S-matrix Bootstrap and non-invertible symmetries

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Motivation

Symmetries and Amplitudes

- Symmetries and their anomalies are key concepts in physics.

  → Give constraints on dynamics of QFTs *(terms in EFT Lagrangians, RG flows, …)*

- Concepts refined and generalized in last years *(topological operators, non-invertible & higher-form syms.)*

What are the implications of generalized symmetries on scattering amplitudes?

Punchline 1: Crossing symmetry of S-matrix is modified in the presence of certain non-invertible symmetries

[Copetti, LC, Komatsu 2403.04835]

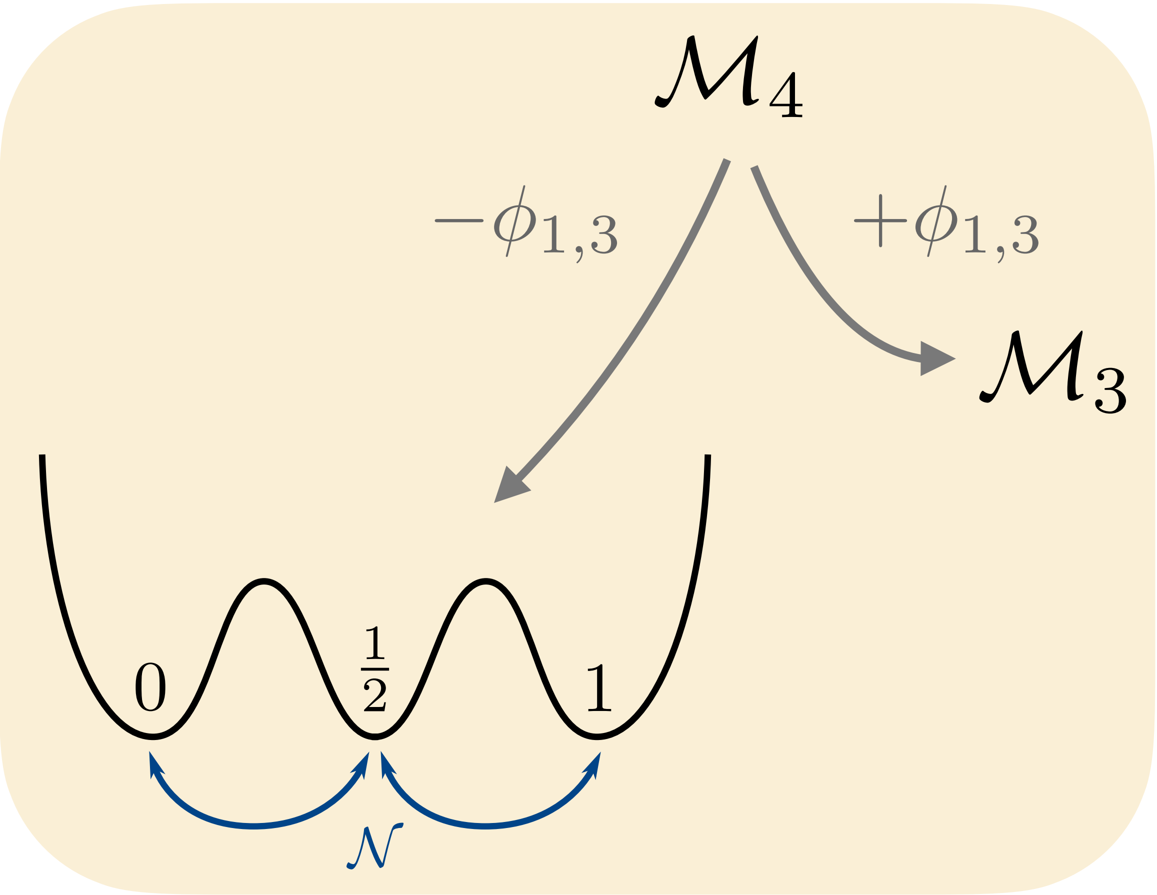
Punchline 2: We can use these symmetries to put concrete non-perturbative bounds on the space of QFTs

[Copetti, LC, Komatsu 2408.13132 +LC in progress]

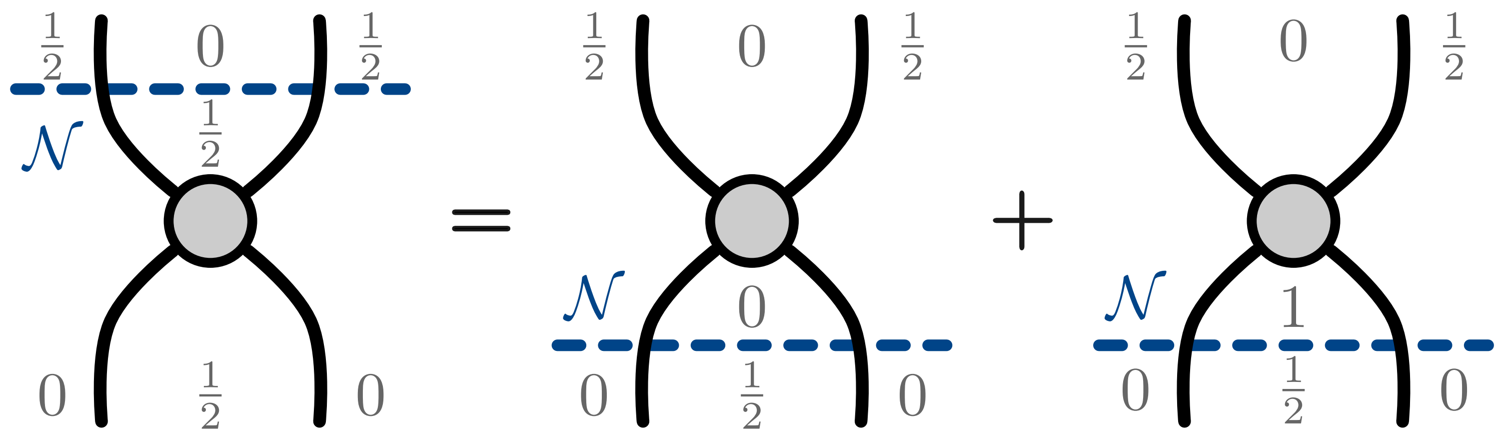
Explicit checks and predictions in 1+1d RG flows

Appetizer: Tricritical Ising → gapped$_3$

Tricritical Ising deformed by $\phi_{1,3}$ $(\varepsilon'_{\frac{3}{5},\frac{3}{5}})$

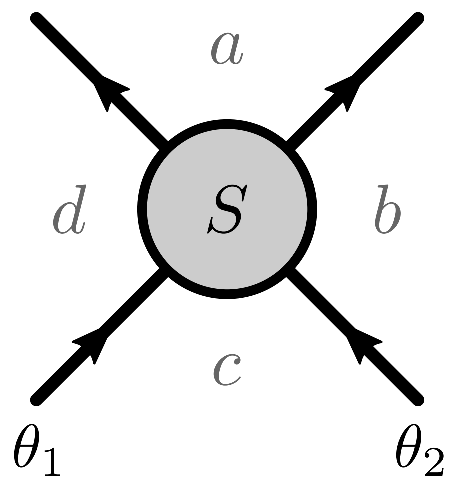


Non-invertible symmetry $\mathcal{N}$



Scattering kinks interpolating between vacua K_{ab}

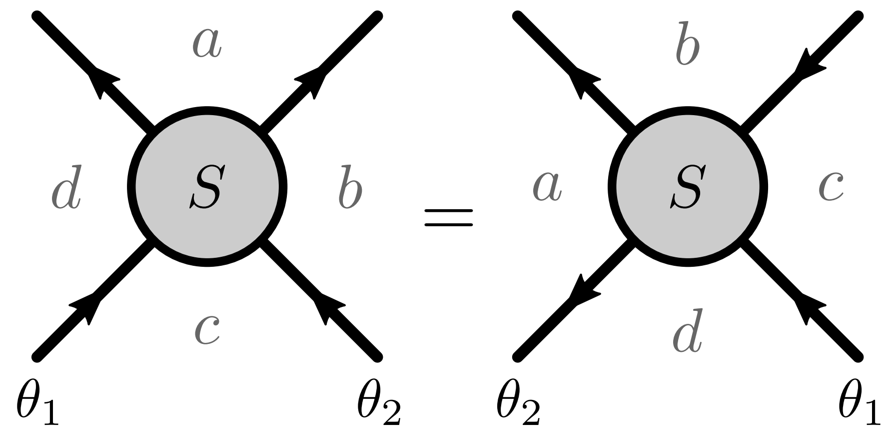
$S^{ab}_{dc}(\theta) =$



$a = 0, \frac{1}{2}, 1$

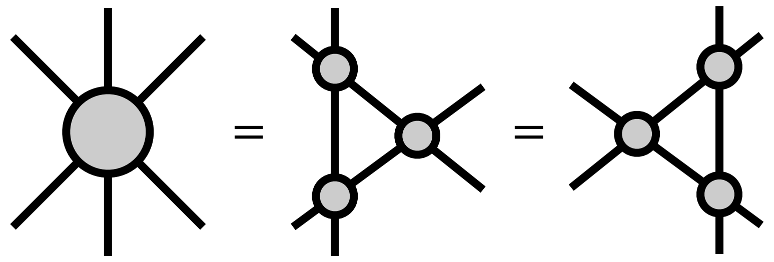
$s = (p_1 + p_2)^2 = 4m^2 \cosh^2(\theta/2)$

$S^{ab}_{dc}(\theta) =$ [diagram] $=$ [diagram] $= S^{bc}_{ad}(i\pi - \theta)$



Crossing $(s \leftrightarrow t)$

- S-matrix bootstrapped from unitarity+crossing+integrability
  [Zamolodchikov '89]



$$\widehat{S}^{ab}_{dc}(\theta) = Z(\theta)\left(\frac{\mathrm{d}_a\mathrm{d}_c}{\mathrm{d}_b\mathrm{d}_d}\right)^{\frac{i\theta}{2\pi}}\left[\sqrt{\frac{\mathrm{d}_a\mathrm{d}_c}{\mathrm{d}_b\mathrm{d}_d}}\sinh\left(\frac{\theta}{n}\right)\delta_{bd} + \sinh\left(\frac{i\pi-\theta}{n}\right)\delta_{ac}\right]$$

$(n = 4, \quad \mathrm{d}_0 = \mathrm{d}_1 = 1, \quad \mathrm{d}_{1/2} = \sqrt{2})$

→ ~~non-invertible sym.~~

- S-matrix bootstrapped from unitarity+integrability+non-invertible sym.

$$S^{ab}_{dc}(\theta) = Z(\theta)\left[\sqrt{\frac{\mathrm{d}_a\mathrm{d}_c}{\mathrm{d}_b\mathrm{d}_d}}\sinh\left(\frac{\theta}{n}\right)\delta_{bd} + \sinh\left(\frac{i\pi-\theta}{n}\right)\delta_{ac}\right]$$

→ **Modified crossing!**

$$S^{ab}_{dc}(\theta) = \sqrt{\frac{\mathrm{d}_a\mathrm{d}_c}{\mathrm{d}_b\mathrm{d}_d}}\, S^{bc}_{ad}(i\pi - \theta)$$

Outline

1. Fusion Categories and Minimal Models
2. Scattering Amplitudes and Modified Crossing
3. S-matrix Bootstrap and non-invertible syms
    - $\mathscr{A}_n$, Fibonacci, Haagerup $\mathscr{H}_3$
4. Final Remarks

Fusion Categories and Minimal Models

Generalized Symmetries

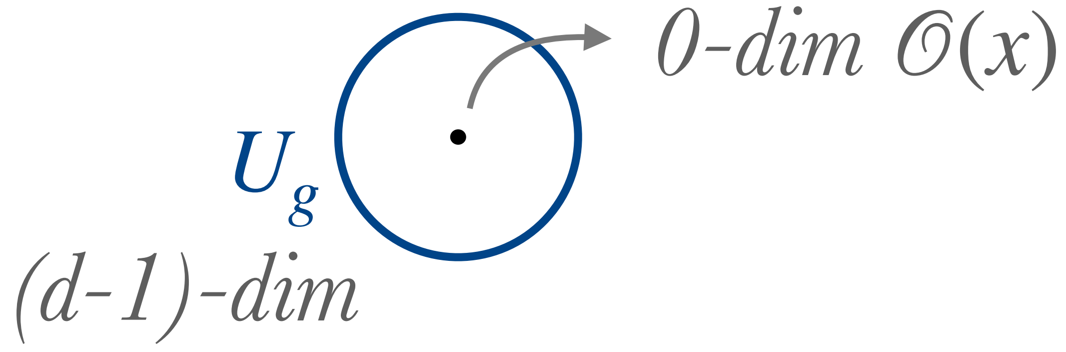
- (global) Symmetries in QFT ↔ Topological operators

[Gaiotto, Kapustin, Seiberg, Willett '14]

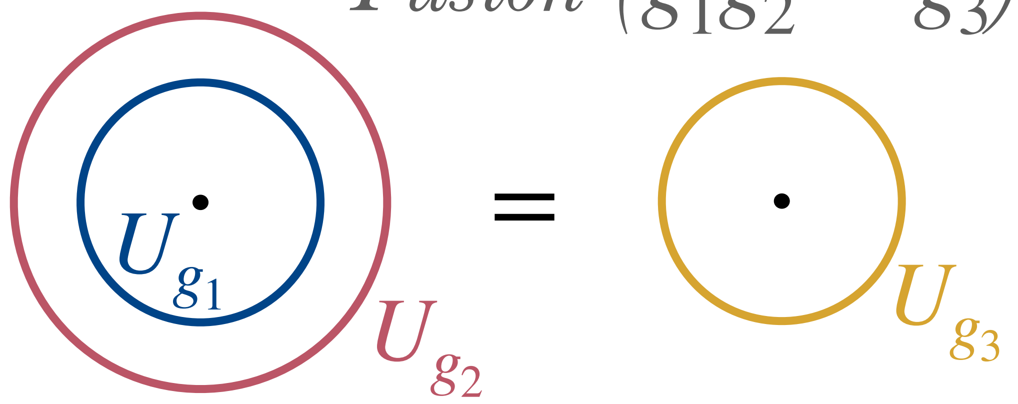
Move in t → charge conservation

e.g. from Noether current j_μ: $U = \exp\left(i\alpha \oint d^{d-1}x\, j_0(x)\right)$

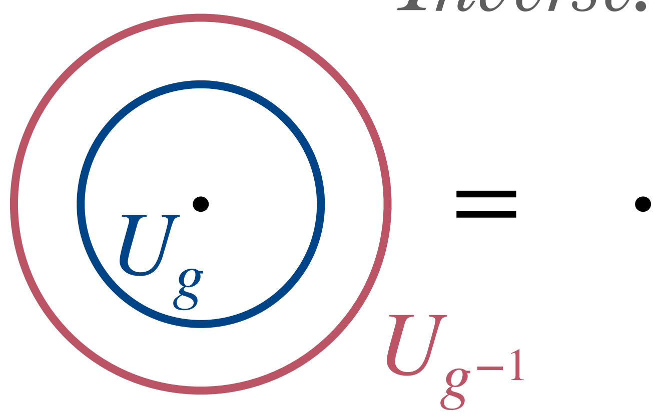
- Usual group symmetry: 0-form, invertible



Fusion $(g_1 g_2 = g_3)$



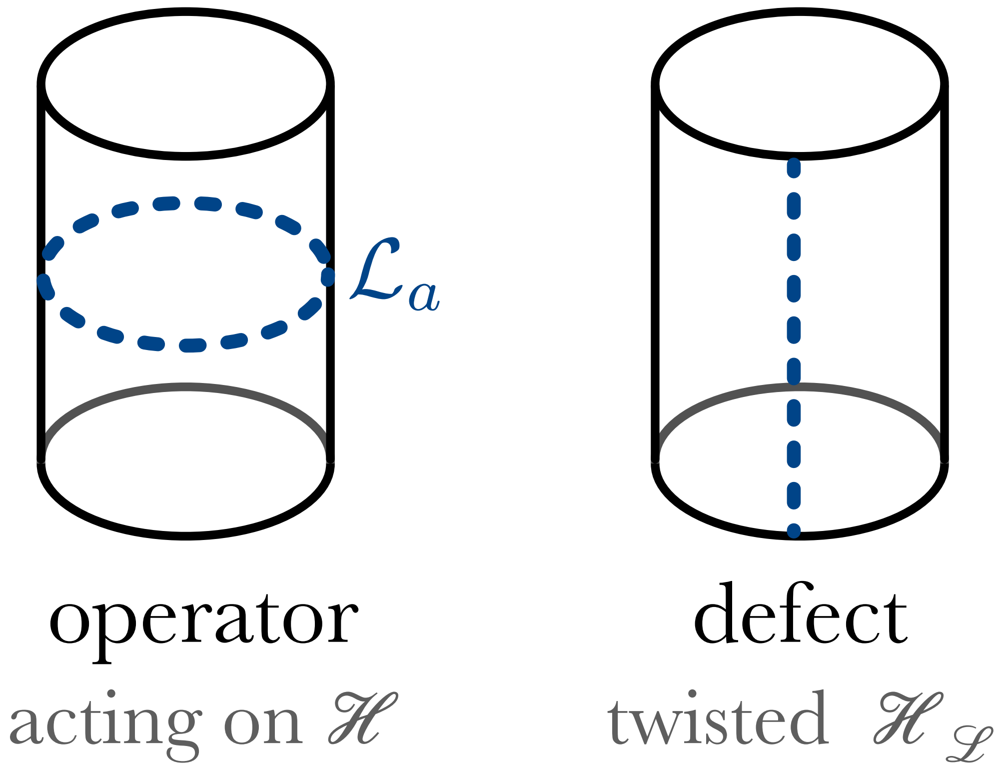
Inverse: $g^{-1}g = 1$



- Generalizations:
  - Higher-form symmetries. *q-form sym, (d-q-1)-dim topological ops.*
  - Non-invertible symmetries. g^{-1} *not necessary*

Here: 0-form symmetries in 1+1d (including non-invertible)

topological lines $\mathscr{L}_a$ → **Fusion categories**

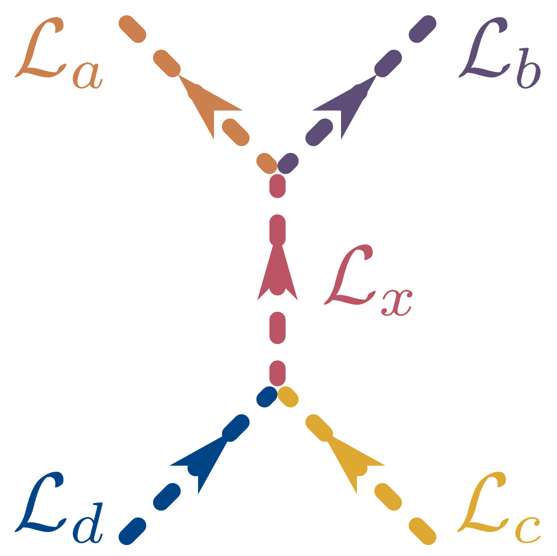


Fusion Categories

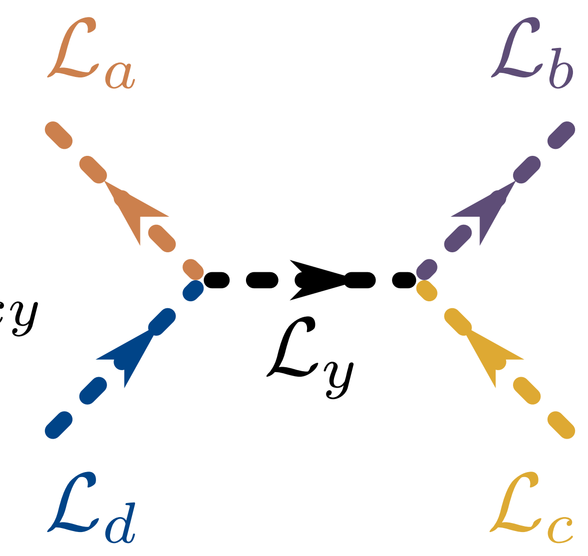
- **Objects** *(1, finite number)*

  $\mathcal{L}_a$

- **F-symbols**

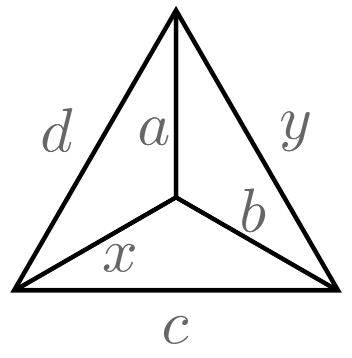


$$= \sum_y F^{abc}_{d\ \ xy}$$



Tetrahedral

$$\begin{bmatrix} a & b & x \\ c & d & y \end{bmatrix} = \frac{1}{\sqrt{\mathrm{d}_x \mathrm{d}_y}} F^{abc}_{d\ \ xy}$$



- **Fusion coefficients** $N_{ab}{}^{c} \in \mathbb{Z}_{\geq 0}$

$$\mathcal{L}_a \mathcal{L}_b = \sum_{\mathcal{L}_c} N_{ab}{}^{c} \mathcal{L}_c$$

vs group $\mathcal{L}_{g_1}\mathcal{L}_{g_2} = \mathcal{L}_{g_3}$, $\mathcal{L}_g \mathcal{L}_{g^{-1}} = 1$

quantum dimension

max eigenvalue $(N_a)_{bc}$

$\langle \mathcal{L}_a \rangle =$

$= \mathrm{d}_a$

Examples

- $\mathbb{Z}_2$

  $\{\mathscr{L}\} = 1, \eta$ $\eta^2 = 1$ (invertible) $\mathrm{d}_\eta = 1$

- $\mathbb{Z}_2$ Tambara-Yamagami (TY)

  $\{\mathscr{L}\} = 1, \eta, \mathcal{N}$ $\mathcal{N}^2 = 1 + \eta$ $\mathrm{d}_\mathcal{N} = \sqrt{2}$

- Fibonacci

  $\{\mathscr{L}\} = 1, W$ $W^2 = 1 + W$ $\mathrm{d}_W = \frac{1+\sqrt{5}}{2}$

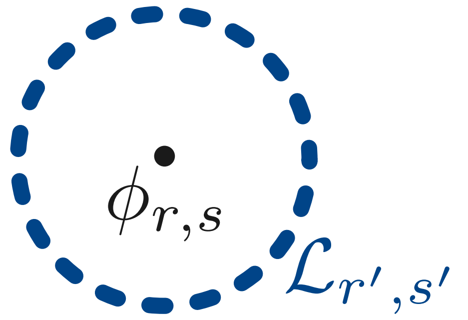
- Non-invertible symmetries with $\mathrm{d}_a \notin \mathbb{Z}$ are incompatible with trivially gapped phase [Chang, Lin, Shao, Wang, Yin '18]

  Here: RG flows with anomalous symmetries leading to **degenerate vacua** in IR

Minimal Models and deformations

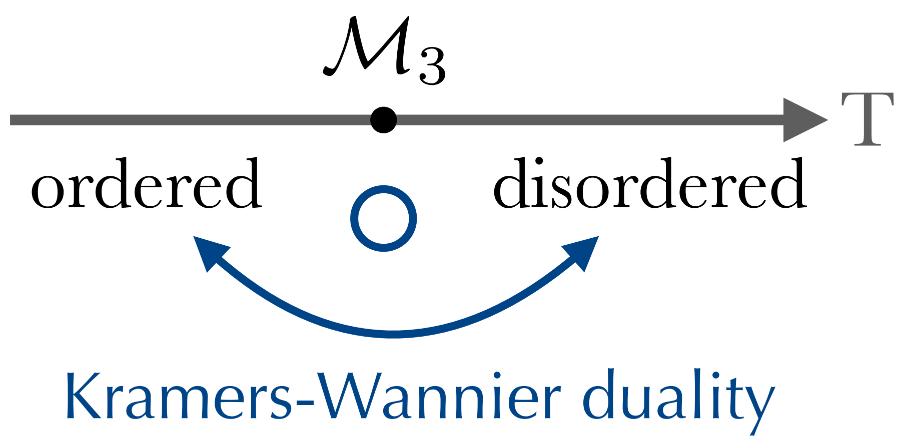
- **Verlinde lines** $\mathscr{L}_{r,s}$: Topological lines with same fusion algebra as primaries $\phi_{r,s}$

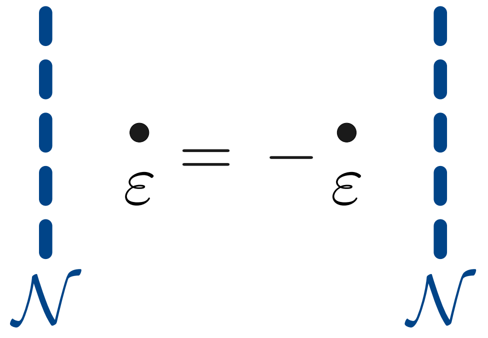
[Petkova, Zuber '00; Chang, Lin, Shao, Wang, Yin '18]



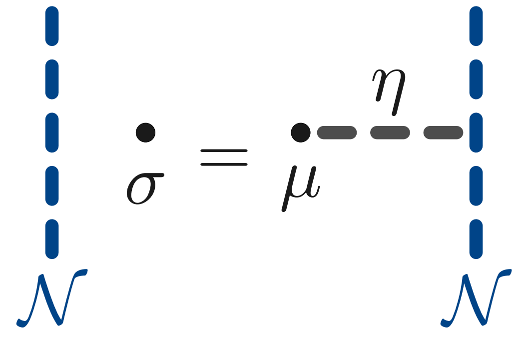
$= \dfrac{\mathcal{S}_{r',s';\,r,s}}{\mathcal{S}_{1,1;\,r,s}} \quad \phi_{r,s}$ $\qquad \mathcal{S}_{r,s}$: modular S-matrix

- **Critical Ising** $\mathscr{M}_3$ $\quad \to$ sym. lines: $\{\mathscr{L}\} = 1, \eta, \mathcal{N}$. $\quad \mathcal{N}^2 = 1+\eta$. $\quad$ ($\mathbb{Z}_2$ Tambara-Yamagami)





$\to \langle \varepsilon\varepsilon\varepsilon \rangle_{\mathscr{M}_3} = 0$

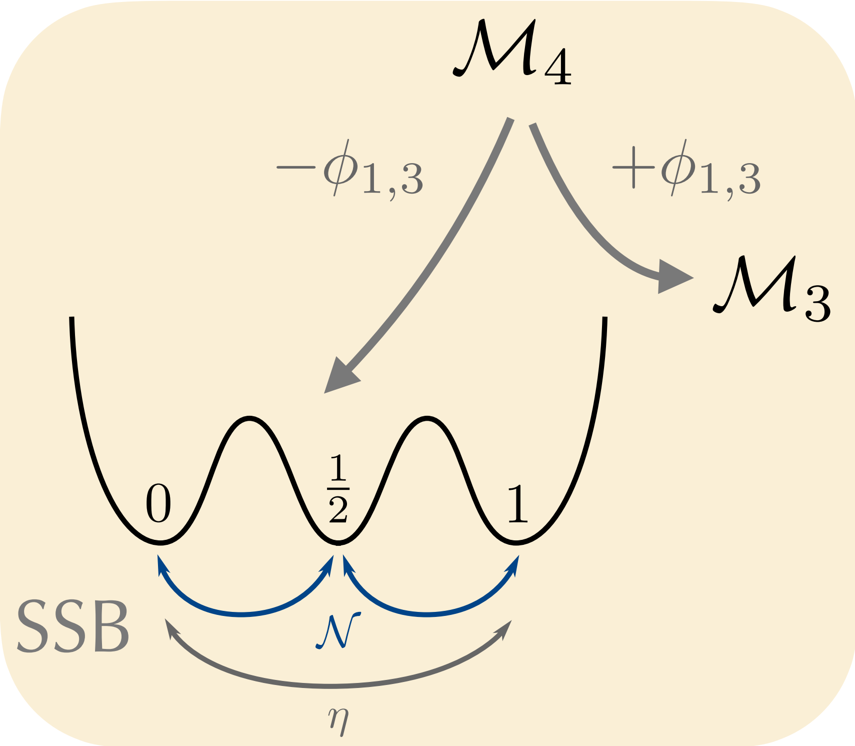


σ (local) $\leftrightarrow$ μ (twisted)

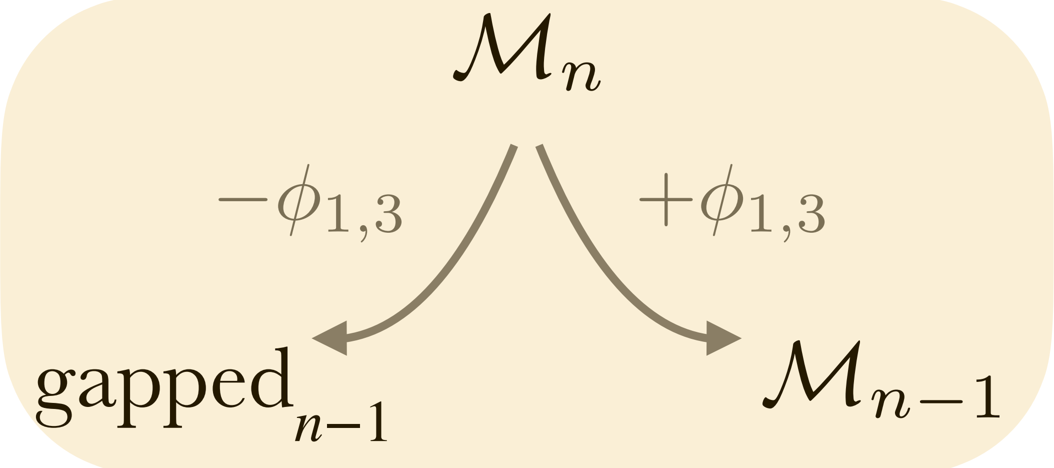
- **Tricritical Ising** $\mathscr{M}_4$

$\mathbb{Z}_2$ TY $\quad (\phi_{1,3} = \varepsilon'$ def.)

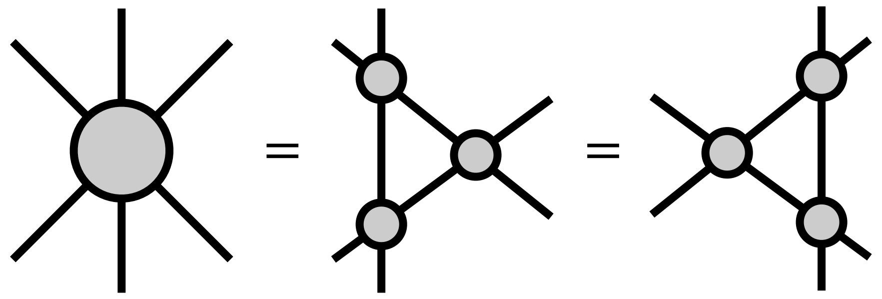
Fibonacci ($\phi_{2,1} = \sigma'$ def.)	1	η	$\mathcal{N}$
	W	W'	Z



- $\mathscr{M}_n \pm \phi_{1,3}$ $\quad \mathscr{L}_{r,1} \to \mathscr{A}_n$ fusion category

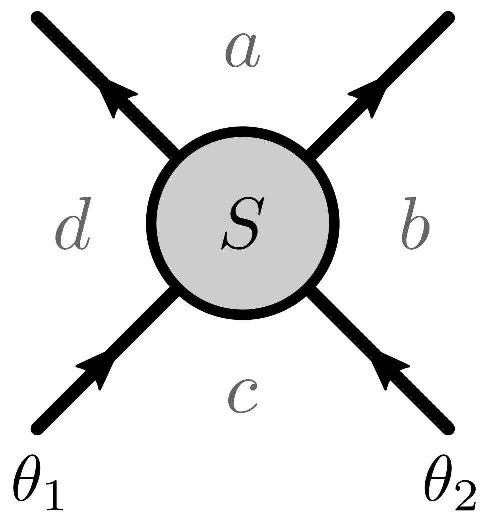


- $\phi_{1,3}$, $\phi_{2,1}$, $\phi_{1,2}$ deformations are **integrable**. Exact S-matrix



Scattering Amplitudes and Modified Crossing

Scattering amplitudes in 1+1d

$S^{ab}_{dc}(\theta) =$ 

Massive kinks interpolating between neighbouring vacua

$\theta = \theta_1 - \theta_2$
$s = 4m^2 \cosh^2(\theta/2)$
$t = 4m^2 - s$

- Example: $\mathscr{M}_n - \phi_{1,3}$ RG flow

U+C+I give: [Zamolodchikov '89]

$a = 1, \ldots, n-1 \quad \mathrm{d}_a = \frac{\sin \pi a/n}{\sin \pi/n}$

$$\widehat{S}^{ab}_{dc}(\theta) = Z(\theta) \left(\frac{\mathrm{d}_a \mathrm{d}_c}{\mathrm{d}_b \mathrm{d}_d}\right)^{\frac{i\theta}{2\pi}} \left[\sqrt{\frac{\mathrm{d}_a \mathrm{d}_c}{\mathrm{d}_b \mathrm{d}_d}} \sinh\left(\frac{\theta}{n}\right) \delta_{bd} + \sinh\left(\frac{i\pi - \theta}{n}\right) \delta_{ac}\right]$$

note oscillations

→ does not commute w/ non-invertible syms!

U+I+S give:

$$S^{ab}_{dc}(\theta) = Z(\theta) \left[\sqrt{\frac{\mathrm{d}_a \mathrm{d}_c}{\mathrm{d}_b \mathrm{d}_d}} \sinh\left(\frac{\theta}{n}\right) \delta_{bd} + \sinh\left(\frac{i\pi - \theta}{n}\right) \delta_{ac}\right]$$

→ **Modified crossing (mC)**
[Copetti, LC, Komatsu '24]

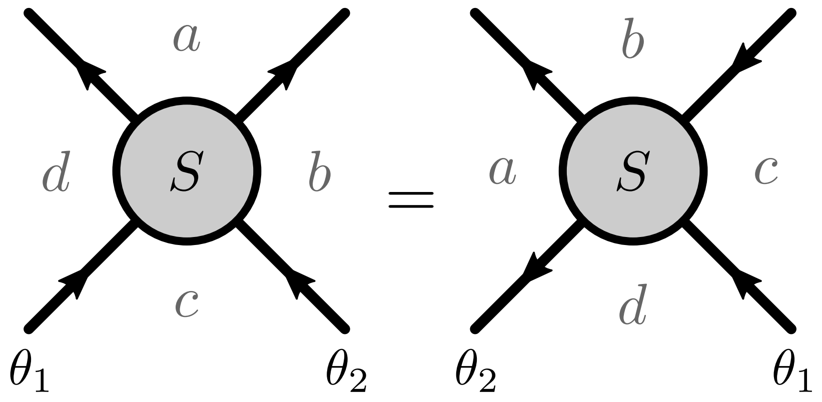
$$S^{ab}_{dc}(\theta) = \sqrt{\frac{\mathrm{d}_a \mathrm{d}_c}{\mathrm{d}_b \mathrm{d}_d}}\, S^{bc}_{ad}(i\pi - \theta)$$

AXIOMS

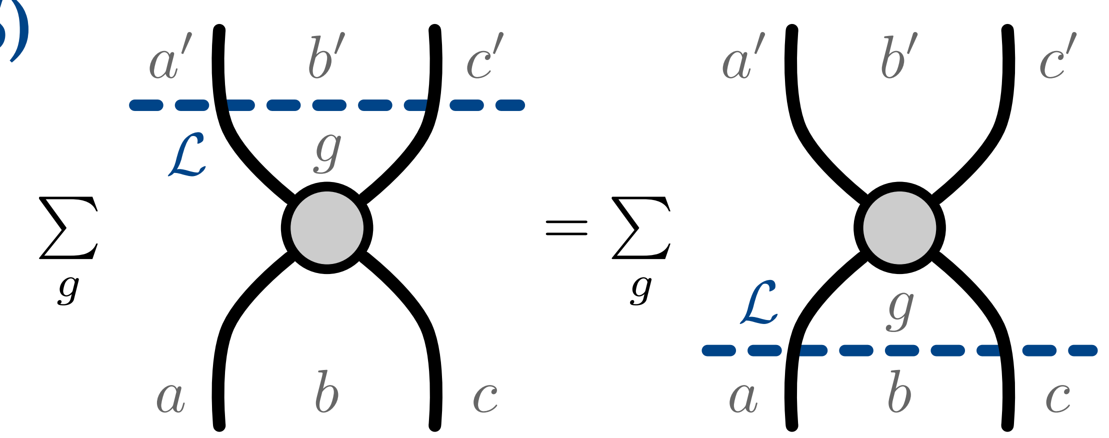
Unitarity (U)

$$|S^{ab}_{dc}(\theta \geq 0)|^2 \leq 1$$

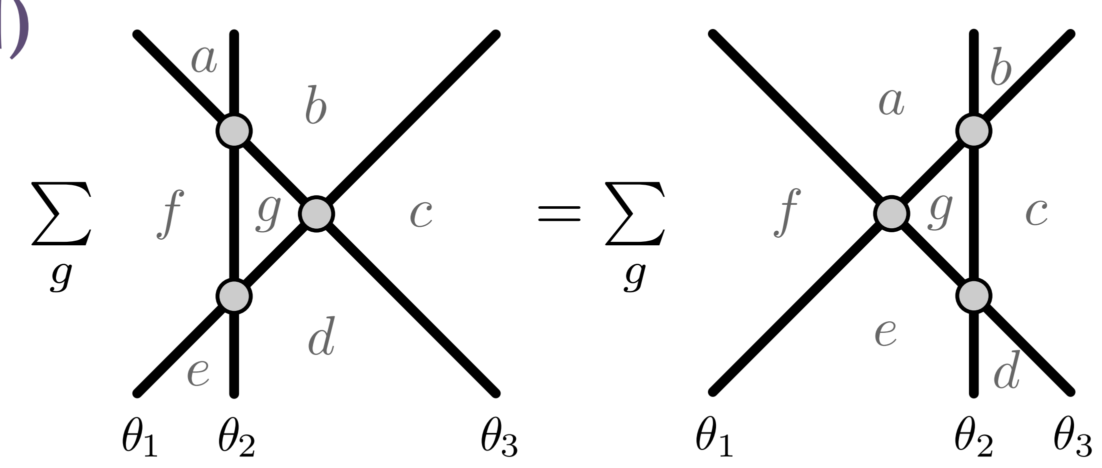
Crossing (C)

$S^{ab}_{dc}(\theta) =$  $= S^{bc}_{ad}(i\pi - \theta)$

Symmetries (S)



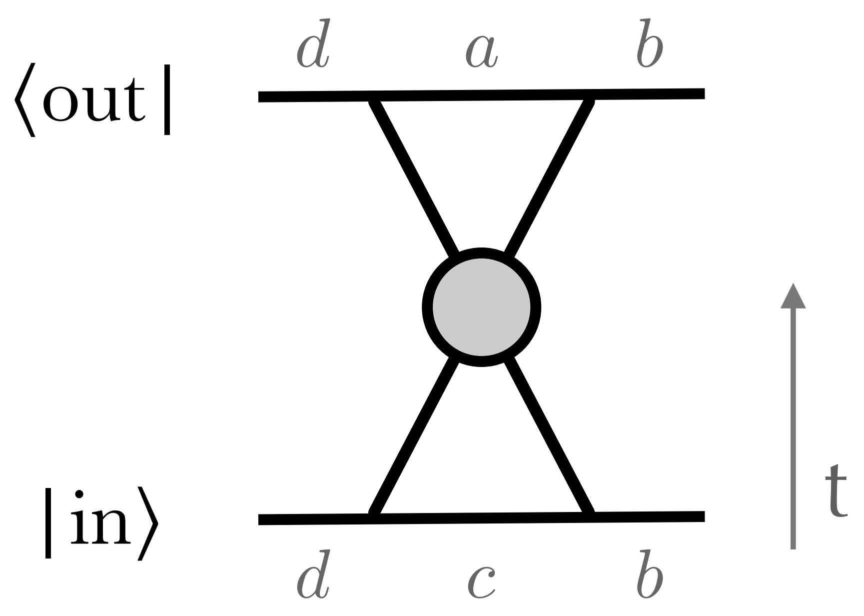
***Integrability (I)**



Factorized scattering, no particle production

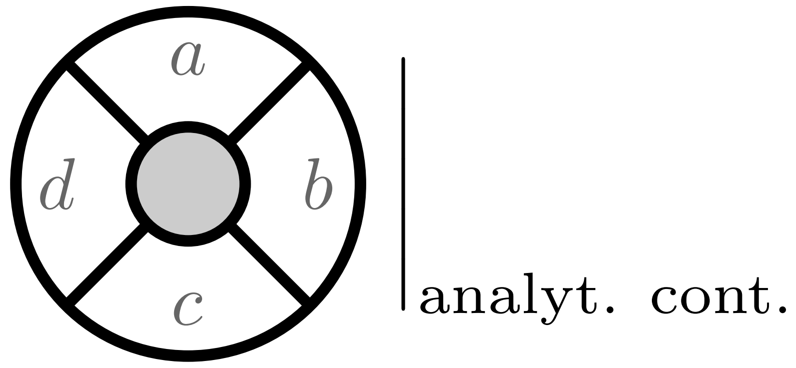
Modified Crossing

- New crossing rules can be understood from proper normalization of $|\text{in}\rangle$ and $|\text{out}\rangle$ states, taking into account topological degrees of freedom.



$$\langle \text{out}|\text{in}\rangle = S^{ab}_{dc}(\theta)\,\underbrace{(2\pi)^2\,2\sqrt{s}\sqrt{s-4m^2}\,\delta^2(p_1+p_2-p_1'-p_2')}_{\text{` }\delta^2(\,\cdot\,)\text{ '}}$$

$S^{ab}_{dc}(\theta) \propto$



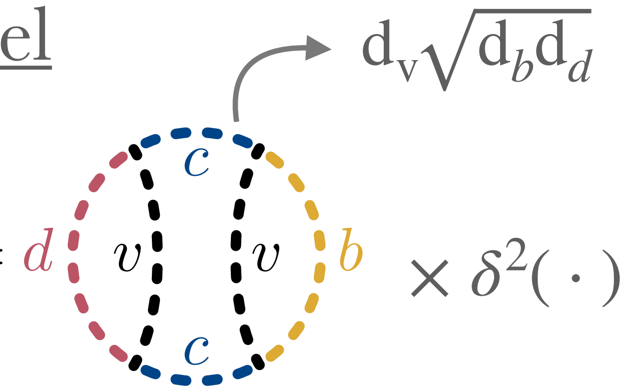
naively crossing symmetric

- In order to have unitarity $S(\theta)S(-\theta) \leq 1$, normalize states using TQFT data:
  vacua identified with symmetry lines *(regular module category)*, kink acts as symmetry line v *(e.g. $\mathcal{N}$ for $\mathcal{M}_4 \to \text{gapped}_3$)*.

→ **Modified crossing** [Copetti, LC, Komatsu '24]

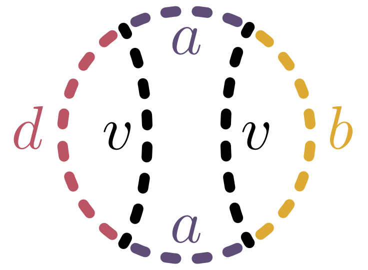
s-channel

$\langle \text{in}|\text{in}\rangle_s =$



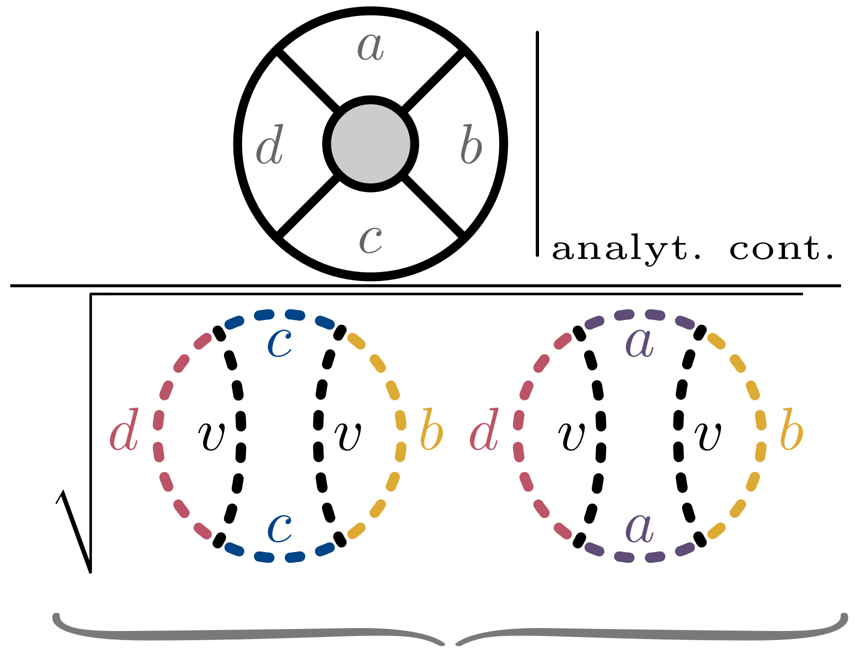
$\times\, \delta^2(\,\cdot\,)$

$\langle \text{out}|\text{out}\rangle_s =$



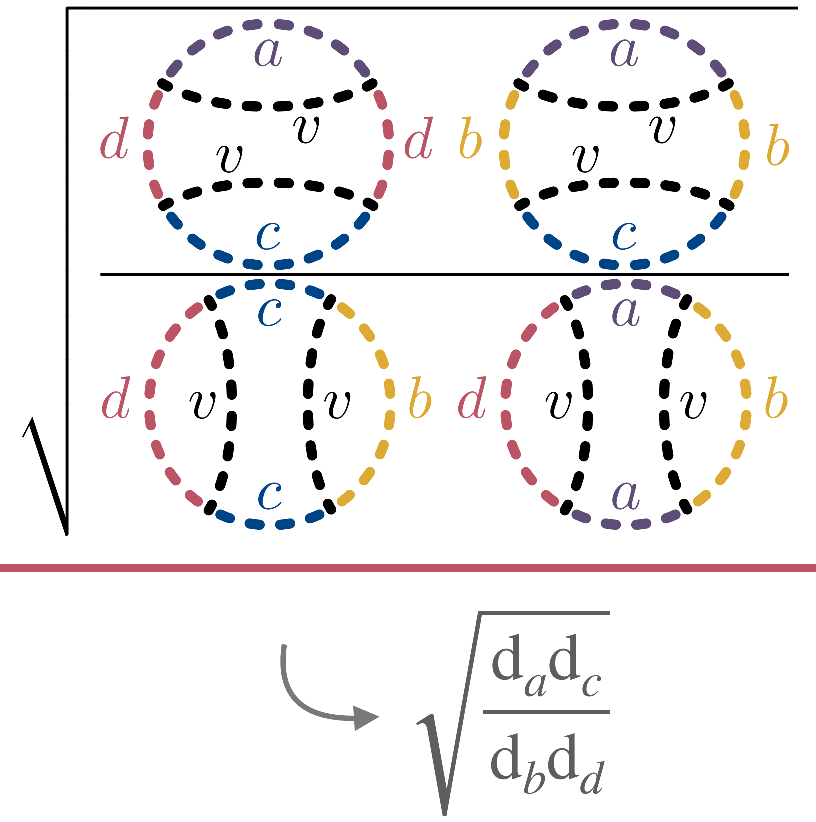
$\times\, \delta^2(\,\cdot\,)$

$S^{ab}_{dc}(\theta) =$



different in t-channel

$S^{ab}_{dc}(\theta) =$



$S^{bc}_{ad}(i\pi - \theta)$

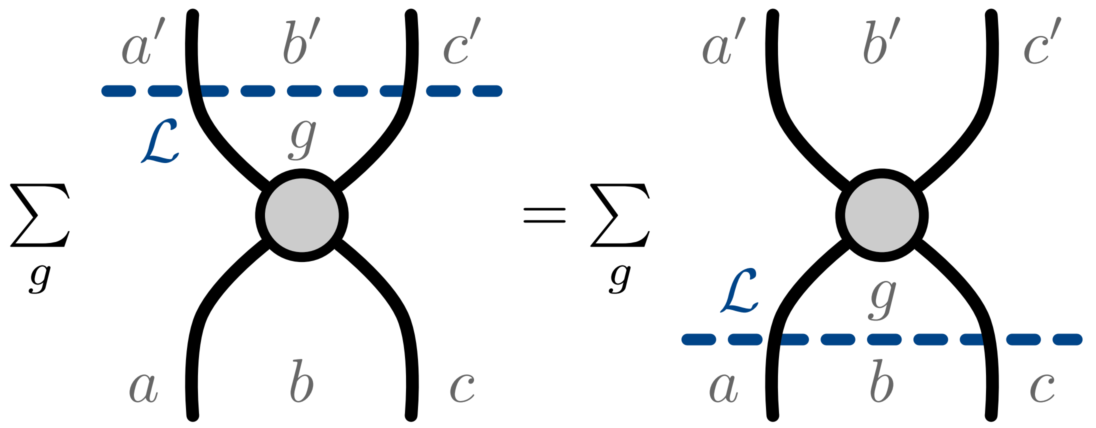
S-matrix Bootstrap and non-invertible symmetries

S-matrix Bootstrap with non-invertible syms.

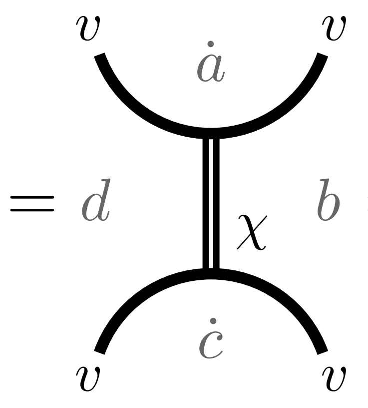
Find the space of consistent 2 → 2 kink scattering amplitudes $S^{ab}_{dc}(s)$ with a given fusion category

- **Symmetries** $\{\mathscr{L}\}$ → Projector into fusion channel χ

SSB all syms → regular representation



$$S^{ab}_{dc}(s) = \sum_{\chi} A_\chi(s)(P_\chi)^{ab}_{dc}$$



$$(P_\chi)^{ab}_{dc} = d \quad b = \sqrt{\mathrm{d}_a \mathrm{d}_c}\, \mathrm{d}_\chi \begin{bmatrix} v & v & \chi \\ d & b & a \end{bmatrix} \begin{bmatrix} v & v & \chi \\ d & b & c \end{bmatrix}$$

[see also Aasen, Fendley, Mong '20]

→ Kinks form a symmetry multiplet, so degenerate mass
[C. Córdova, García-Sepúlveda, Holfester '24]

- **Unitarity** $|A_\chi(s)|^2 \leq 1 \quad (s \geq 4m^2)$

- **Modified crossing**

$$S^{ab}_{dc}(s) = \sqrt{\frac{\mathrm{d}_a \mathrm{d}_c}{\mathrm{d}_b \mathrm{d}_d}}\, S^{bc}_{ad}(4m^2 - s) \implies A_\chi(s) = \sum_{\chi'} \overbrace{\mathrm{d}_{\chi'} \begin{bmatrix} v & v & \chi \\ v & v & \chi' \end{bmatrix}}^{C_{\chi\chi'}} A_{\chi'}(4m^2 - s)$$

- **Analyticity** Spectrum fixes analytic properties of $A_\chi(s)$.
Possible singularities are bound state poles, multi-particle cuts…

Bootstrap problem

Given a fusion category,
What is the space of allowed $A_\chi(s)$?

1. Write ansatz that trivializes **A+mC+S**
2. Impose **U** numerically for $s_j \geq 4m^2$
3. Bound parameters, i.e. max functionals

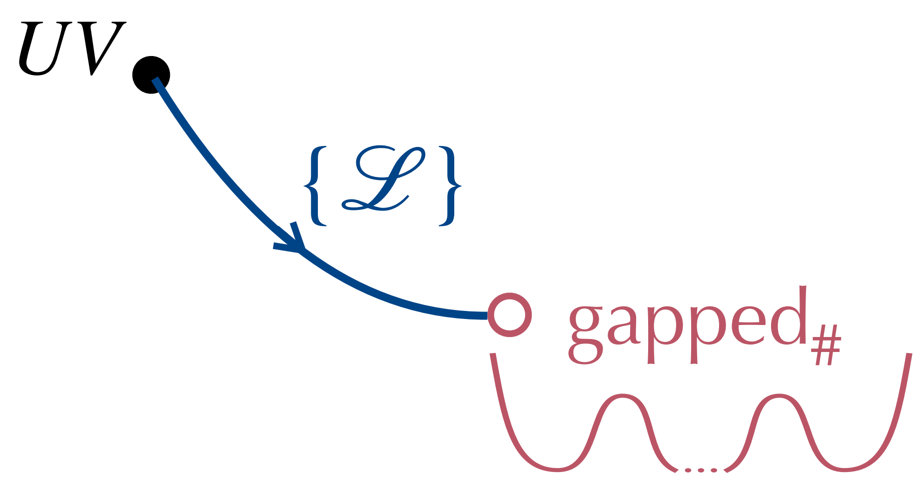
e.g. $\mathscr{F}[A_\chi] = \sum_\chi n_\chi A_\chi(s_*)$

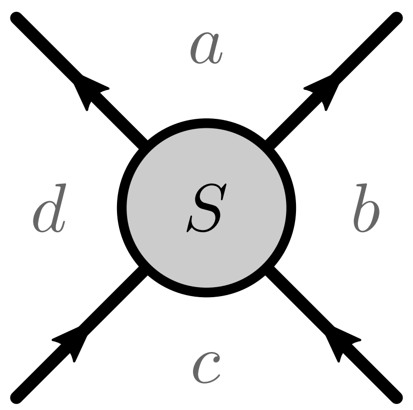
or dual $\mathscr{F}[A_\chi] \leq \mathscr{F}_d[K_\chi] = \int \sum_\chi |K_\chi(s)|$

Example 1: $\mathscr{A}_n$ category

Set-up:

QFT with categorical sym.





$2 \to 2$ Kink scattering

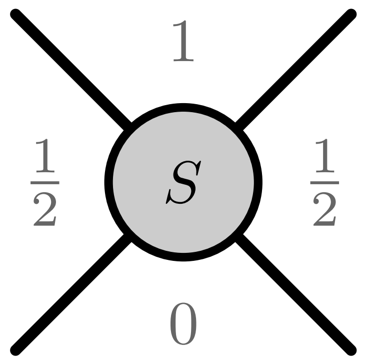
Bootstrap from **A**+**mC**+**U**+**S** assuming:

- minimum spectrum required by symmetry
- SSB all symmetries (#vacua=#$\{\mathscr{L}\}$)

[Copetti, LC, Komatsu '24]

- (n-1) vacua $a = 0, 1/2, \ldots, (n-1)/2$
- Minimum spectrum: $K_{a,a\pm1/2}$

  no bound states

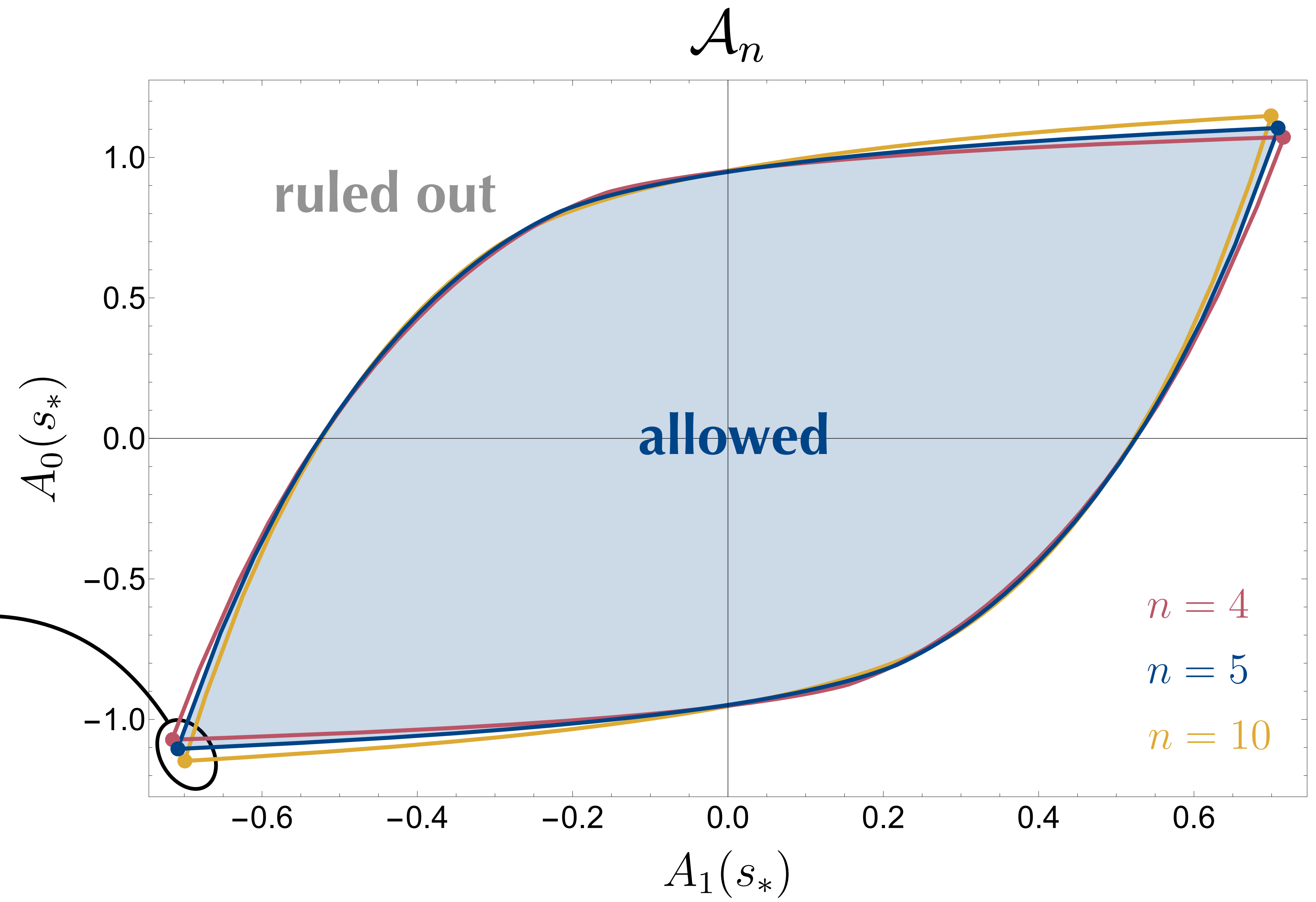


- 2 fusion channels $A_0(s)$, $A_1(s)$

  $\mathscr{L}_{1/2}{}^2 = \mathscr{L}_0 + \mathscr{L}_1$

  $C_{\chi\chi'} = \frac{1}{d_{1/2}} \begin{pmatrix} 1 & d_1 \\ 1 & -1 \end{pmatrix}$ $\quad d_a = \frac{\sin \pi(2a + 1/n)}{\sin \pi/n}$

Numerical Bootstrap $A_{0,1}(s_* = 3.41m^2) \Longrightarrow$



$\mathcal{M}_n - \phi_{1,3}$

$A_0 = Z(\theta)\sinh\left(\frac{i\pi+\theta}{n}\right)$

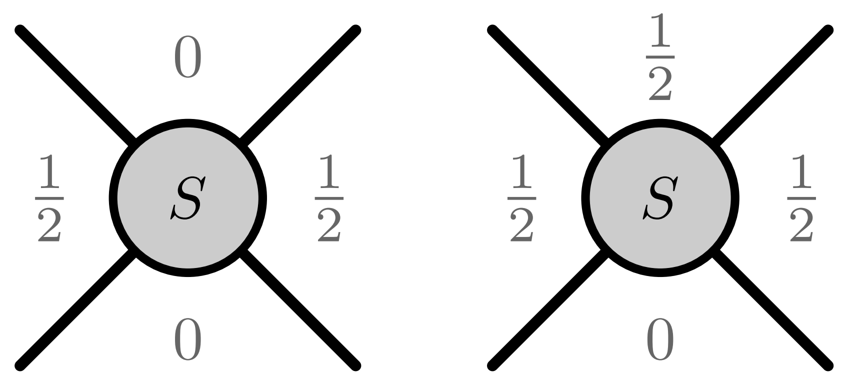
$A_1 = Z(\theta)\sinh\left(\frac{i\pi-\theta}{n}\right)$

- Integrability at vertices.
- No oscillations $(\cdot)^{i\theta}$, good large energy behaviour c.f. $O(N)$, $\mathbb{Z}_{2,4}$, $U(N)$.

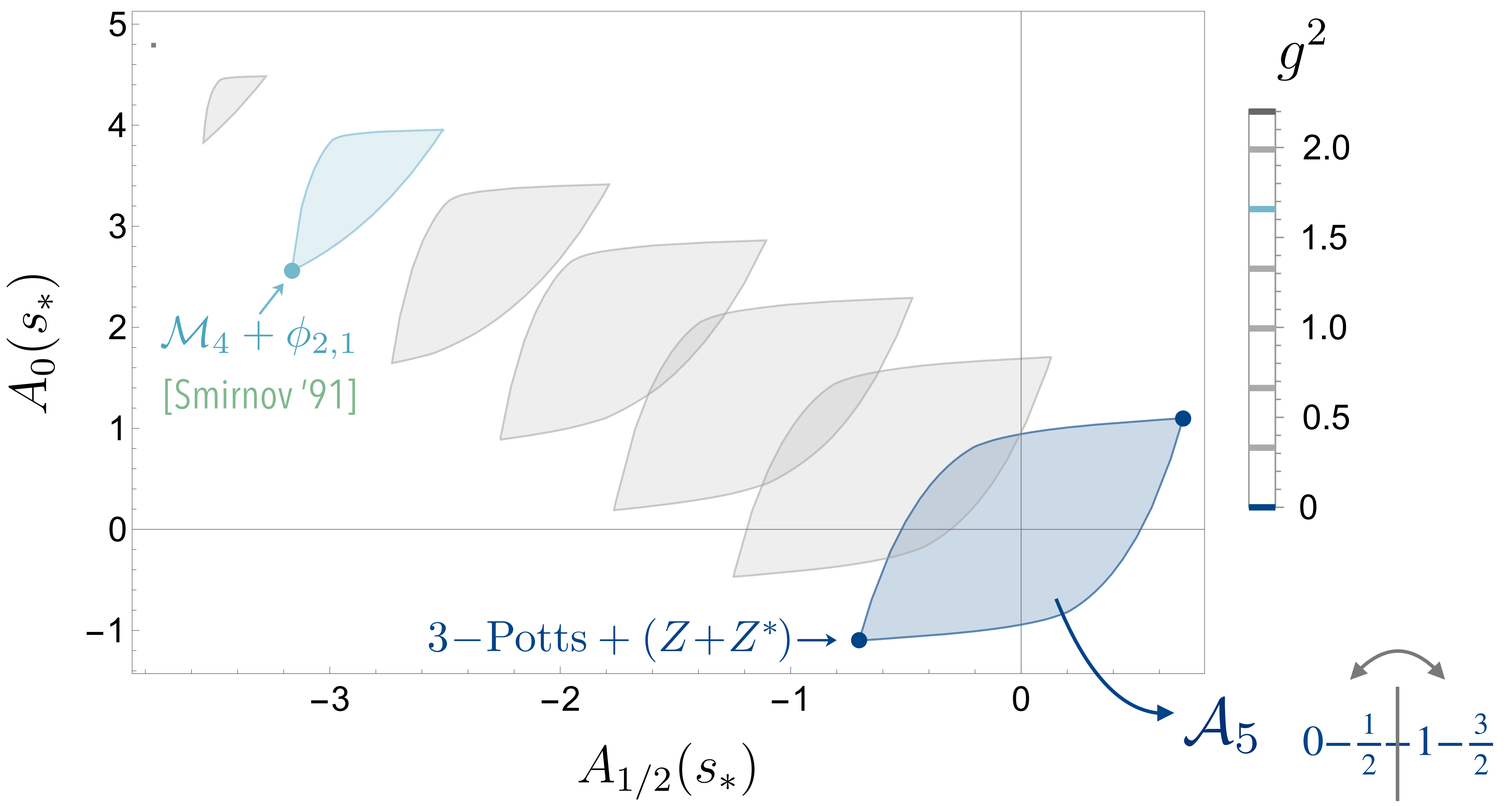
Example 2: Fibonacci

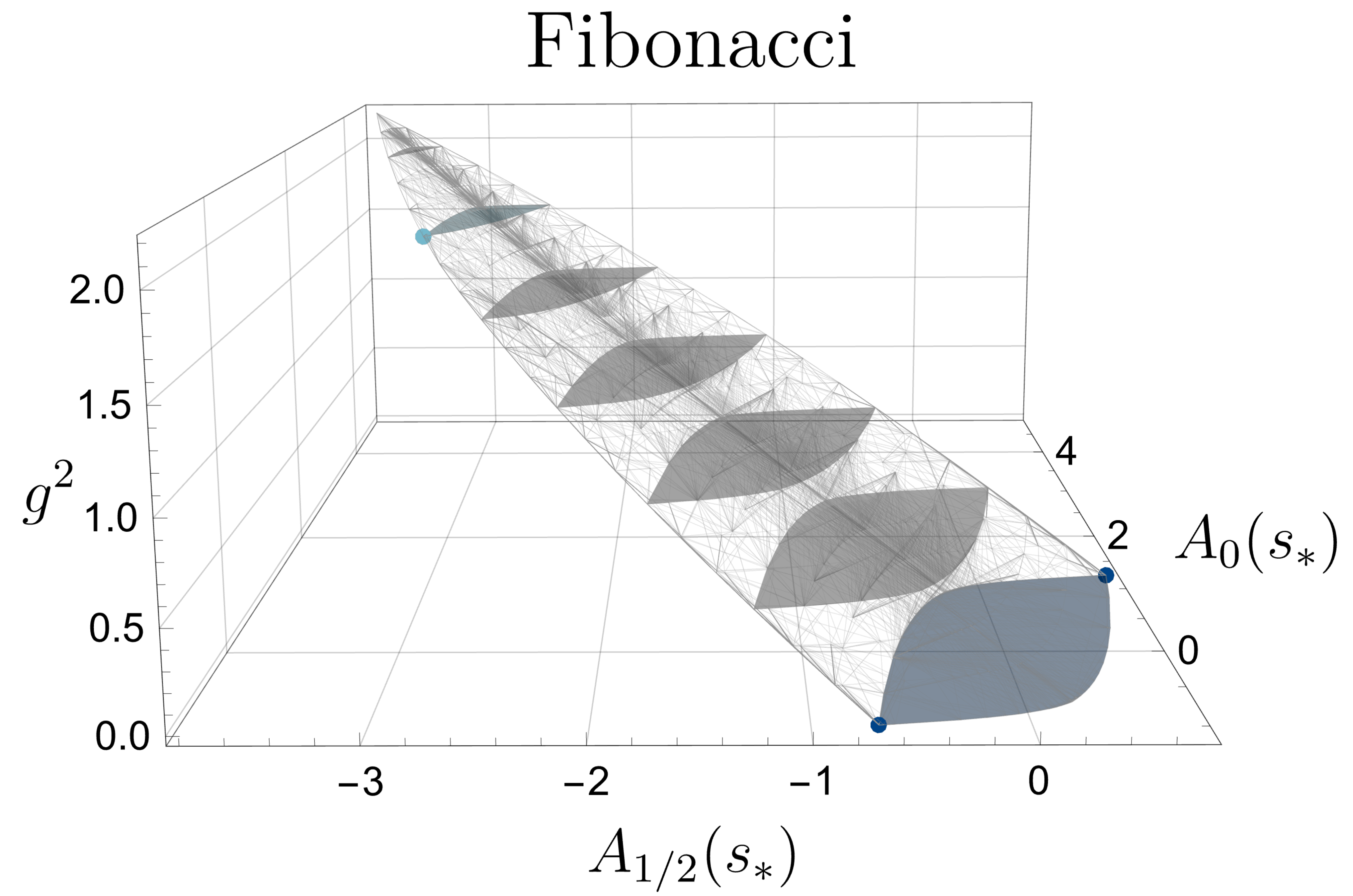
- 2 vacua $a = 0, 1/2$; 2 fusion channels $A_0(s)$, $A_{1/2}(s)$ $\quad \mathscr{L}_{1/2}{}^2 = \mathscr{L}_0 + \mathscr{L}_{1/2}$
- Minimum spectrum: Kink $K_{0,1/2}$, antikink $K_{1/2,0}$ and breather $K_{1/2,1/2}$

  Breather as bound state of kink/antikink → cubic coupling g $\quad A_{1/2}(s) \sim \dfrac{g^2}{s - m^2}$



Numerical Bootstrap $\{A_{0,1/2}(s_* = 3.41m^2), g^2\} \Longrightarrow$



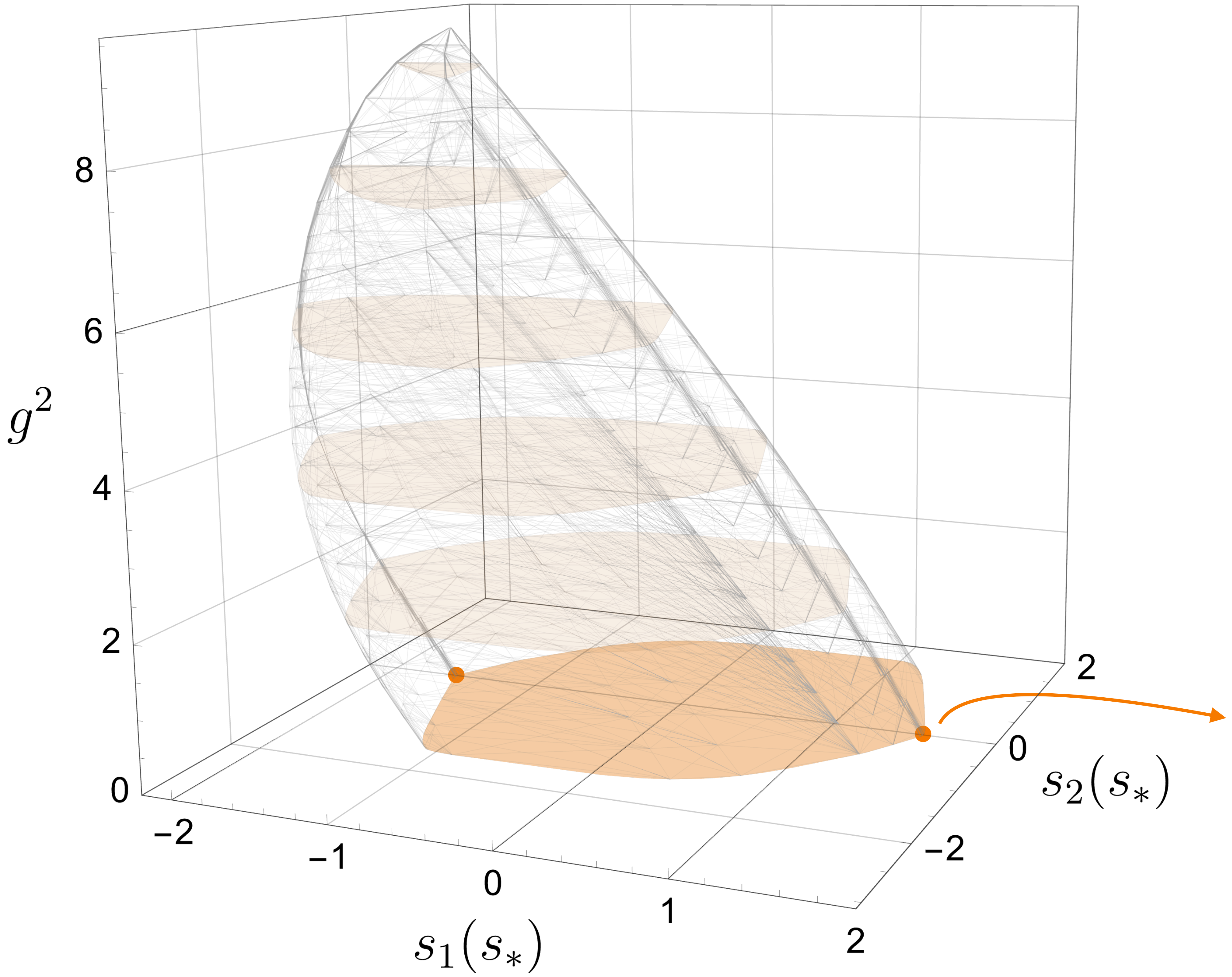


- ‣ Integrability at marked vertices, no oscillations $(\cdot)^{i\theta}$.
- ‣ Enhanced symmetry only at $g^2 = 0$ (breather cannot be interpreted as bound state in Fib $\times\, \tilde{\mathbb{Z}}_2$).

Haagerup $\mathcal{H}_3$: uncharted territory

- 6 topological lines: $\{\underbrace{1,\alpha,\bar\alpha}_{d_{\alpha_i}=1},\underbrace{\rho,\alpha\rho,\bar\alpha\rho}_{d_{\rho_i}=\frac{1}{2}(3+\sqrt{13})=\zeta}\}$. Fusion rules: $\rho\times\bar\alpha=\alpha\rho\,,\quad \rho\times\rho=1+\rho+\alpha\rho+\bar\alpha\rho$
- No known field theory realization! *Hints for CFTs with* $c\sim 2, 3/2$ [Vanhove, Lootens, Van Damme, Wolf, Osborne, Haegeman, Verstraete '21; Huang, Lin, Ohmori, Tachikawa,Tezuka '21; Corcoran, de Leeuw '24]

Haagerup $\mathcal{H}_3$ [LC, in progress]

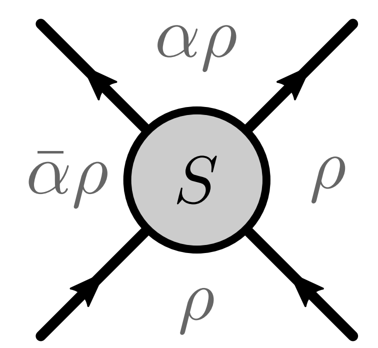


- 6 vacua; 4 fusion channels $A_1(s)$, $A_\rho(s)$, $A_{\alpha\rho}(s)$, $A_{\bar\alpha\rho}(s)$.

  Projectors and modified crossing from F-symbols in [Huang, Lin '21]

$$C=\begin{pmatrix}\zeta^{-1}&\zeta^{-1}&\zeta^{-1}&\zeta^{-1}\\ 1&x&y_1&y_2\\ 1&y_1&y_2&x\\ 1&y_2&x&y_1\end{pmatrix}\qquad x=\frac{2-\sqrt{13}}{3},\quad y_{1,2}=\frac{1}{12}\left(5-\sqrt{13}\mp\sqrt{6}\sqrt{1+\sqrt{13}}\right).$$

- Many kinks $K_{a,b}$ (15), including breather $K_{\rho,\rho}$



$$A_\rho(s)\sim\frac{g^2}{s-m^2}\,,$$

$\{s_1,s_2\}$: crossing sym. combinations of $A_\chi(s)$

- Vertices have $A_\rho=A_{\alpha\rho}=A_{\bar\alpha\rho}$

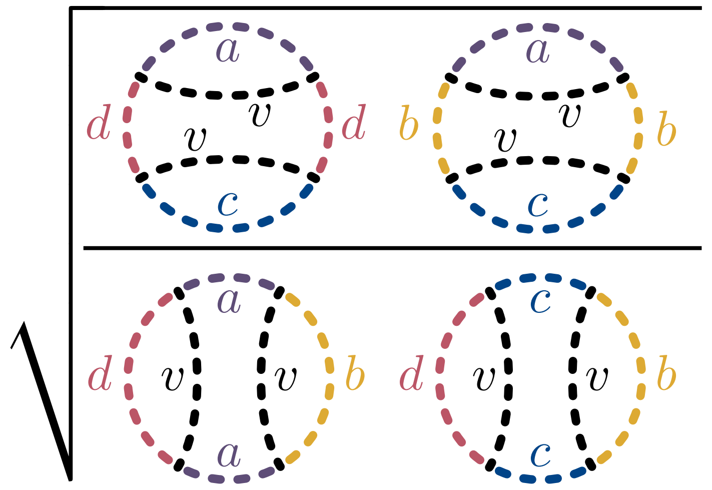
  Analytic solution: $\dfrac{A_\rho(\theta)}{A_1(\theta)}=\dfrac{\alpha+i\tanh(\mu\,\theta/\pi)}{\alpha-i\tanh(\mu\,\theta/\pi)}$

- Seems no integrable solutions…

Final Remarks

Summary

- Non-invertible symmetries and anomalies in 1+1d gapped RG flows lead to **modified crossing**.

$S^{ab}_{dc}(s) =$  $S^{bc}_{ad}(t)$

- Generic for when IR described by non-trivial TQFT.
- Modification from corrections to norms of $|\mathrm{in}\rangle, |\mathrm{out}\rangle$ states.

- **S-matrix Bootstrap** Categorical symmetry $\mathscr{C}$ can be used to explore the space of consistent $\mathscr{C}$-sym QFTs.
  - $\mathscr{A}_n$ and Fibonacci: known integrable models appear at vertices.
  - Haagerup $\mathscr{H}_3$: new models expected.

- Other symmetry breaking patterns: *Non-regular representation. Module category* $\mathscr{M}$, $d_a \to g_a$ *relative Euler terms,* $F \to \varphi^*$ *dual boundary F-symbol*

Future directions

- S-matrix Bootstrap for other categories.
- Form Factors and their inclusion in Bootstrap. [Karateev, Kuhn, Penedones '19]
  → *More hints for field theory realization of Haagerup?*
- Higher d? Chern-Simons+matter [...,Mehta, Patel, Prakash, Minwalla, Sharma '22]
  Monopole scattering [Csaki, Hong, Shirman, Telem, Terning, Waterbury '20]
  [van Beest, Boyle Smith, Delmastro, Komargodski || Mouland, Tong '23]
- Toy model for understanding soft dynamics in gravity and gauge theories?

Thank you!