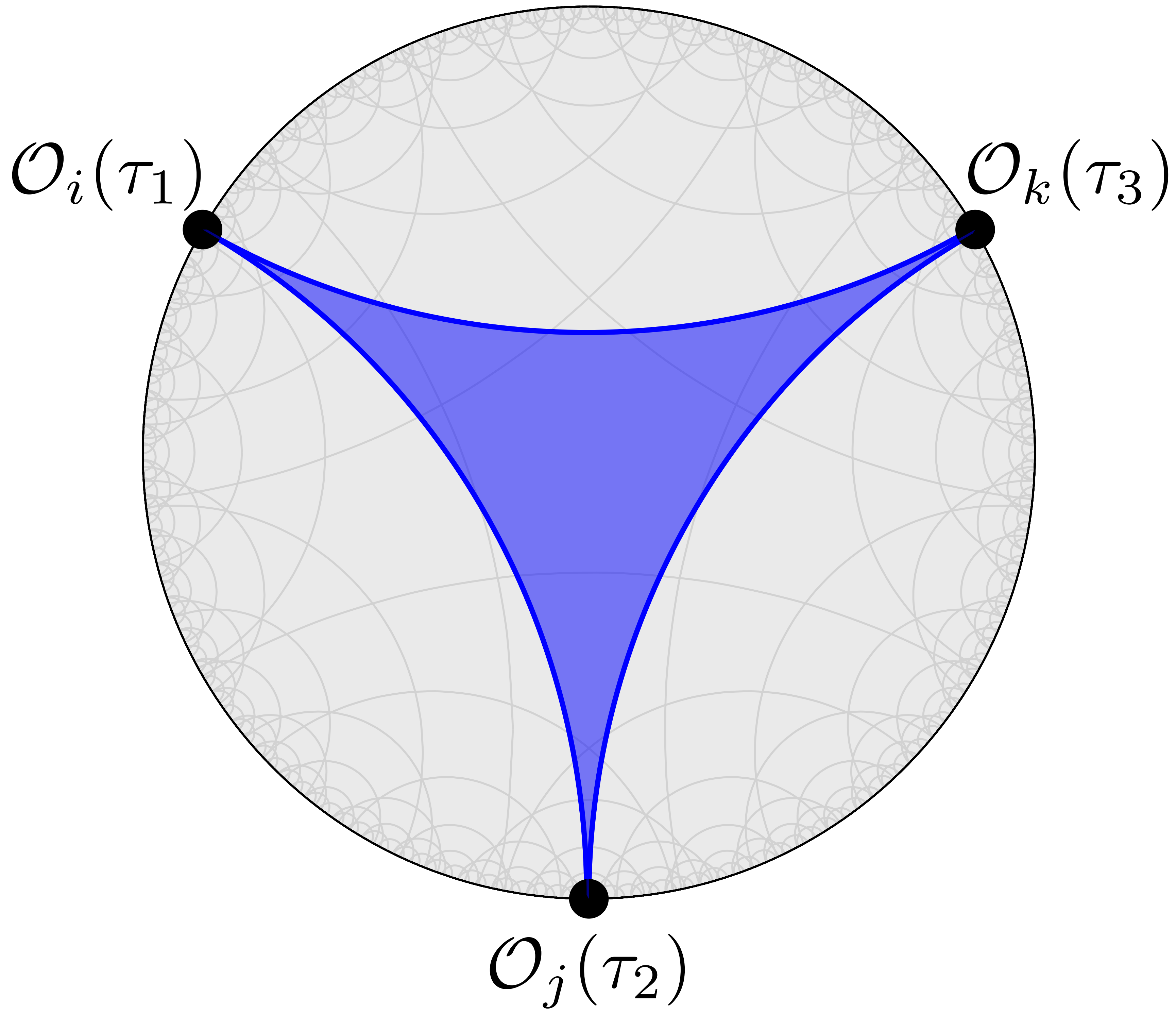
- Maybe eclipse is rare, we can try to improve in some smarter way?



What’s wrong?

OPE convergence

- Maybe eclipse is rare, we can try to improve in some smart way?
- Not true for bulk + 3 boundary



Improve OPE convergence

- We have to find a better basis to expand correlators.
- Conformal block ⟶ local block

[A. Hamilton, D.N. Kabat, G. Lifschytz and D.A. Lowe]

[M. Paulos, N. Levine] [M. Meineri, J. Penedones, T. Spirig]

$$\langle\hat{\Phi}(\tau,z)\mathcal{O}_i(\tau_1)\mathcal{O}_j(\tau_2)\rangle=\sum_l b_l^{\hat{\Phi}}C_{lij}G_{\Delta_l}^{\Delta_i\,\Delta_j}(\tau,z,\tau_1,\tau_2)=\sum_l b_l^{\hat{\Phi}}C_{lij}G_{\Delta_l}^{\Delta_i\,\Delta_j\,(\alpha)}(\tau,z,\tau_1,\tau_2)\,.$$

- Claim: in a generic unitary theory, the new sum converges uniformly when

$$\alpha>\frac{1}{2}(\Delta_i+\Delta_j+\Delta_{\hat{\Phi}})$$

- The final result does not depend on α

Now we get an improved set of ODEs:

$$\frac{d\Delta_i}{d\lambda} = \sum_l b_l^{\hat{\Phi}} C_{iil} \mathcal{I}^{(\alpha_i)}(\Delta_l)$$

$$\frac{db_i^{\hat{\Phi}}}{d\lambda} = \sum_{j,l} b_l^{\hat{\Phi}} b_j^{\hat{\Phi}} \frac{C_{lji} + C_{lij}}{2} \mathcal{J}_{\Delta_i}^{(\alpha_{ij})}(\Delta_l, \Delta_j)$$

$$\frac{dC_{ijk}}{d\lambda} = \sum_{m,l} \left(b_l^{\hat{\Phi}} C_{lim} C_{mjk} \mathcal{K}_{\Delta_i \Delta_j \Delta_k}^{(\alpha_{im})}(\Delta_l, \Delta_m) + b_l^{\hat{\Phi}} C_{ilm} C_{mjk} \mathcal{K}_{\Delta_i \Delta_k \Delta_j}^{(\beta_{im})}(\Delta_l, \Delta_m) \right)$$

$$+ \quad (ijk) \to (jki) \quad + \quad (ijk) \to (kij)$$

Play with it!

Play with

$$\frac{d\Delta_i}{d\lambda} = \sum_l b_l^{\hat{\Phi}} C_{iil} \mathcal{I}^{(\alpha_i)}(\Delta_l)$$

$$\mathcal{I}(\Delta_l) \quad \longrightarrow \quad \mathcal{I}^{(\alpha_i)}(\Delta_l) \sim \frac{2^{\Delta_l}}{\Delta_l} \Delta_l^{3/2-2\alpha_i} \sin(\alpha_i - \Delta_l/2)$$

Check absolute convergence

$$\sum_{\Delta_l \leqslant \Delta_{max}} \left| b_l^{\hat{\Phi}} C_{iil} \mathcal{I}^{(\alpha_i)}(\Delta_l) \right| \sim \int^{\Delta_{max}} d\Delta_l \, \Delta_l^{2\Delta_i + \Delta_{\hat{\Phi}} - 2\alpha_i - 1}$$

↕ same

Recall the condition for uniform convergence: $\alpha_i > \Delta_i + \frac{1}{2}\Delta_{\hat{\Phi}}$

- ~~We~~ My collaborators also checked many free-theory examples, also for the other two equations.
- Standard conformal block expansion ✗
- Local block expansion ✓

- Strongly recommend: talk by Manuel Loparco at Bootstrap 2025 (on Youtube)

Level repulsion

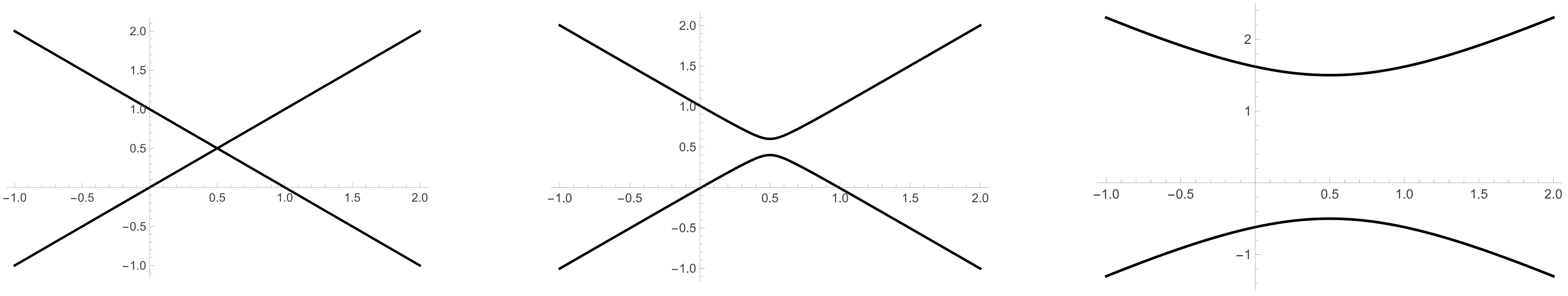
- A continuous family of theories (Hamiltonians)
- Unitarity + no symmetry $\xrightarrow{\text{generic}}$ no level crossing

  [von Neumann, Wigner]

- The simplest toy example is

$$M(t,\epsilon) = \begin{pmatrix} t & \epsilon \\ \epsilon & 1-t \end{pmatrix}$$

$$\Rightarrow \quad \lambda_+ - \lambda_- = \sqrt{(1-2t)^2 + 4\epsilon^2}$$



$\epsilon = 0$ $\epsilon = 0.1$ $\epsilon = 1$

$\mathbb{Z}_2$ symmetry

$$[\sigma_3, M(t, \epsilon = 0)] = 0$$

Back to our ODEs:

$$\frac{d\Delta_i}{d\lambda} = \sum_l b_l^{\hat{\Phi}} C_{iil} \mathcal{I}^{(\alpha_i)}(\Delta_l)$$

$$\frac{db_i^{\hat{\Phi}}}{d\lambda} = \sum_{j,l} b_l^{\hat{\Phi}} b_j^{\hat{\Phi}} \frac{C_{lji} + C_{lij}}{2} \mathcal{J}_{\Delta_i}^{(\alpha_{ij})}(\Delta_l, \Delta_j)$$

$$\frac{dC_{ijk}}{d\lambda} = \sum_{m,l} \left(b_l^{\hat{\Phi}} C_{lim} C_{mjk} \mathcal{K}_{\Delta_i \Delta_j \Delta_k}^{(\alpha_{im})}(\Delta_l, \Delta_m) + b_l^{\hat{\Phi}} C_{ilm} C_{mjk} \mathcal{K}_{\Delta_i \Delta_k \Delta_j}^{(\alpha_{im})}(\Delta_l, \Delta_m) \right)$$

$$+ \quad (ijk) \to (jki) \quad + \quad (ijk) \to (kij)$$

Back to our ODEs:

$$\frac{d\Delta_i}{d\lambda} = \sum_l b_l^{\hat{\Phi}} C_{iil} \mathcal{I}^{(\alpha_i)}(\Delta_l)$$

$$\sim \frac{\mathcal{I}^{(\alpha_{ij})}(\Delta_l)}{\Delta_i - \Delta_j} \quad \text{as } \Delta_i \to \Delta_j$$

$$\frac{db_i^{\hat{\Phi}}}{d\lambda} = \sum_{j,l} b_l^{\hat{\Phi}} b_j^{\hat{\Phi}} \frac{C_{lji} + C_{lij}}{2} \mathcal{J}_{\Delta_i}^{(\alpha_{ij})}(\Delta_l, \Delta_j)$$

$$\frac{dC_{ijk}}{d\lambda} = \sum_{m,l} \Big(b_l^{\hat{\Phi}} C_{lim} C_{mjk} \mathcal{K}_{\Delta_i \Delta_j \Delta_k}^{(\alpha_{im})}(\Delta_l, \Delta_m) + b_l^{\hat{\Phi}} C_{ilm} C_{mjk} \mathcal{K}_{\Delta_i \Delta_k \Delta_j}^{(\alpha_{im})}(\Delta_l, \Delta_m) \Big)$$

$$+ \quad (ijk) \to (jki) \quad + \quad (ijk) \to (kij)$$

Back to our ODEs:

$$\frac{d\Delta_i}{d\lambda} = \sum_l b_l^{\hat{\Phi}} C_{iil} \mathcal{I}^{(\alpha_i)}(\Delta_l)$$

$$\sim \frac{\mathcal{I}^{(\alpha_{ij})}(\Delta_l)}{\Delta_i - \Delta_j} \quad \text{as } \Delta_i \to \Delta_j$$

$$\frac{db_i^{\hat{\Phi}}}{d\lambda} = \sum_{j,l} b_l^{\hat{\Phi}} b_j^{\hat{\Phi}} \frac{C_{lji} + C_{lij}}{2} \mathcal{J}_{\Delta_i}^{(\alpha_{ij})}(\Delta_l, \Delta_j)$$

$$\frac{dC_{ijk}}{d\lambda} = \sum_{m,l} \left(b_l^{\hat{\Phi}} C_{lim} C_{mjk} \mathcal{K}^{(\alpha_{im})}_{\Delta_i \Delta_j \Delta_k}(\Delta_l, \Delta_m) + b_l^{\hat{\Phi}} C_{ilm} C_{mjk} \mathcal{K}^{(\alpha_{im})}_{\Delta_i \Delta_k \Delta_j}(\Delta_l, \Delta_m) \right)$$

$$+ \quad (ijk) \to (jki) \quad + \quad (ijk) \to (kij)$$

$$\sim \frac{1}{2} \frac{\mathcal{I}^{(\alpha_{im})}(\Delta_l)}{\Delta_i - \Delta_m} \quad as\ \Delta_i \to \Delta_m$$

Level repulsion from ODEs

Suppose Δ_1 and Δ_2 are very close.

$$\frac{d^2\Delta_1}{d\lambda^2} \approx \frac{\Omega_{12}^2 + \Omega_{12}(b_1^{\hat{\Phi}} C_{211} + b_2^{\hat{\Phi}} C_{111})(\mathcal{I}^{(\alpha)}(\Delta_1) - \mathcal{I}^{(\alpha)}(\Delta_2))}{2\Delta_{12}}$$

$$\text{where} \quad \Omega_{12} := \sum_l b_l^{\hat{\Phi}} (C_{12l} + C_{21l}) \mathcal{I}^{(\alpha)}(\Delta_l)$$

Level repulsion from ODEs

Suppose Δ_1 and Δ_2 are very close.

$$\frac{d\Omega_{12}}{d\lambda} \approx \frac{\Omega_{12} \sum_l b_l^{\hat{\Phi}} (C_{22l} - C_{11l}) \mathcal{I}^{(\alpha)}(\Delta_l)}{\Delta_{12}} = -\frac{\Omega_{12}}{\Delta_{12}} \frac{d\Delta_{12}}{d\lambda}$$

$$\Rightarrow \quad \Omega_{12} \sim \frac{c}{\Delta_{12}}$$

Level repulsion from ODEs

Suppose Δ_1 and Δ_2 are very close.

$$\frac{d^2\Delta_1}{d\lambda^2} \approx \frac{\Omega_{12}^2 + \Omega_{12}(b_1^{\hat{\Phi}} C_{211} + b_2^{\hat{\Phi}} C_{111})(\mathcal{I}^{(\alpha)}(\Delta_1) - \mathcal{I}^{(\alpha)}(\Delta_2))}{2\Delta_{12}}$$

$$\text{where} \quad \Omega_{12} := \sum_l b_l^{\hat{\Phi}} (C_{12l} + C_{21l}) \mathcal{I}^{(\alpha)}(\Delta_l) \qquad \Omega_{12} \sim \frac{c}{\Delta_{12}}$$

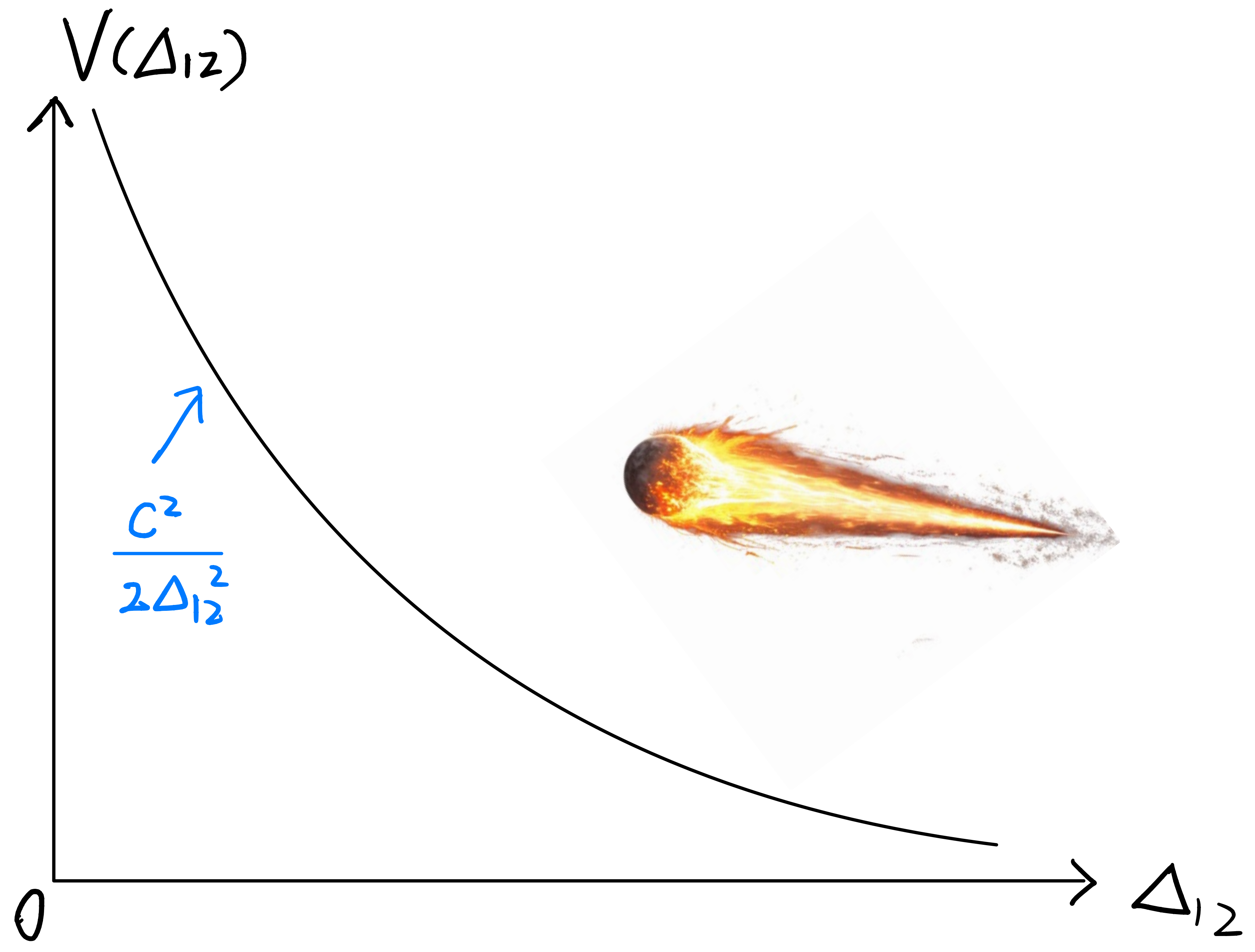
$$\frac{d^2\Delta_{12}}{d\lambda^2} = \frac{c^2}{\Delta_{12}^3} + O\left(\frac{1}{\Delta_{12}^2}\right)$$

Level repulsion from ODEs

Suppose Δ_1 and Δ_2 are very close.

$$\frac{d^2\Delta_{12}}{d\lambda^2} = \frac{c^2}{\Delta_{12}^3} + O\left(\frac{1}{\Delta_{12}^2}\right)$$

Similar result in 1D CFT [Behan]



where $\Omega_{12} := \sum_l b_l^{\hat{\Phi}}(C_{12l} + C_{21l})\mathcal{I}^{(\alpha)}(\Delta_l)$ $\Omega_{12} \sim \frac{c}{\Delta_{12}}$

Summary

- QFT in AdS “=” $\Delta_i,\ C_{ijk},\ b_i^{\hat{\Phi}}$
- Its relevant deformation is described by ODEs.
- Implicitly assumed unitarity + some very mild condition.
- No assumption of parity symmetry.
- Nonperturbative.
- Mechanism of level repulsion.
- No free lunch: need CFT data as initial conditions.

What I didn’t discussed…

- Analyticity and upper bound of the AdS correlators
- Many numerical checks in free theories
- (Integrated) local blocks
- New locality sum rules
- Bulk UV divergence
- …

Outlooks

- Self-consistency (crossing, bulk locality, unitarity, etc…)
- Flat-space limit
- Gradient flow? Mononicity?
- Numerics
- …

Thank you!