The Gauge Theory Bootstrap: Predicting Pion Dynamics from QCD

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Based on arXiv: 2309.12402 and 2403.10772 and 2505.19332

with Yifei He (ENS, Paris)

PRL 133, 191601 (2024); PRD 110, 096001 (2024).

See also recent work arXiv: 2511.11513 by R. Cordoba (ENS, Paris).

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Summary

We propose the Gauge Theory Bootstrap approach to study the strongly coupled region of QCD.

- **Problem and Main idea to solve it**
- **Review of S-matrix bootstrap**
- **Gauge Theory Bootstrap setup, new iterative procedure**
- **Results for phase shifts, form factors and low energy parameters**
- **Discussion and future work (N_c, m_q)**

QCD = asymptotically free $SU(N_c)$ gauge theory

N_f massive quarks $m_q \ll \Lambda_{\rm QCD}$ fundamental representation of gauge group

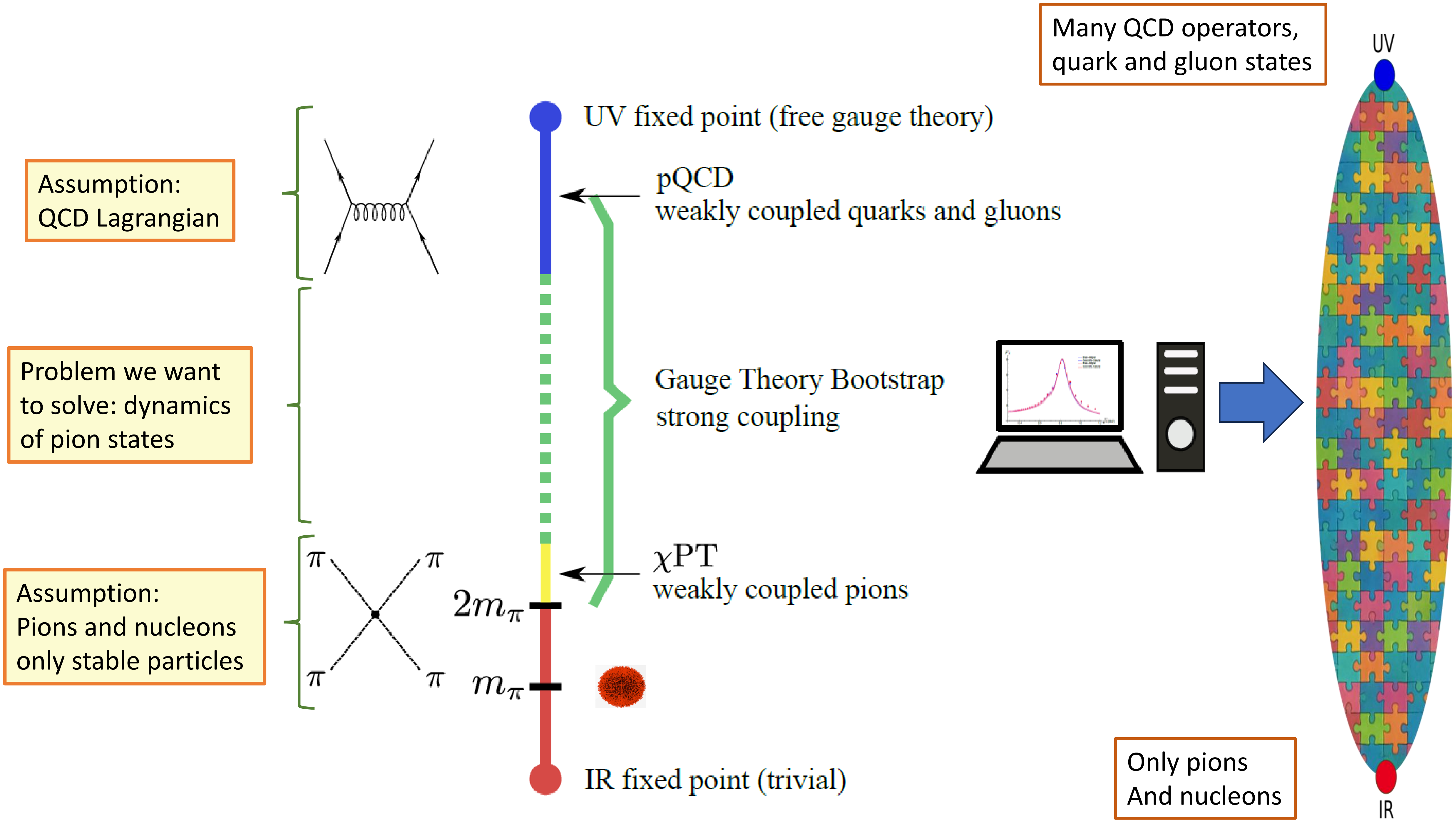
$$\mathcal{L} = i\sum_j^{N_f} \bar{q}_j \not{D} q_j - \sum_j^{N_f} m_q \bar{q}_j q_j - \frac{1}{4} G_a^{\mu\nu} G^a_{\mu\nu} + \text{gauge fixing} + \text{ghost}$$

gauge theory parameters: $N_c \quad N_f \quad m_q \quad \Lambda_{\rm QCD}$

In this talk:
N_c=3 (mostly)
N_f=2

What is the strongly coupled physics?

Chiral symmetry breaking and confinement:
Only QCD stable particles are pions and nucleons
[stable ignoring weak and e.m. interaction]
Even then: Mean life π^+, π^- : 26ns, π^0 : 8.5e-17 s, ρ : 4.5e-24 s



Assumption:
QCD Lagrangian
Problem we want
to solve: dynamics
of pion states
Assumption:
Pions and nucleons
only stable particles
π
π
π
π
$2m_\pi$
m_π
UV fixed point (free gauge theory)
pQCD
weakly coupled quarks and gluons
Gauge Theory Bootstrap
strong coupling
χPT
weakly coupled pions
IR fixed point (trivial)
Many QCD operators,
quark and gluon states
UV
Only pions
And nucleons
IR

The EFT approach to pion physics

$N_f = 2$

very low energy effective Lagrangian (lowest order):

$$\mathcal{L} = \frac{f_\pi^2}{4}\left\{\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + m_\pi^2 \mathrm{Tr}\left(U + U^\dagger\right)\right\} \qquad U = e^{i\frac{\vec{\tau}\cdot\vec{\pi}}{f_\pi}}$$

$$\mathcal{L}_2^{2\pi} = \frac{1}{2}\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} - \frac{1}{2}m_\pi^2\vec{\pi}^2 \qquad \mathcal{L}_2^{4\pi} = \frac{1}{6f_\pi^2}\Big((\vec{\pi}\cdot\partial_\mu\vec{\pi})^2 - \vec{\pi}^2(\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi})\Big) + \frac{m_\pi^2}{24f_\pi^2}(\vec{\pi}^2)^2 \qquad \ldots$$

$$\begin{aligned}\mathcal{L}_4 \;=\; & \frac{l_1}{4}\left\{\mathrm{Tr}[D_\mu U(D^\mu U)^\dagger]\right\}^2 + \frac{l_2}{4}\mathrm{Tr}[D_\mu U(D_\nu U)^\dagger]\mathrm{Tr}[D^\mu U(D^\nu U)^\dagger] \\ & + \frac{l_3}{16}[\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger)]^2 + \frac{l_4}{4}\mathrm{Tr}[D_\mu U(D^\mu\chi)^\dagger + D_\mu\chi(D^\mu U)^\dagger] \\ & + l_5\left[\mathrm{Tr}(f_{\mu\nu}^R U f_L^{\mu\nu} U^\dagger) - \frac{1}{2}\mathrm{Tr}(f_{\mu\nu}^L f_L^{\mu\nu} + f_{\mu\nu}^R f_R^{\mu\nu})\right] \\ & + i\frac{l_6}{2}\mathrm{Tr}[f_{\mu\nu}^R D^\mu U(D^\nu U)^\dagger + f_{\mu\nu}^L(D^\mu U)^\dagger D^\nu U] \\ & - \frac{l_7}{16}[\mathrm{Tr}(\chi U^\dagger - U\chi^\dagger)]^2\end{aligned}$$

[Gasser, Leutwyler, 1983]

unknown l_j coefficients

determined from experiments

should be computed from UV gauge theory

Problem we want to solve

But most importantly, it is not valid at higher energies where pions are strongly coupled

Summary: Assuming confinement and chiral symmetry breaking implying the low energy physics is dominated by pions (Lagrangian given by symmetry) and using,

$$\underbrace{N_c \quad N_f \quad m_q \quad \alpha_s}_{\textit{defining gauge theory}} \qquad \underset{\text{set the units}}{m_\pi} \qquad \underset{\text{size of pion}}{f_\pi}$$

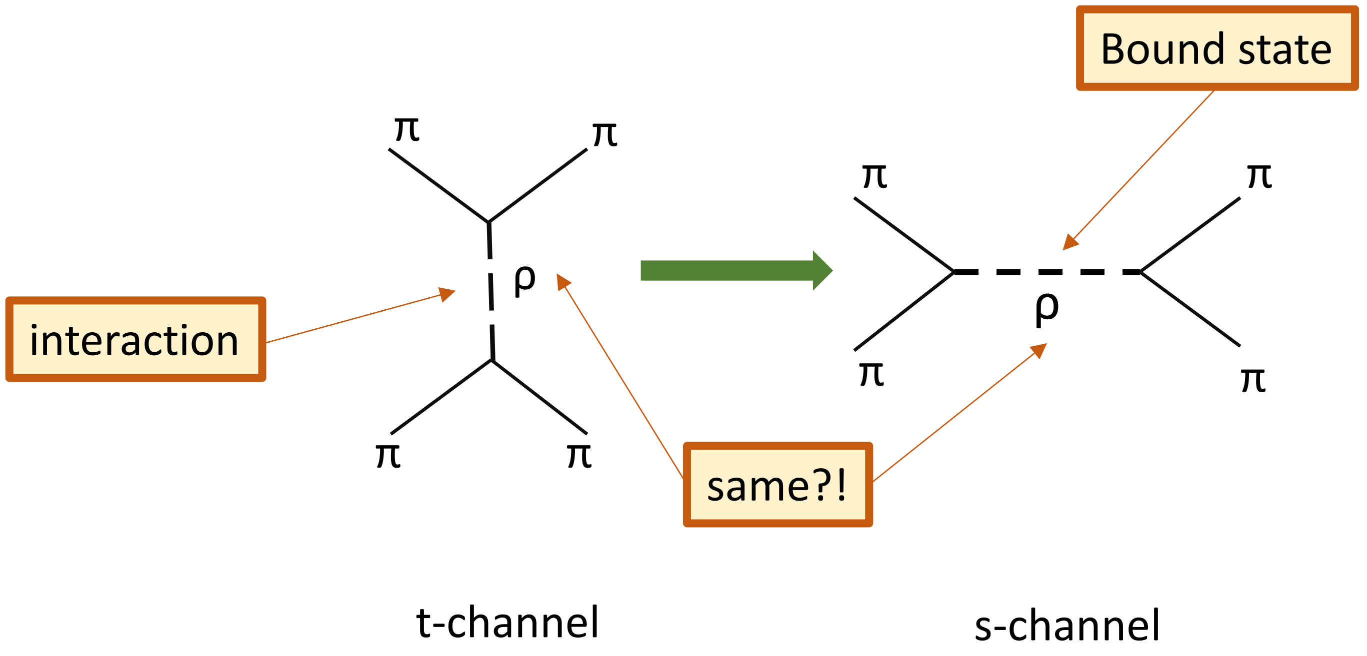
We want to compute, in the strongly coupled region:

- Scattering phase shifts: $\delta_\ell^I(s)$
- Low energy parameters in pion Lagrangian: $l_1, l_2, l_3, l_4, l_5, l_6, \ldots$
- Pion form factors for various currents: $\langle \pi^a(p_1) | j_\mu^b(x) | \pi^c(p_2) \rangle$, $\langle \pi^a(p_1) | T_{\mu\nu}(x) | \pi^c(p_2) \rangle$
- Spectral densities of currents from: $\langle 0 | \hat{T} \left\{ j_\mu^a(x) j_\nu^b(y) \right\} | 0 \rangle$, $\langle 0 | \hat{T} \left\{ T_{\mu\nu}(x) T_{\alpha\beta}(y) \right\} | 0 \rangle$

Theoretical/numerical prediction
- not using experimental scattering data as input
- no assumption on spectrum

Origins of the bootstrap idea

[G. Chew, The Analytic S Matrix: A Basis for Nuclear Democracy, 1966]



More recently those ideas are used to study pion phenomenology. For example analyzing π π scattering using the Roy eq. [Pelaez, Yndurain 2005, Colangelo, Gasser, Leutwyler, 2001]

But: In modern language, QCD at low energy has a mass gap and the physics is dominated by pion scattering that has only one continuous parameter m_π/Λ_{QCD} (having fixed two discrete ones N_c and N_f). Massless quarks have no continuous parameters. So it is somewhat unique!.

Some related S-matrix bootstrap developments and applications

Revival: Use all constraints on the S-matrix **(symmetries, crossing, unitarity, analyticity)** to define a general space of allowed theories. No unique theory will arise in that way, but exploring such space allows one to put bounds on various couplings. [Paulos, Penedones, Toledo, van Rees, Vieira, 2016 & 2017]

Form factors: Other relevant quantities that can be incorporated to the bootstrap are **form factors** and **current correlators.** This is the Form Factor Bootstrap. It was used for example to put bounds on the central charge of certain 2d theories. [Karateev, Kuhn, Penedones, 2019]

Pion bounds and phenomenology: In the context of pion physics (and more general scalar theories) it was used to study the landscape of consistent pion theories and put **bounds on various low energy quantities** such as scattering lengths, Wilson coefficients, values of the amplitudes and its derivatives e.g. at the symmetric point. Also used to fit phenomenological data. [A. Guerrieri, J. Penedones, and P. Vieira, 2019, 2021, Guerrieri, Haring, Su, 2024].

Large-N pions: It was also used to (possible) identify the pion theory corresponding to **large-N gauge theory** by bounding the Wilson coefficients of the non-linear sigma model and finding special points at the boundary of the allowed space. [Albert, Rastelli, 2022, 2023, Albert, Henriksson, Rastelli, Vichi, 2023].

Some other related work is the flux tube bootstrap [Miro, Guerrieri, Hebbar, Penedones, Vieira, Gabai, Gorbenko, Qiao, Albert, Homrich, Dubovsky, ...] and loop equation bootstrap in lattice gauge theory [Anderson, M.K., Kazakov, Zheng, ...].

S-matrix bootstrap revival, example of result:

Paulos, Penedones, Toledo, van Rees, Vieira, 2016 & 2017

Consider the scalar coupling: (it is bounded Lopez-Mennesier 1977)

$$\lambda = \frac{1}{32\pi} \mathcal{M}(\pi^0\pi^0 \to \pi^0\pi^0)\Big|_{s=t=u=\frac{4}{3}m_\pi^2} = \frac{3\pi}{4} A\left(\frac{4}{3}m_\pi^2, \frac{4}{3}m_\pi^2, \frac{4}{3}m_\pi^2\right)$$

For <u>any</u> QFT that contains a scalar field: $-8.02 \leq \lambda \leq 2.661$

But for pions (from non-linear sigma model or Weinberg model):

$$\lambda \simeq \frac{m_\pi^2}{32\pi f_\pi^2} \simeq 0.023$$

We need <u>*dynamical*</u> input to restrict λ, here we use dynamical input from QCD via SVZ sum rules and asymptotic form factors.

Reminder, pion scattering:

$$\pi_a(p_1) + \pi_b(p_2) \to \pi_c(p_3) + \pi_d(p_4)$$

$$T_{ab,cd} = A(s,t,u)\delta_{ab}\delta_{cd} + A(t,s,u)\delta_{ac}\delta_{bd} + A(u,t,s)\delta_{ad}\delta_{bc}$$

Gauge Theory Bootstrap setup:

[2309.12402, 2403.10772, 2505.19332 w/ Yifei He] github: hyfysics/gauge-theory-bootstrap

- Match 2->2 pion scattering with non-linear sigma model at low energy.
  At high energy we cannot compute pion scattering from QCD so we use form factors and spectral densities instead.

- Match form factors and spectral densities at low (free pions) and high energy (SVZ sum rules+ asymptotic form factor).

- Impose positivity of 3x3 matrix for s>4:

2 pion states

Current

$$
\begin{array}{c} \\ \langle \text{in}|_{P',I,\ell} \\ \langle \text{out}|_{P',I,\ell} \\ \langle 0|\mathcal{O}^\dagger_{P',I,\ell} \end{array}
\begin{array}{c} |\text{in}\rangle_{P,I,\ell} \quad |\text{out}\rangle_{P,I,\ell} \quad \mathcal{O}_{P,I,\ell}|0\rangle \\ \begin{pmatrix} 1 & S_\ell^I(s) & \mathcal{F}_\ell^I \\ S_\ell^{I*}(s) & 1 & \mathcal{F}_\ell^{I*} \\ \mathcal{F}_\ell^{I*} & \mathcal{F}_\ell^I & \rho_\ell^I(s) \end{pmatrix} \succeq 0 \end{array}
$$

- We look for saturation of inequality.

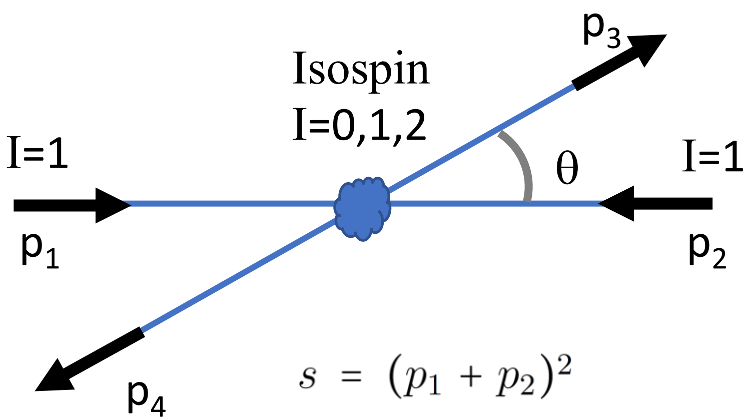
Variables

- Pion Scattering

variables

$$A(s,t,u)=\frac{1}{\pi^2}\int_{4m_\pi^2}^{\infty}dx\int_{4m_\pi^2}^{\infty}dy\left[\frac{\rho_1(x,y)}{(x-s)(y-t)}+\frac{\rho_1(x,y)}{(x-s)(y-u)}+\frac{\rho_2(x,y)}{(x-t)(y-u)}\right]$$

+subtractions



$$s = (p_1+p_2)^2$$
$$t = (p_1-p_3)^2$$
$$u = (p_1-p_4)^2$$

Crossing and (Mandelstam) analyticity ✓

Impose unitarity via partial waves

- Form factors and spectral densities

$$\langle \pi^a(p_1)|j_\mu^b(x)|\pi^c(p_2)\rangle \Rightarrow F(s)=\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}dx\frac{\mathrm{Im}F(x)}{x-s}$$

variables

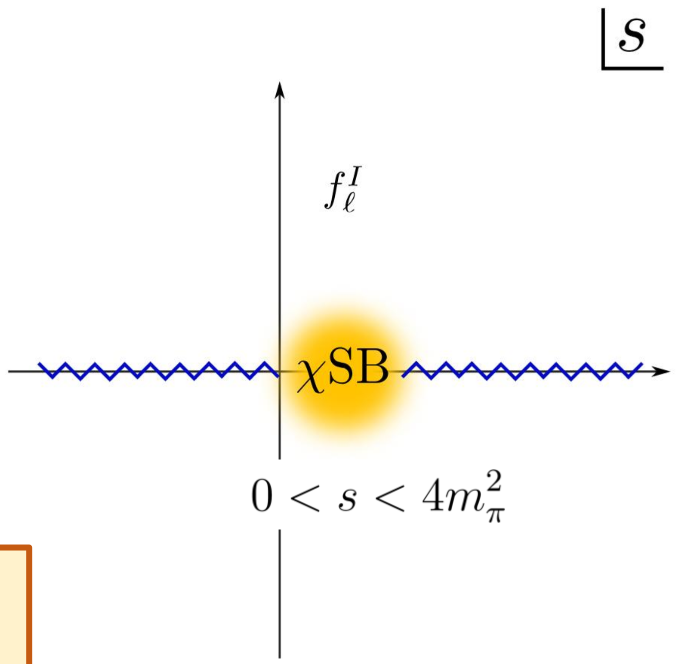
Dispersion relation

$$\Pi_\ell^I(s)=i\int\frac{d^4x}{(2\pi)^4}e^{iPx}\langle 0|\hat{T}\left\{j_\ell^{I\dagger}(x)j_\ell^I(0)\right\}|0\rangle \Rightarrow \rho(s)=2\mathrm{Im}\Pi(s+i\epsilon)\qquad s>4m_\pi^2$$

Low Energy Input

approximate linear behavior of scattering amplitude at very low energy [Weinberg, 1966]

S0: $f^0_{0,\text{tree}}(s) = \frac{2}{\pi}\frac{2s - m_\pi^2}{32\pi f_\pi^2}$ P1: $f^1_{1,\text{tree}}(s) = \frac{2}{\pi}\frac{s - 4m_\pi^2}{96\pi f_\pi^2}$ S2: $f^2_{0,\text{tree}}(s) = \frac{2}{\pi}\frac{2m_\pi^2 - s}{32\pi f_\pi^2}$



requires p.w. in the bootstrap match the tree level non-linear sigma model p.w. in unphysical region

I

$$\frac{f_0^2(s)}{f_1^1(s)} \simeq \frac{3(2m_\pi^2 - s)}{s - 4m_\pi^2} \qquad \frac{f_0^0(s)}{f_1^1(s)} \simeq \frac{3(2s - m_\pi^2)}{s - 4m_\pi^2}$$

no f_π input here:
low energy pion coupling will be a prediction

II

Free pion Lagrangian $\mathcal{L}_2^{2\pi} = \frac{1}{2}\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} - \frac{1}{2}m_\pi^2\vec{\pi}^2$

$I=0, \ell=0\ (S0)$ $\quad j_S \simeq \frac{1}{2}m_\pi^2\pi^a\pi^a + \mathcal{O}(\pi^4)$

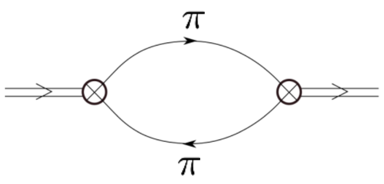
$I=1, \ell=1\ (P1)$ $\quad j_V^\mu \simeq \epsilon^{abc}\pi^b\partial_\mu\pi^c + \mathcal{O}(\pi^4)$

$I=0, \ell=2\ (D0)$ $\quad T^{\mu\nu} \simeq \partial_\mu\pi^a\partial_\nu\pi^a - \frac{1}{2}(\partial_\alpha\pi^a\partial^\alpha\pi^a - m_\pi^2\pi^a\pi^a)\eta_{\mu\nu} + \mathcal{O}(\pi^4)$

$$\rho_0^0(s) \simeq \frac{m_\pi^4}{(2\pi)^4}\frac{3}{16\pi}\left(1-\frac{4m_\pi^2}{s}\right)^{\frac{1}{2}}$$

$$\rho_1^1(s) \simeq \frac{1}{(2\pi)^4}\frac{s}{24\pi}\left(1-\frac{4m_\pi^2}{s}\right)^{\frac{3}{2}}$$

$$\rho_2^0(s) \simeq \frac{1}{(2\pi)^4}\frac{s^2}{160\pi}\left(1-\frac{4m_\pi^2}{s}\right)^{\frac{5}{2}}$$



[Gasser, Leutwyler, 1983]

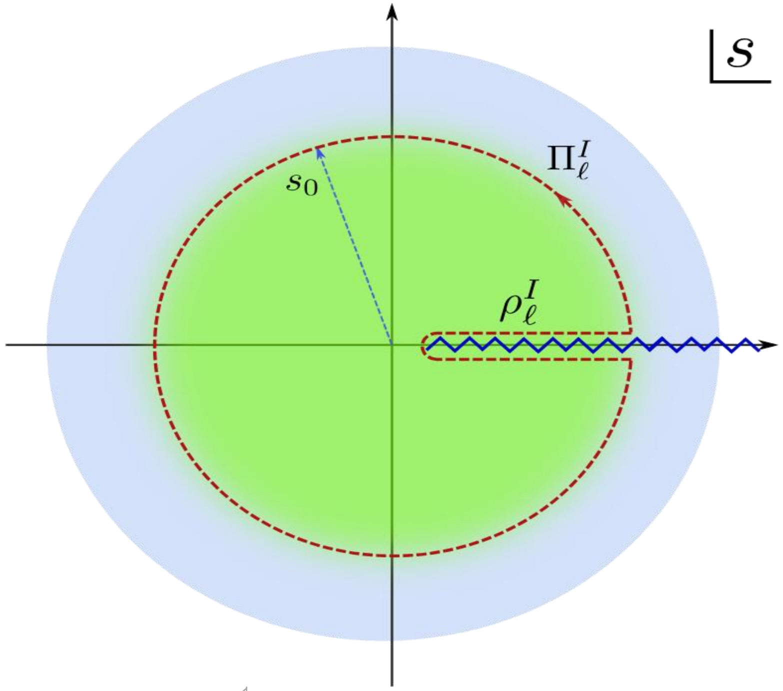
High Energy Input

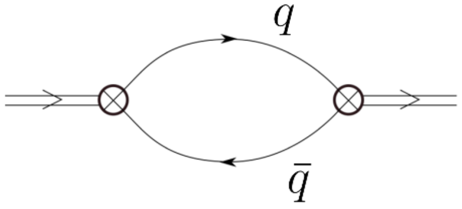
SVZ sum rules

$$\Pi_\ell^I(s) = i\int \frac{d^4x}{(2\pi)^4} e^{iPx}\langle 0|\hat{T}\left\{j_\ell^{I\dagger}(x) j_\ell^I(0)\right\}|0\rangle$$

$$\int_4^{s_0} \rho(x) x^n dx = -s_0^{n+1}\int_0^{2\pi} e^{i(n+1)\varphi}\Pi(s_0 e^{i\varphi})d\varphi$$ III

[Shifman, Vainshtein, Zakharov, 1979]



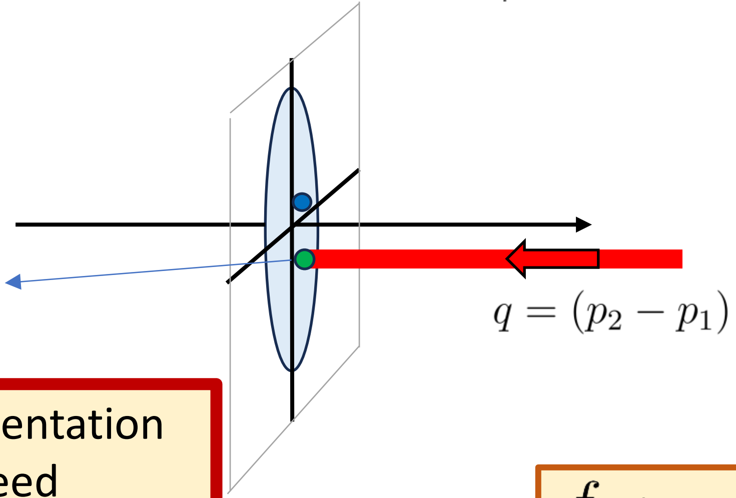


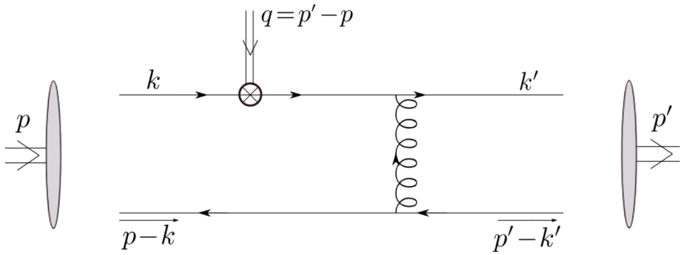
Asymptotic form factors

$$\langle\pi(p_2)|J^\mu_{\text{em}}(0)|\pi(p_1)\rangle = (p_1^\mu + p_2^\mu)F_\pi(q^2)$$

$$F_\pi(s) \simeq -\frac{16\pi\alpha_s(s)f_\pi^2}{s}$$ IV

[Lepage, Brodsky, 1979]
[Mueller, Phys. Rep. 1981]





In current implementation the coefficients need rescaling, too small. Improvement needed.

f_π **input here, equivalent to the pion size as seen by the high energy probe**

Example:

$D0$

$$j_2^0 = T^{\mu\nu}\Delta_\mu\Delta_\nu$$

Choose operator(s)

$$\langle \pi^+(p_1)\pi^-(p_2)|j_2^0|0\rangle = 4|\vec{p}_1|^2\sqrt{\frac{2\pi}{15}}\,Y_{22}(\hat{p}_1)\ F_2^0(s)$$

Define form factor

$$F_2^0(0) = 1$$

Can we compute it at s=0?

$$\langle I=0, P\ell\sigma|j_2^0|0\rangle = \delta_{\ell 2}\delta_{\sigma 2}\mathcal{F}_2^0(s)$$

$$\mathcal{F}_2^0(s) = \frac{1}{4\pi^3}\sqrt{\frac{3\pi}{15}}\frac{1}{s^{\frac{1}{4}}}\left(\frac{s-4}{4}\right)^{\frac{5}{4}}\ F_2^0(s)$$

$$\Pi_2^0(s) \simeq -\frac{1}{(2\pi)^4}\frac{1}{8\pi^2}\left(\frac{11}{10}-\frac{17}{18}\frac{\alpha_s}{\pi}\right)\ s^2\ \ln\left(-\frac{s}{\mu^2}\right)$$

Operator 2-point function

$$\frac{1}{s_0^{n+3}}\int_4^{s_0}\rho_2^0(x)x^n dx \simeq \frac{1}{(2\pi)^4}\frac{1}{4\pi}\frac{1}{n+3}\left(\frac{11}{10}-\frac{17}{18}\frac{\alpha_s}{\pi}\right),\ n\geq -2$$

$$|F_2^0(s)| \simeq \frac{48\pi\alpha_s f_\pi^2}{s},\ (s\to\infty)$$

Asymptotic value

$$\rho_2^0 = \frac{1}{(2\pi)^4}\frac{s^2}{160\pi}\left(1-\frac{4}{s}\right)^{\frac{5}{2}}\quad [\text{low energy}]$$

Procedure: We construct and look for saturation of positivity for a matrix of pion states and QCD operators [He, M.K., 2025]

$$A_{ij} = \langle \psi_i | \psi_j \rangle, \quad \xi_i^* \langle \psi_i | \psi_j \rangle \xi_j = || \xi_i | \psi_i \rangle ||^2 \geq 0 \quad \Rightarrow \quad A \succeq 0$$

2 pion state

2 pion state

Current (QCD operators)

$$\begin{array}{c} \langle \mathrm{in}|_{P',I,\ell} \\ \langle \mathrm{out}|_{P',I,\ell} \\ \langle 0|\mathcal{O}^\dagger_{P',I,\ell} \end{array} \begin{array}{c} \begin{array}{ccc} |\mathrm{in}\rangle_{P,I,\ell} & |\mathrm{out}\rangle_{P,I,\ell} & \mathcal{O}_{P,I,\ell}|0\rangle \end{array} \\ \begin{pmatrix} 1 & S_\ell^I(s) & \mathcal{F}_\ell^I \\ S_\ell^{I*}(s) & 1 & \mathcal{F}_\ell^{I*} \\ \mathcal{F}_\ell^{I*} & \mathcal{F}_\ell^I & \rho_\ell^I(s) \end{pmatrix} \end{array} \succeq 0$$

One can add more pion states (4 pions etc.) and more operators.

Currently we reach (close to) saturation by moving along the gradient of the constraints using an iterative procedure. Why saturation?

Intuition: saturating the matrix of state overlaps

Ignoring other interactions, pion (and nucleon) are the only stable particles in QCD

The Hilbert space is spanned by n-pion (|in> or |out>) states:

$$|\pi\rangle,\ |\pi\pi\rangle,\ |\pi\pi\pi\rangle,\ |\pi\pi\pi\pi\rangle,\ \ldots$$

To use the QCD information, we also consider states created by QCD operators:

$$\mathcal{O}_1|0\rangle,\ \ \mathcal{O}_2|0\rangle,\ \ \mathcal{O}_3|0\rangle,\ \ \ldots$$

$\langle\pi|$, $\langle\pi\pi|$, $\langle\pi\pi\pi|$, $\langle\pi\pi\pi\pi|$, $\ldots$, $\langle\mathcal{O}_1|$, $\langle\mathcal{O}_2|$, $\langle\mathcal{O}_3|$, $\ldots$

- **Scattering is unitary**
- **State created by local operator should be representable by pion states**

matrix should have zero modes

$$\succeq 0$$

should saturate

We look for null states and later we improve by adding more operators and/or states

Simple example, uncertainty principle

$$\Delta x\, \Delta p \geq \frac{\hbar}{2}$$

$$\begin{aligned} 1 &= \langle\psi|\psi\rangle = \int |\psi(x)|^2\, dx \\ 0 &= \langle\psi|x|\psi\rangle = \int x|\psi(x)|^2\, dx \\ 0 &= \langle\psi|p|\psi\rangle = \int \psi^*(x)\psi'(x)\, dx \end{aligned}$$

$$\Rightarrow \quad \int x^2|\psi(x)|^2\, dx \times \int |\psi'(x)|^2\, dx - \frac{1}{4} \geq 0$$

Can this inequality saturate (=0)?
It seems there are many possibilities

$$\mathcal{O}_\lambda = \hat{x} + i\lambda\hat{p} \quad \Rightarrow \quad \langle\psi|\mathcal{O}_\lambda^\dagger \mathcal{O}_\lambda|\psi\rangle \geq 0 \quad \Rightarrow \quad (\Delta x)^2 - \frac{\hbar^2}{4(\Delta p)^2} \geq 0$$

$$\lambda = \frac{\hbar}{2(\Delta p)^2}$$

Saturation gives a differential equation!

$$\mathcal{O}_\lambda|\psi\rangle = 0 \quad \Rightarrow \quad x\psi(x) + \lambda\hbar\psi'(x) = 0 \quad \Rightarrow \quad \psi(x) = A\, e^{-\frac{x^2}{2\lambda\hbar}}$$

Null state

The shape of the function is completely determined by the saturation condition

Iteration procedure currently used to look for null states

Saturation of the 3x3matrix implies (KKP):

$$\begin{cases} |S_\ell^I(s)| = 1 \\ e^{i\delta_\ell^I(s)} = e^{i\delta_{\ell,F}^I(s)} \end{cases}$$

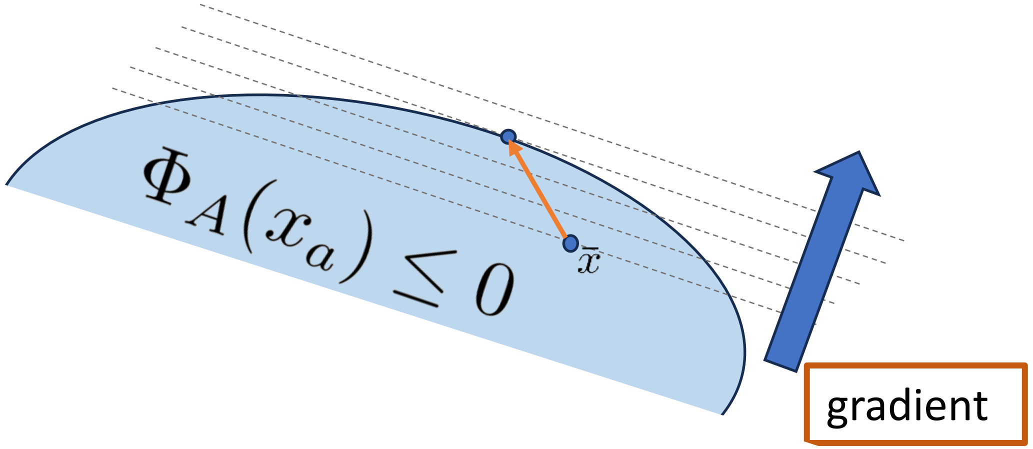
Watson's theorem

Phase shift

Phase of form factor $F_\ell^I(s) = |F_\ell^I(s)|\, e^{i\delta_{\ell,F}^I(s)}$

Assume now we want to improve the saturation of the matrix.

In general, for a constraint $\Phi_A(x_a) \leq 0$



x_a Variables

$$\mathcal{F} = \sum_{A,a} \frac{\partial \Phi_A}{\partial x_a}\bigg|_{\bar{x}_a} x_a$$

New functional

In our case we consider the unitarity constraint

$$(S_\ell^I)^* \, S_\ell^I \leq 1 \qquad \Phi_A \;\Rightarrow\; (S_\ell^I)^* \, S_\ell^I - 1 \leq 0$$

Assuming a linear parameterization of the amplitude:

$$S_\ell^I(s) = 1 + i h_\ell^I(s) = 1 + i \sum_a h_{\ell,a}^I(s) \, x_a$$

This iterative procedure was already used in arXiv:1805.02812 w/ Yifei He and A. Irrgang to get precise results for the 2d bosonic O(N) model scattering matrix.

We get a new functional that we use in the iteration:

$$\mathcal{F} = \sum_{I,\ell} \int ds \left[(S_\ell^I)^* \underbrace{\frac{\partial S_\ell^I(s)}{\partial x_a} x_a}_{i h_{\ell,\text{new}}^I} + \text{c.c.} \right]$$

$$= \sum_{I,\ell} \int ds \left[h_{\ell,\text{new}}^I (h_{\ell,\text{old}}^I)^* + (h_{\ell,\text{new}}^I)^* h_{\ell,\text{old}}^I - 2 \, \text{Im}(h_{\ell,\text{new}}^I) \right]$$

However now we have form factors and we want to include Watson's theorem without complicating the numerics.

The simplest but clear idea we found was to replace, in the functional, the phase of the partial wave by the phase of the form factor:

$$h^I_{\ell,\mathrm{old}} \rightarrow |h^I_{\ell,\mathrm{old}}| e^{i\delta^I_{\ell,F}}$$

The iterative procedure converges to a solution that saturates unitarity (when possible) and satisfies Watson's theorem (when possible).

Intuitively we can think of a chain of information as follows:

1) QCD sum rules put QCD info into the spectral density ρ_ℓ^I

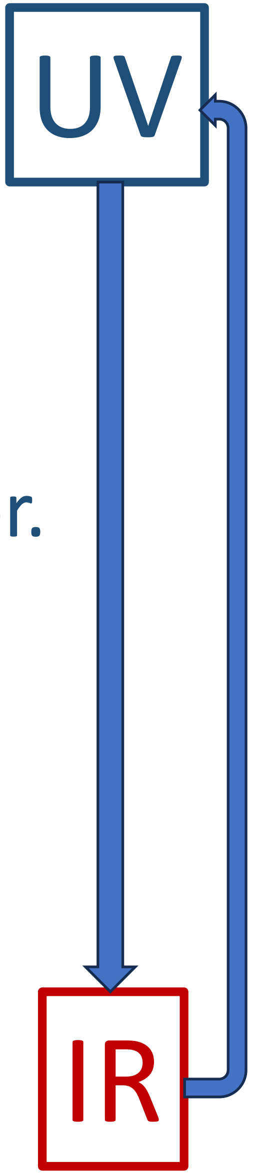
2) Saturation implies $\rho_\ell^I = |\mathcal{F}_\ell^I|^2$.

3) Analyticity relates the modulus and the phase of the form factor.

$$\ln F_\ell^I = \ln |F_\ell^I| + i\,\delta_{\ell,F}^I$$

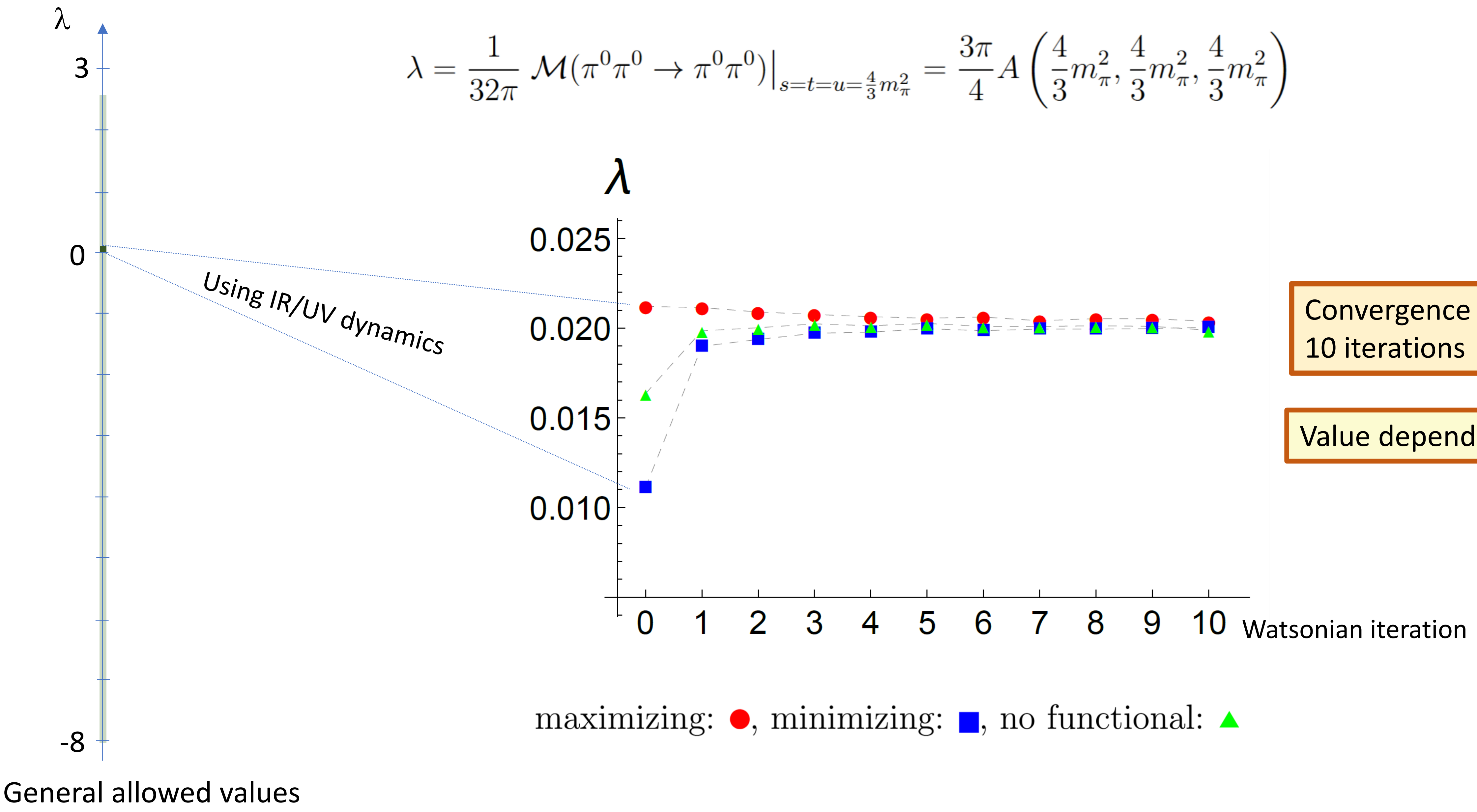
4) Watson's theorem implies $\delta_{\ell,F}^I = \delta_\ell^I$

5) The phase shifts are required to be consistent for a low energy amplitude, and the procedure makes everything consistent.



Test of Iterative procedure: Computing pion quartic coupling

$$\lambda = \frac{1}{32\pi}\,\mathcal{M}(\pi^0\pi^0 \to \pi^0\pi^0)\Big|_{s=t=u=\frac{4}{3}m_\pi^2} = \frac{3\pi}{4} A\left(\frac{4}{3}m_\pi^2, \frac{4}{3}m_\pi^2, \frac{4}{3}m_\pi^2\right)$$



Convergence after 10 iterations

Value depends on UV input

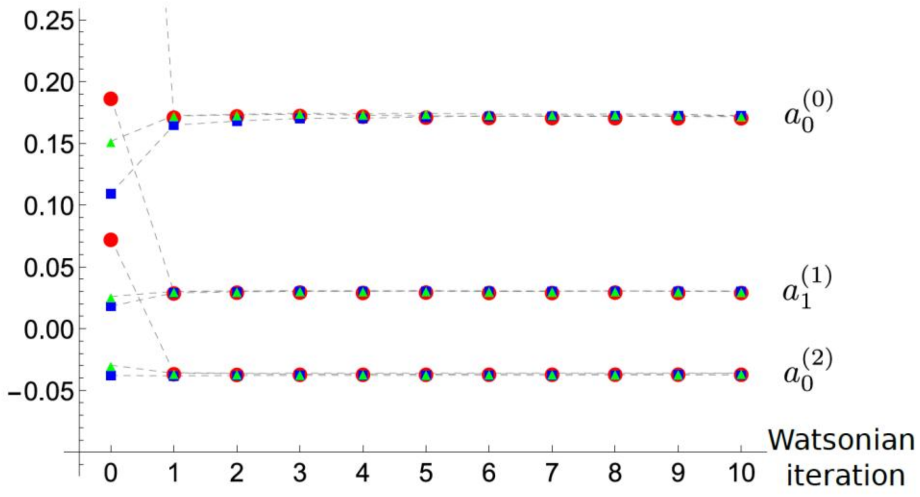


Figure 4: Convergence of max-min $a_\ell^{(I)}$ with Watsonian iterations. maximizing: ●, minimizing: ■, no functional: ▲

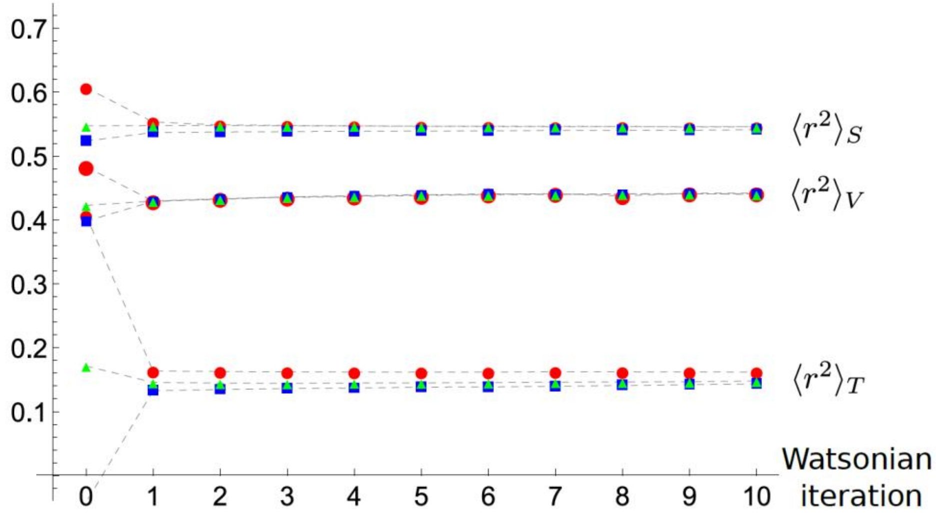


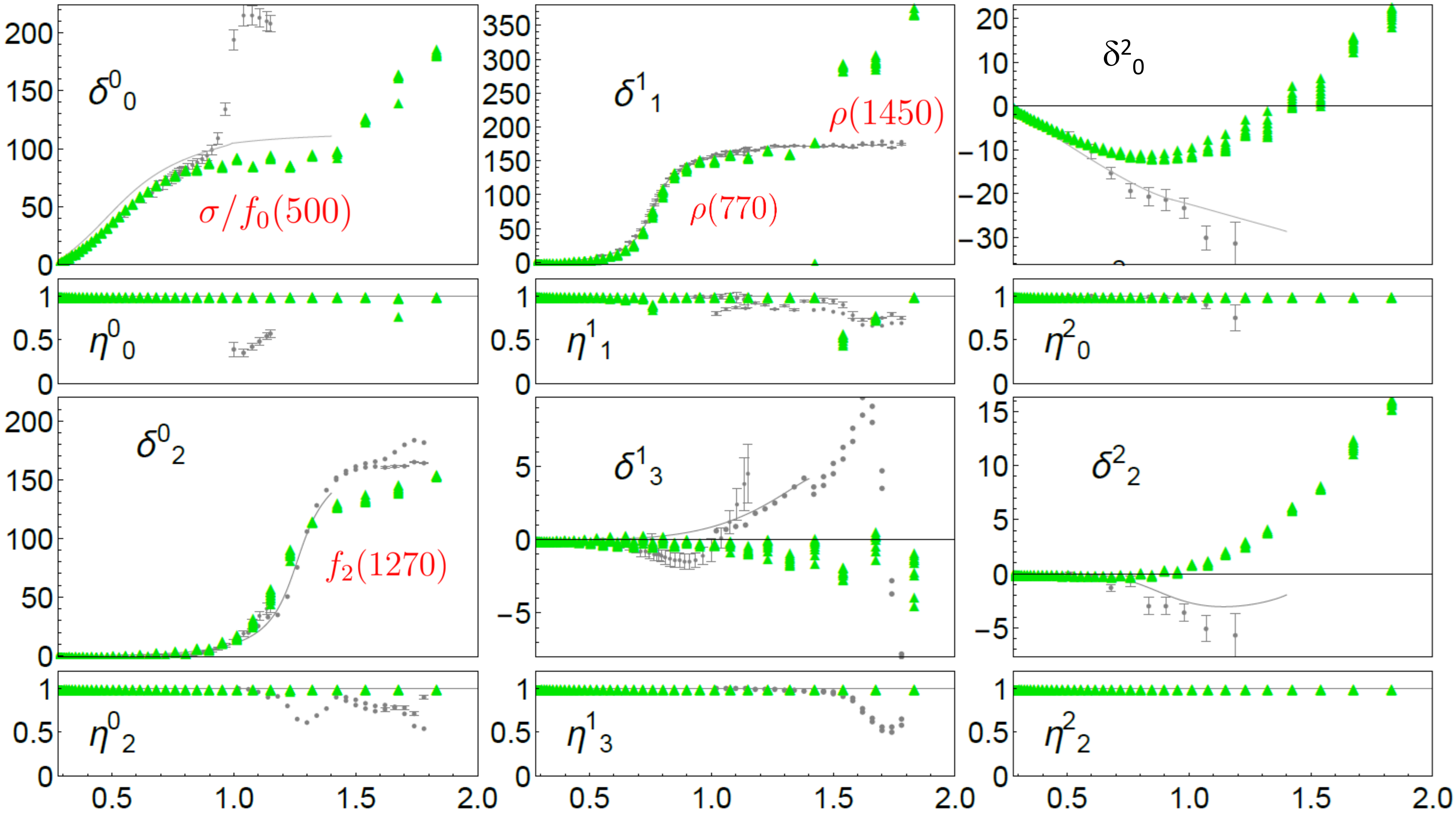
Figure 5: Convergence of max-min $\langle r_\pi^2 \rangle_\ell^I$ with Watsonian iterations. maximizing: ●, minimizing: ■, no functional: ▲. As conventional we denote $\langle r_\pi^2 \rangle_0^0 = \langle r_\pi^2 \rangle_S$, $\langle r_\pi^2 \rangle_1^1 = \langle r_\pi^2 \rangle_V$, $\langle r_\pi^2 \rangle_2^0 = \langle r_\pi^2 \rangle_T$, the scalar vector and tensor radii.

The same convergence is seen
in scattering lengths and pion radii

Results for phase shifts, form factors and low energy parameters.

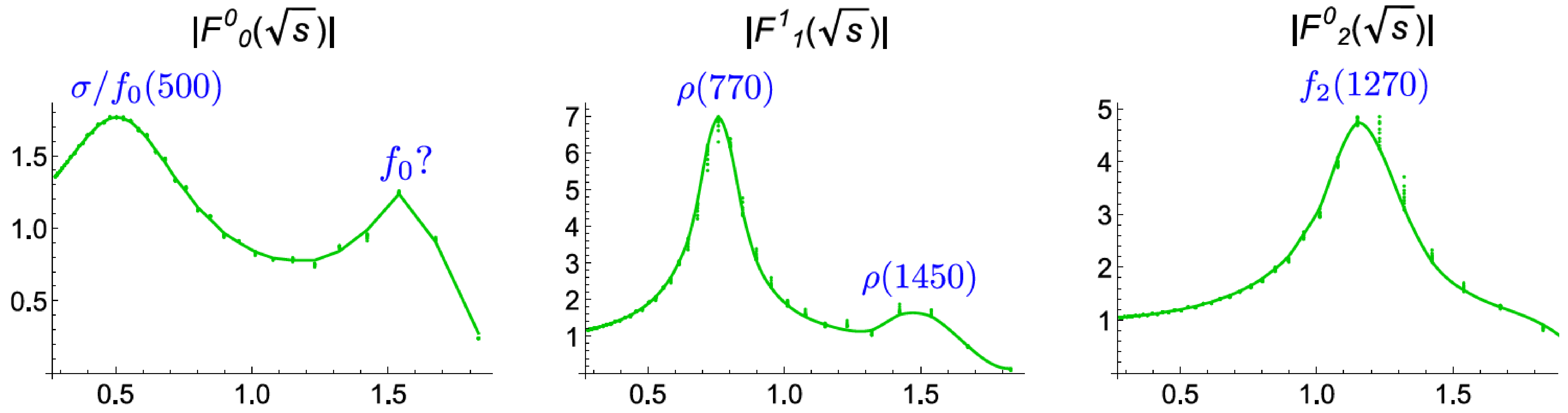
Partial waves from 1-10 Watsonian iterations

starting from
a generic
feasible point



experimental data
(gray dots)
[Protopopescu et al, 1973]
[Losty et al, 1974]
pheno fit
(gray line)
[Pelaez, Yndurain, 2005]

Form factors



Low energy (near threshold) results

Computed low energy data

	GTB	GL	PY	W
✓ $a_0^{(0)}$	0.172	0.220 ± 0.005	0.230 ± 0.010	0.16
✓ $a_0^{(2)}$	-0.0366	-0.0444 ± 0.0010	-0.0422 ± 0.0022	-0.046
✓ $b_0^{(0)}$	0.251	0.280 ± 0.001	0.268 ± 0.010	0.18
✓ $b_0^{(2)}$	-0.063	-0.080 ± 0.001	-0.071 ± 0.004	-0.092
✓ $10^3 \times a_1^{(1)}$	32.1	37.0 ± 0.13	38.1 ± 1.4	31
$10^3 \times b_1^{(1)}$	2.69	5.67 ± 0.13	4.75 ± 0.16	0
$10^4 \times a_2^{(0)}$	12.9	17.5 ± 0.3	18.0 ± 0.2	0
$10^4 \times a_2^{(2)}$	-1.1	1.70 ± 0.13	2.2 ± 0.2	0

		Exp. fits
✓ $\langle r^2 \rangle_S^\pi$ (fm^2)	0.55	0.61 ± 0.04
✓ $\langle r^2 \rangle_V^\pi$ (fm^2)	0.441	0.439 ± 0.0087
$\langle r^2 \rangle_T^\pi$ (fm^2)	0.146	?

Prediction?

Gasser-Leutwyler
Pelaez-Yndurain
Weinberg

Scalar: $F_0^0(s) = F_0^0(0)\left[1 + \frac{1}{6} s \langle r^2 \rangle_S^\pi + \ldots\right]$

Vector : $F_1^1(s) = 1 + \frac{1}{6} s \langle r^2 \rangle_V^\pi + \ldots$

Gravitational: $F_2^0(s) = 1 + \frac{1}{6} s \langle r^2 \rangle_T^\pi + \ldots$

π size

0.8 fm

0.6 fm

0.4 fm

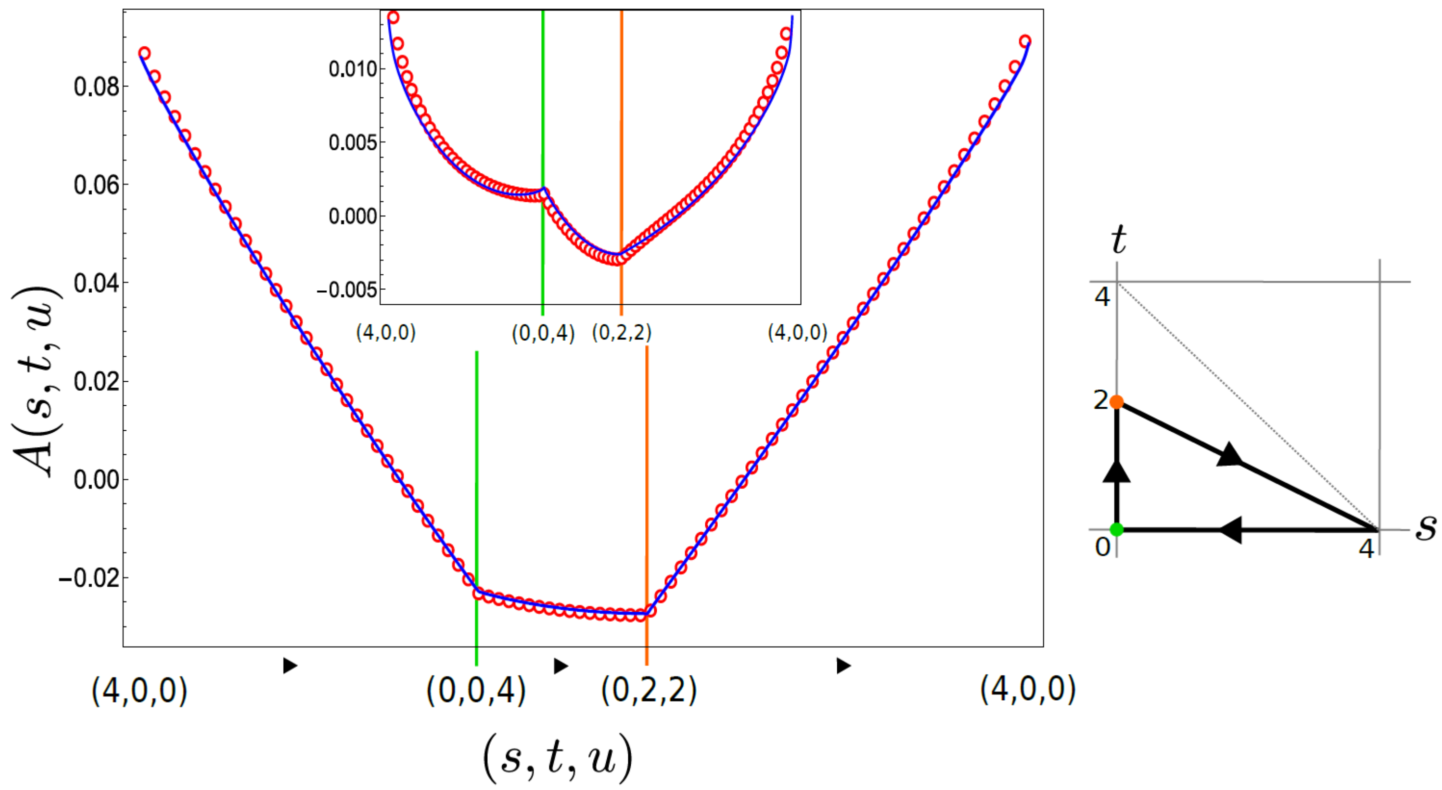
Computing LEC

		GTB	GL	Bij	CGL
	$\bar{\ell}_1$	1.6	-2.3 ± 3.7	-1.7 ± 1.0	-0.4 ± 0.6
✓	$\bar{\ell}_2$	5.5	6.0 ± 1.3	6.1 ± 0.5	4.3 ± 0.1
	$\bar{\ell}_3$	7.8	2.9 ± 2.4		
✓	$\bar{\ell}_4$	4.7	4.3 ± 0.9	4.4 ± 0.3	4.4 ± 0.2
✓	$\bar{\ell}_6$	14.3	18.7 ± 1.1	16.0 ± 0.5 ± 0.7	
			Exp.	**W**	
✓	λ	0.02		0.023	
✓	f_π (MeV)	101	92		

pion coupling

No experimental input, only from SVZ sum rules and asymptotic form factors

Comparison with one-loop χ_{PT}



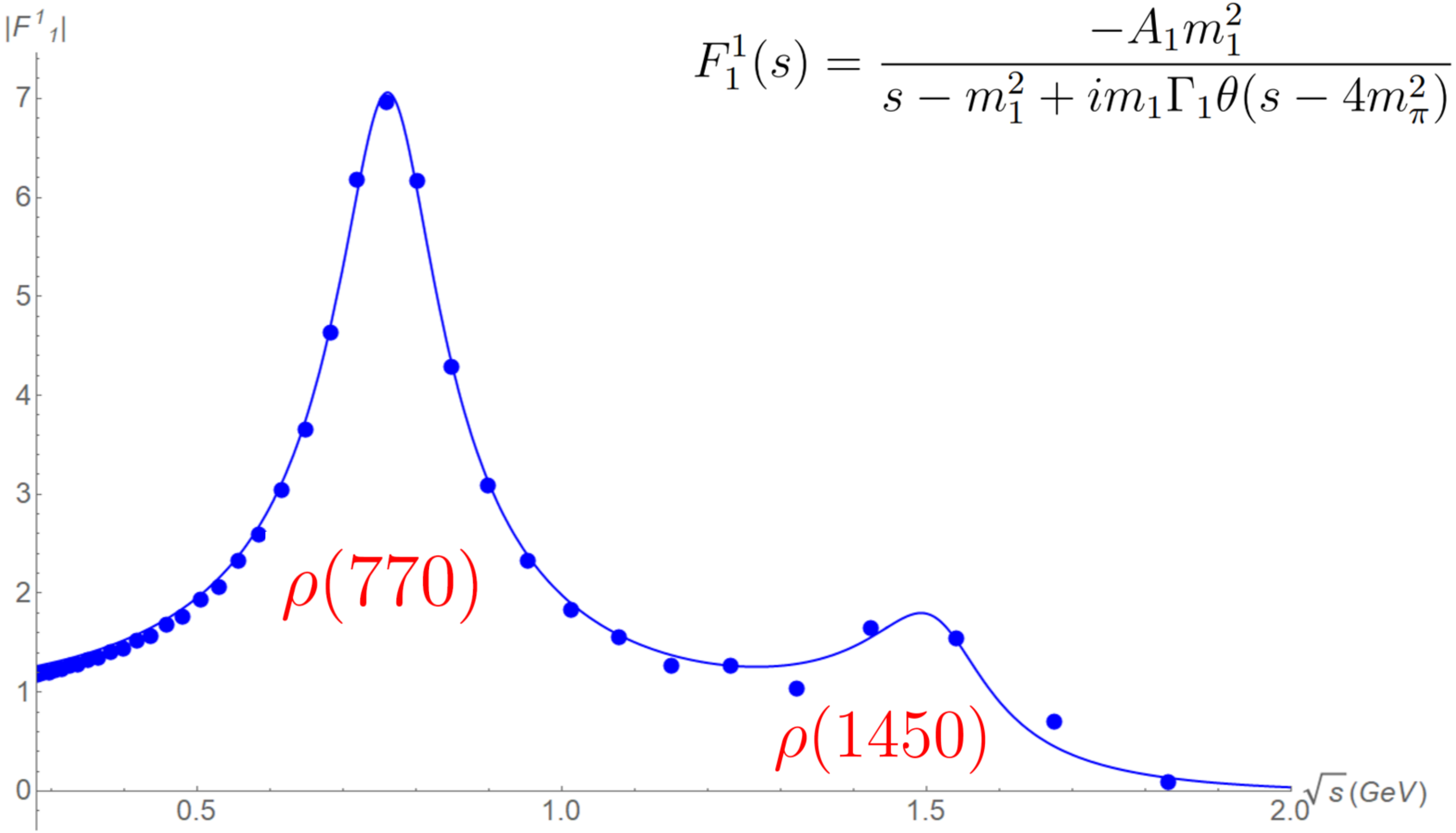
Inside the Mandelstam triangle

Figure 6: We follow the value of the pion scattering amplitude $A(s,t,u) = A(s,u,t)$ along the path indicated on the right in the (s,t) plane. The blue line is the prediction of χ_{PT} using tree-level plus one-loop with the values of the constants extracted from our result as shown in table 1. The red circles are the results of the GTB converged solution. The inset depicts the same with the tree-level part removed.

Intermediate energy results

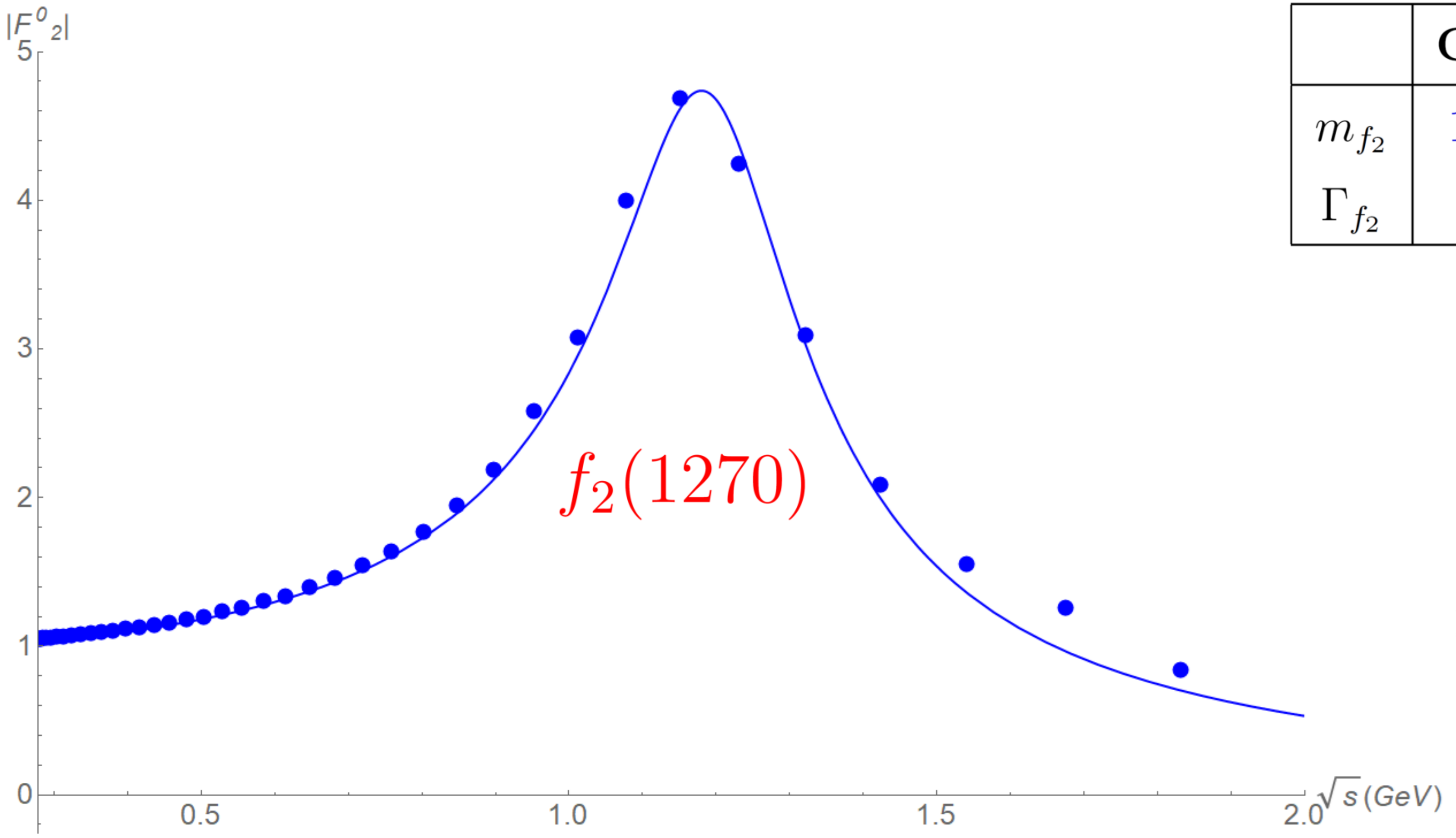
Vector form factor

$$F_1^1(s) = \frac{-A_1 m_1^2}{s - m_1^2 + i m_1 \Gamma_1 \theta(s - 4m_\pi^2)} + \frac{A_2 m_2^2}{s - m_2^2 + i m_2 \Gamma_2 \theta(s - 4m_\pi^2)}$$

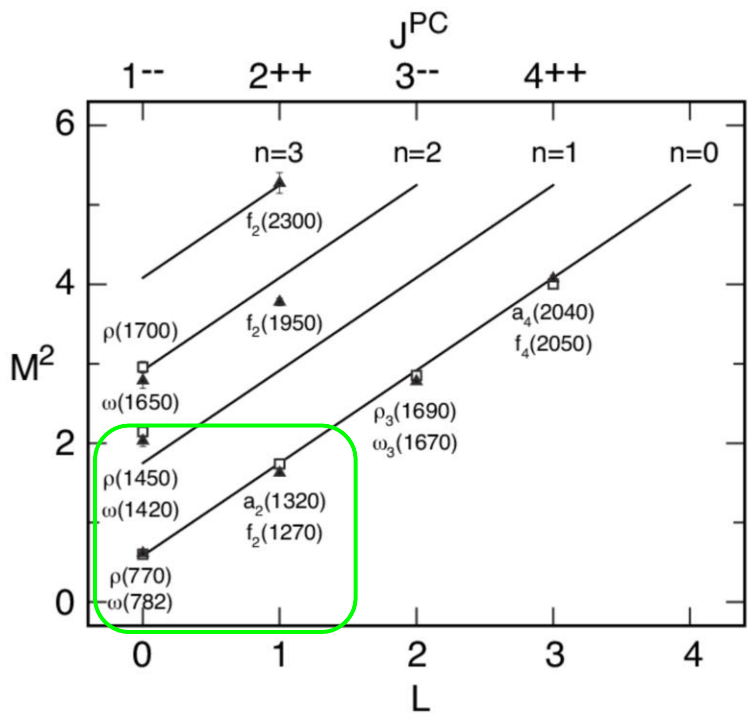


	GTB	PDG
m_1	758	775 ± 0.23 MeV
Γ_1	137	149.1 ± 0.8 MeV
m_2	1514	1465 ± 25 MeV
Γ_2	162	400 ± 60 MeV
A_1	1.28	
A_2	0.18	

Gravitational form factor



	GTB	PDG
m_{f_2}	1180	1275.4 ± 0.6 MeV
Γ_{f_2}	249	186.6 ± 2.3 MeV

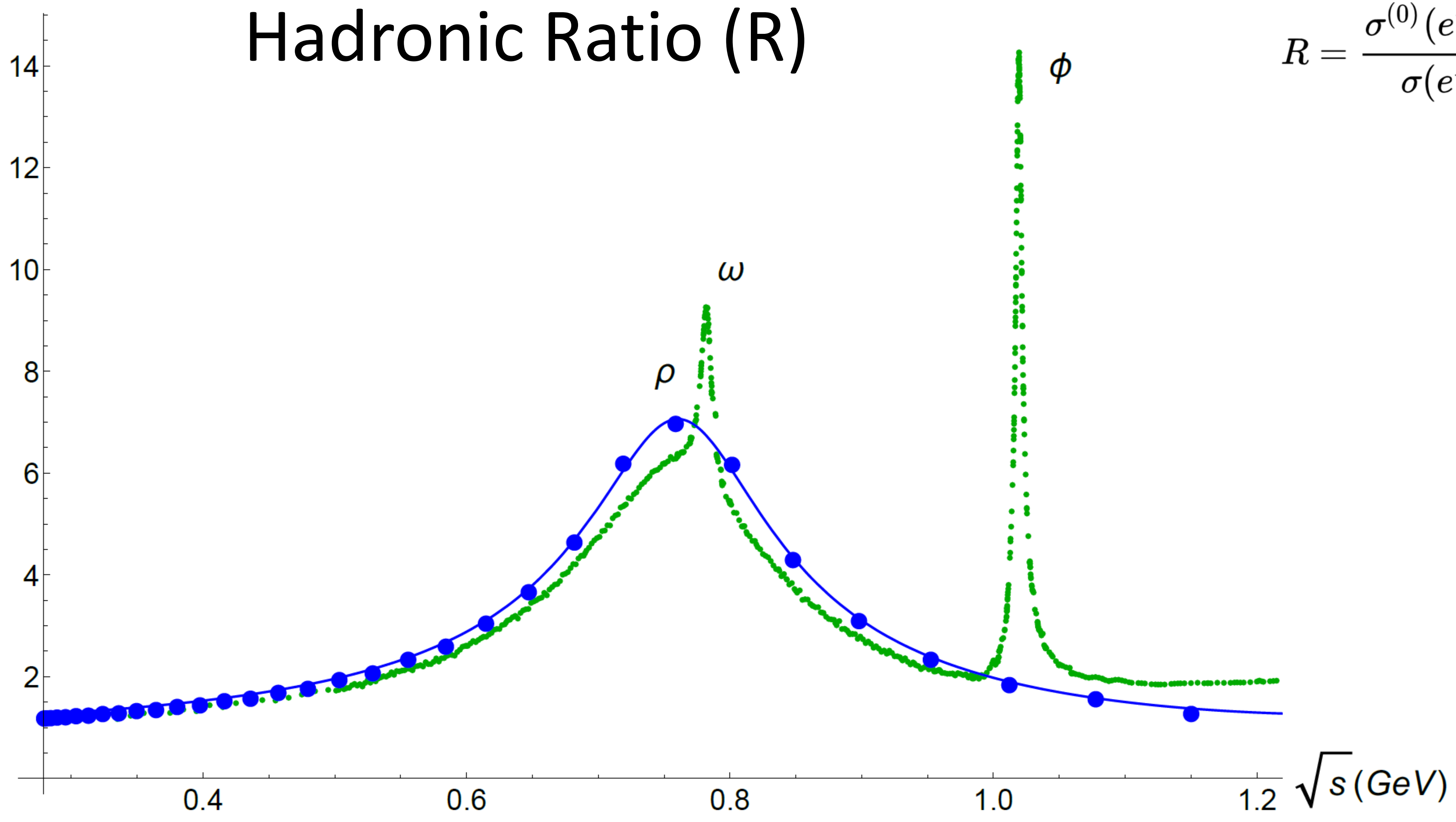


Summary of computed spectrum

	GTB	PDG	
m_ρ	758	775 ± 0.23	MeV
Γ_ρ	137	149.1 ± 0.8	MeV
$m_{\rho'}$	1514	1465 ± 25	MeV
$\Gamma_{\rho'}$	162	400 ± 60	MeV
m_{f_2}	1180	1275.4 ± 0.8	MeV
Γ_{f_2}	249	186.6 ± 2.3	MeV

Hadronic Ratio (R)

$$R = \frac{\sigma^{(0)}(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

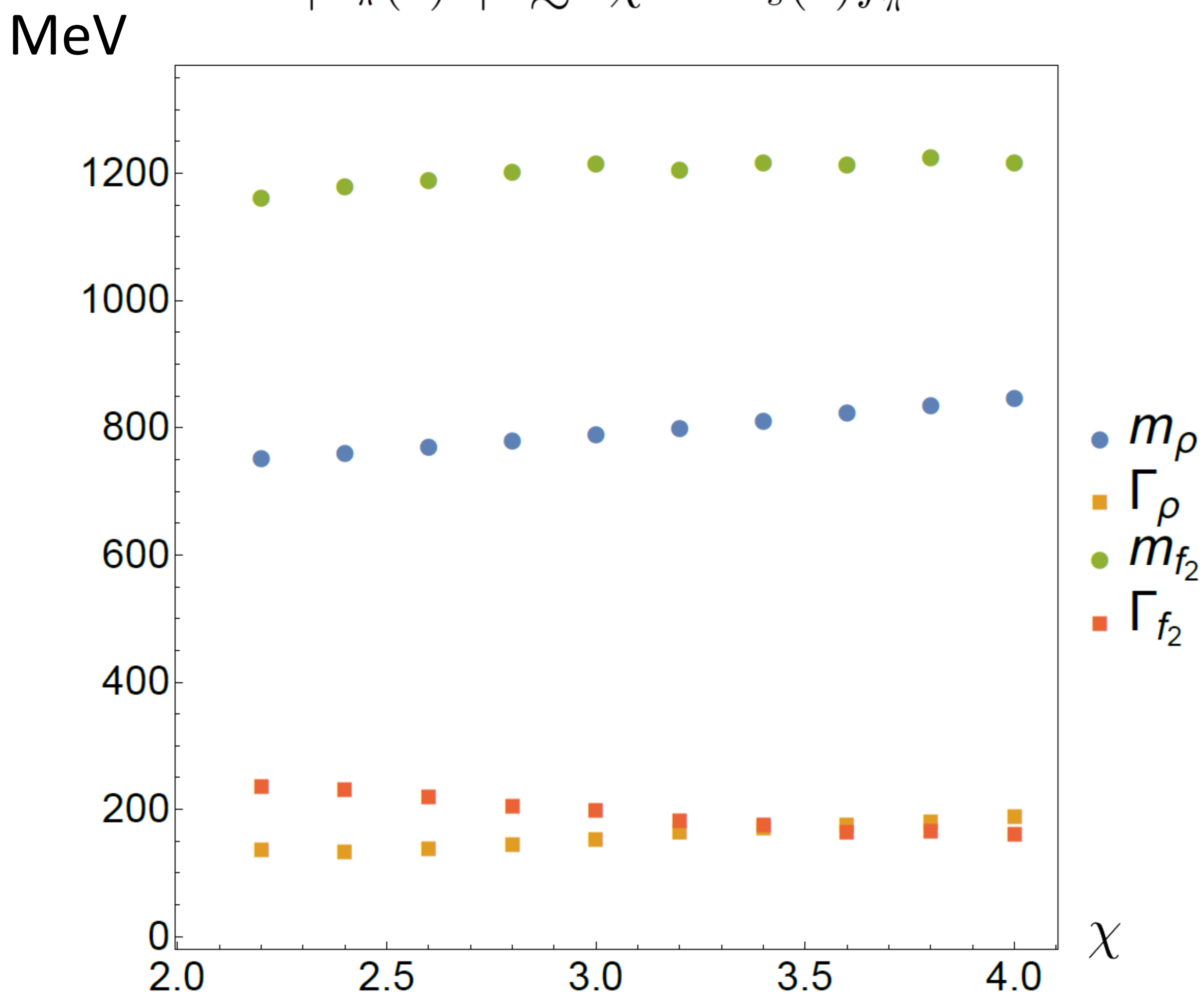


See also: A. Zahed, "Bootstrapping leading hadronic muon anomaly," arXiv:2412.00187 [hep-th].

Muon (g-2) problem

Testing input errors from asymptotic FF

$$|F_\pi(s)s| \overset{s>s_0}{\lesssim} \chi 16\pi\alpha_s(s) f_\pi^2$$



Thermodynamics of dilute interacting pion gas

deviation from ideal gas dominated by binary collisions

consider cluster expansion

$$\Xi = e^{\beta p V} = 1 + z \sum_{\nu, N_\nu = 1} e^{-\beta E_\nu} + z^2 \sum_{\nu, N_\nu = 2} e^{-\beta E_\nu} + \mathcal{O}(z^3)$$

$$\beta P = \lim_{V \to \infty} \frac{1}{V} \ln \Xi = \sum_{n=1}^{\infty} b_n(T) z^n$$

fugacity $z = e^{\beta \mu}$

second virial coefficient encodes corrections from interaction

can be related to phase shift:

[Beth, Uhlenbeck, 1937]
[Dashen, Ma, Bernstein, 1969]

can now do this with a theoretical computation

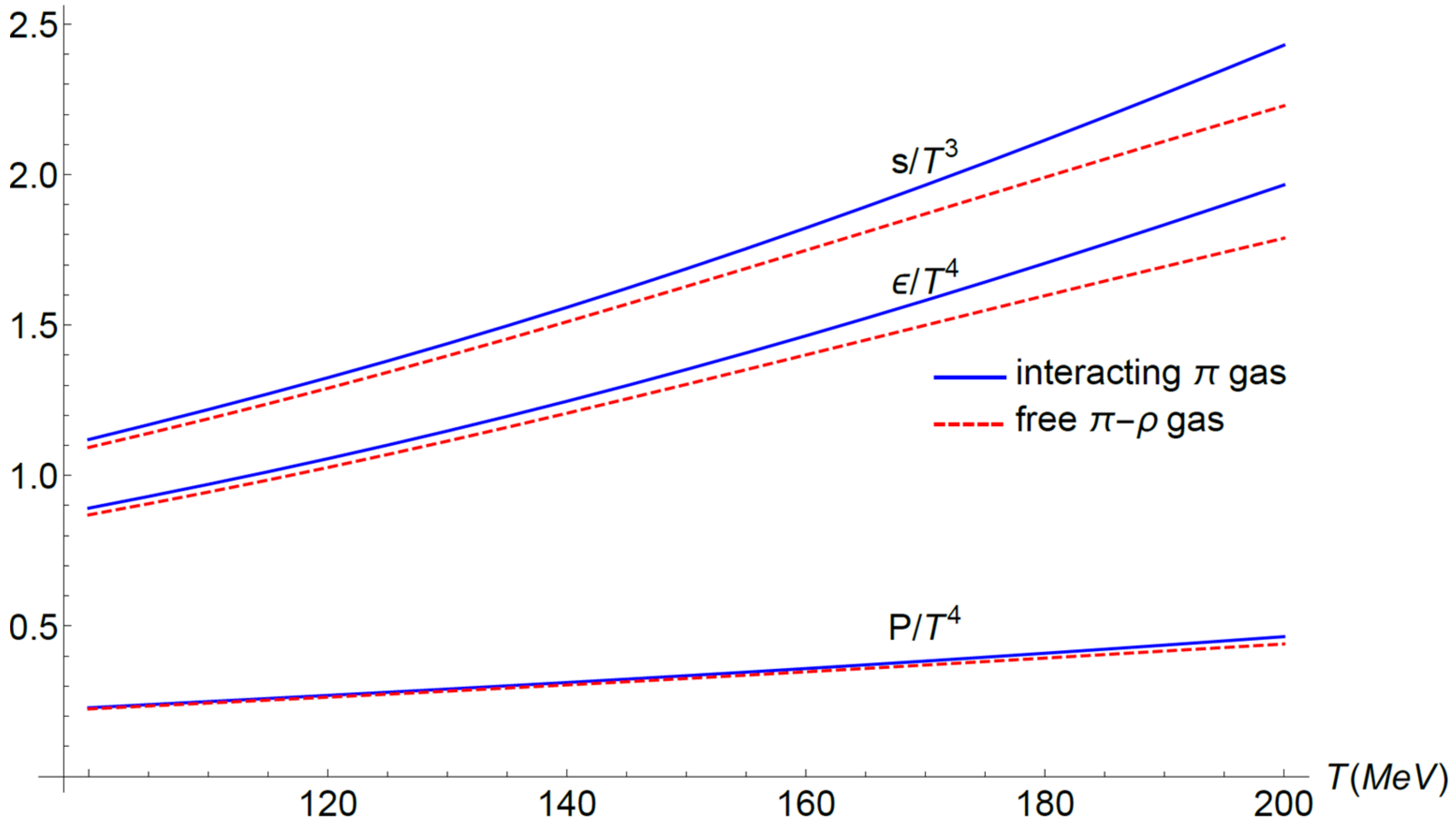
recent application to QCD string: [Baratella, Miro, Gendy, 2024]

$$b_2 = \frac{1}{2\pi^3 \beta} \int_{2m_\pi}^{\infty} dM \, M^2 K_2(\beta M) \sum_{I\ell}{}' (2I+1)(2\ell+1) \frac{\partial \delta_\ell^I}{\partial M}$$

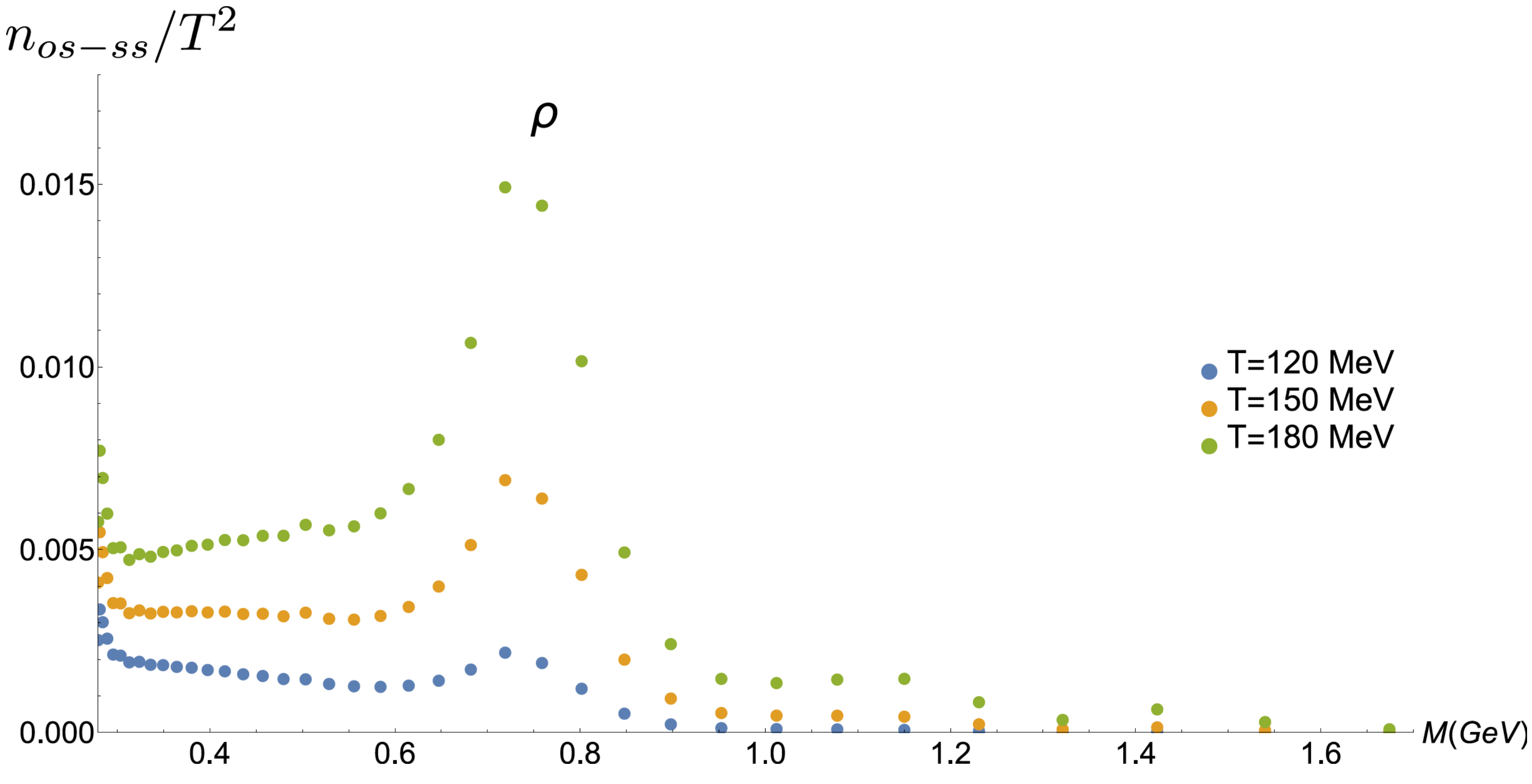
[Venugopalan, Parakash, 1992]

Thermodynamic quantities for dilute pion gas

$$P_{\text{int}} = Tb_2 \qquad \epsilon_{\text{int}} = -\frac{\partial b_2}{\partial \beta} \qquad s_{\text{int}} = \left[b_2 \left(1 - 2\mu\beta\right) - \beta \frac{\partial b_2}{\partial \beta} \right]$$



Pairs of pions with opposite minus same sign (charge) in the dilute gas. (distributed by invariant mass)



Current work (very preliminary)

Changing N_c : w/Yifei He, Rafael Cordoba, Derek Ping,
Emil Alegria, Jignesh Mohanty.

Changing m_q : w/Yifei He, Nick Evans, Konstantinos Rigatos,
Wanxiang Fan.

Changing N$_c$ (keeping $\alpha_s N_c$ fixed):

w/Y. He, R. Cordoba, D. Ping, E. Alegria, J. Mohanty.

N$_c$ is QCD (high energy) information and therefore it appears in the sum rules and also in the asymptotic form factors.

$$\Pi_0^0(s) = -\frac{s}{(2\pi)^4}\frac{N_f m_q^2}{16\pi^2}\left\{2N_c \ln\left(-\frac{s}{\mu^2}\right) + \frac{\alpha_s}{\pi}\left[s_1 \ln\left(-\frac{s}{\mu^2}\right) + s_2 \ln^2\left(-\frac{s}{\mu^2}\right)\right]\right.$$
$$\left. + \left(\frac{\alpha_s}{\pi}\right)^2\left[s_3 \ln\left(-\frac{s}{\mu^2}\right) + s_4 \ln^2\left(-\frac{s}{\mu^2}\right) + s_5 \ln^3\left(-\frac{s}{\mu^2}\right)\right] + O(\alpha_s^3)\right\}$$

$$\Pi_1^1 = -\frac{s}{(2\pi)^4}\frac{N_f N_c}{48\pi^2}\left\{\ln\left(-\frac{s}{\mu^2}\right) + r_1\frac{\alpha_s}{\pi}\ln\left(-\frac{s}{\mu^2}\right)\right.$$
$$\left. + \left(\frac{\alpha_s}{\pi}\right)^2\left[r_2 \ln\left(-\frac{s}{\mu^2}\right) + r_3 \ln^2\left(-\frac{s}{\mu^2}\right)\right] + O(\alpha_s^3)\right\}$$

e.g. Narison: QCD as a theory of hadrons

$$\Pi_2^0(s) = \frac{1}{(2\pi)^4}\frac{s^2}{8\pi^2}\left(t_1 \ln\left(-\frac{s}{\mu^2}\right) + t_2\frac{\alpha_s}{\pi}\ln\left(-\frac{s}{\mu^2}\right) + O(\alpha_s^2)\right)$$

$$t_1 = \frac{1}{20}\left(-2N_c^2 - N_c N_f + 2\right)$$

$$t_2 = \frac{1}{288}\left(N_c^2 - 1\right)\left(16N_c - 7N_f\right)$$

Zoller, Chetyrkin, JHEP 2012, Caron-Huot, Pokraka, Zahraee, JHEP01(2024)195

A crude way to understand expected N_c dependence from sum rule is the narrow width approx.

$$\int_4^{s_0} \rho(x) x^n dx = -s_0^{n+1} \int_0^{2\pi} e^{i(n+1)\varphi}\Pi(s_0 e^{i\varphi}) d\varphi$$

$$\rho \simeq \frac{A(m_\rho)}{(s-m_\rho^2)^2 + m_\rho^2\Gamma_\rho^2} \quad \Longrightarrow \quad \rho \simeq \frac{A}{m_\rho \Gamma_\rho}\delta(s - m_\rho^2)$$

$$\begin{aligned} m_0 &= \int \rho \, ds = \frac{A}{m_\rho \Gamma_\rho} \\ m_1 &= \int s\rho \, ds = m_\rho^2 \frac{A}{m_\rho \Gamma_\rho} \end{aligned}$$

$$\frac{m_1}{m_0} \simeq m_\rho^2, \quad \Gamma_\rho \sim \frac{1}{m_0}$$

Mass is independent of overall factor but width is inversely proportional

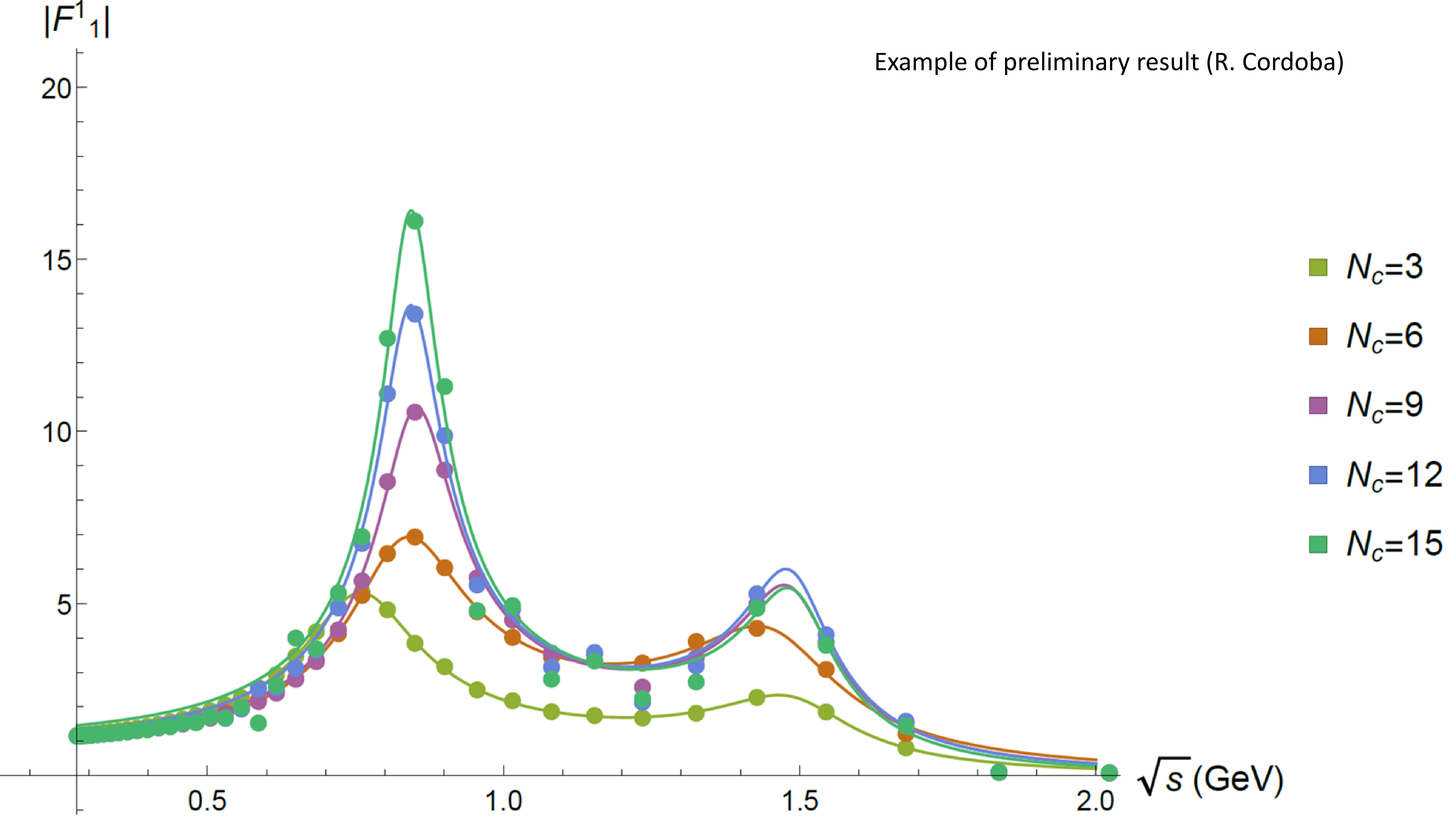
Form factors

$$F_1^1 \simeq -\frac{36\pi\alpha_s f_\pi^2}{s}\frac{N_c^2-1}{2N_c^2}$$

$$\alpha_s N_c \, \frac{f_\pi^2}{N_c} \simeq$$ Constant with N_c

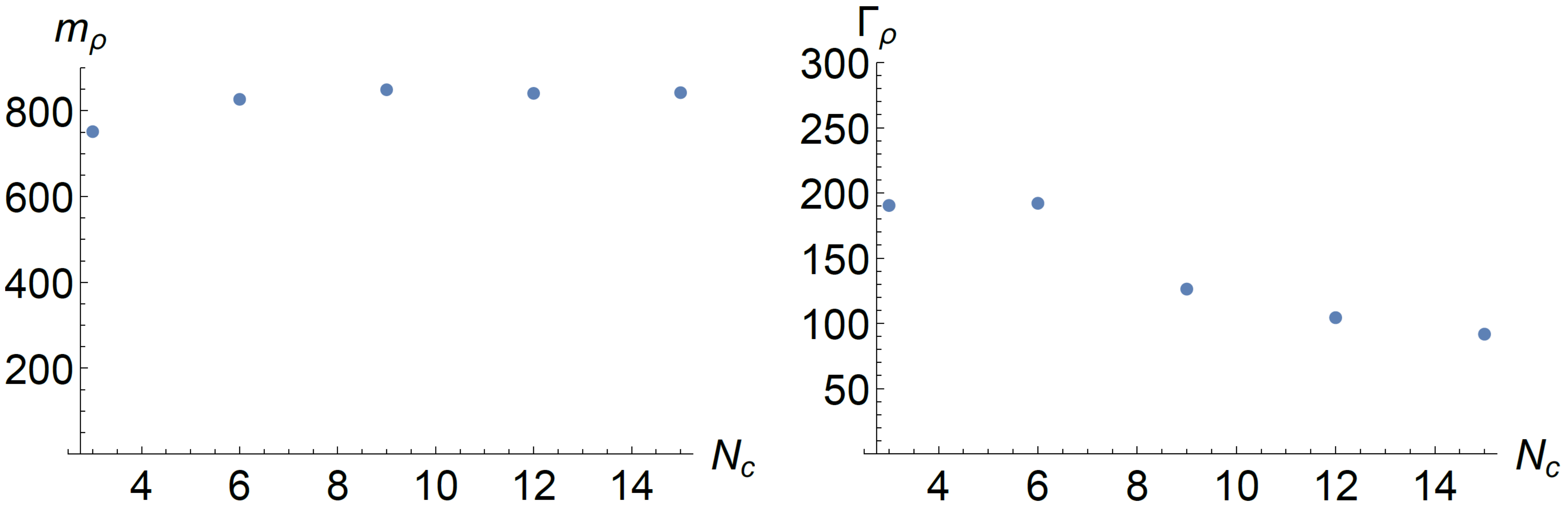
$$\langle 0|A_\mu^a(x)|\pi_b(p)\rangle = i\delta^{ab}p_\mu f_\pi e^{-ipx} \quad \Longrightarrow \quad f_\pi \sim \sqrt{N_c}$$

The same happens with the other form factors, including gravitational, so we keep the asymptotic form factors the same



$|F^1{}_1|$
Example of preliminary result (R. Cordoba)
20
15
10
5
0.5
1.0
1.5
2.0
$\sqrt{s}$ (GeV)
N_c=3
N_c=6
N_c=9
N_c=12
N_c=15

Mass and width dependence with N_c



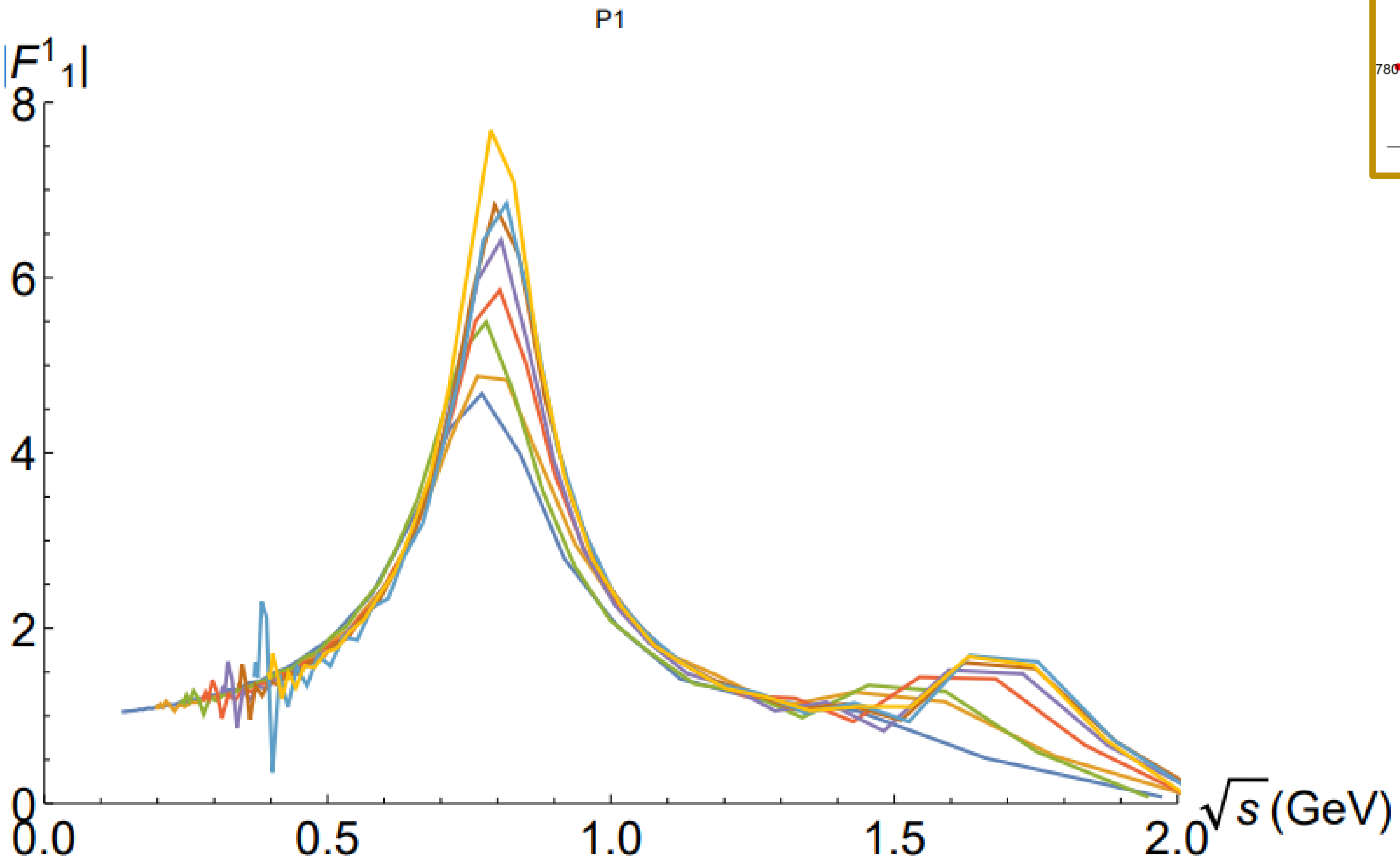
Changing m_q (N_c=3) w/Y. He, N. Evans, K. Rigatos, W. Fan

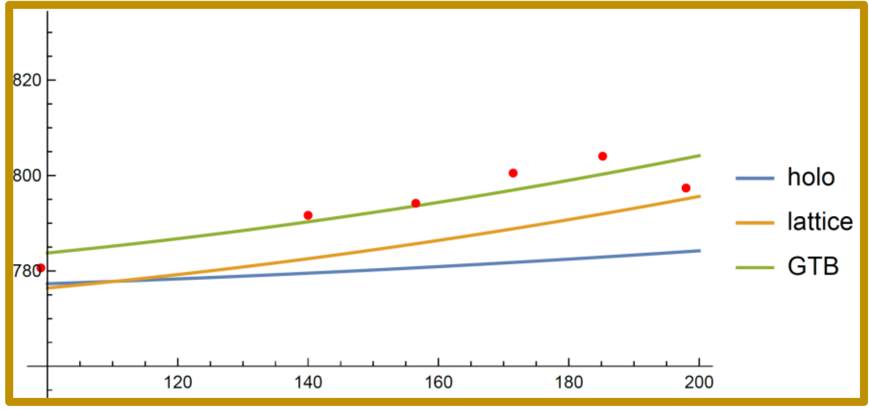
The quark mass appears in the S0 sum rule and also determines the pion mass. Here the main effect is the change in pion mass from GMOR relation:

$$m_\pi^2 = \frac{m_q}{f_\pi^2} |\langle 0|\bar{q}q|0\rangle|$$

If we change the pion mass, the decay width of the rho meson changes due to phase space available. In particular if the pion is too heavy, rho is stable.

Preliminary results by Wanxiang Fan





m_ρ = 777 MeV + 0.00068 m_π^2/MeV

Lattice: m_ρ = 770MeV+0.00064 m_π^2/MeV

Holographic model: Erdmenger, Evans, Porod and Rigatos, JHEP 02 (2021), 058 2010.10279 [hep-ph]

Conclusions

- Gauge Theory Bootstrap:

  using only $N_c \; N_f \; m_q \; \Lambda_{\rm QCD}$ (*gauge theory parameters*) — m_π set the units — f_π size of pion

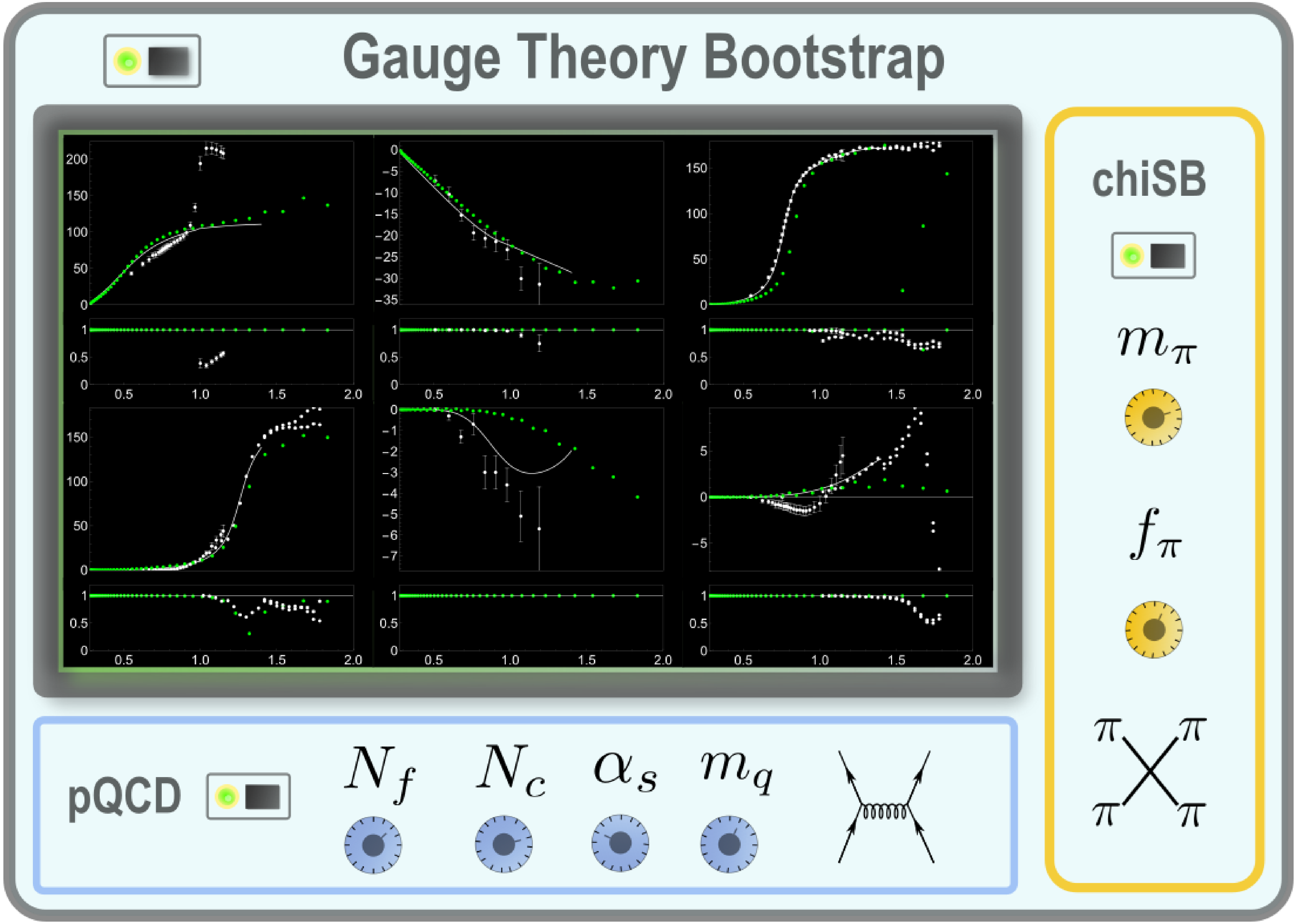
  strongly coupled low energy physics of asymptotically free gauge theories

- Numerical test with $N_f = 2 \;\; N_c = 3$ and find good agreement with experiments

  Unique GTB solution (within errors), predicting pion dynamics from QCD

  We are on the right track for ***solving QCD*** (gauge theories)

- Not precision age yet: need more robust computations, set error bars, etc.... a lot of work to do. github: hyfysics/gauge-theory-bootstrap


Gauge Theory Bootstrap
chiSB
m_π
f_π
π π π π
pQCD
N_f
N_c
α_s
m_q