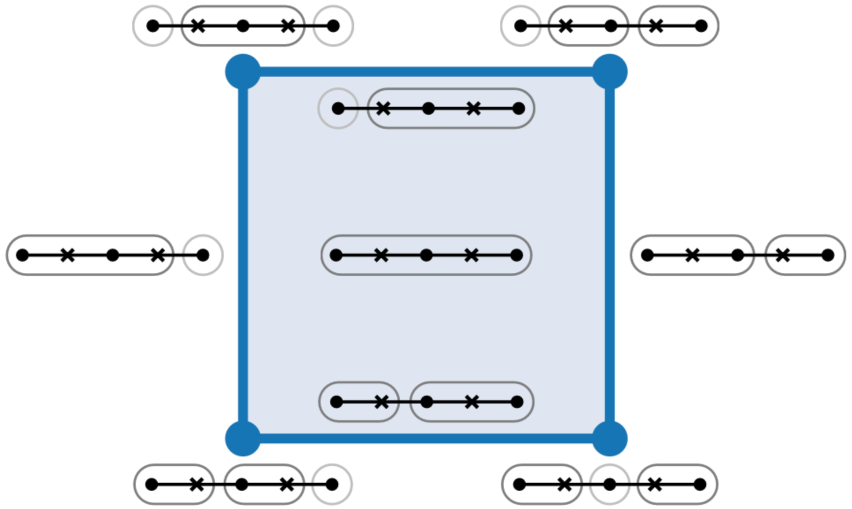
A Hidden Pattern in Cosmological Correlators

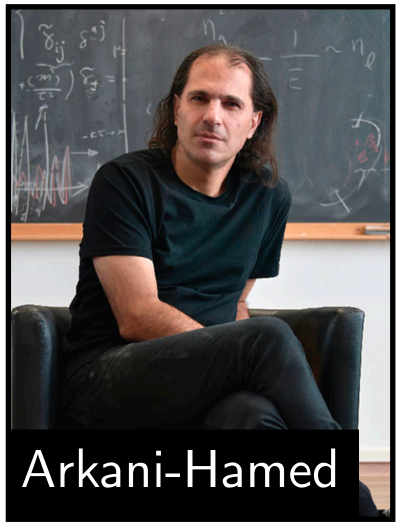


Hayden Lee

University of Pennsylvania

[Progress of Theoretical Bootstrap, YITP, 18 November 2025]

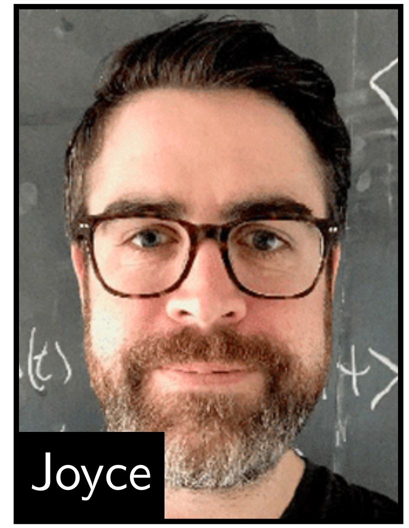
This is based on work with:



IAS

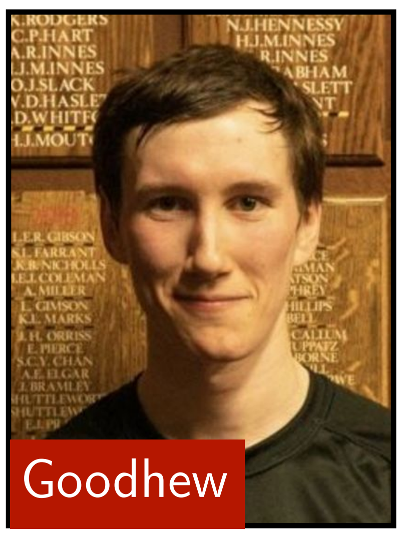
Amsterdam

Caltech



Chicago

SNS



NTU

SNS

Amsterdam

IGFAE

Massive Kinematic Flow [work in progress]

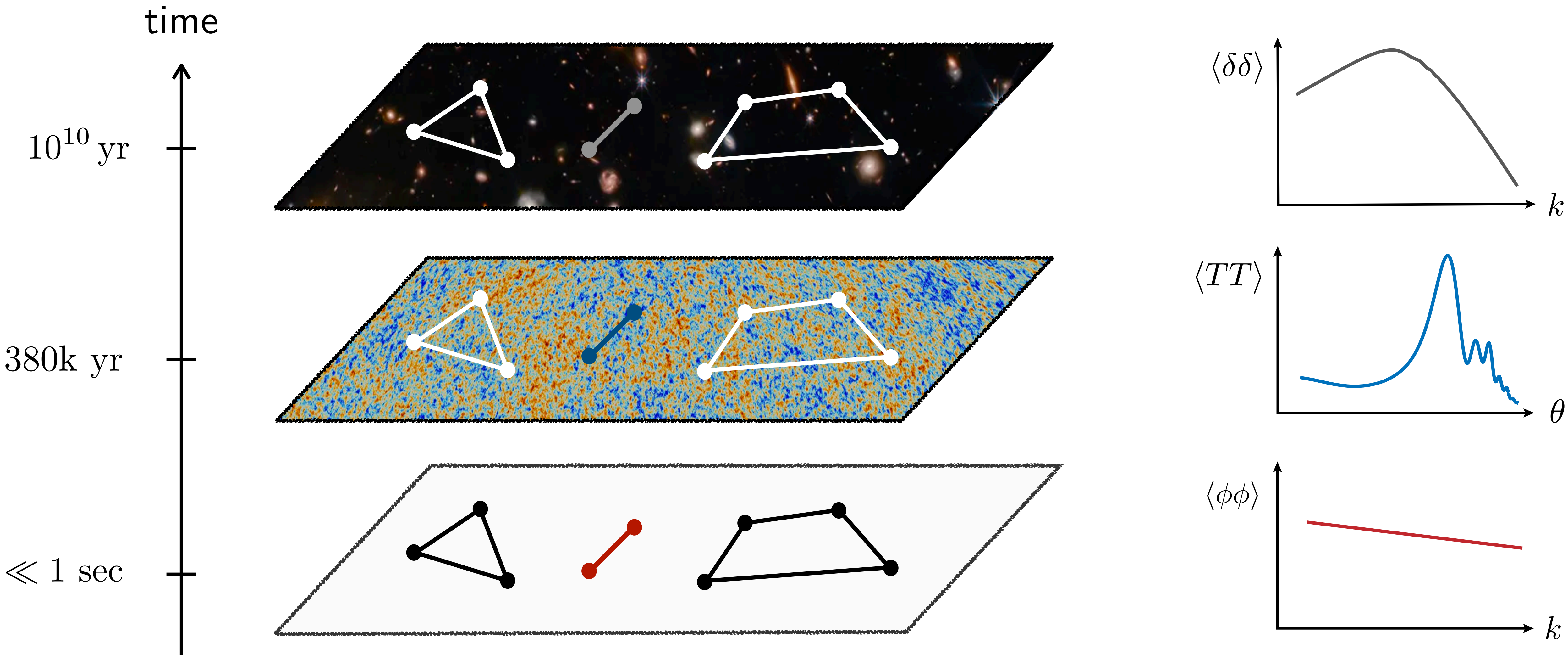
Geometry of Kinematic Flow [2504.14890]

Kinematic flow for cosmological loop integrands [2410.17994]

Kinematic Flow and the Emergence of Time [2312.05300]

Differential Equations for Cosmological Correlators [2312.05303]

Cosmological Correlations

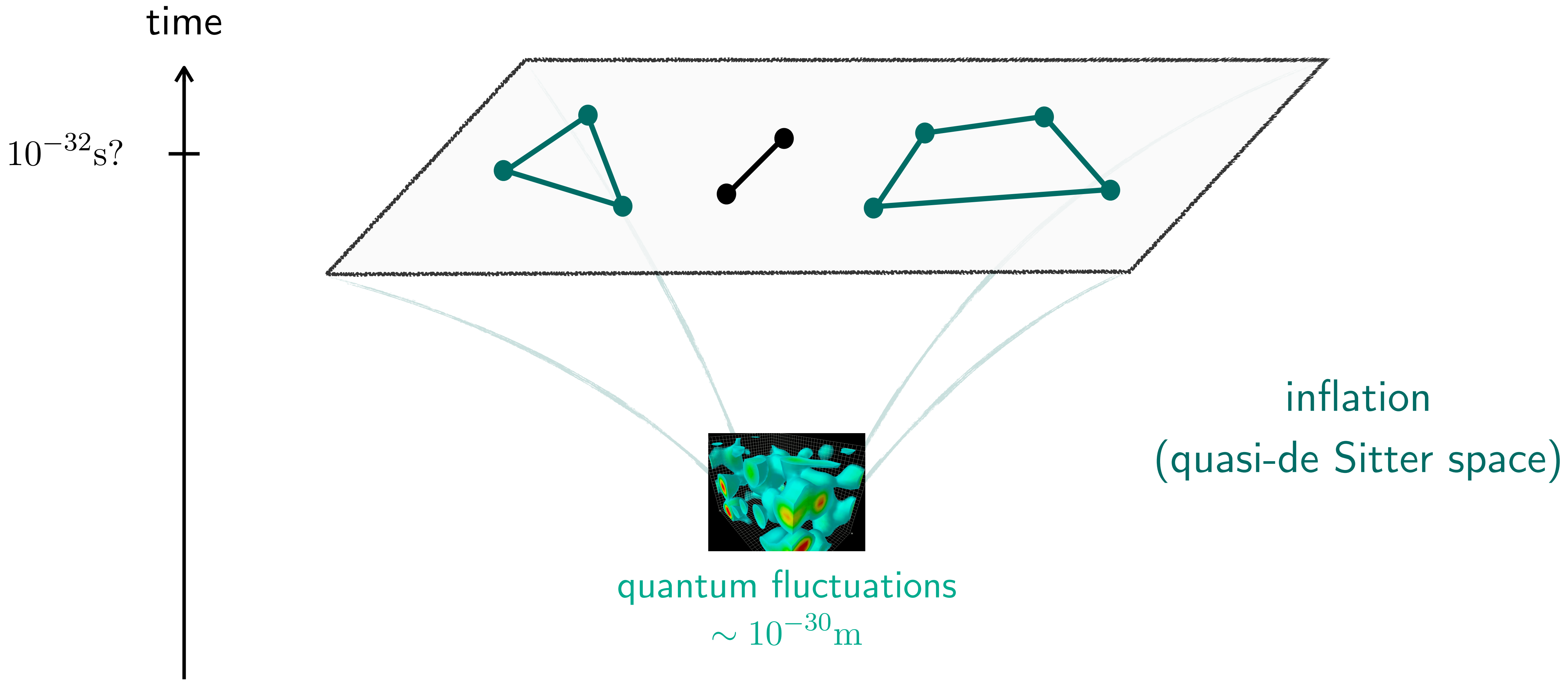


By measuring cosmological correlations, we learn both about the

evolution of the Universe and its **initial conditions.**

[See also G. Pimentel's talk]

Inflationary Correlations

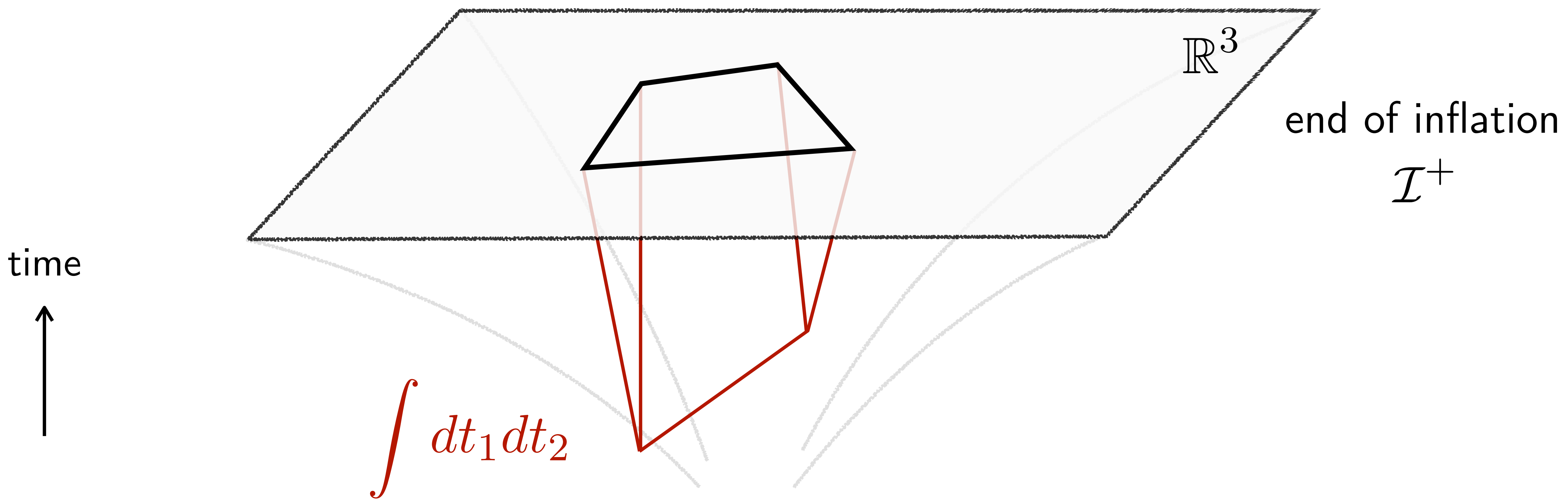


During inflation, correlators arise from quantum fluctuations.

Dynamical information is encoded in **higher-point correlations**.

Direct Calculation

In the standard approach, correlators are computed with **time integrals**.

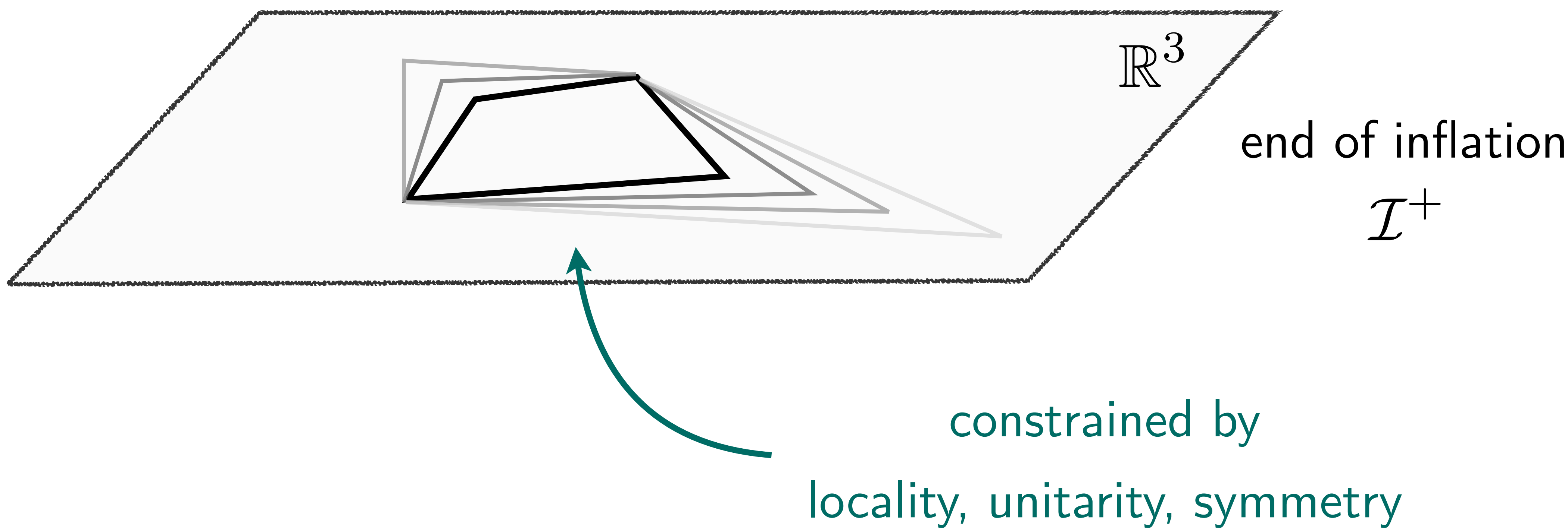


While the relevant time integrals are hard to compute,

the **final answers** are often remarkably simple.

Bootstrap Approach

Goal: to directly **bootstrap** correlators on the spatial boundary.

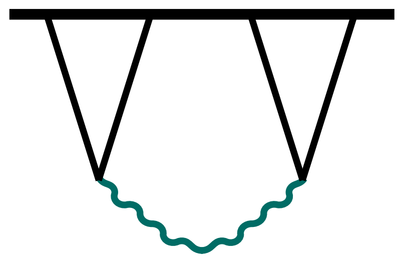


practical advantage: **simplifies** computations

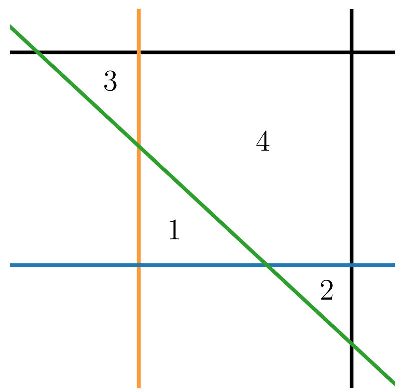
conceptual advantage: reveals **hidden structures**

Outline

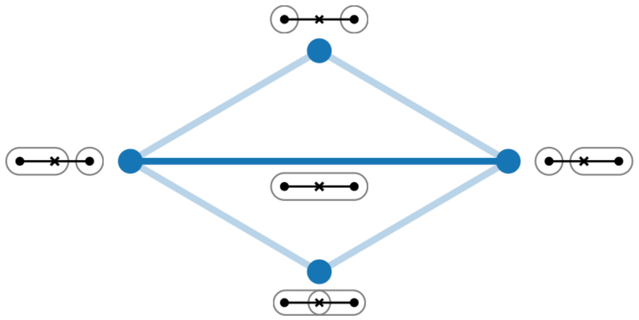
1. Correlators in dS



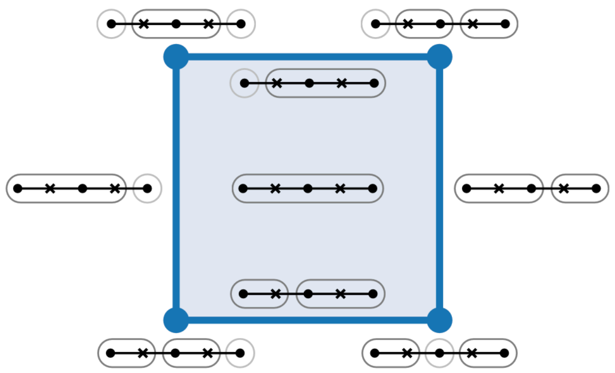
2. Correlators in Toy Universe



4. A Hidden Pattern in dS

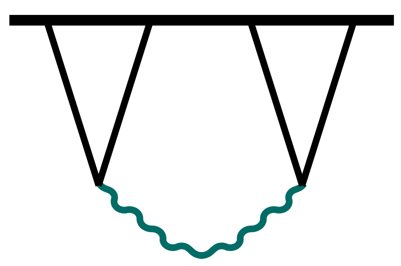


3. A Hidden Pattern in Toy Universe



Outline

1. Correlators in dS



2. Correlators in Toy Universe

4. A Hidden Pattern in dS

3. A Hidden Pattern in Toy Universe

Symmetries in dS

Boundary correlators in dS are constrained by the conformal group SO(4,1).

$$\langle \phi_{\vec{k}_1}\phi_{\vec{k}_2}\phi_{\vec{k}_3}\phi_{\vec{k}_4}\rangle = \psi(\vec{k}_1, \cdots, \vec{k}_4)\delta^{(3)}(\vec{k}_1 + \cdots + \vec{k}_4)$$

- 3 translations
- 3 rotations

} momentum conservation in Fourier space

ψ is a function of "energy" variables $k_i = |\vec{k}_i|$

- 1 dilatation
- 3 SCTs

} $D = \vec{k}\cdot\vec{\partial}_k + 3 - \Delta$

$\vec{K} = \vec{k}(\vec{\partial}_k \cdot \vec{\partial}_k) - 2(\vec{k}\cdot\vec{\partial}_k + 3 - \Delta)\vec{\partial}_k$

$[m^2 = \Delta(3-\Delta)H^2]$

Conformal Symmetry

For conformally coupled scalars, the **conformal Ward identity** takes the form

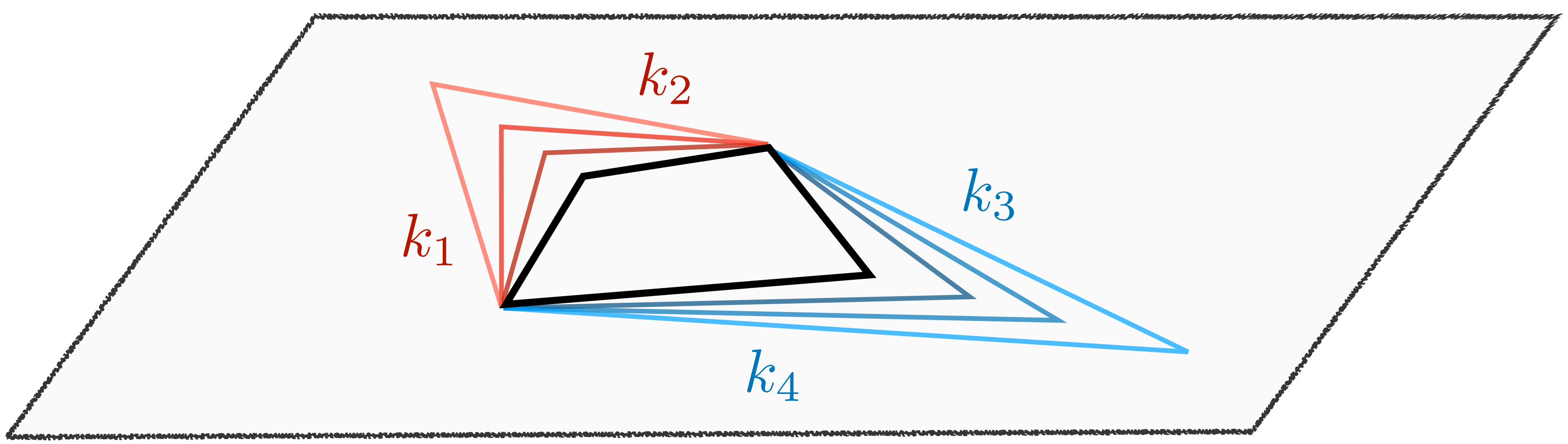
$$\boxed{(\Delta_{X_1} - \Delta_{X_2})\psi = 0}$$

conformal Casimir

$$\Delta_X = (X^2 - 1)\partial_X^2 - 2X\partial_X$$

$$X_1 \equiv k_1 + k_2 \qquad X_2 \equiv k_3 + k_4$$

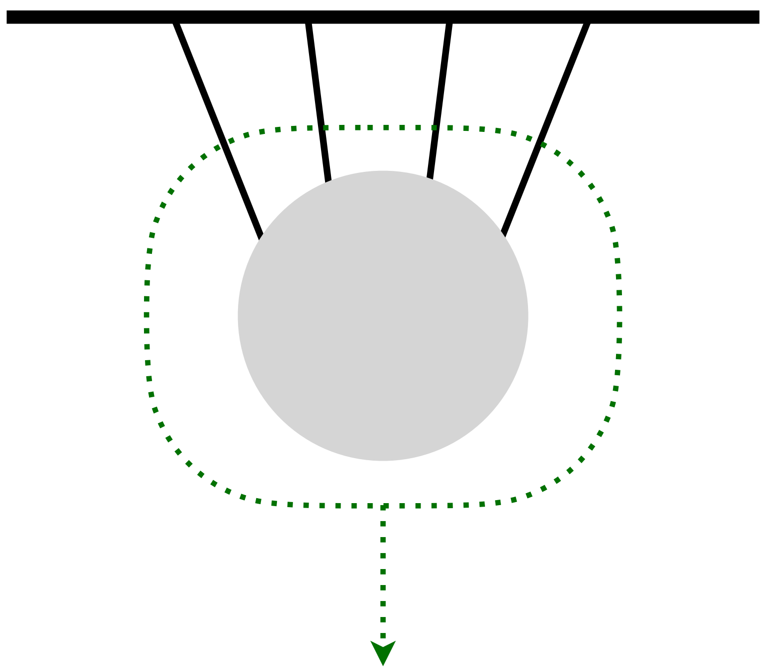
This determines the strength of the correlation as its shape is deformed.



Singularity

Cosmological correlators have a singularity when the **total energy** vanishes.

$\lim_{E \to 0}$



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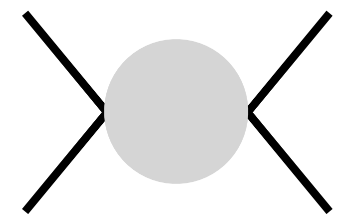


$\overline{}$

$E^{\#}$

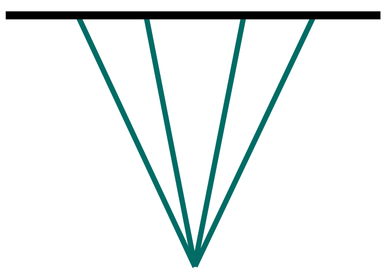
$$(E = X_1 + X_2 = \textstyle\sum_a k_a)$$

Amplitudes thus live in a kinematic subspace of cosmological correlators.

Tree-Level Correlators

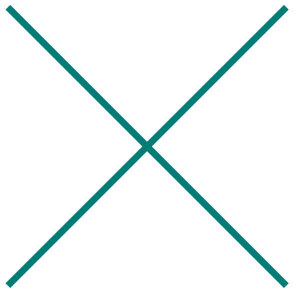
Contact diagrams have the simplest analytic structure.

$$(\Delta_{X_1} - \Delta_{X_2})\frac{1}{X_1 + X_2} = 0 \qquad \Rightarrow$$



$$= \sum_n c_n \Delta_{X_1}^n \left(\frac{1}{X_1 + X_2}\right)$$

This is the analog of



$$= \sum_n c_n s^n$$

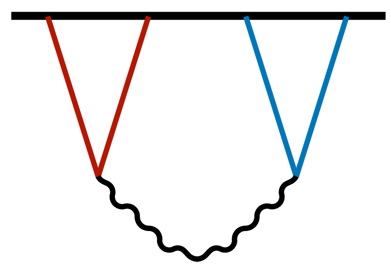
Arkani-Hamed, Maldacena [2015]
Arkani-Hamed, Baumann, HL, Pimentel [2018]

Tree-Level Correlators

Exchange diagrams have contact diagrams as the source term.

$$\lim_{\mu\to\infty}\psi_{\rm ex}=\frac{1}{\mu^2}\psi_{\rm con}\qquad\Rightarrow$$

$$(\Delta_{X_1}-\mu^2)$$
$$(\Delta_{X_2}-\mu^2)$$



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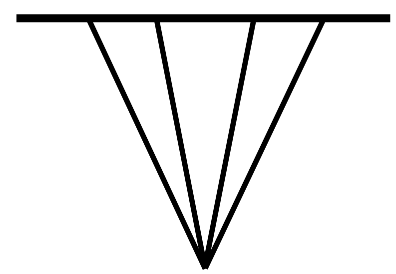
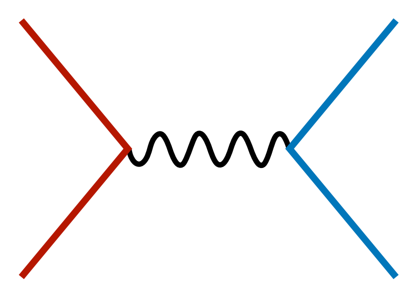


$[\mu^2=m^2/H^2]$

This is the analog of

$$(s_{12}-M^2)$$
$$(s_{34}-M^2)$$



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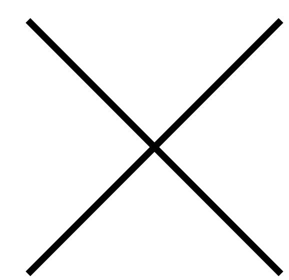
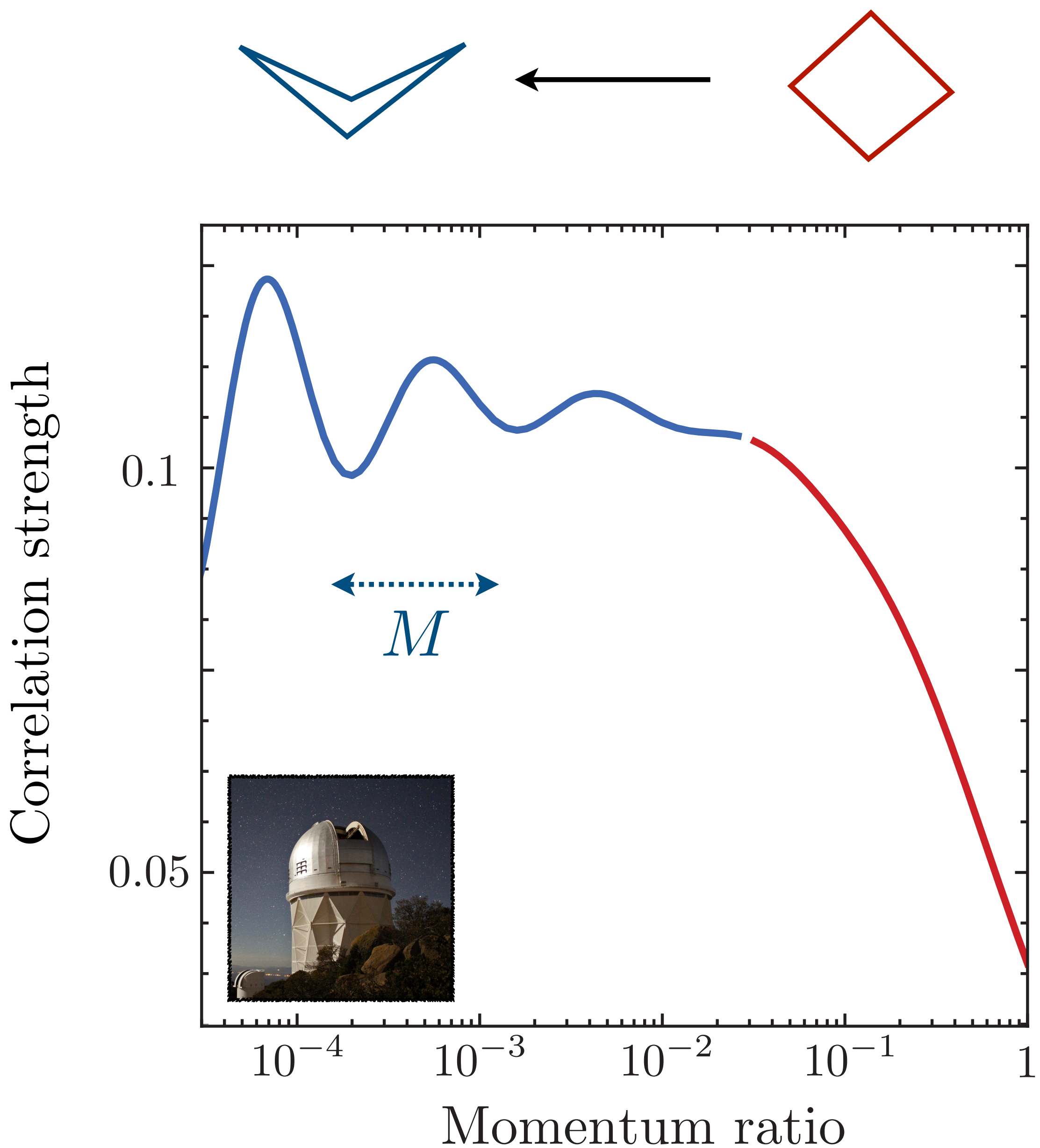
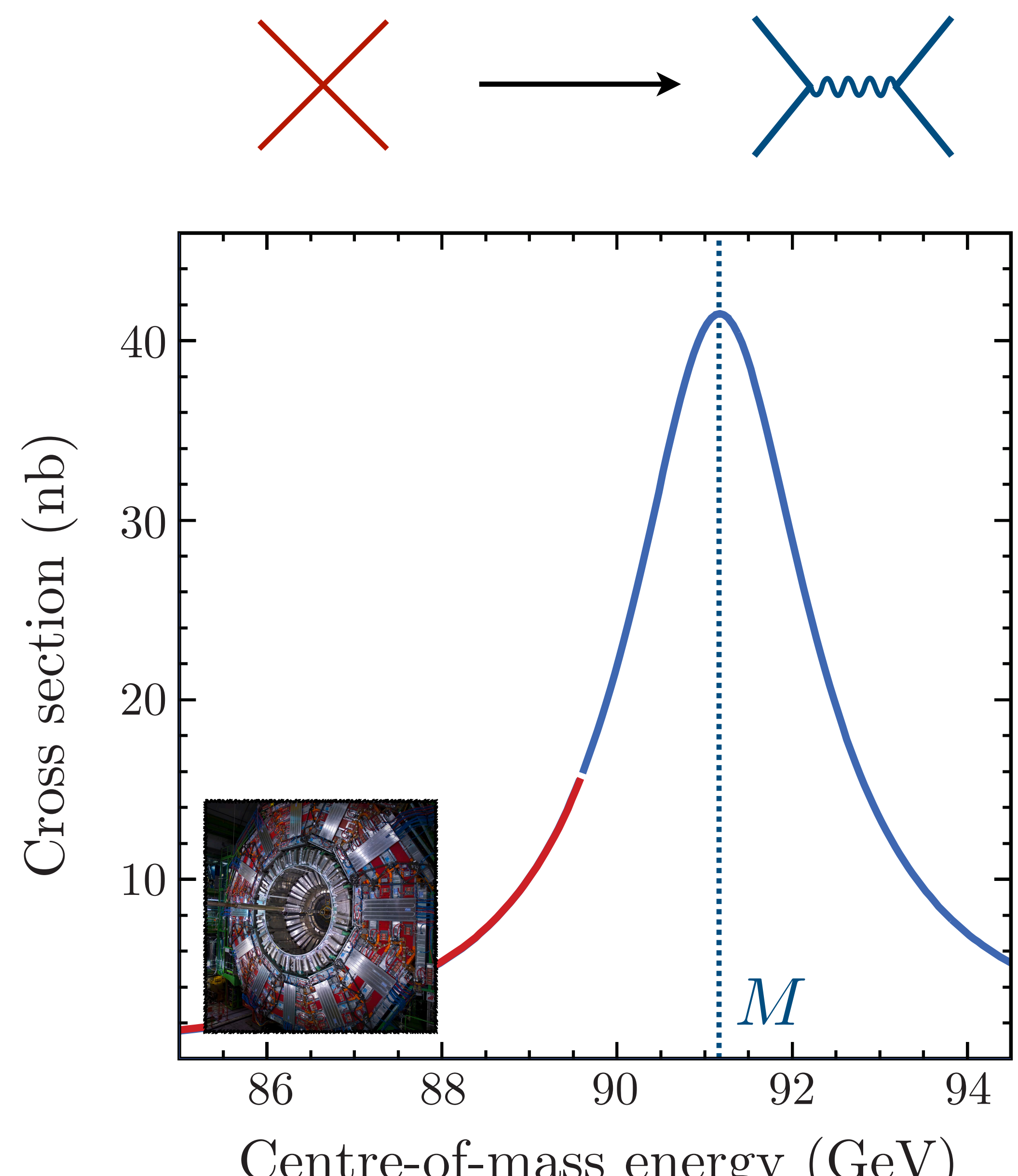


Arkani-Hamed, Maldacena [2015]
Arkani-Hamed, Baumann, HL, Pimentel [2018]

Cosmological Collider Physics

The solution captures the effect of **particle production** during inflation.



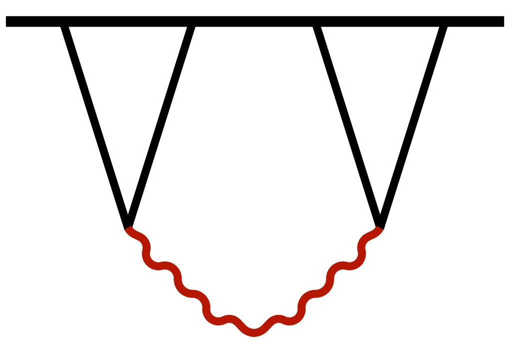


[See also Z. Xianyu's and S. Aoki's talks]

Boundary Equations Beyond dS?

In dS, **conformal symmetry** leads to **differential equations** satisfied by boundary correlators.

$$(\Delta_{X_1} - \mu^2)$$



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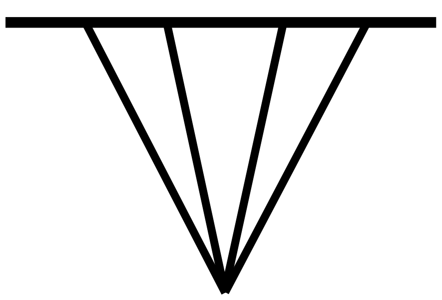
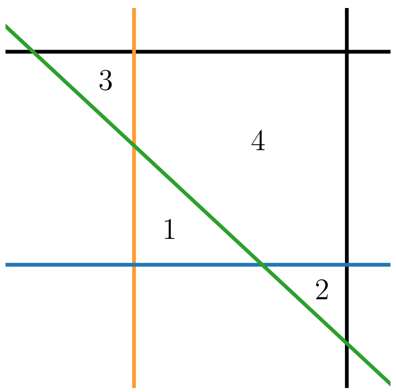


conformal Casimir

Are there similar boundary equations **beyond exact dS**?

1. Correlators in dS

2. Correlators in Toy Universe



4. A Hidden Pattern in dS

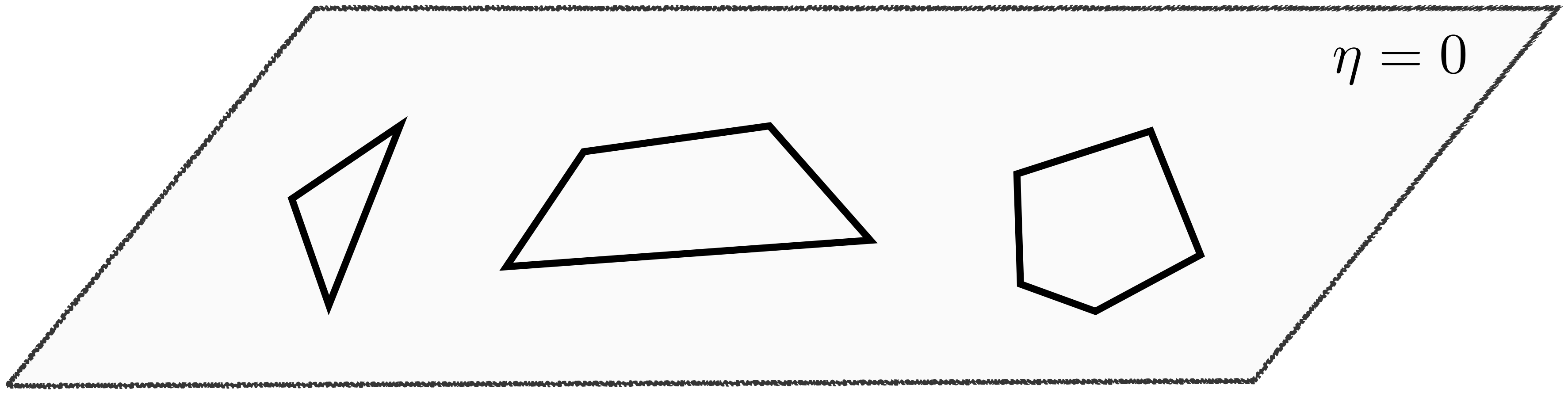
3. A Hidden Pattern in Toy Universe

Power-Law Cosmology

Consider an FRW spacetime expanding as a **power law**:

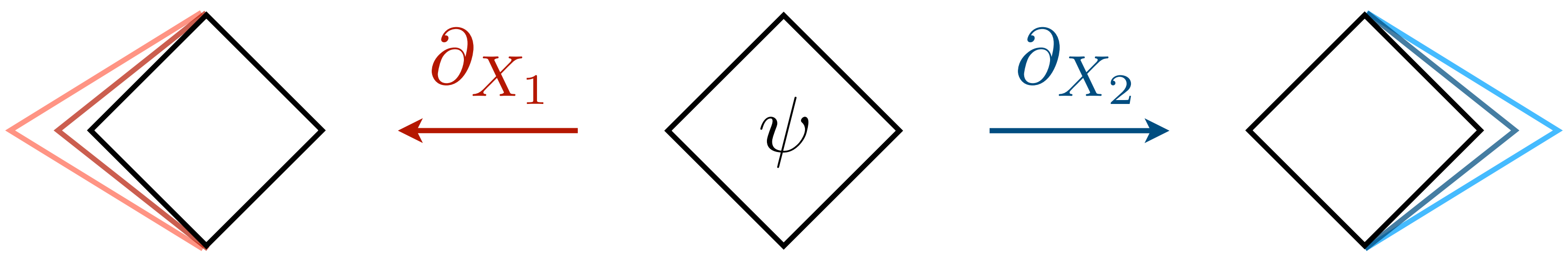
$$\mathrm{d}s^2 = a^2(\eta)[-\mathrm{d}\eta^2 + \mathrm{d}\vec{x}^2] \qquad a(\eta) \propto \frac{1}{\eta^{1+\varepsilon}} \quad \begin{cases} \varepsilon = 0: & \text{dS} \\ \varepsilon = -1: & \text{flat} \\ \varepsilon = -2: & \text{radiation} \\ \varepsilon = -3: & \text{matter} \end{cases}$$

We will study general n-point functions in this spacetime.

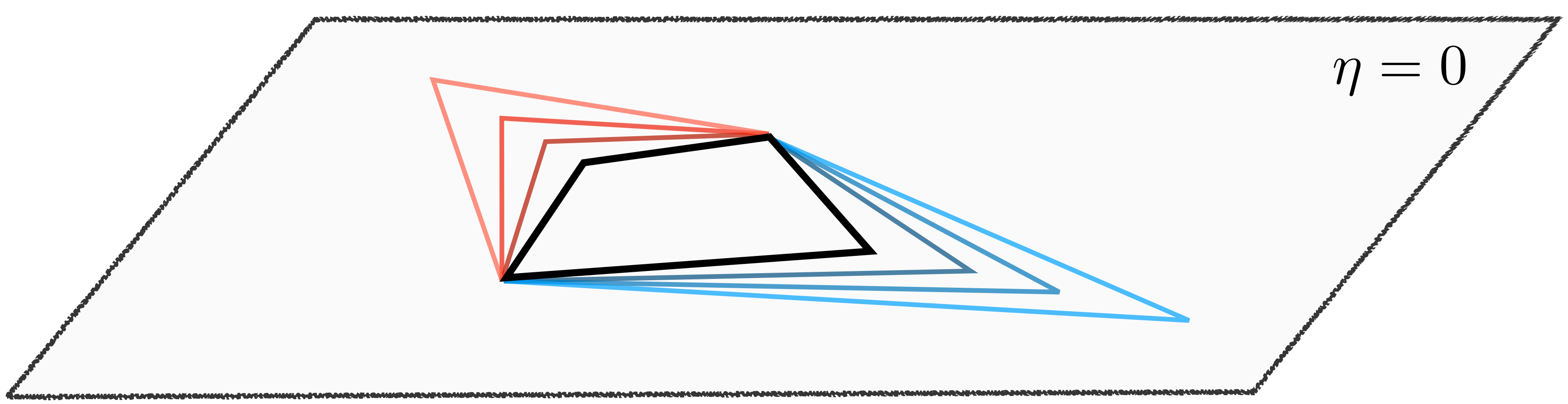


Differential Equations

The goal is to derive the differential equations in kinematic space.



This determines the strength of the correlation as its shape is deformed.



Toy Model

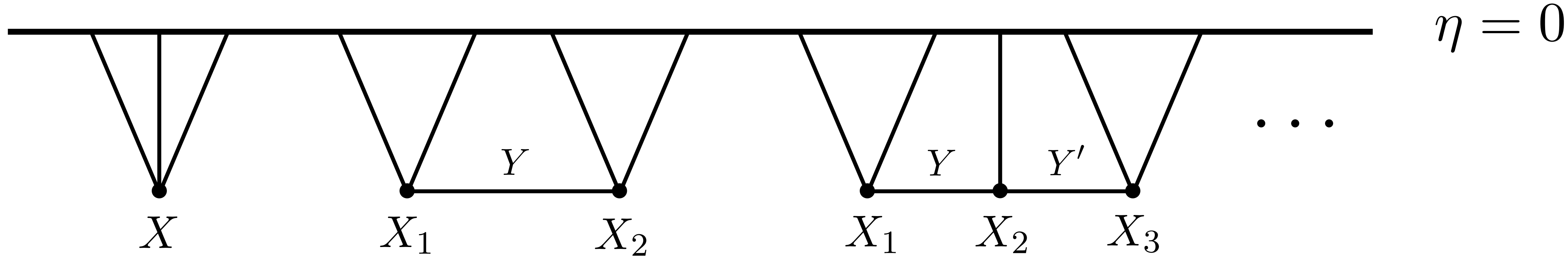
Consider a conformally coupled scalar with polynomial interactions:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{12}R\phi^2 - \frac{\lambda}{3!}\phi^3\right] \qquad a(\eta) \propto \frac{1}{\eta^{1+\varepsilon}}$$

conformal mass

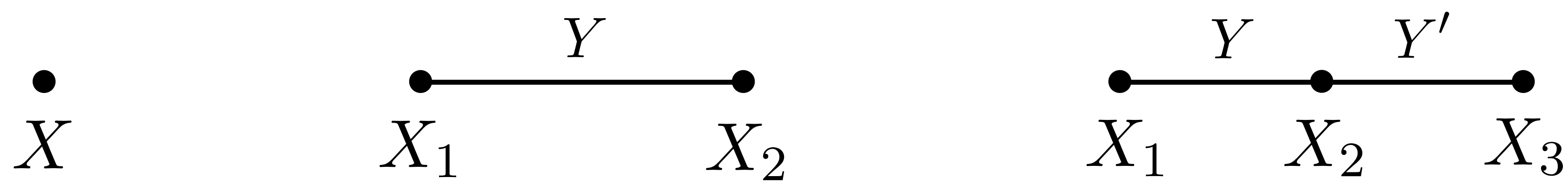
non-conformal interaction

In perturbation theory, correlators can be computed using Feynman rules.

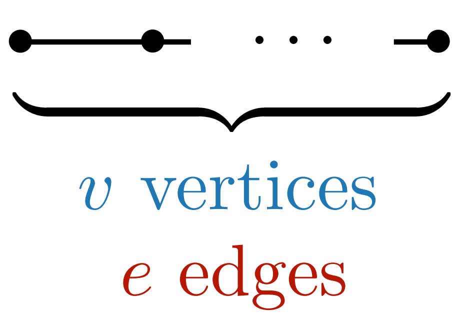


Graphs

Correlators can be represented by graphs with truncated external lines.



In the standard approach, these are computed from time integrals:



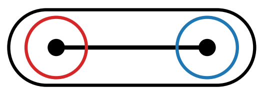
$$= \int_{-\infty}^{0} \prod_v \mathrm{d}\eta_v \, e^{iX_v\eta_v} \prod_e G_{Y_e}(\eta_{v_e}, \eta_{v'_e})$$

$$G_Y(\eta, \eta') = \frac{1}{2Y}\left[e^{-iY(\eta-\eta')}\theta(\eta-\eta') + \eta \leftrightarrow \eta' - e^{iY(\eta+\eta')}\right]$$

Correlators in Flat Space

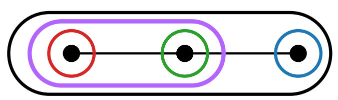
In flat space, results of time integrals can be reproduced from **graph tubings**:

$$\psi^{(2)}_{\text{flat}} =$$

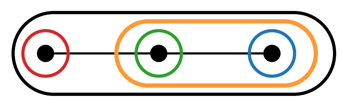


$$= \frac{1}{(X_1+X_2){\color{red}(X_1+Y)}{\color{blue}(X_2+Y)}}$$

$$\psi^{(3)}_{\text{flat}} =$$



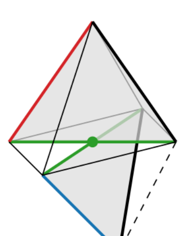
$$+$$

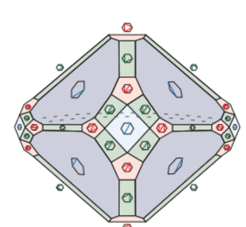


$$= \frac{1}{(X_1+X_2+X_3){\color{red}(X_1+Y)}{\color{green}(X_2+Y+Y')}{\color{blue}(X_3+Y')}}\left(\frac{1}{{\color{purple}X_1+X_2+Y'}}+\frac{1}{{\color{orange}X_2+X_3+Y}}\right)$$

This originates from interesting **combinatorial** & **geometrical** structures

("cosmological polytopes", "cosmohedra").



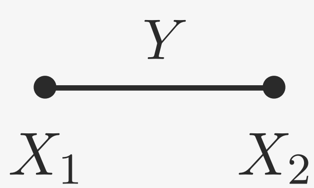


Arkani-Hamed, Benincasa, Postnikov [2017]
Arkani-Hamed, Figueiredo, Vazao [2024]

Correlators in FRW

In **FRW**, the correlators are given by integrals of the flat-space results.

depends on cosmology



$$= \int_0^\infty \mathrm{d}x_1 \mathrm{d}x_2 \frac{(x_1 x_2)^{\varepsilon}}{(X_1 + x_1 + Y)(X_2 + x_2 + Y)(X_1 + X_2 + x_1 + x_2)}$$

flat-space correlator

This integral is not so trivial! How do we do integrals of this type?

Family of Integrals

It turns out to be useful to consider the most general integral with the same singularities:

$$(n_i \in \mathbb{Z})$$

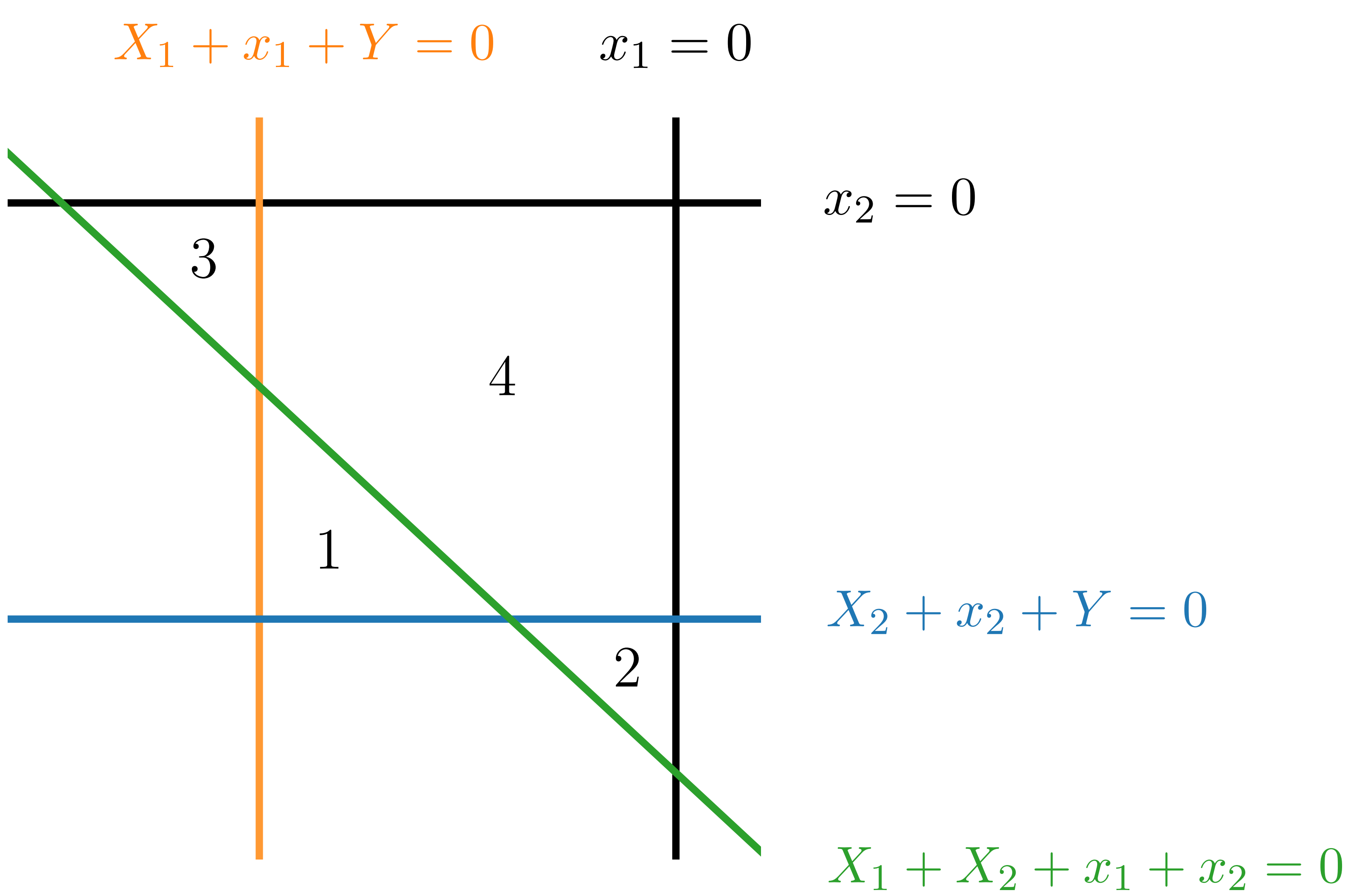
$$I_{n_1 \cdots n_5} = \int_0^\infty \mathrm{d}x_1 \mathrm{d}x_2 \frac{x_1^{\varepsilon+n_1} x_2^{\varepsilon+n_2}}{(X_1+x_1+Y)^{n_3}(X_2+x_2+Y)^{n_4}(X_1+X_2+x_1+x_2)^{n_5}}$$

Lesson from amplitudes: these integrals belong to a **finite** vector space and satisfy a **differential equations** of the form

$$\mathrm{d}\vec{I} = \varepsilon A \vec{I}$$

Hyperplanes

FRW integrals are naturally associated to hyperplane arrangements.



independent integrals

=

bounded regions.

De, Pokraka [2023]

Arkani-Hamed, Baumann, Hillman, Joyce, HL, Pimentel [2023]

Differential Equations

Consequently, there exists a **differential equation** of the form:

$$\boxed{\mathrm{d}\vec{I} = \varepsilon A \vec{I}}$$

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \mathrm{d}\log(X_1 + X_2) + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \mathrm{d}\log(X_1 + Y) + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{d}\log(X_2 + Y)$$

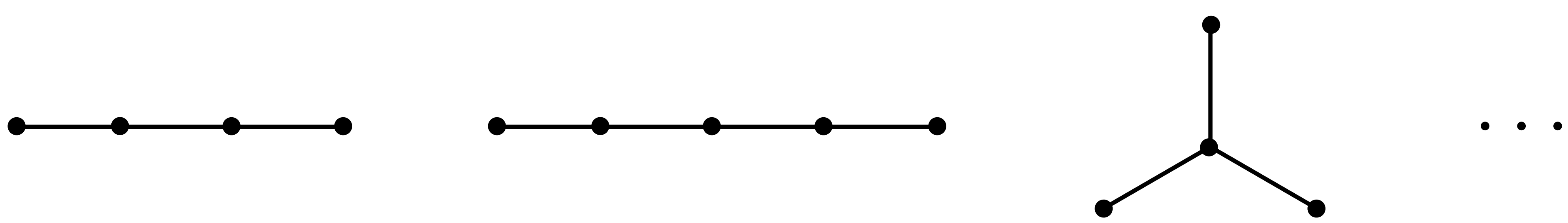
$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{d}\log(X_1 - Y) + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathrm{d}\log(X_2 - Y)$$

singularities (letters)

More Complicated Graphs

One can proceed algorithmically and derive equations for arbitrary graphs.

However, the underlying physics becomes quite obscure.



It turns out to be much more useful to represent equations **graphically**.

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2. Correlators in Toy Universe

4. A Hidden Pattern in dS

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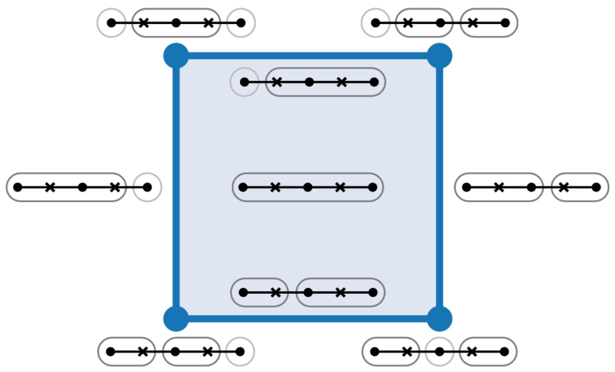
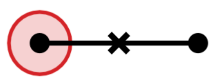
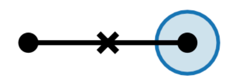


Letters

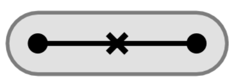
The **singularities** (**letters**) of the equations can be represented as



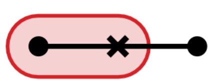
$$\mathrm{d}\log(X_1 + Y)$$



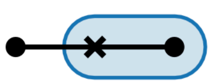
$$\mathrm{d}\log(X_2 + Y)$$



$$\mathrm{d}\log(X_1 + X_2)$$



$$\mathrm{d}\log(X_1 - Y)$$



$$\mathrm{d}\log(X_2 - Y)$$

with <u>connected tubings</u> of a graph.

Functions

Similarly, the **basis functions** can be represented as

$$\vec{I} \;=\; (\quad , \quad , \quad , \quad)$$

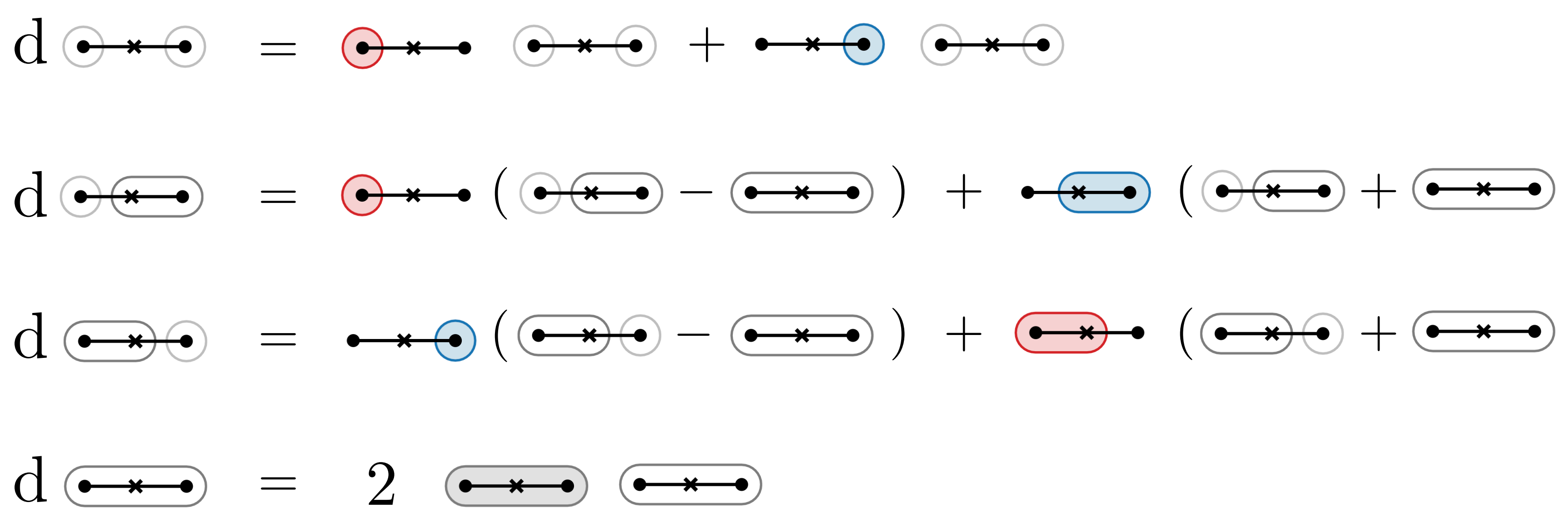
I_1 $\quad$ I_2 $\quad$ I_3 $\quad$ I_4

with <u>complete</u> (possibly disconnected) <u>tubings</u> of a graph.

Differential Equations

In terms of pictures, the differential equations take the form

$$\mathrm{d}\vec{I} = \varepsilon A \vec{I} \quad \Leftrightarrow$$

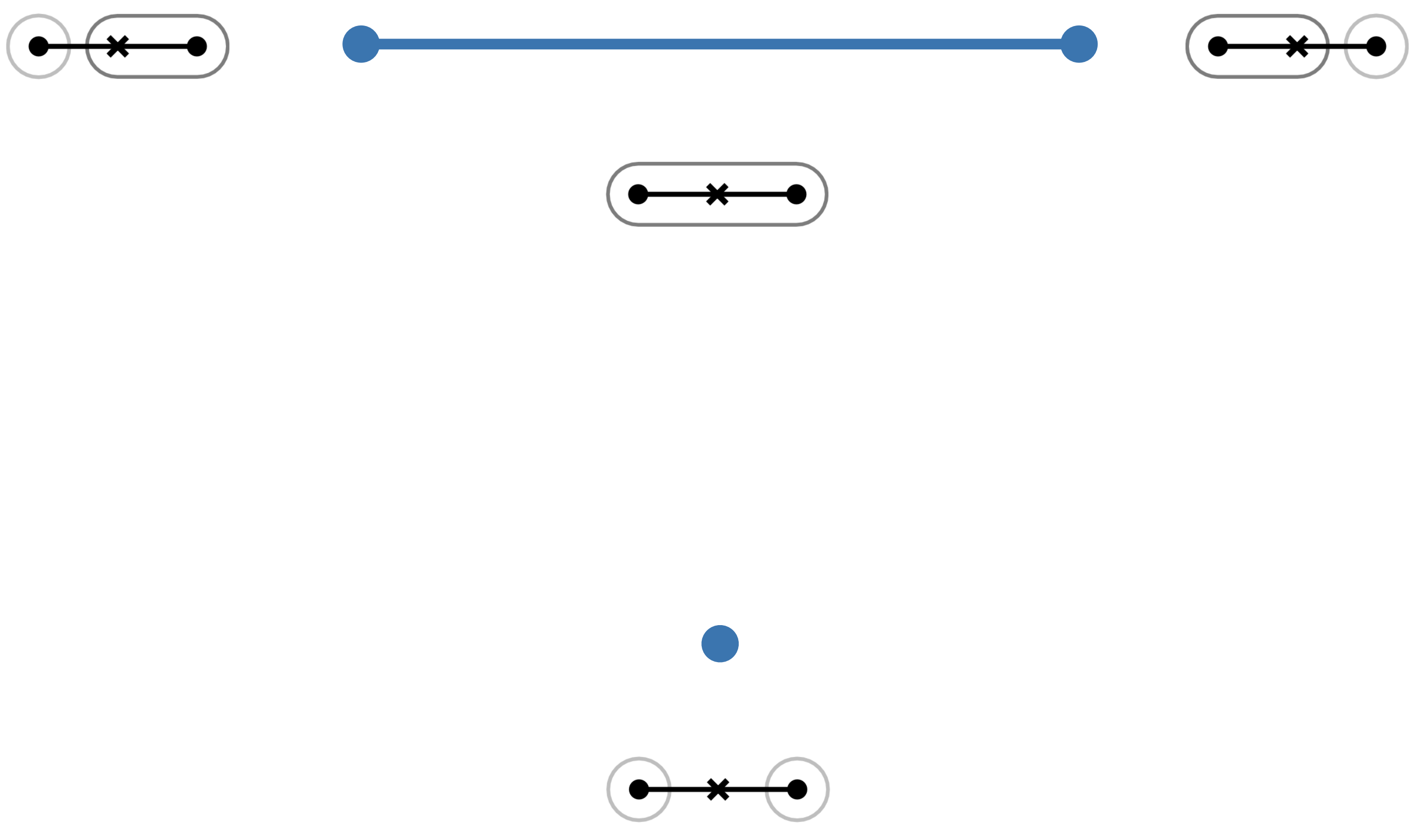


Remarkably, there exists an underlying **geometry** that explains this pattern.

Baumann, Goodhew, Joyce, HL, Pimentel, Westerdijk [2025]
Glew, Pokraka [2025]

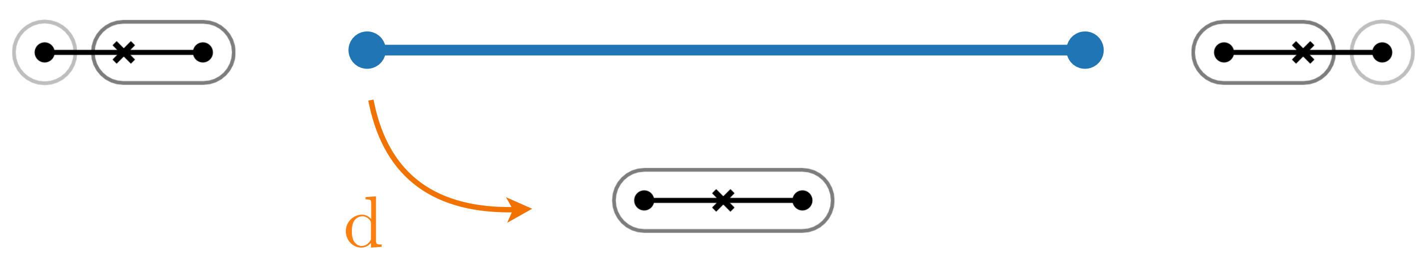
Geometric Arrangement

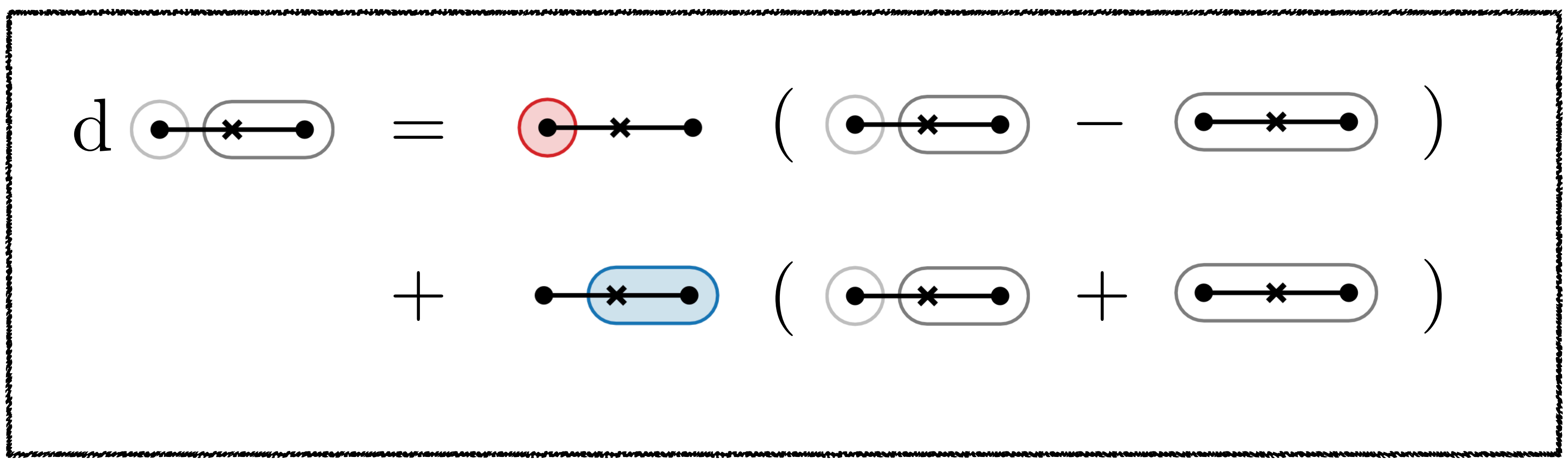
The basis functions can be arranged geometrically as



Kinematic Flow

The differential then becomes **local** in this geometric space of functions.





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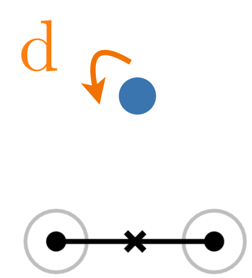
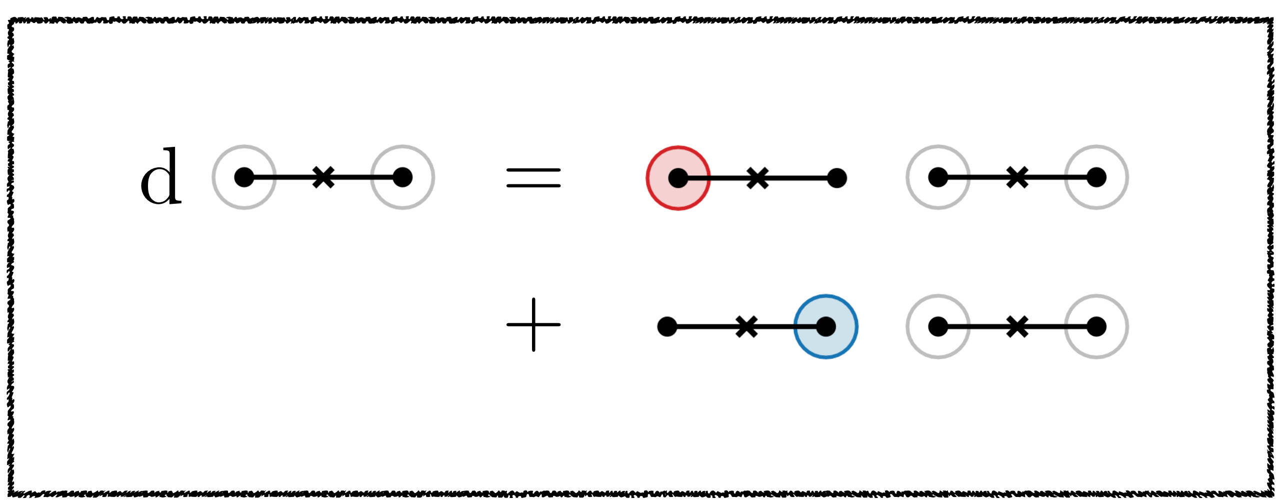
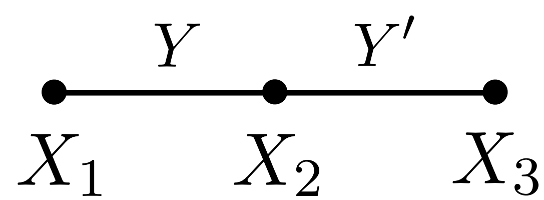
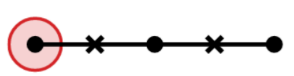


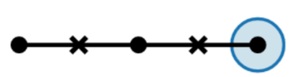


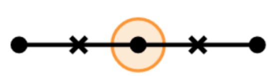
More Complicated Graphs

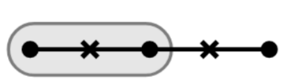


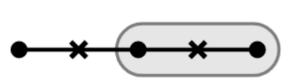
Letters: connected tubings of a marked graph.

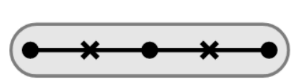
 $\equiv \mathrm{d}\log(X_1 + Y)\,,$

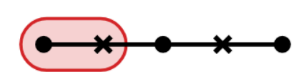
 $\equiv \mathrm{d}\log(X_3 + Y')\,,$

 $\equiv \mathrm{d}\log(X_2 + Y + Y')\,,$

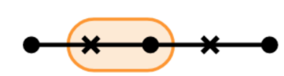
 $\equiv \mathrm{d}\log(X_1 + X_2 + Y')\,,$

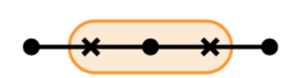
 $\equiv \mathrm{d}\log(X_2 + X_3 + Y)\,,$

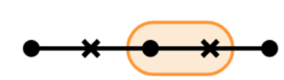
 $\equiv \mathrm{d}\log(X_1 + X_2 + X_3)\,.$

 $\equiv \mathrm{d}\log(X_1 - Y)\,,$

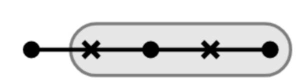
 $\equiv \mathrm{d}\log(X_3 - Y')\,,$

 $\equiv \mathrm{d}\log(X_2 - Y + Y')\,,$

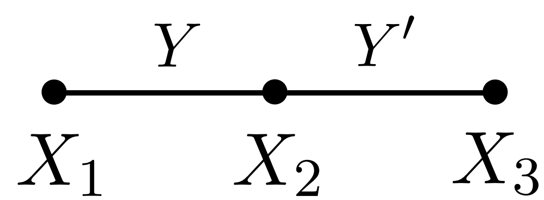
 $\equiv \mathrm{d}\log(X_2 - Y - Y')\,,$

 $\equiv \mathrm{d}\log(X_2 + Y - Y')\,,$

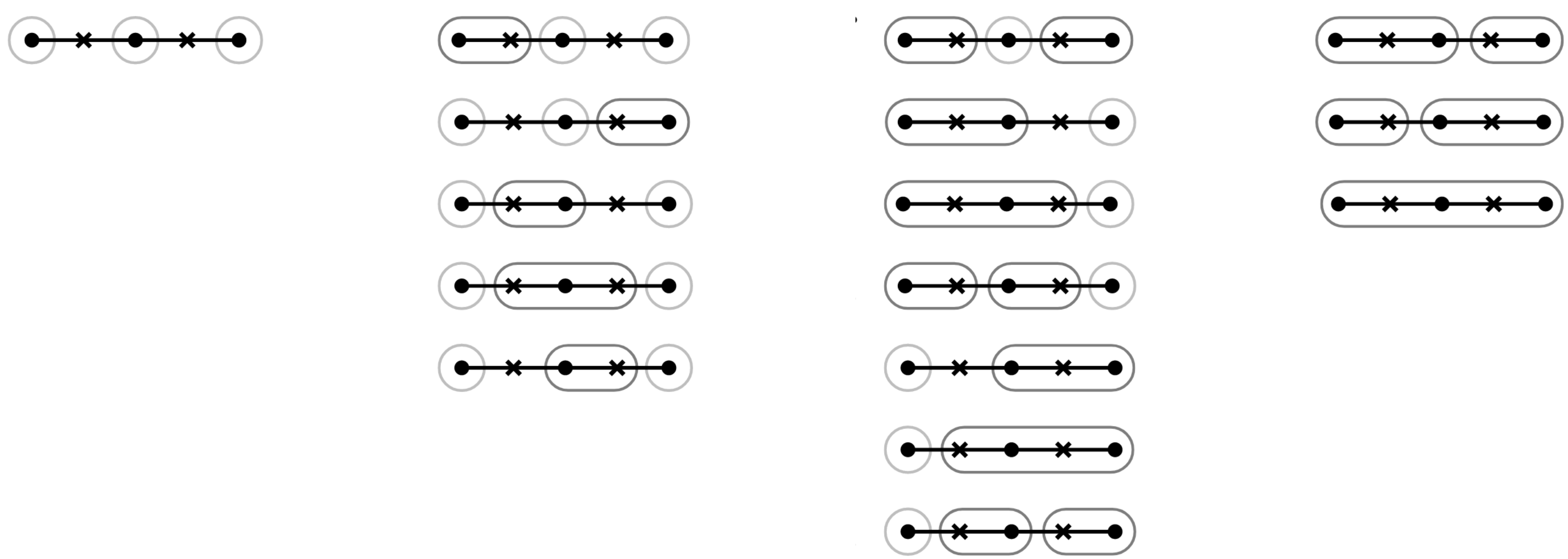
 $\equiv \mathrm{d}\log(X_1 + X_2 - Y')\,,$

 $\equiv \mathrm{d}\log(X_2 + X_3 - Y')\,,$

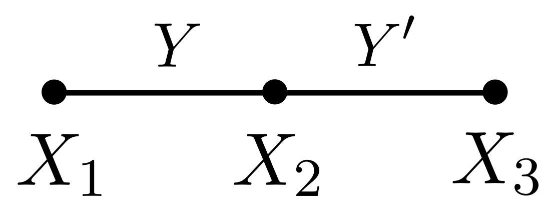
More Complicated Graphs



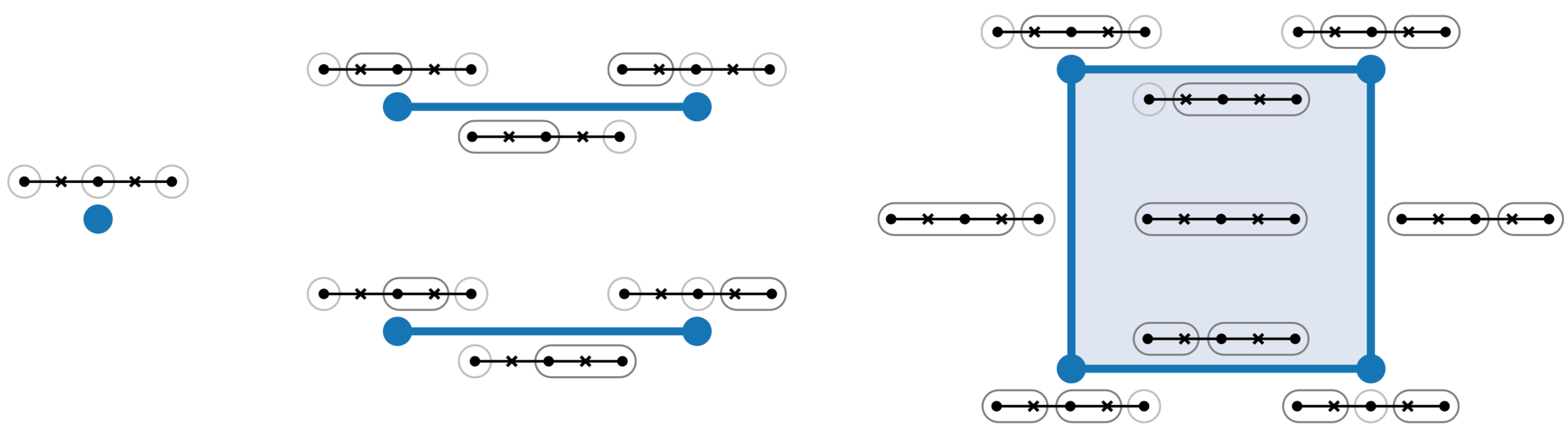
Functions: complete tubings of a marked graph.



More Complicated Graphs

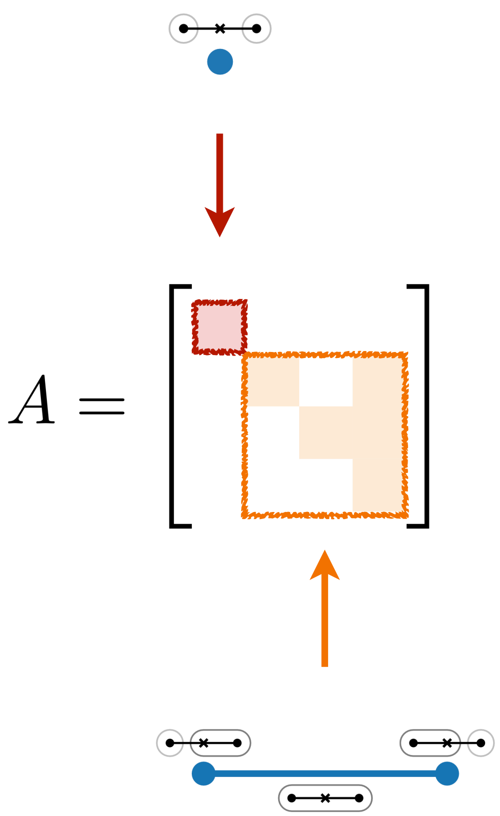


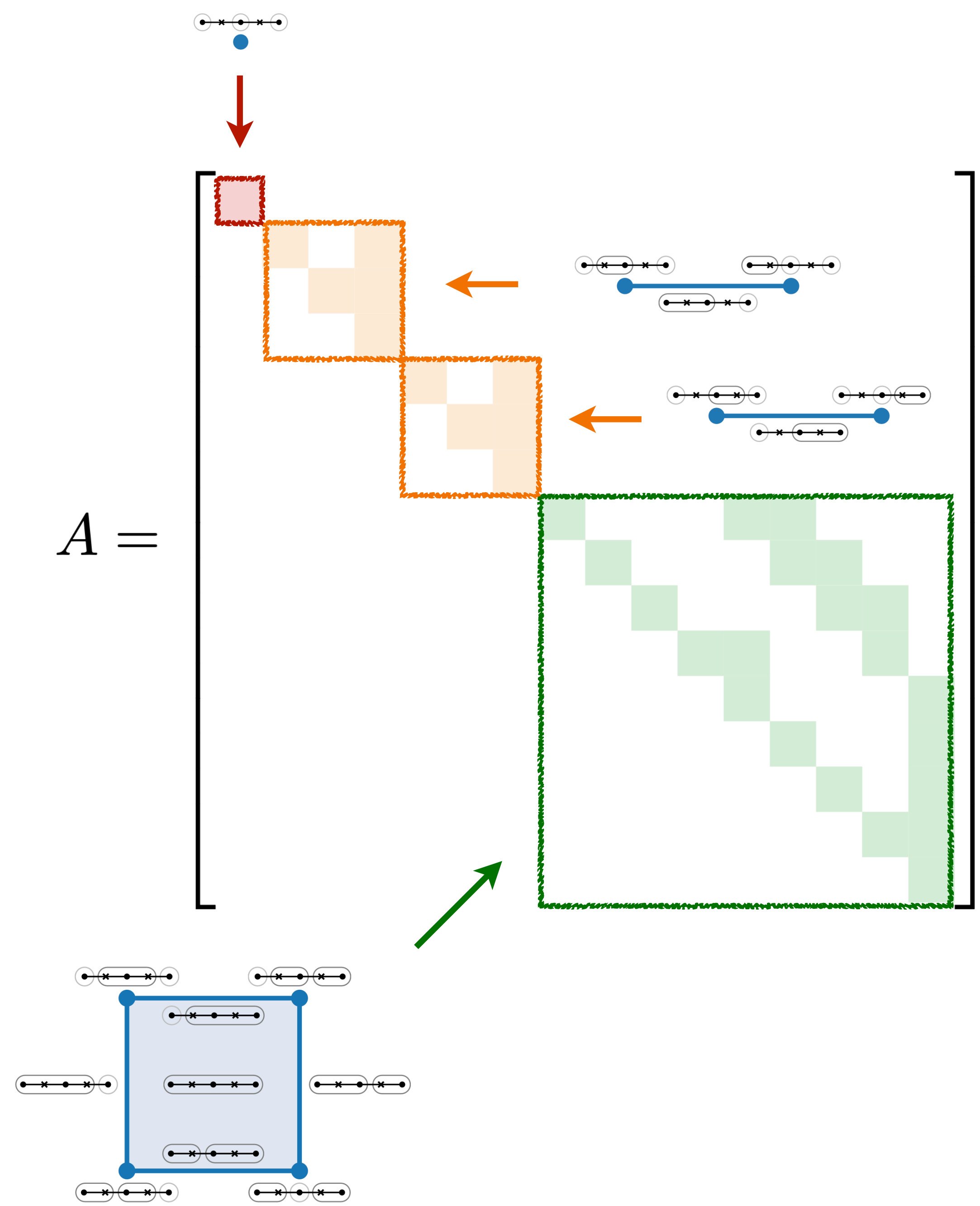
Functions: can be arranged into a point, two intervals, and a square.



Differential Equations

All differential equations can be easily derived using the kinematic flow rules.





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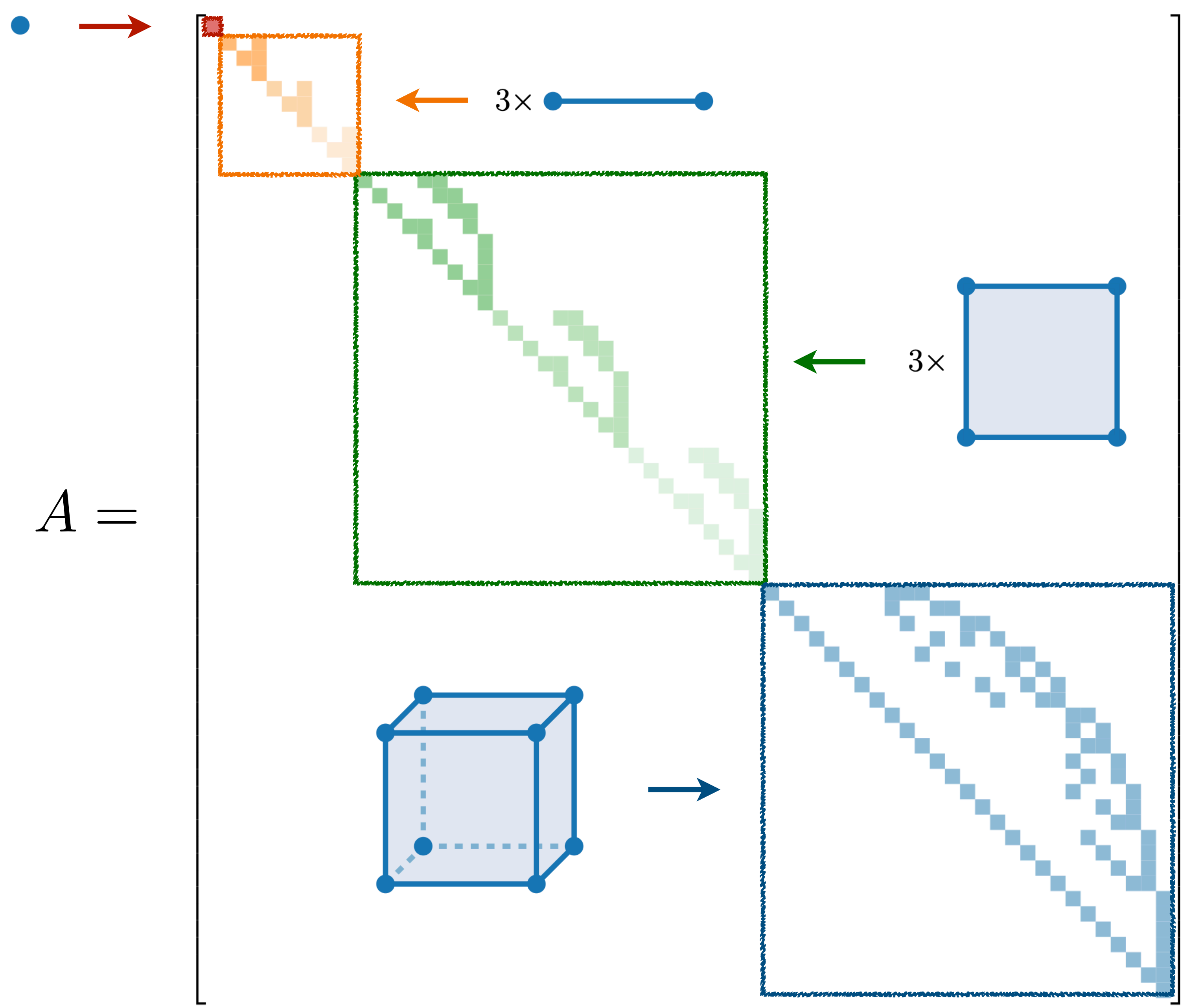
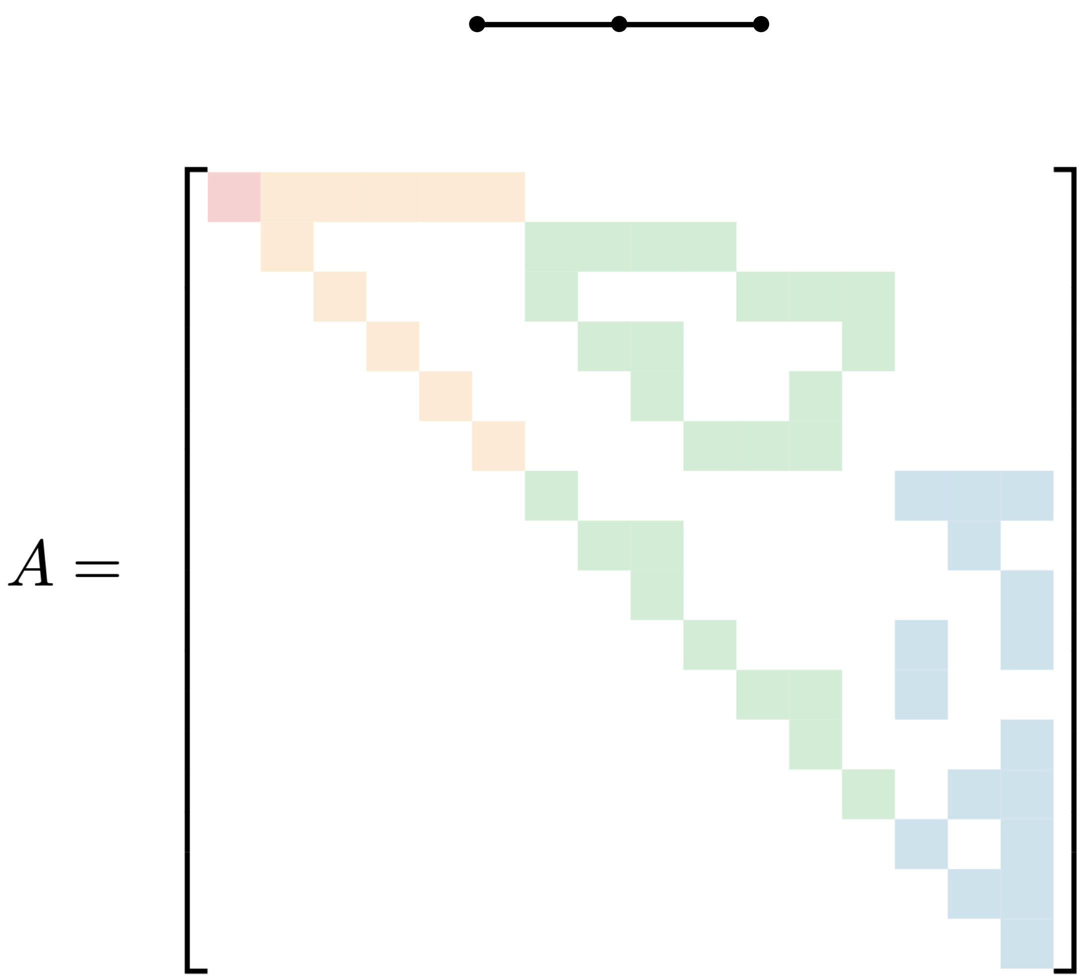
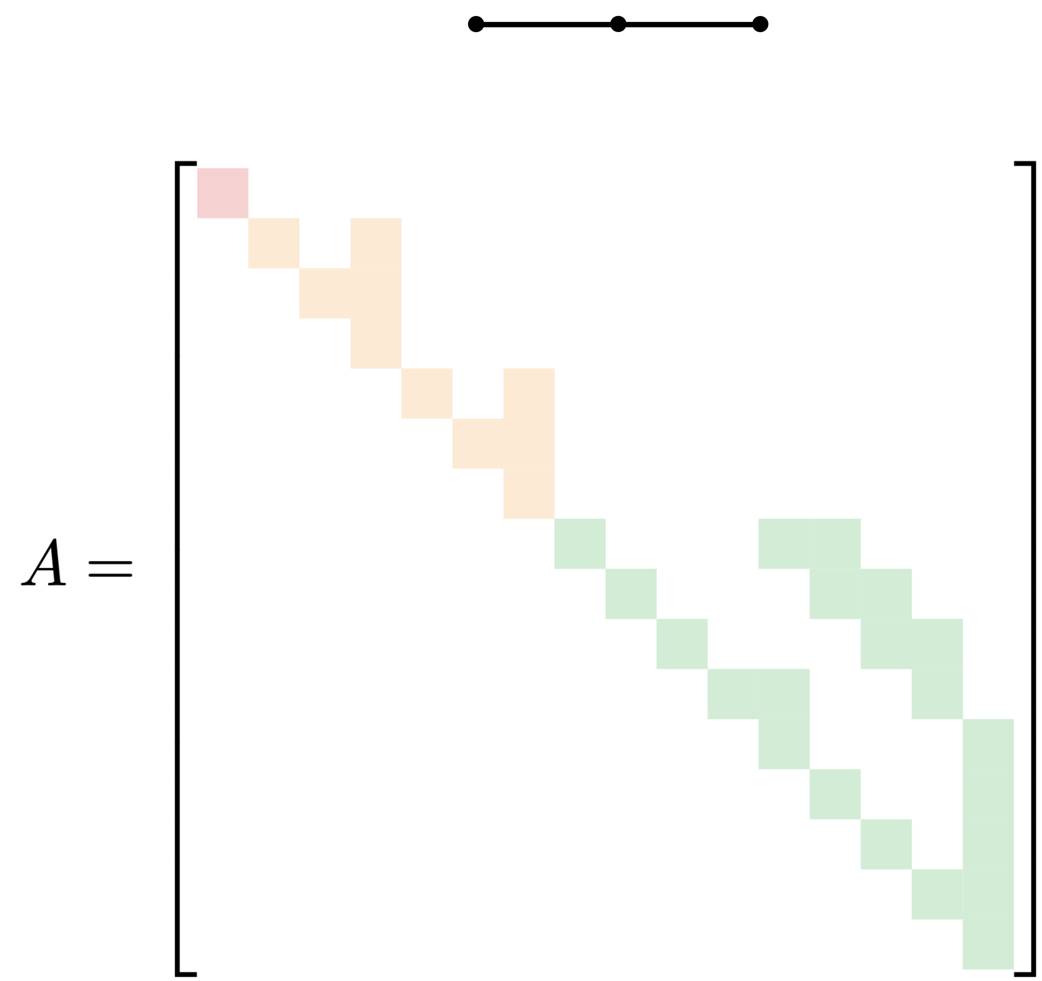


Differential Equations

This geometric organization of tubings dramatically simplifies the equations.



[2312.05303]



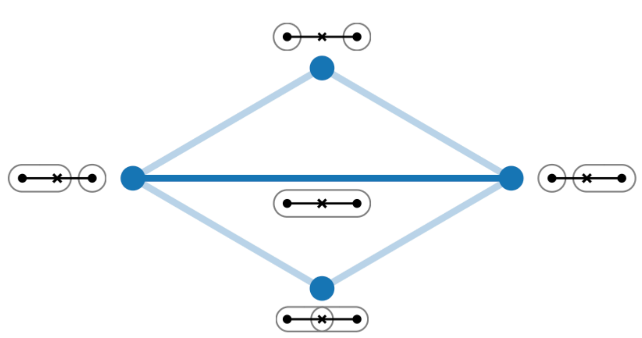
[2504.14890]

[See also H. Goodhew's talk]

1. Correlators in dS

2. Correlators in Toy Universe

4. A Hidden Pattern in dS

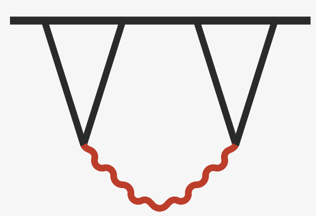


3. A Hidden Pattern in Toy Universe

Beyond Toy Model

Processes involving massive particles in dS can be understood in a similar way.

mass parameter
(deviation from cc scalar)



$$\supset \int_0^\infty \frac{\mathrm{d}x_1 \mathrm{d}x_2}{(x_1 x_2)^{1+\alpha}} \int_1^\infty \mathrm{d}s_1 \mathrm{d}s_2 \frac{x_1^{-\xi} x_2^{-\xi} (s_1^2 - 1)^{\xi} (s_2^2 - 1)^{\xi}}{(x_1 + x_2 + X_1 + X_2 + Y(s_1 - s_2))(x_2 + X_2 + Y)}$$

d-dependent parameter

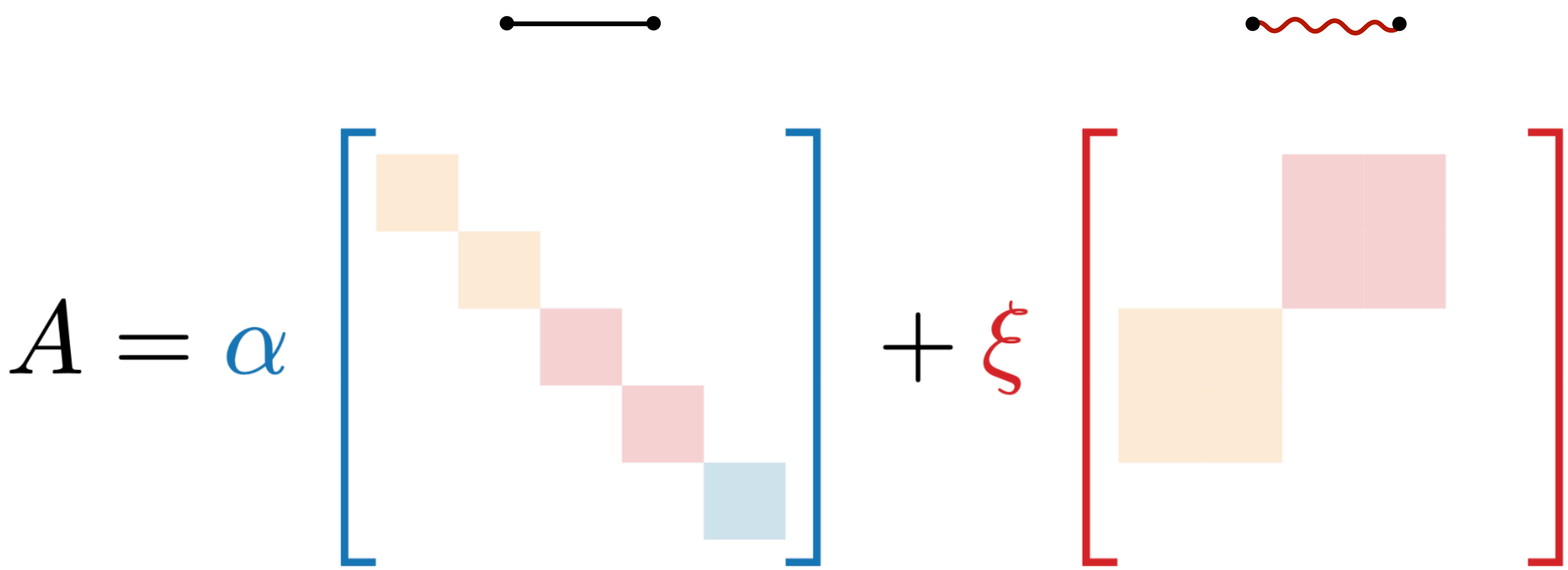
two extra integrals

This leads to a larger integral basis, but similar structures remain.

Chen, Feng, Tao [2024]
Gasparotto, Mazloumi, Xu [2024]

Massive DEQ

The resulting DEQ has the following schematic structure:



conformally coupled
scalar in dS

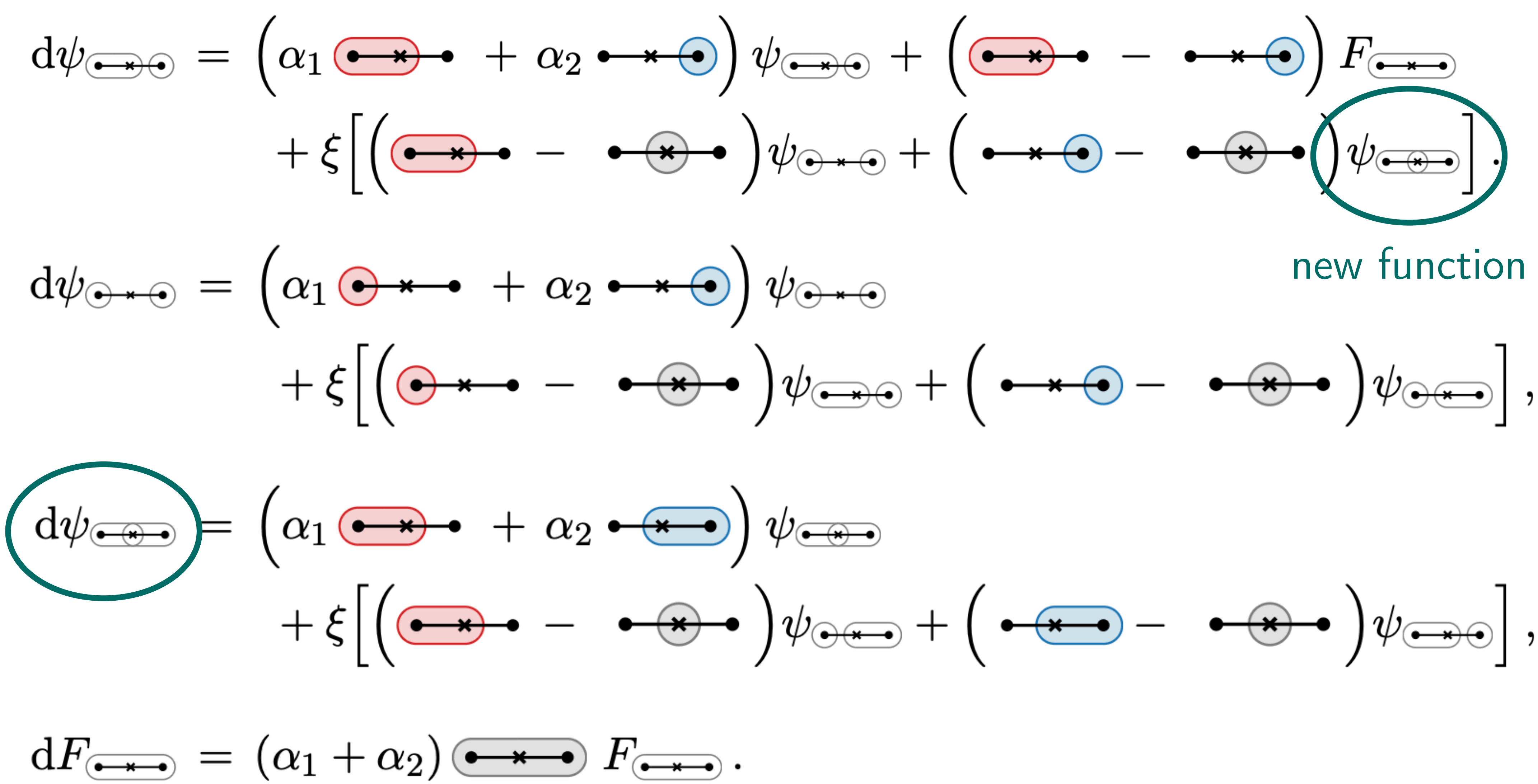
mass dependence in dS
(exact in ξ)

This can be solved in the small (near cc) or large (heavy mass) ξ limits.

[work in progress]

Massive Kinematic Flow

We found an analogous kinematic flow pattern that controls these new DEQ.



[work in progress]

Conclusion

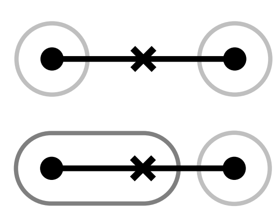
Conclusion

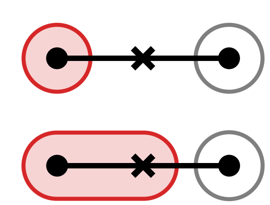
We have started to uncover a **combinatorial and geometric** pattern underlying the differential equations for cosmological correlators.

$$\mathrm{d}\vec{I} = A\vec{I}$$

correlators as graph tubings

predicted by kinematic flow





Simple graphical rules allow us to easily derive the differential equations.

Open Problems

There are many outstanding questions:

- Why does this work?
- What is the geometric origin of the massive kinematic flow?
- How general is this structure?
- Does it generalize to loop amplitudes and holographic correlators?