# Thermal partition function from the S-matrix reloaded

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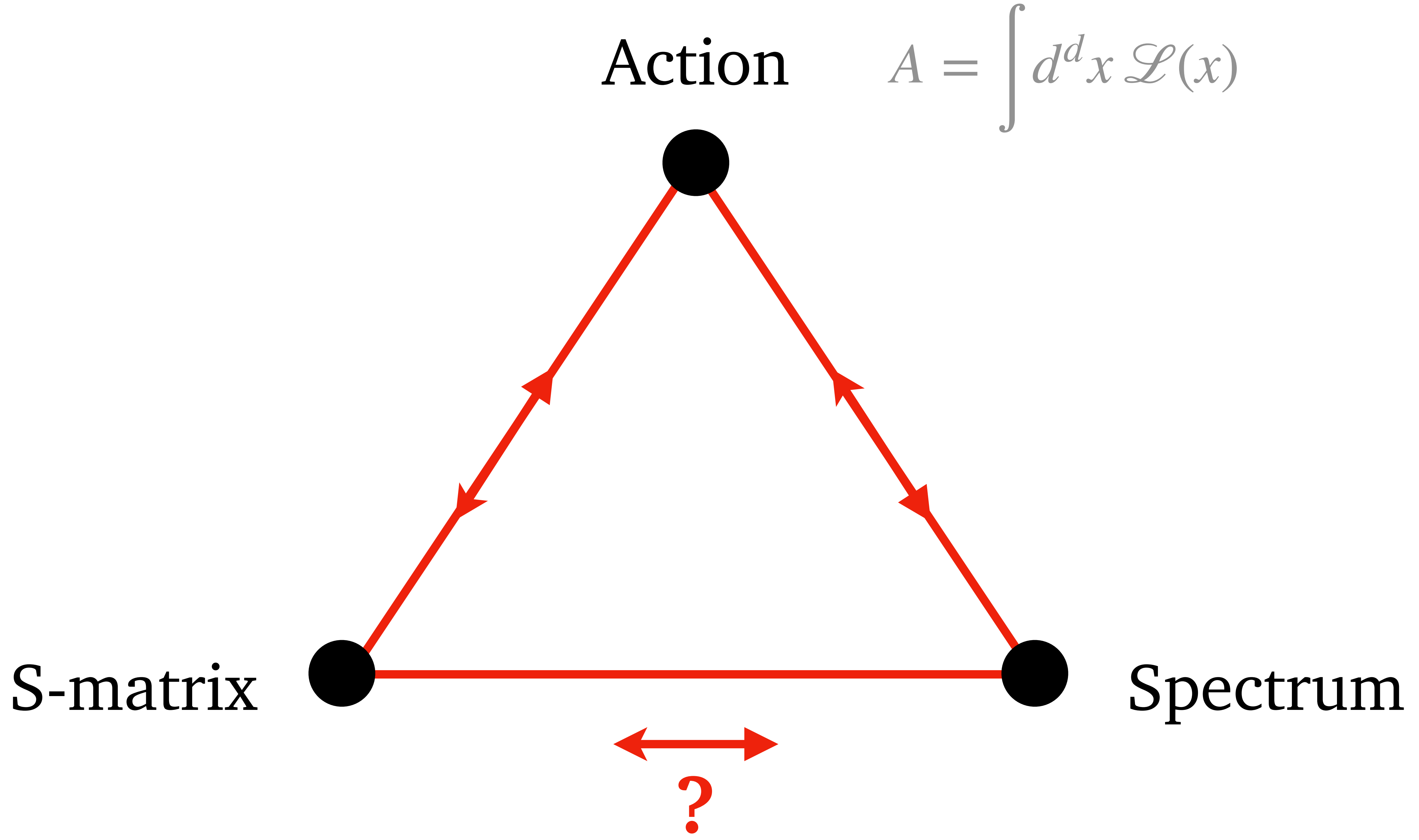
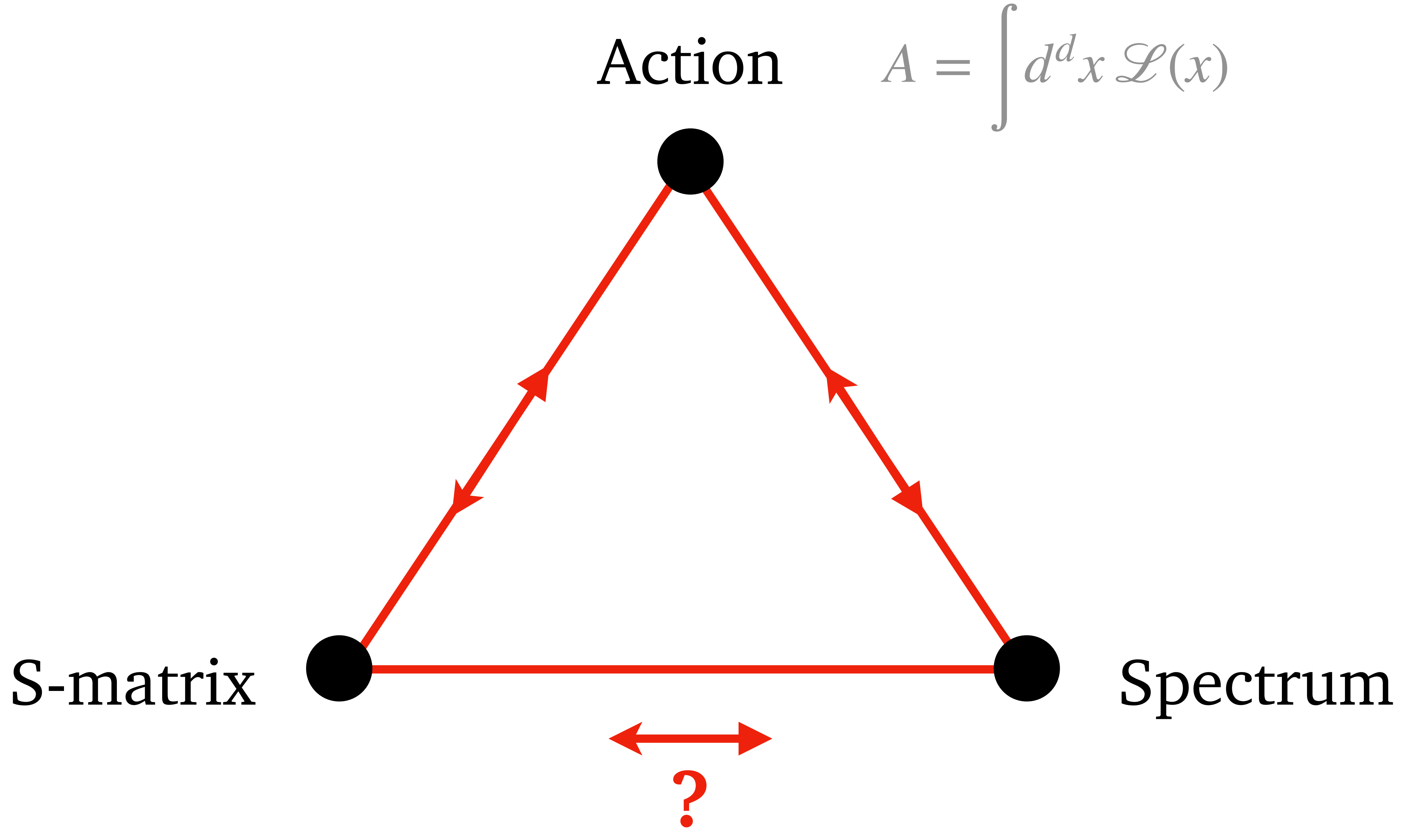
Based on work hep-th/2408.06729 with P. Baratella and E. Gendy

+work in progress with P. Baratella

See also the recent work by D. Schubring hep-th/2408.00729


Action
$A = \int d^d x \, \mathscr{L}(x)$
S-matrix
Spectrum
?

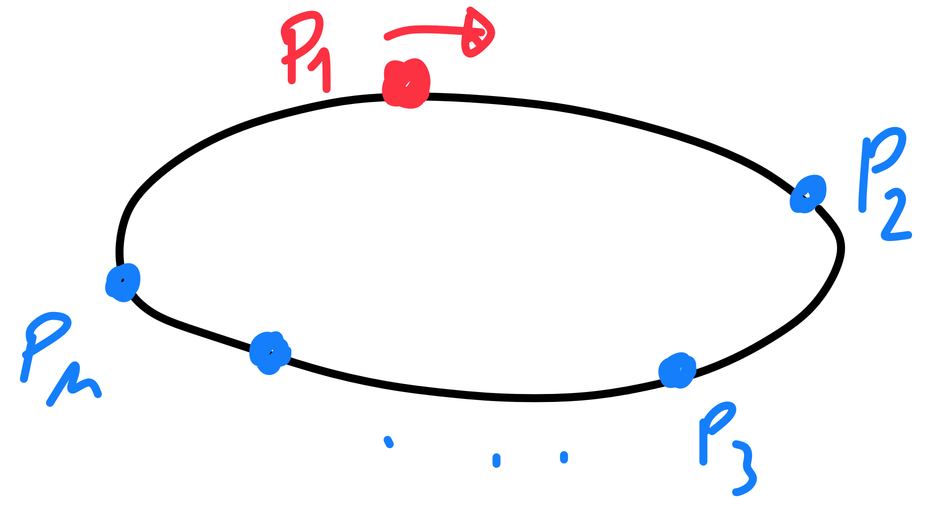



$$A = \int d^d x\, \mathscr{L}(x)$$

In the elastic regime ABA / Lüscher's Formula, and TBA

1.- consider $x_1 \ll x_2$ (mutatis mutandis $x_1 \gg x_2$)

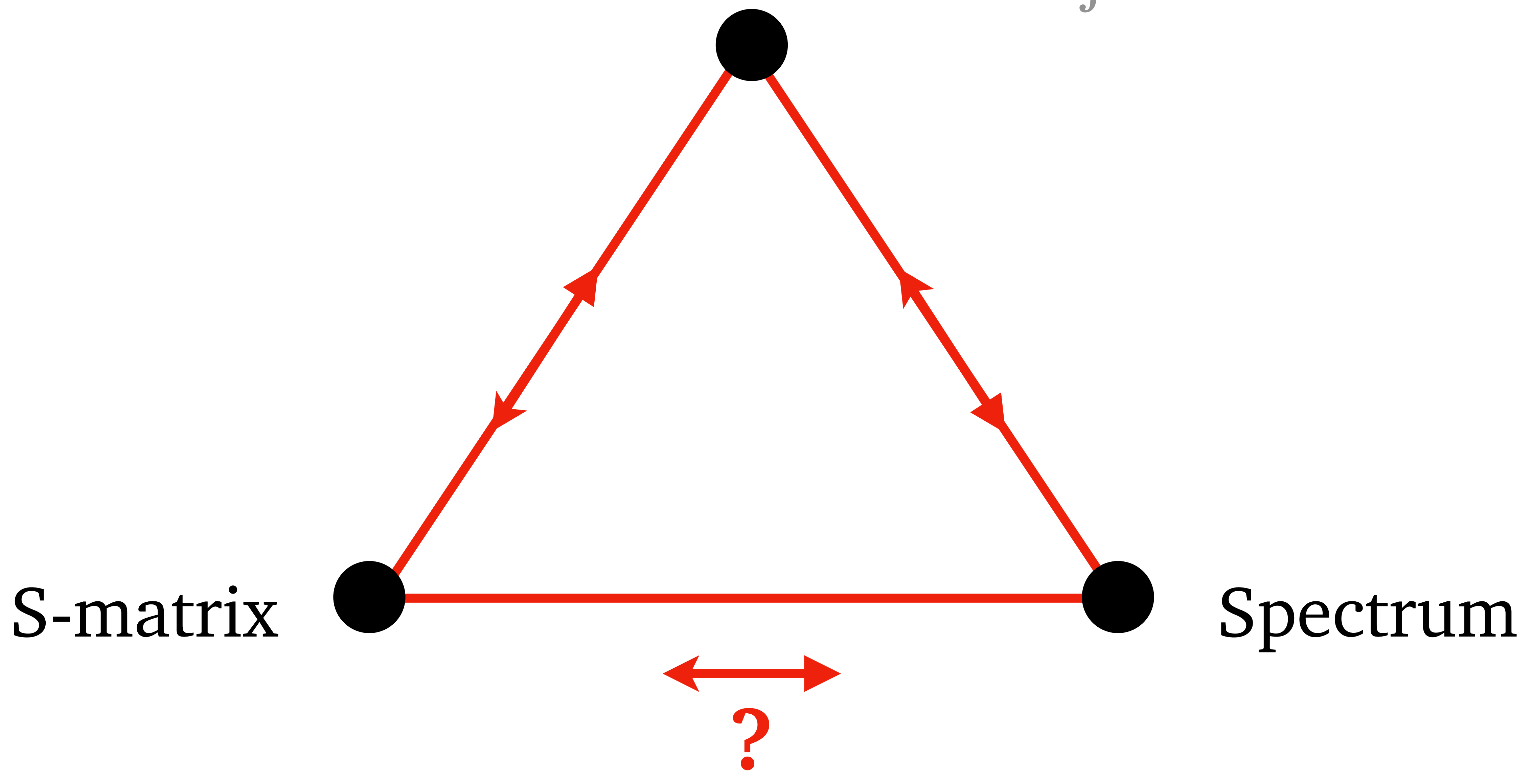
$$\psi(x_1 \gg x_2) = e^{iP_1 x_1 + iP_2 x_2} + e^{iP_1 x_2 + iP_2 x_1} S_{12}(P_1, P_2)$$




2.- Place the theory at finite volume w/ $x_1 \ll x_2 \ll x_1 + R$

$$\psi(x_1, x_2) = \psi(x_1 + R, x_2) \Rightarrow e^{iP_1 R} S_{12}(P_1, P_2) = 1$$

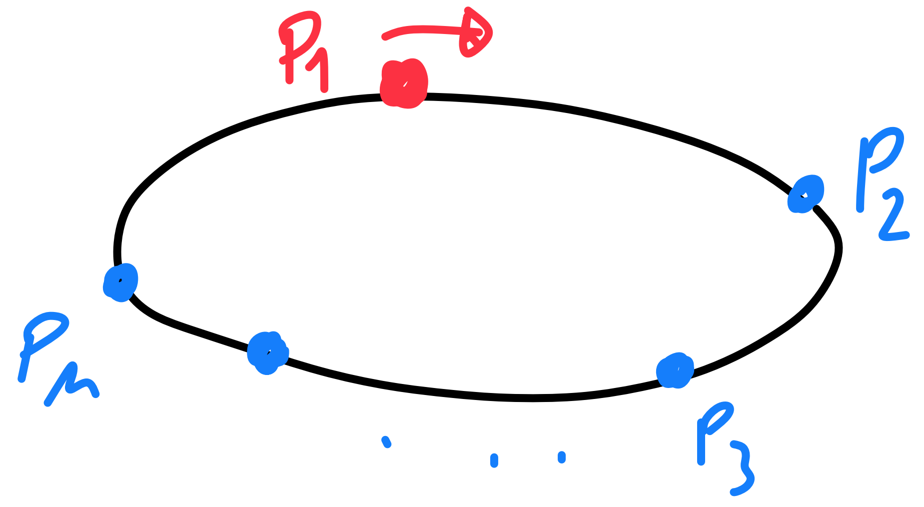
$$A = \int d^d x\, \mathscr{L}(x)$$



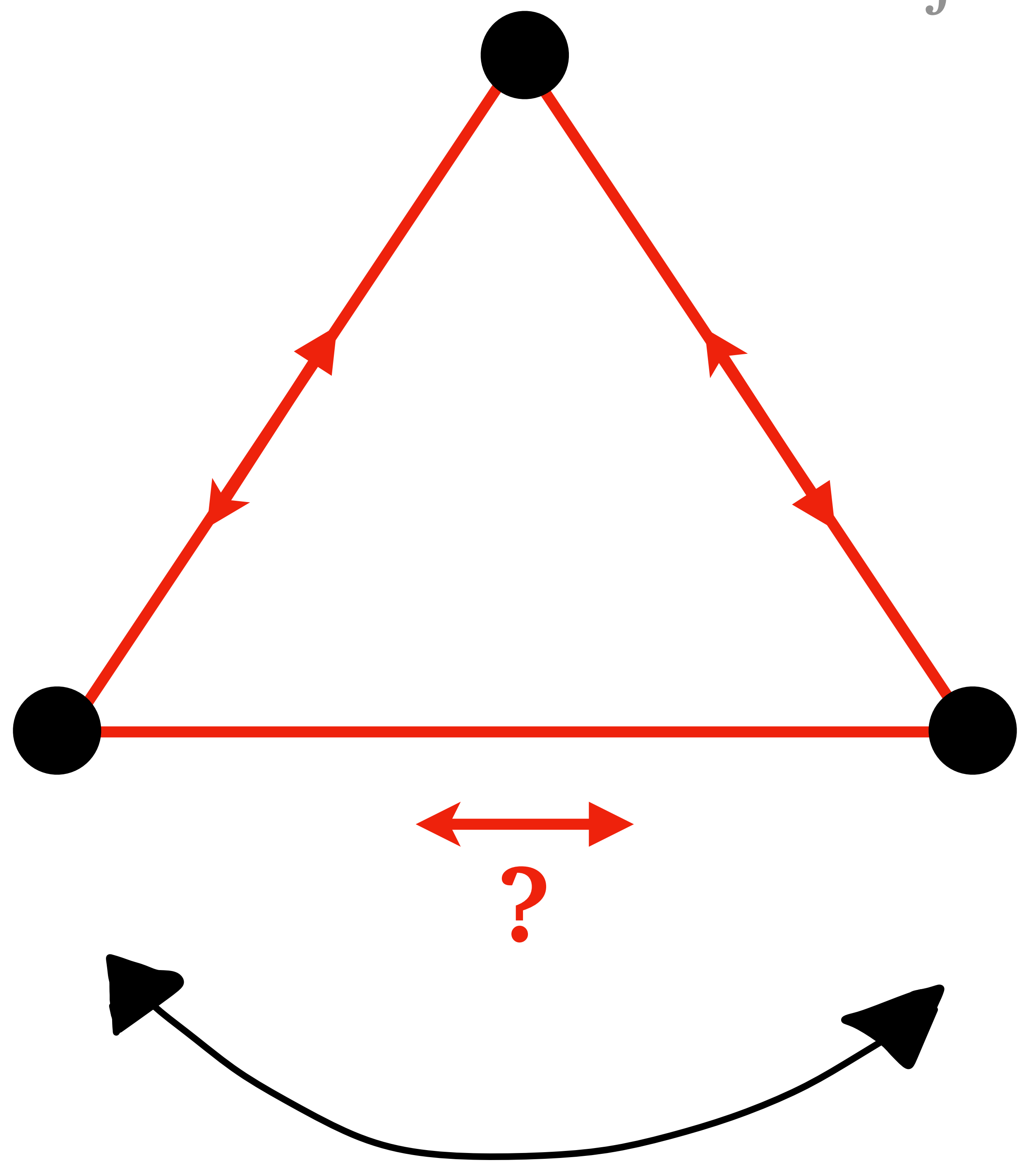

In the elastic regime ABA / Lüscher's Formula, and TBA

Same logic for multiple rescatterings

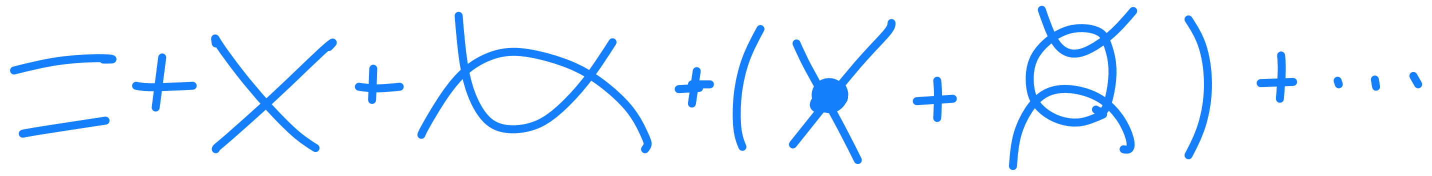
$$p_i R + \sum_j{}' \, 2\,\delta_{ij}(p_i, p_j) = 2\pi N_i$$



Action $A = \int d^d x\, \mathscr{L}(x)$

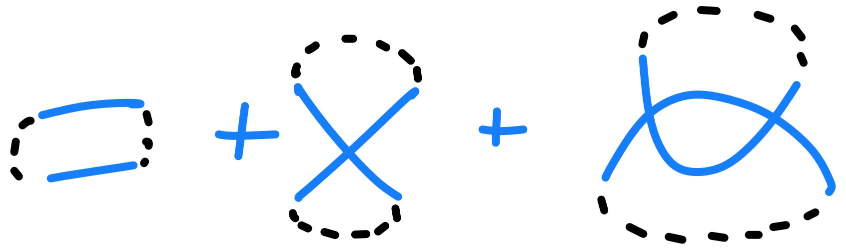



S-matrix $S(s) = 1 + i\frac{s}{4}\ell_s^2 - \frac{s^2}{32}\ell_s^4 + i(\gamma_3 - \frac{1}{384})s^3\ell_s^6 + \cdots$

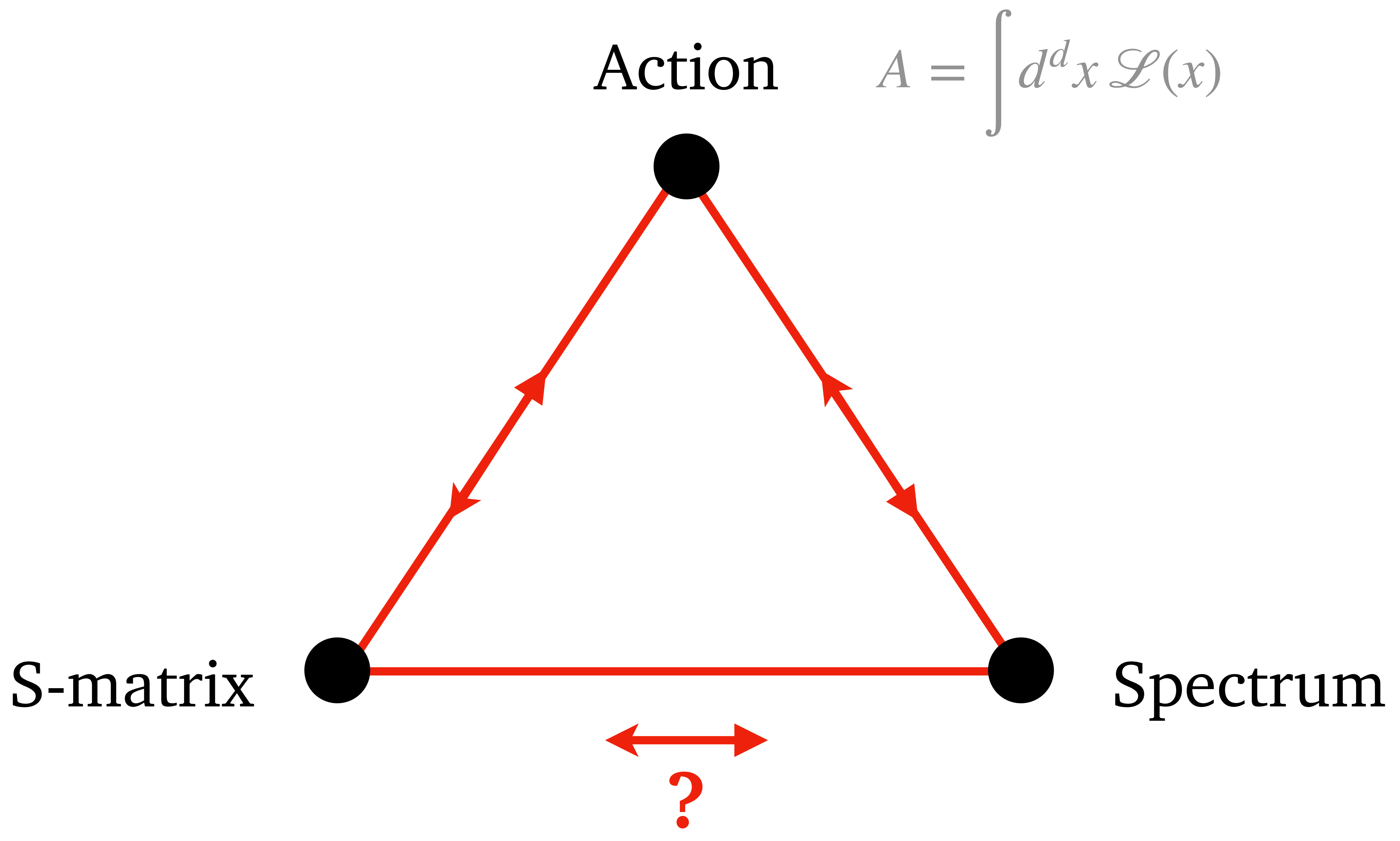


Spectrum $E_0(R) = \frac{R}{\ell_s^2}\sqrt{1 - \frac{\pi\ell_s^2}{3R^2}} - \frac{32\pi^6}{225}\frac{\gamma_3\ell_s^6}{R^7} + O(R^{-9})$

some sort of tracing involved.



Intuition indeed realized!

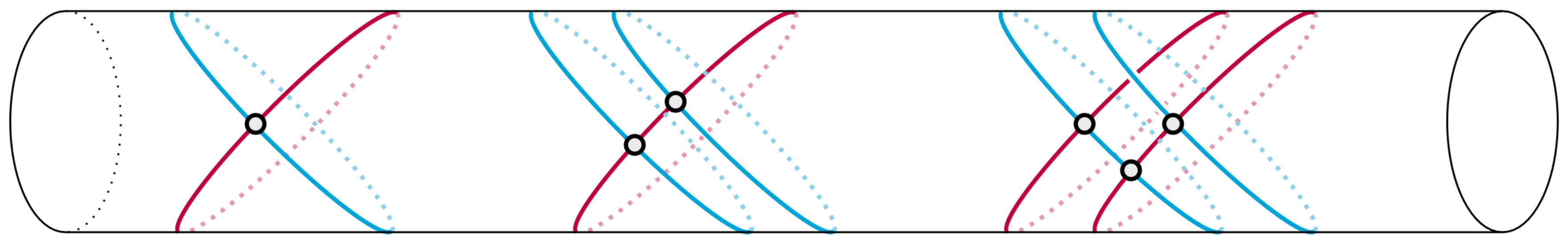



**DMB equation** — Dashen, Ma, Bernstein [Phys.Rev. 197 (1969) 345-350]

$$Z(\beta) = Z_0(\beta) - \frac{\beta}{2\pi i} \int_0^\infty dE e^{-\beta E} \mathrm{Tr} \log S(E)$$

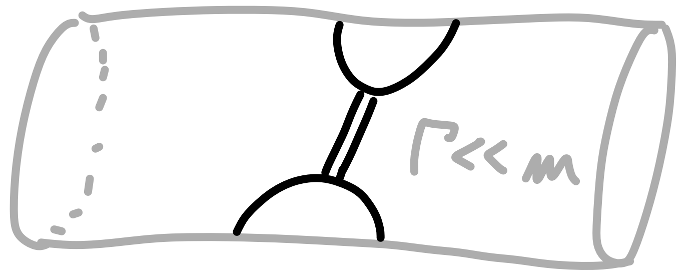
$$Z(\beta) = Z_0(\beta) - \frac{\beta}{2\pi i}\int_0^\infty dE e^{-\beta E}\mathrm{Tr}\log S(E+i\epsilon)$$

1. Purely imaginary $\mathrm{Tr}\log S = \log \overbrace{\det S}^{e^{i\phi}}$
2. Fully (disconnected) S-matrix $\langle\alpha| S(E=E_\alpha)|\beta\rangle = \mathbb{I} + 2\pi i\,\delta(E_\alpha - E_\beta)\,T_{\alpha\beta}$
3. Note that the particle density is given by $\rho(\epsilon) = \rho_0(\epsilon) + \frac{1}{2\pi i}\partial_\epsilon \mathrm{Tr}\log S(\epsilon)$
4. $S(E)$ is the vacuum S-matrix, Trace + integral with Boltzmann weights generates thermal physics. Pictorially:



$+ \cdots$

5. Thermal partition function. Derivation formally valid on any $S_1 \times \Sigma$. We will take $R^d$.
6. Long lived particles count as degrees of freedom

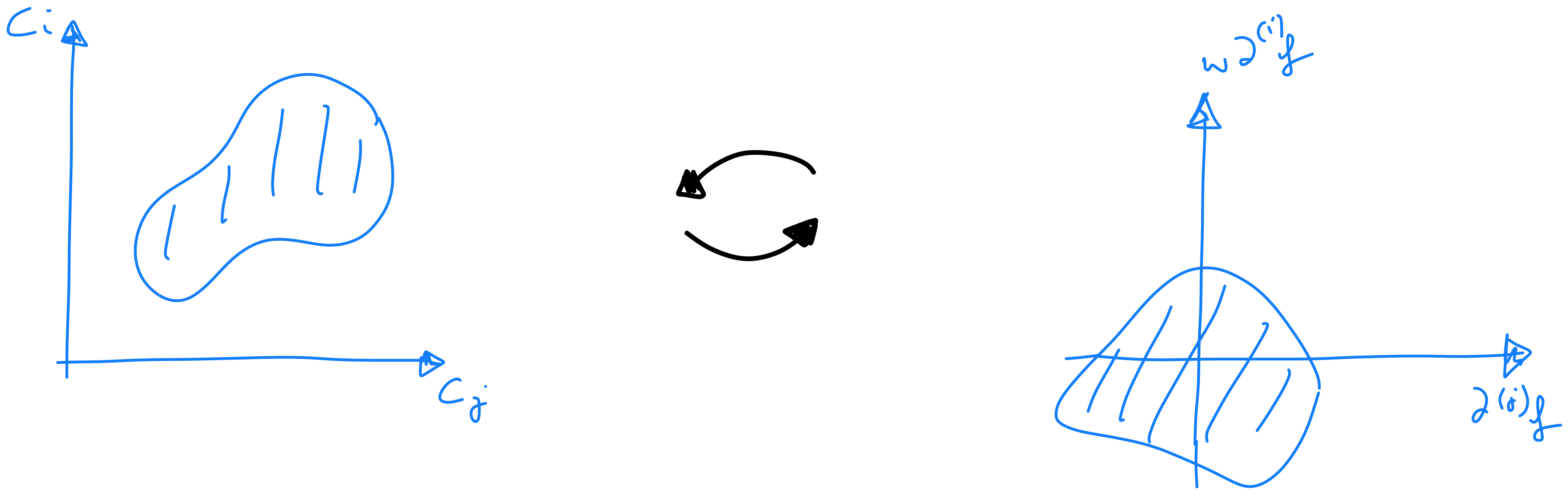



7. DMB mostly used in non-relativisitc context, at tree-level with 2-particle states — leading order.

↳ Why now?

Many new results constraining the space of EFT's, typically in terms of S-matrix w/o reference to $\mathscr{L}$.

How can we transfer this info to other physical observables?



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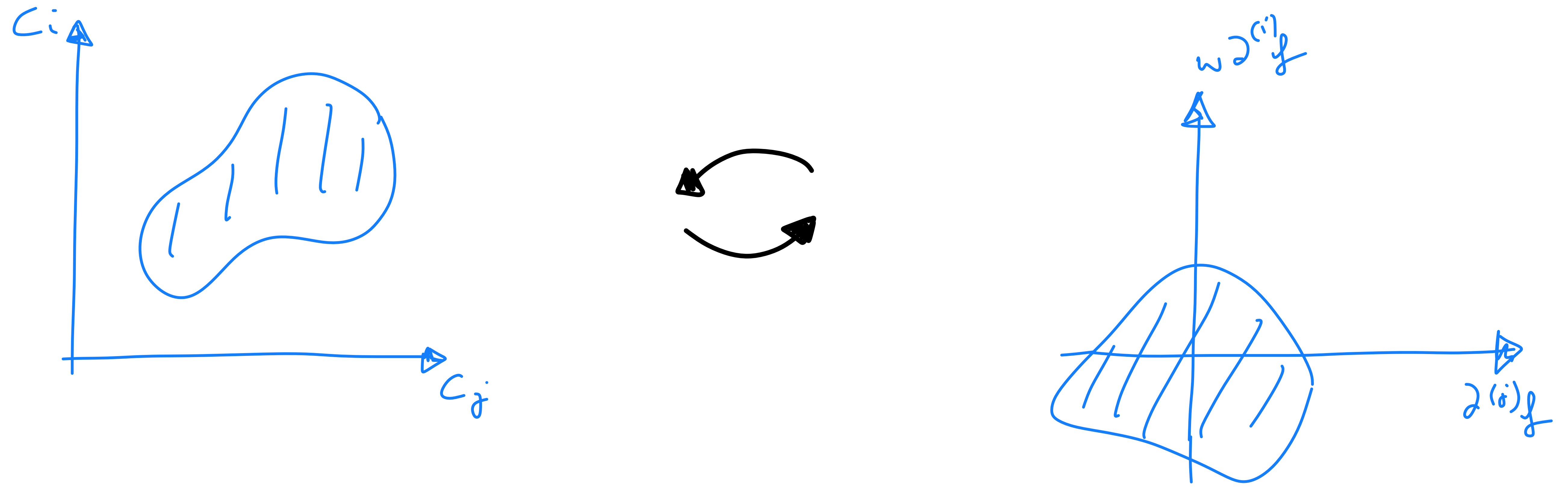
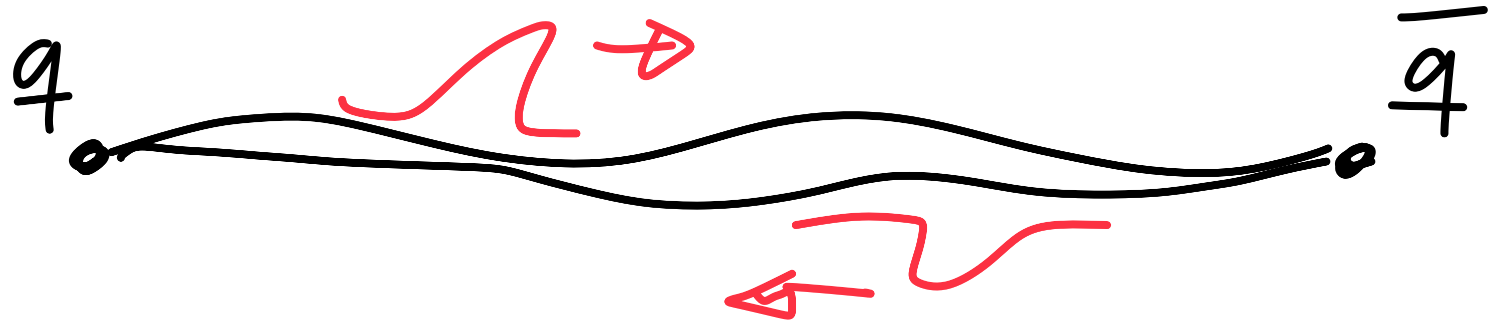



e.g. "Ghost condensation" on $P(X=(\partial\phi)^2)$ theories.

on $X$ = cte background $\quad c_s = \dfrac{P'(X)}{P'(X)+2XP''(X)} < 1 \Rightarrow P(X)$ convex.

Bounds on Flux Tubes




$$V(r) = \sigma R + \frac{\pi(D-2)}{R} + \dots = \sqrt{R^2 - \frac{\pi}{3}(D-2)} + \frac{\delta(D)}{R^7} + O(1/R^9)$$

$\delta(D)$ is bounded by S-matrix bootstrap

$$S(s) = e^{i\left[s + \text{coeff.}\times\delta(D)\times s^3 + O(s^5)\right]}$$

- With the development of on-shell S-matrix methods, we have access to precise Matrix elements with arbitrary number of particles. Bad convergence of QCD- New calculation scheme (no ghost, no gauge redundancies). Alternative treatment of the QCD equation of state based on an IR safe S-matrix?

- The method shines when the path integral formulation is not available, yet we may have access to the S-matrix.

**Plan of the talk**

1.- Introduction

2.- Formal derivation of DMB

3.- Applications & recent developments

## 2.- Formal derivation

straight forward to account for. But I'll omit from now on.

$$Z(\mu,\beta) = \mathrm{Tr}\, e^{-\beta(H-\mu Q)}$$

$$= \int_0^\infty dE\, e^{-\beta E}\, \mathrm{Tr}\underbrace{\left[\frac{1}{E-H+i0^+} - \frac{1}{E-H-i0^+}\right]}_{\equiv G^* - G}$$

Recall that $\lim_{\epsilon\to 0} \frac{i\pi}{x+i\epsilon} = \delta(x)$

Next we need to relate this Kernel to the S-matrix

Define $G_0(\epsilon) \equiv \dfrac{1}{\epsilon - H_0 + i0^+}$ & $G(\epsilon) \equiv \dfrac{1}{\epsilon - H + i0^+}$

## 2.- Formal derivation

1.- $S(E) = G_0^* (G^*)^{-1} G \, G_0^{-1}$

From Lippman - Schwinger

$$\psi_\pm = \phi + G_0(E \pm i\epsilon) V \psi_\pm \rightsquigarrow \psi^\pm = \Omega^\pm \phi \qquad \text{w/} \ (\Omega^\pm)^{-1} = (1 - G_0 V)^{-1}$$

Then $S(E) = (\Omega^-)^+ \Omega^+$

2.- $\mathrm{Tr}\, S^{-1} \partial_E S = \mathrm{Tr}[-G_0^* + G^* - G + G_0]$

simple algebra

$$\Rightarrow Z(\beta) = \frac{1}{2\pi i} \int_0^\infty dE \, e^{-\beta E} \left\{ \mathrm{Tr}\left[ S^{-1}(E) \partial_E S(E) \right] - \mathrm{Tr}\left[ G_0(E) - G_0^*(E) \right] \right.$$

$$= Z_0(\beta) - \beta \int_0^\infty \frac{dE}{2\pi i} e^{-\beta E} \, \mathrm{Tr} \log S(E) \equiv e^{-\beta F} \quad //.$$

Free energy density $f = \lim_{V \to \infty} \frac{F}{V}$. Upon taking the 'log' we get an extensive quantity, i.e. a single power of $V$. $\log Z = f \cdot V \times \beta$.

$$\boxed{F(\beta) = F_0(\beta) - \frac{1}{2\pi i} \int_0^\infty dE \, e^{-\beta E} \, \mathrm{Tr}_c \log S(E)}$$

connected.

... from the free energy $F = E - TS$ thermodynamic quantities follow:

↳ e.o.s. $w = \frac{P}{\rho}$ ; $P = -f$ ; $\rho = \frac{\partial(\beta f)}{\partial \beta}$ ; $S = -\partial_T F$.

↳ or derived quantities such as $c_v = T \frac{\partial^2 p}{\partial T^2}$ ; $c_s^2 = \frac{\partial \rho}{\partial p}$ , etc.

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# 3.- Applications & recent developments

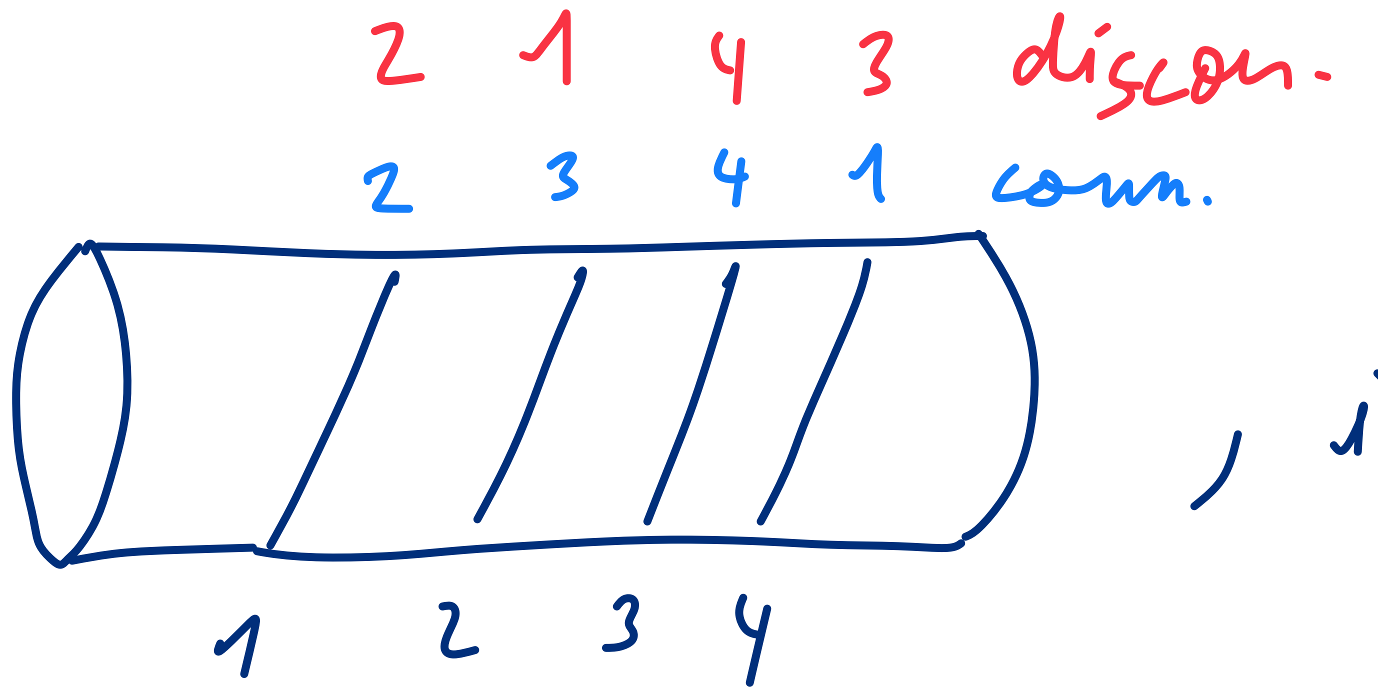
## Simplest things first: free theory

$$-\beta F_0 = \mathrm{Tr}_{\mathcal{H}}\, e^{-\beta H_0} = \sum_{N=0}^{\infty} \frac{1}{N!} \int d^3k_1 \cdots d^3k_N \, \langle k_1, \dots, k_N | \, e^{-\beta H_0} \, | k_1 \dots k_N \rangle_c$$

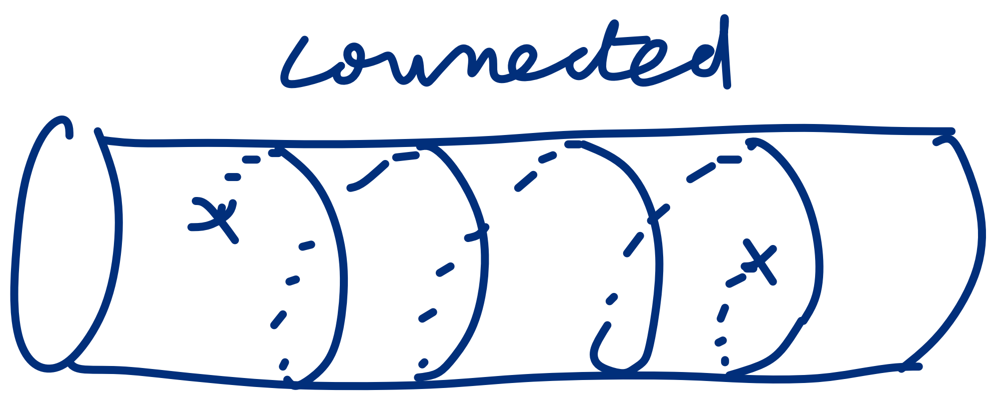
↳ Next we use that $H_0$ is diagonal and $\langle k_1 \dots k_N | k_1' \dots k_N' \rangle = \delta(k_1 - k_1') \cdots \delta(k_N - k_N') \pm$ perms.

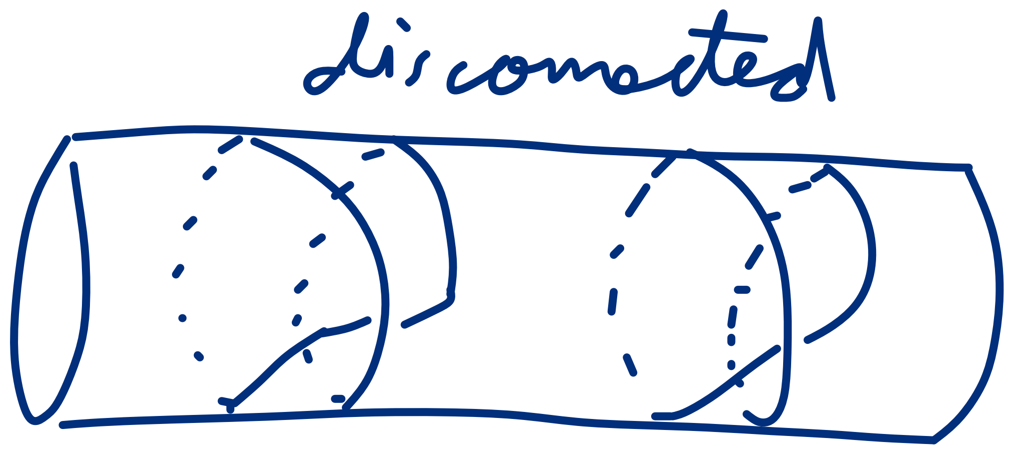
↳ when we identify $k_i = k_i'$ through the trace we may get many $\delta(0)$'s, up to $\delta(0)^N$

↳ Here "c" enters crucially, leaving terms with a single $\delta(0)$. (no $\delta(0)$ for $N=0$).




, i.e.



For a given connected history there are $(N-1)!$ equivalent ones, due to cyclic perms.

All in all

$$-\beta F_0 = 1 + \delta(0)\int d^3k \sum_{N=1}^{\infty} \frac{1}{N}\, e^{-N\beta E(k)} = 1 - \delta(0)\int d^3k \log\left(1-e^{-\beta E(k)}\right)$$

$\delta(0) \to V/(2\pi)^3$ and therefore

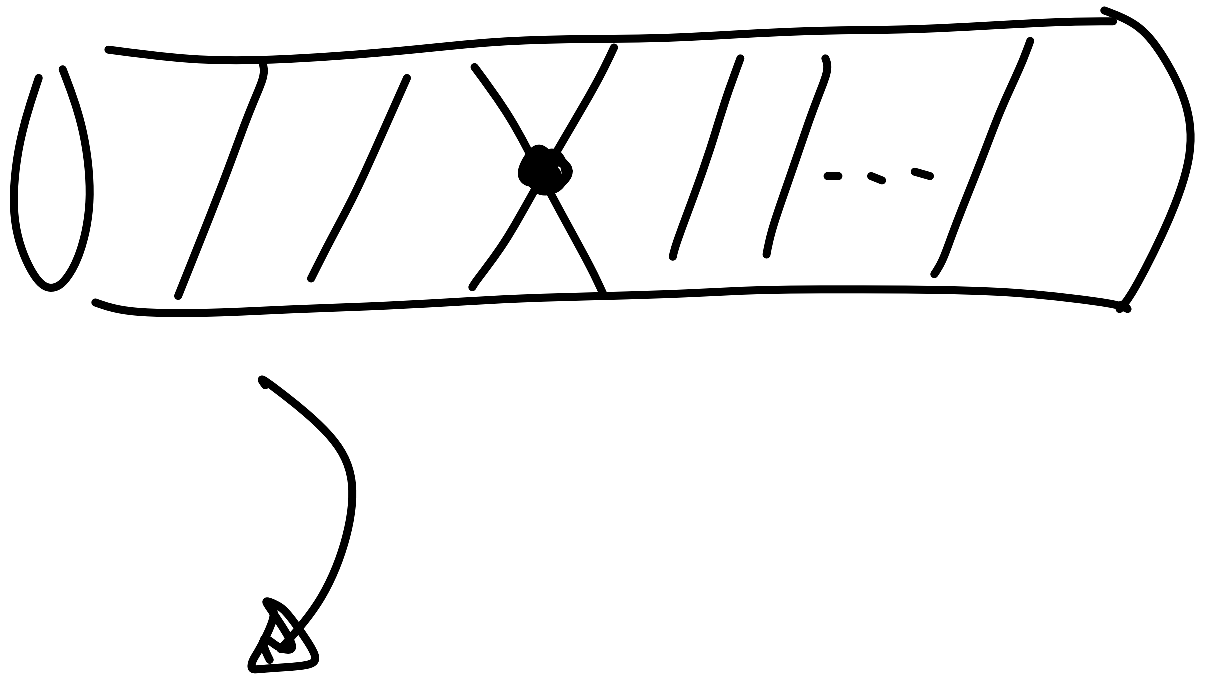
$$f_0 = \frac{T}{2\pi^3}\int_0^\infty dk\, k^2 \ln\left(1-e^{-\beta k}\right) = -\frac{\pi^2 T^4}{90} \times 2(N_c^2-1)$$

# polarizations × # gluons

( aside, to be compared with text book calculation:

 $+$  $= -\frac{\pi^2 T^4}{90}(N_c^2-1)\left[-2+4\right]$ )

## * Interactions at leading order:



$$F - F_0 = \int dE \, e^{-\beta E} \, \mathrm{Tr}_c \left[ T \, \delta(E - H) \right] + O(T^2)$$

a bit of algebra

$$= \underbrace{(2\pi)^3 \delta(0)}_{V} \sum_{N=2}^{\infty} \frac{1}{N!} \int \prod_{i=1}^{N} d^3\tilde{k}_i \, n(k_i) \langle k_1 \dots k_n | T | k_1 \dots k_n \rangle + O(T^2)$$

w/ $d^3\tilde{k}_i = \dfrac{d^3 k_i}{(2\pi)^3 \, 2\pi E_i}$ & $n(k_i) = \dfrac{1}{e^{\beta E_k} - 1}$ the number density.

$f \equiv F/V$ , free energy density.

## * QCD at $O(\alpha_s)$

We need the $2 \to 2$ S-matrix element:

all incoming: $$\mathcal{M}(1^a_-, 2^b_+, 3^c_+, 4^d_-) = 2 g_s^2 \langle 14\rangle^2 [23]^2 \left( \frac{f^{abe} f^{cde}}{s_{12}\, s_{14}} + \frac{f^{ace} f^{bde}}{s_{13}\, s_{14}} \right)$$

contains all dynamical info for $++ \to ++$, $-- \to --$ & $+- \to +-$

Forward limit very simple for all of them:

$$\lim_{\text{forward}} \mathcal{M} = -2 g_s^2 f^{abc} f^{abc} = -2 g_s^2 N_c (N_c^2 - 1)$$

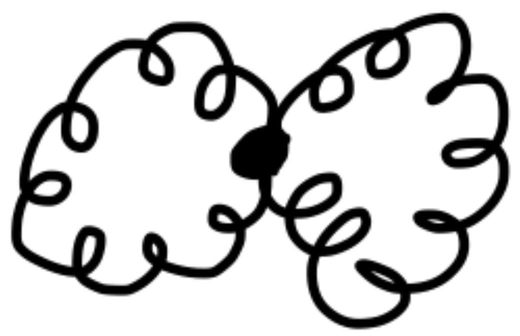
Therefore

$$f - f_0 = 2 g_s^2 \Big( \overset{++}{\tfrac{1}{2}} + \overset{--}{\tfrac{1}{2}} + \overset{+-}{1} \Big) N_c (N_c^2 - 1) \left[ \int_0^\infty \frac{4\pi k^2 dk}{(2\pi)^3\, 2k\,(e^{\beta k} - 1)} \right]^2 + O(g_s^4)$$

$$= \alpha_s N_c (N_c^2 - 1) \frac{\pi T^4}{36}$$

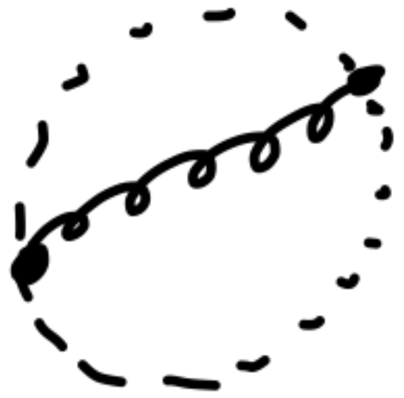
state of the art is $g_s^4$ + resummations.

To be compared w/ text book computation, e.g. Ch. 4.5 of Laine & Vuorinen, "Basics of Thermal Field Th.":



$$= \frac{g^2}{4}\, 3 N_c \;\sum\!\!\!\!\!\!\!\int_K \frac{\delta^{ac} g_{\mu\nu}}{K^2} \;\sum\!\!\!\!\!\!\!\int \frac{1}{K^2} = 3 g^2 N_c (N_c^2-1) \frac{T^4}{(12)^2}$$

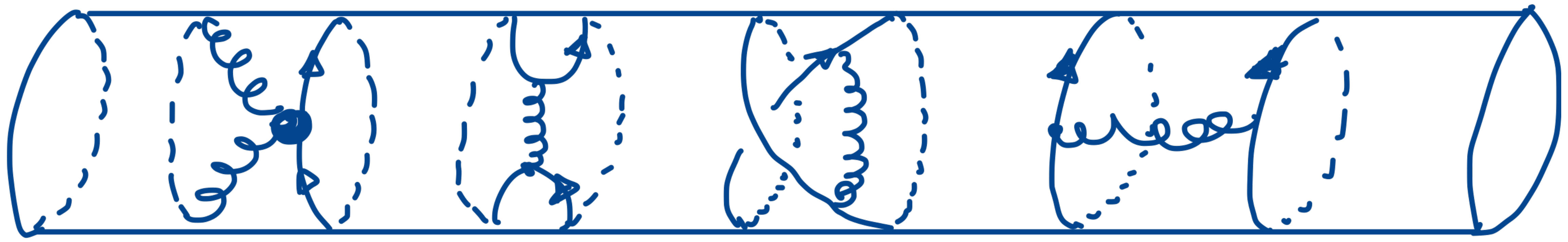
$$= -\frac{3 g^2 N_c}{12} \;\sum\!\!\!\!\!\!\!\int_K \frac{\delta^{aa}}{K^2} \;\sum\!\!\!\!\!\!\!\int_P \frac{K^2 + (K-P)^2 + P^2}{P^2 (K-P)^2} = -\frac{9}{4} g^2 N_c (N_c^2-1) \left(\frac{T^2}{12}\right)^2$$



$$= -\frac{1}{2}\left(-g^2 N_c\right) \;\sum\!\!\!\!\!\!\!\int \frac{\delta^{aa}}{K^2} \;\sum\!\!\!\!\!\!\!\int \frac{P^2 - K\cdot P}{P^2 (K-P)^2} = \frac{g^2}{4} N_c (N_c^2-1) \left(\frac{T^2}{12}\right)^2$$

obscures UV finiteness, gauge invariance, Thermal v.s. vacuum loops.

## * Adding QCD matter + comments



$\times 8$ $\quad$ $\times 2$ $\quad$ $\frac{1}{2!}\times 4$ $\quad$ $=0$

We need two amplitudes:

$$\lim_{\text{forward}} \mathcal{M}(1g^a_-, 3g^b_+, 3\bar{u}_i, 4u_j) = \lim_{\text{forward}} \mathcal{M}(1u_i, 3\bar{u}_+, 2\bar{u}_j, 4u_e) = -g^2(N_c^2-1)$$

$$-\Delta f_{\text{quarks}} = 4\int dN_g\, dN_q \left[\mathcal{M}(g_- u_+ \to g_- u_+) + \mathcal{M}(g_+ u_+ \to g_+ u_+)\right] +$$

$$+ 2\int dN_u\, dN_u\, \mathcal{M}(\bar{u}_+ u_- \to \bar{u}_+ u_-) + 4\int \frac{dN_1\, dN_2}{2} \mathcal{M}[u_- u_- \to u_- u_-]$$

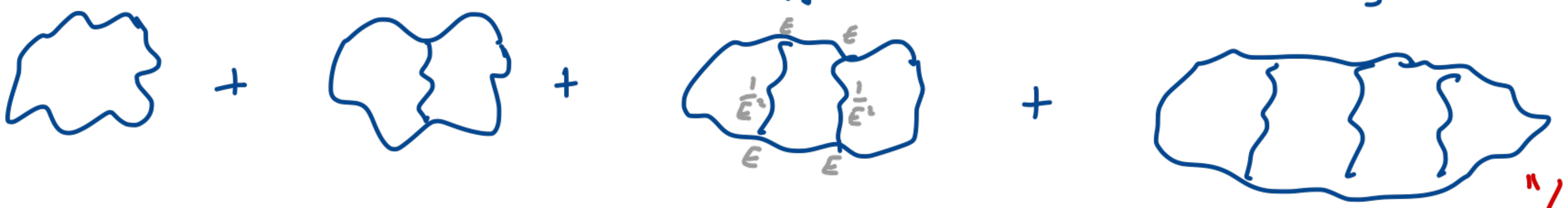
$$= \alpha_s (N_c^2-1) \frac{\pi T^4}{36} \frac{5}{4} N_f .$$

$$\int dN_{g/q} \equiv \int \frac{d^3k}{(2\pi)^3\, 2k\, (e^{\beta k} \mp 1)} = T^2 \begin{cases} \frac{1}{24} & \text{boson} \\ \frac{1}{48} & \text{fermion.} \end{cases}$$

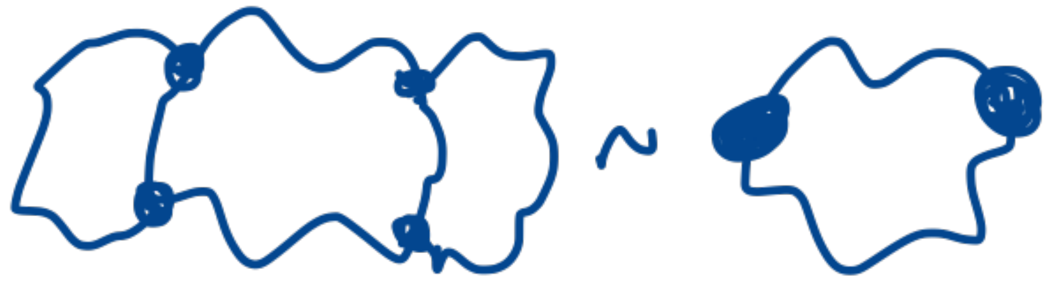
* QCD state of the art: @ fix order.

$$\left[\int d^3k\right]^3 \left(\frac{1}{\epsilon^2}\right)^6 \epsilon^4 = \int d^6\epsilon \, \frac{1}{\epsilon^8} \quad \text{IR div.}$$

$\alpha_s^0 \qquad + \qquad \alpha_s \qquad + \qquad \alpha_s^2 \qquad + \qquad \alpha_s^3$




"Linde problem"! $g^6 T^4 \log\left(T/\Lambda_{IR}\right)$



$$= T \int d^3k \, \frac{(g^2T^2)^2}{(k^2+g^2T^2)^2} \sim \frac{g^4 T^5}{gT} = g^3 T^4 \propto \alpha_s^{3/2}$$

$\Pi_{00} \propto g^2 T^2$

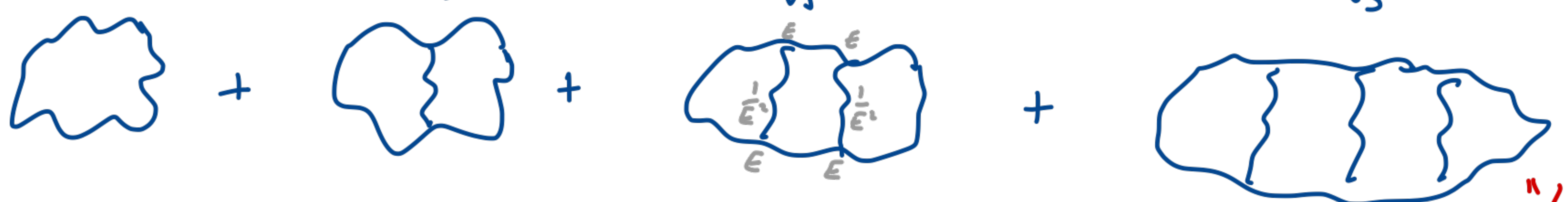
↳ Debye mass, electric component, generates non-integer "$\alpha_s$" powers.

collective phenomena due to modes $k \sim gT \ll T$

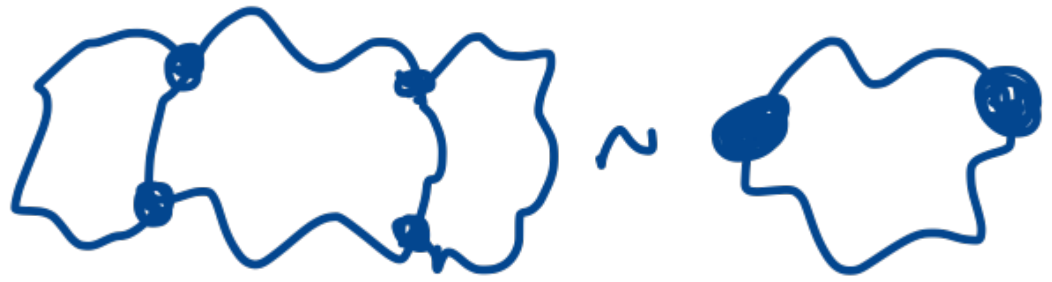
## * QCD state of the art: @ fix order.

$$\left[\int d^3k\right]^3 \left(\frac{1}{\epsilon^2}\right)^6 \epsilon^4 = \int d^6\epsilon \, \frac{1}{\epsilon^8} \quad \text{IR div.}$$

$\alpha_s^0 \qquad \alpha_s \qquad \alpha_s^2 \qquad \alpha_s^3$




"Linde Problem"! $g^6 T^4 \log\left(T/\Lambda_{IR}\right)$



$$\sim \quad = T \int d^3k \, \frac{(g^2T^2)^2}{(k^2+g^2T^2)^2} \sim \frac{g^4 T^5}{gT} = g^3 T^4 \propto \alpha_s^{3/2}$$

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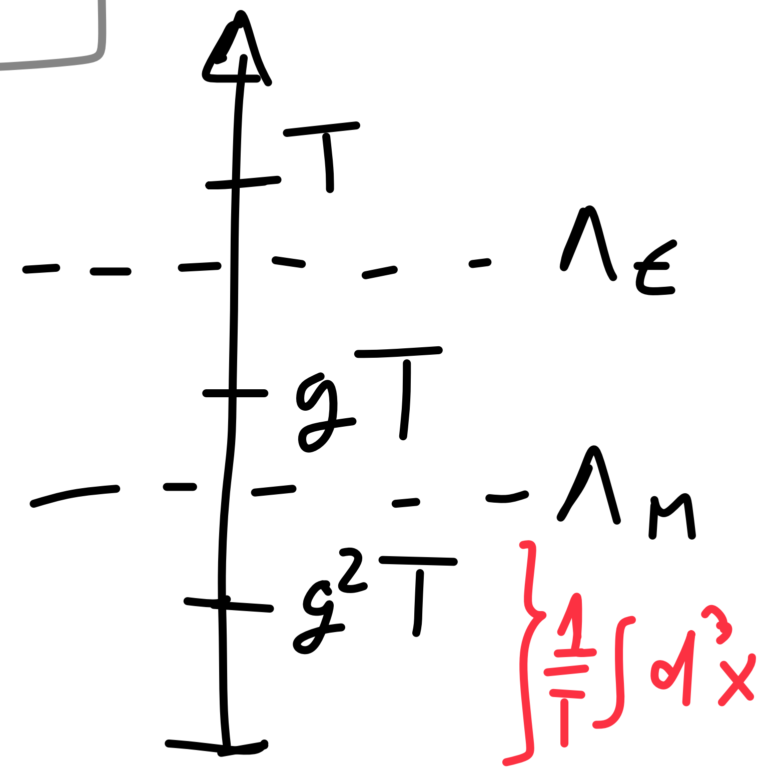


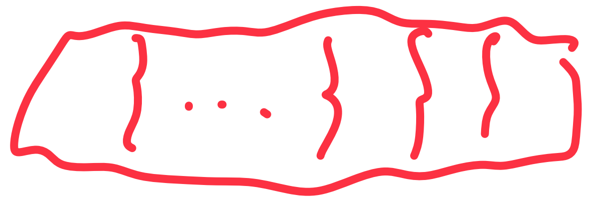

$\int \frac{1}{T}\int d^3x \, F_{ij}F^{ij}$

$\Rightarrow g_M^2 = g^2 T$

$\Rightarrow f = -\frac{\log Z}{V} \propto g_M^6$

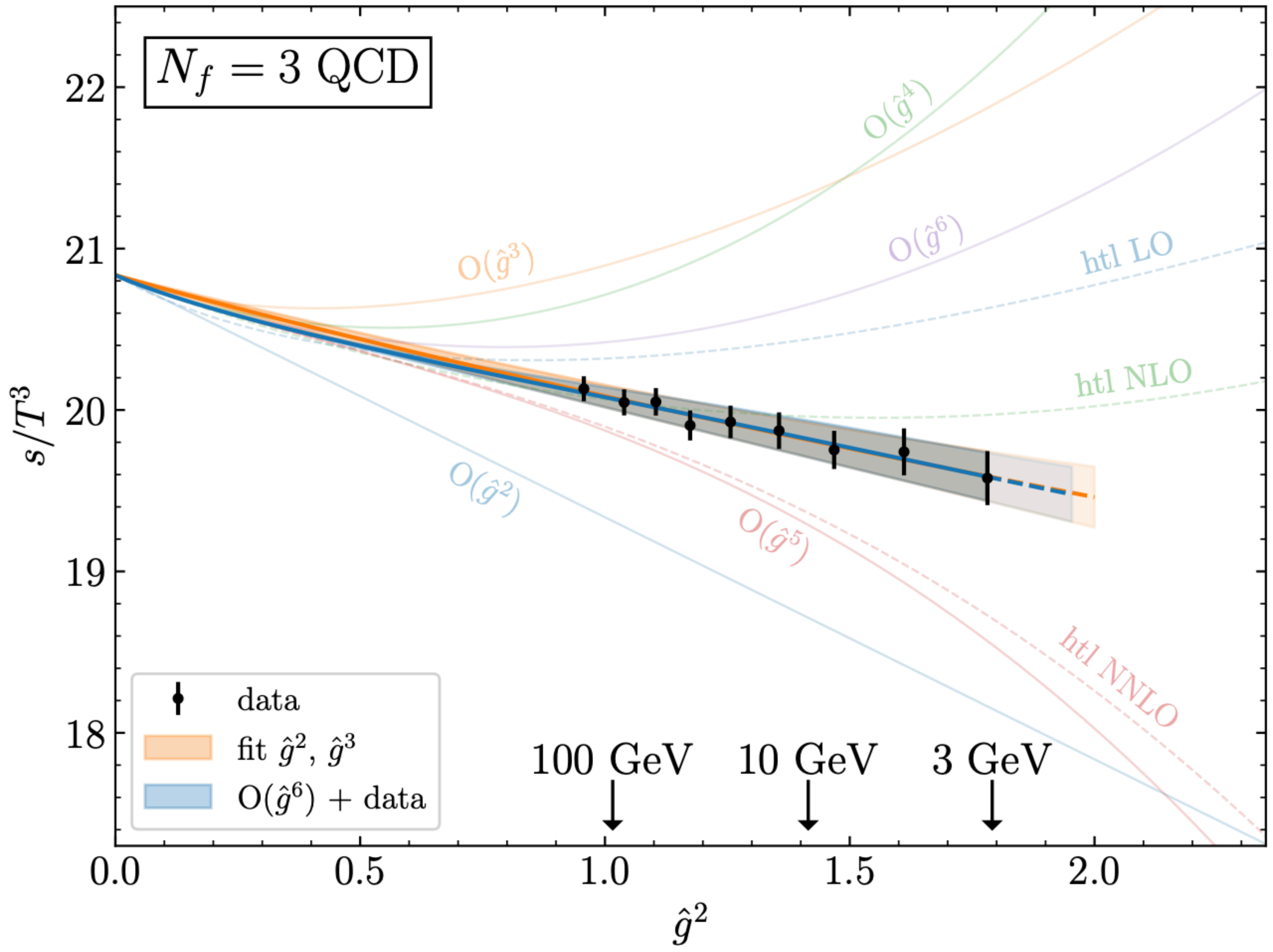
↳ Linde problem: softer modes $k \sim g^2 T \ll T$ are strongly coupled.

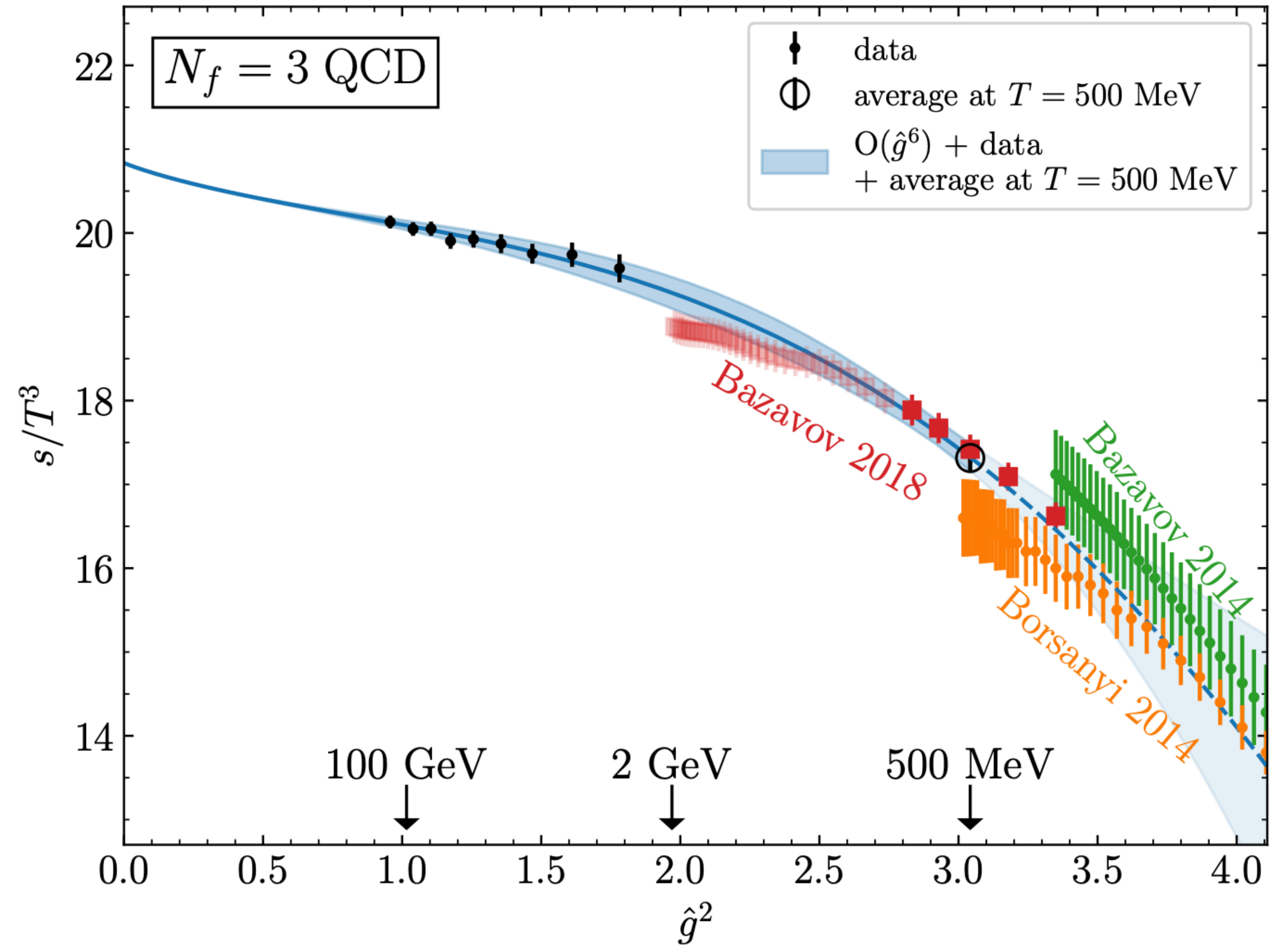
The effective expansion parameter is $\epsilon = \frac{g^2 T}{\pi m} \begin{cases} g/\pi \; ; \text{ for } m \sim gT \text{ Debye mass} \\ 1 \; ; \text{ for color-magnetic fields } m \lesssim g^2 T \end{cases}$



$$\sim g^{2(n-1)} \left(T \int_0^{g^2T} d^3k\right)^n \frac{k^{2(n-1)}}{(k^2+\Lambda_{IR})^{3(n-1)}}$$

Recent Lattice MC determinations of QCD EoS up to very high temperatures vs pQCD




from Bresciani, Dalla Brida, Giusti and Pepe hep-lat/2501.11603

I will explain the DMB calculation up to the same order of htl NNLO.

- Before addressing real QCD, we would like to test DMB to large orders.

- In particular we should address the role of potential singularities arising from the forward limit of the scattering amplitude.

- <u>We need a model for which:</u>

    - The S-matrix & The partition function are known.

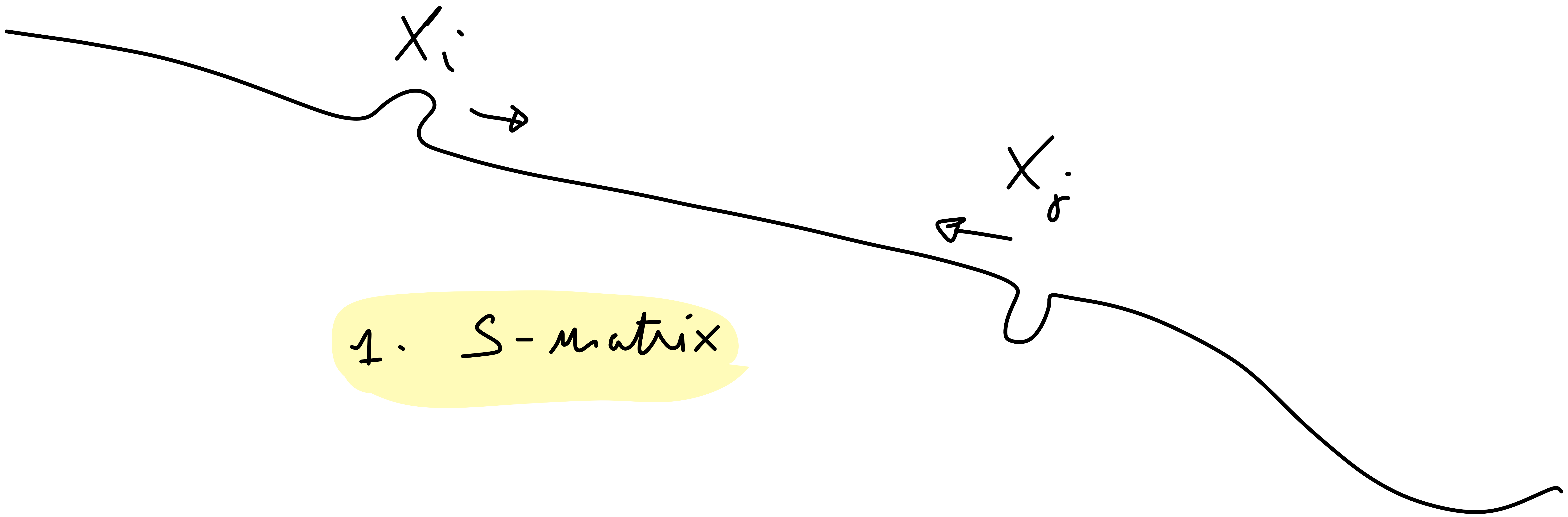
    - No physical IR divs.

    - Ideally we would like the model to be interesting by itself, and that neither the S-matrix nor the partition function are ‘too complicated’.

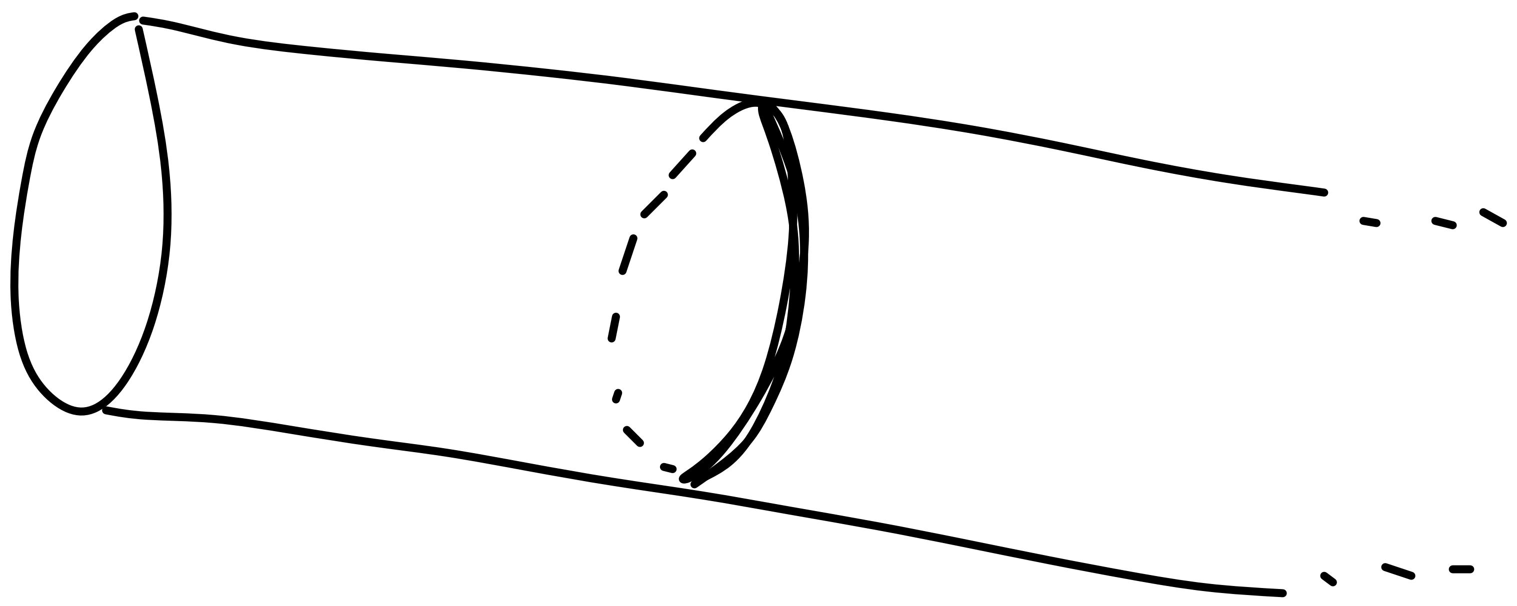
* Simpler set up, Yang-Mills confining flux tubes:

Write down the most general 2d action realizing (non-linearly) the symmetry breaking pattern $ISO(D-1,1) \mapsto ISO(1,1) \times SO(D-2)$. There are "$D-2$" goldstone bosons or transverse excitations. For $D=3$, there is an integrable realization of the system. (i.e. the Wilson coefficients of the action can be fixed to realized the following factorized S-matrix):

E.g. two interesting observables:




1. S-matrix



2. Partition function

* Simpler set up, Yang - Mills confining flux tubes:

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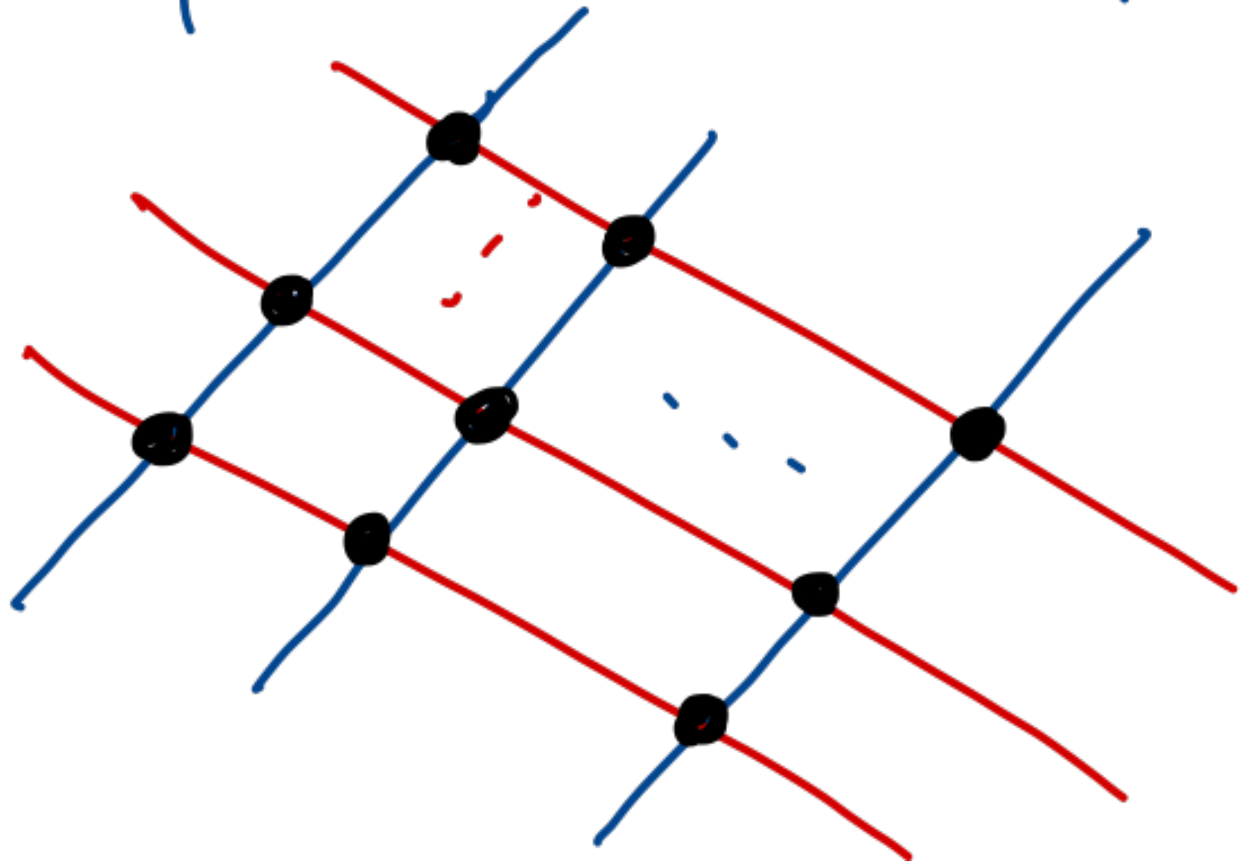
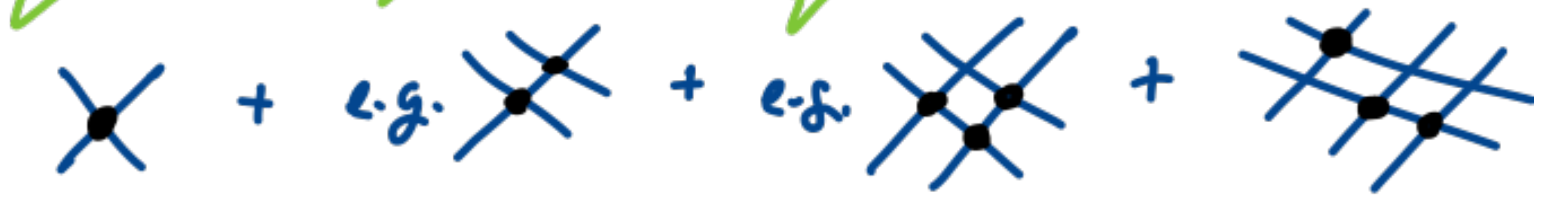
$$S_{\alpha\beta} = e^{\frac{i\ell_s^2}{4}\sum_{i<j} s_{ij}} \langle \alpha | \beta \rangle \longleftrightarrow f(\beta) = \frac{1}{\ell_s^2}\sqrt{1 - \frac{\ell_s^2}{\beta^2}\frac{\pi}{3}}$$

[Dubovsky, Flauger, Gorbenko] 1205.6805



$$= \frac{1}{\ell_s^2} - \frac{\pi}{6\beta^2} - \frac{\ell_s^2\pi^2}{72\beta^4} - \frac{\ell_s^4\pi^3}{432\beta^6} - \frac{5\ell_s^6\pi^4}{10368\beta^8} + \dots$$



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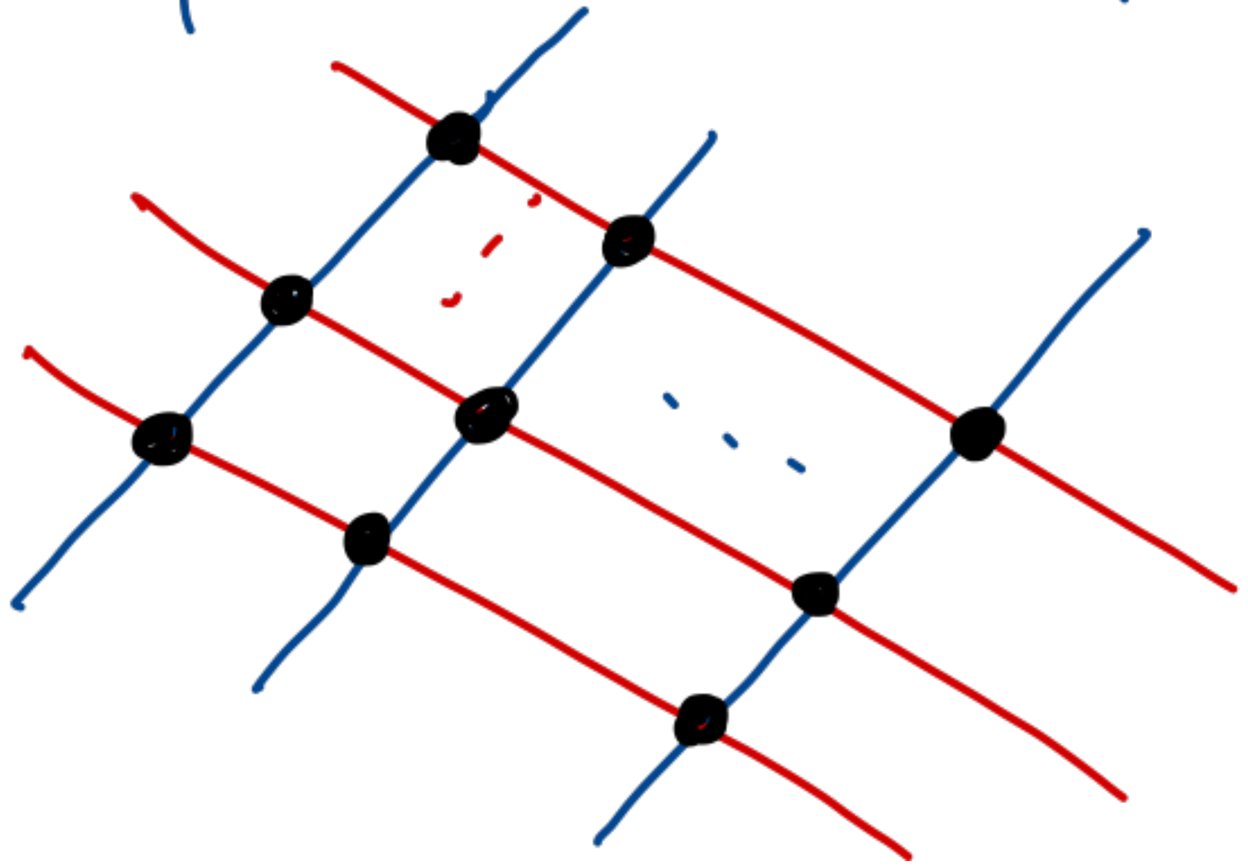

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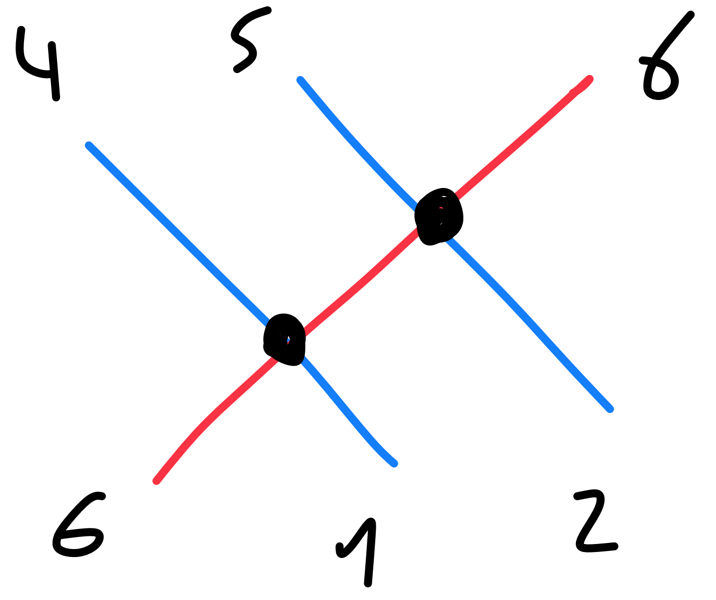
✓ ✓ ✓ ✓ ✓


Clearly $\log S_{\alpha\beta} \propto \ell_s^2$ but $f(\beta)$ is non-linear in $\ell_s^2$!

The paradox is resolved by carefull treatment of $i\epsilon$ and the forward limit!

From $S_{\alpha\beta}$ we get the following singular $3 \to 3$ matrix element:




$$= \mathcal{M}(1_\ell\, 2_\ell\, 3_\nu \to 4_\ell\, 5_\ell\, 6_\nu) = 32\pi i\, \ell_s^4\, E_1^2 E_2^2 E_3^2 \left( \delta(E_1 - E_4) + \delta(E_4 - E_5) \right)$$

in terms of Feynman Diags:

$$\mathcal{M}^{(sing)} = -\ell_s^4 \left( p_3 \cdot p_1\, p_4^\mu + p_3 \cdot p_4\, p_1^\mu \right) \frac{Q_\mu Q_\nu}{Q^2 + i\epsilon} \left( p_6 \cdot p_2\, p_5^\nu + p_6 \cdot p_5\, p_2^\nu \right) + (1 \leftrightarrow 2,\, 4 \leftrightarrow 5).$$

$$\frac{Q_\mu Q_\nu}{Q^2 + i\epsilon} = \delta_\mu^0 \delta_\nu^0 + \frac{1}{2|\vec{Q}|} \left( \frac{q_\mu q_\nu}{Q_0 - |\vec{Q}| + i\epsilon} + \frac{\bar{q}_\mu \bar{q}_\nu}{-Q_0 - |\vec{Q}| + i\epsilon} \right)$$

w/ $q_\mu = (|\vec{Q}|, \vec{Q})$

$\bar{q}_\mu = (|\vec{Q}|, -\vec{Q})$

Next evaluate the diagram on the support of

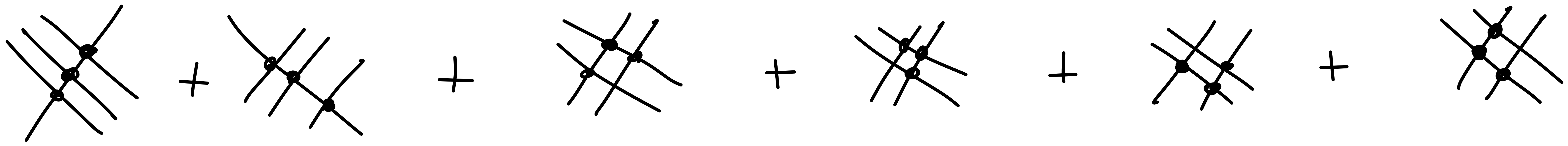
$$\delta(E-E_\alpha)\, \mathrm{Re}\, \frac{1}{E-E_\alpha+i\epsilon} = -\frac{1}{2}\, \partial_\epsilon\, \delta(\epsilon-E_\alpha)$$

All in all

$$f_{N\ell O} = f_{\ell\ell r} + f_{rr\ell} = -\frac{\ell_s^4 \pi^3}{432\,\beta^6}$$

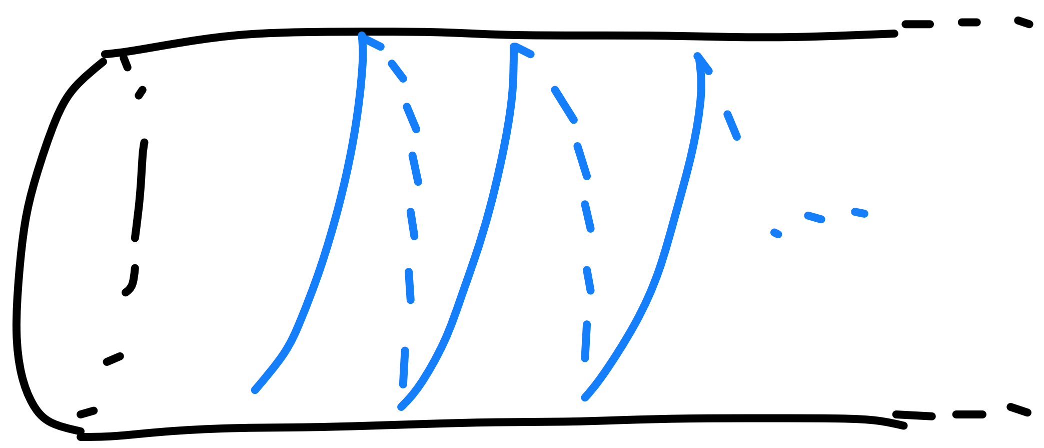
In agreement with $f(\beta) = \frac{1}{\ell_s^2}\sqrt{1-\frac{\ell_s^2\pi}{3\beta^2}}$

In the paper we pushed the method to the next order:



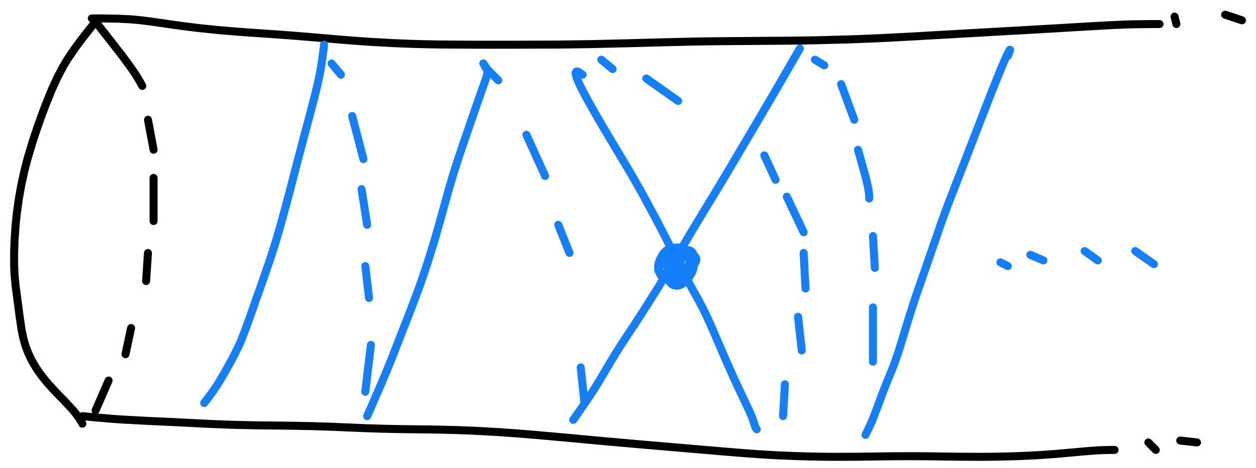
Coming back to perturbative QCD…

* So far I have shown



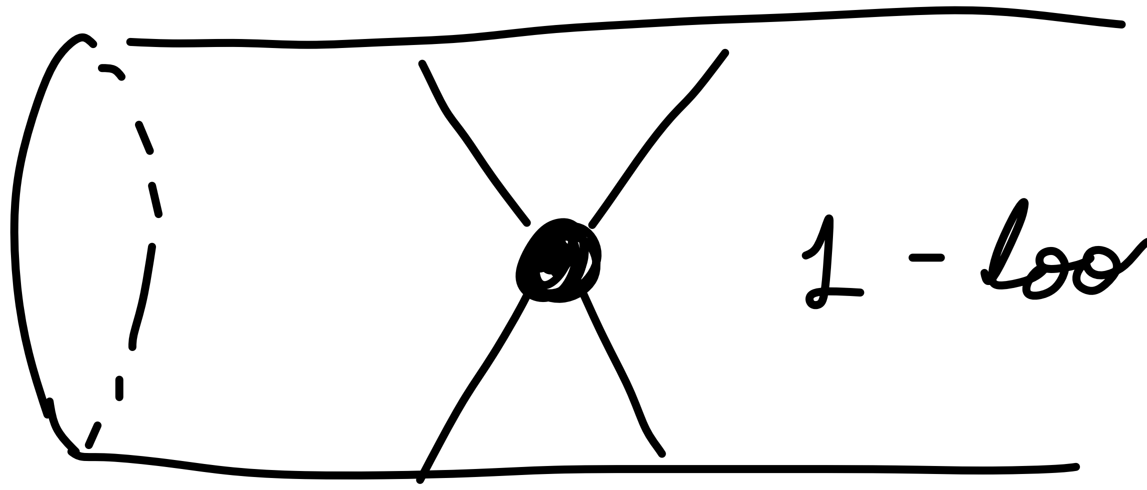
$\alpha_s^0$

+



$\alpha_s$

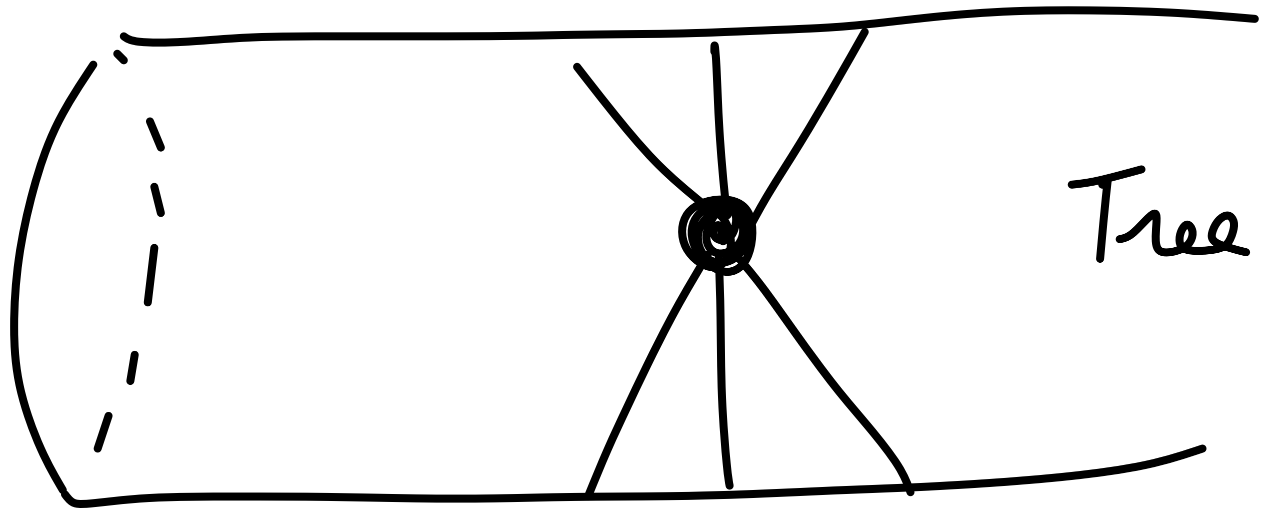
* At order $\alpha_s^2$ we need to compute



1-loop MHV

$\mathcal{M}(++--)$

+



Tree-level NMHV

$\mathcal{M}(+++---)$

↳ Simple, well known amplitudes.

## * How does screening arise in this formalism?

↳ Next I focus on the most singular $\alpha_s^2$ diagrams to show the mechanism.

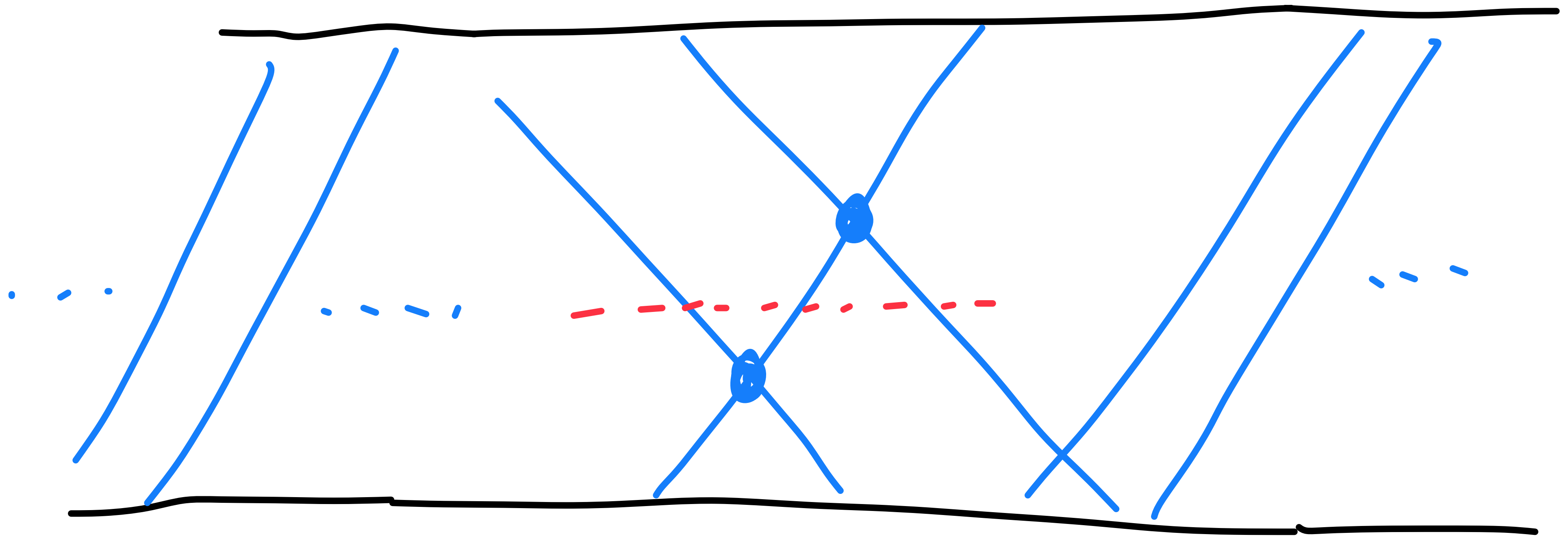
* Recall that

$$\langle 3^{A'}_{\sigma} 4^{B'}_{\tau} | T | 1^{A}_{\sigma} 2^{B}_{\tau} \rangle = (2\pi)^3 \delta^{(3)}(\vec{k}_3 + \vec{k}_4 - \vec{k}_1 - \vec{k}_2)\, 2 g_s^2 \sum_c{}' f^{A'B'C} f^{ABC} + \text{singular}$$

$\frac{s}{t+i\epsilon}$ vanishes in the forward limit $f^{AAC} f^{BBC}$

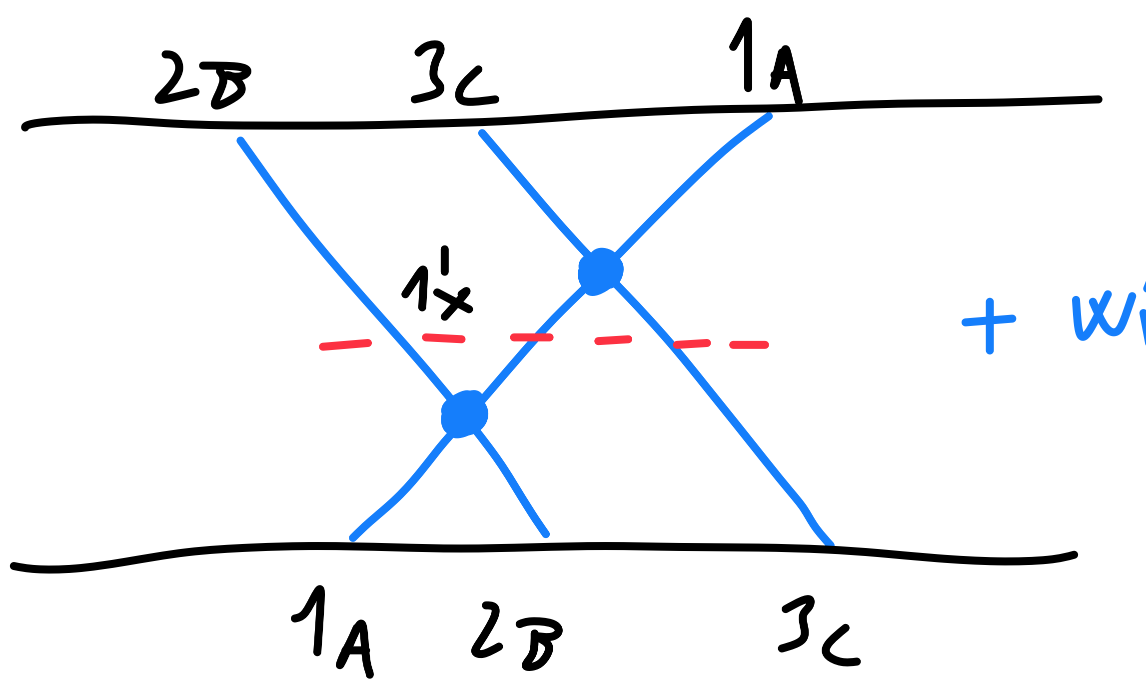
* Next we focus on the most singular diagrams.

They contain the maximal number of on-shell internal propagators.



$\alpha_s^2$

* The diagram is given by



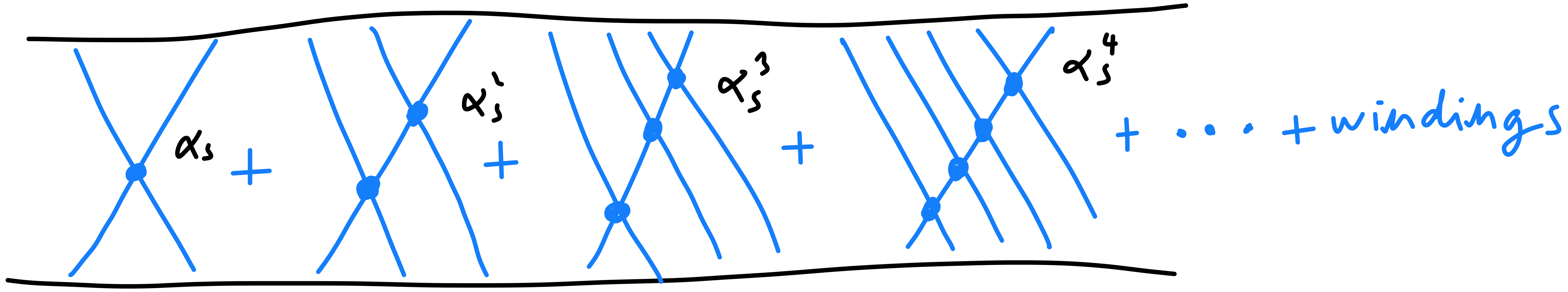

+ windings

$$= \# \left( \int \frac{d^3k \; n(E_k)}{(2\pi)^3 \, 2E_k} \right)^2 \underbrace{\left( \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{n'(E_k)}{4E_k^2} \right)}_{\text{IR divergent}}$$

$$\left[ L^3 (2g_s)^2 N_\sigma^3 N_c^2 (N_c^2 - 1) \right]$$

This IR divergence can be cured via a Debye mass. But in this formalism it seems unnatural to add a thermal mass $m_g \sim g_s T$. We are only tracing over the vacuum S-matrix! .

* summing over all the "comb" diagrams.

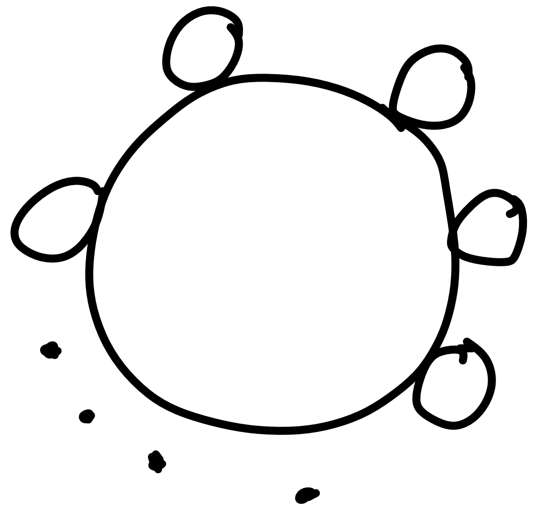



* It turns out

$$\sum_{n=0}^{\infty} f_n^{comb} = N_\sigma (N_c^2 - 1) \int \frac{d^3k}{(2\pi)^3} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{I_T N_\sigma N_c 2 g_s^2}{2 E_k} \right)^n \partial^n n^{(-1)}$$

$$= \beta^{-1} N_\sigma (N_c^2 - 1) \int \frac{d^3k}{(2\pi)^3} \ln \left( 1 - e^{-\beta \sqrt{K^2 + g^2 \frac{N_c}{3\beta^2}}} \right)$$

↳ Finite result. this is nothing but Daisy QCD diagrams



↳ Expected behaviour IR div for $K \sim gT$ modes.

## Summary:

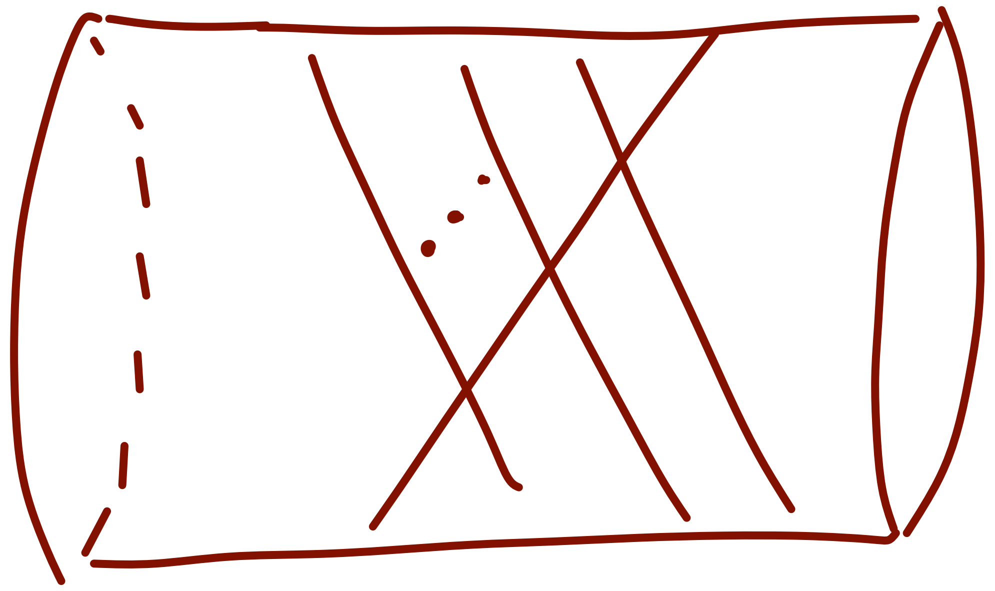
$$Z(\beta) = Z_0(\beta) - \frac{1}{2\pi i}\int dE \sum_n e^{-\beta E}\,\mathrm{Tr}\,[\,T^{(n)}(E+i\epsilon)\,]$$

$$\left[\lim_{\epsilon\to 0}\,,\ \mathrm{Tr}\right] \neq 0$$

$$\left[\int dE\,,\ \sum_n\right] \neq 0$$

$$f(\beta) = \int dE\, e^{-\beta E}\,\mathrm{Tr}\,\log e^{i l_s^2 \sum_{i<j} p_i\cdot p_j}$$

$$= \sqrt{1 - \frac{l_s^2}{\beta^2}\frac{\pi}{3}}$$



IR finite emergent thermal mass.

**Outlook**

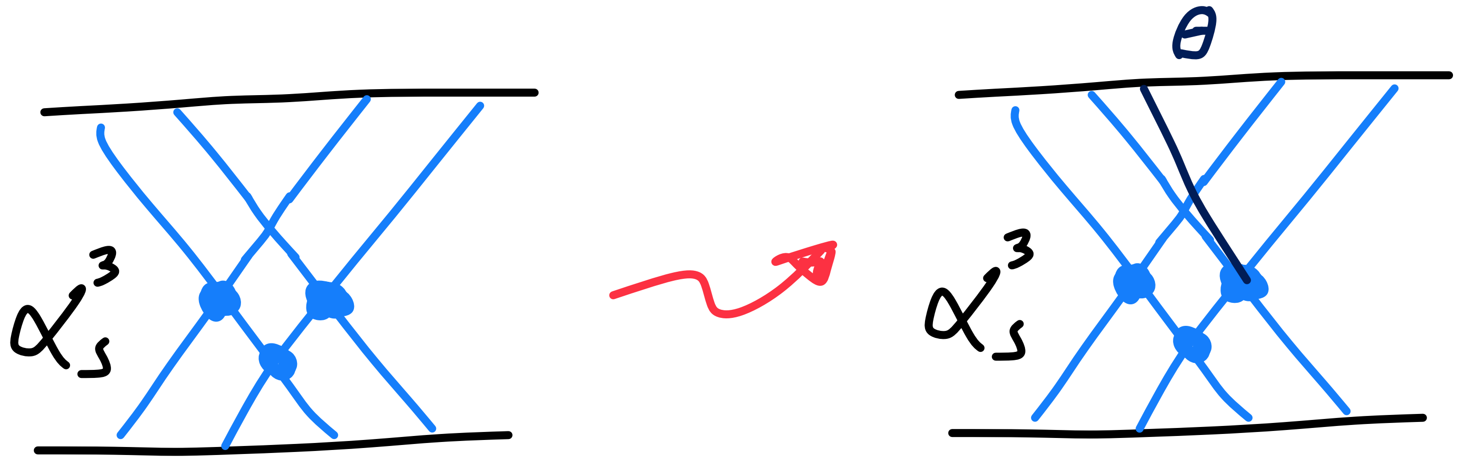
1 - It is possible to generalise to other observables. E.g. finite volume exited states from form factors.

$$S \to S + \int h(x)\,\Theta(x)\,, \quad \text{then} \quad \left.\frac{\delta F[h]}{\delta h(x)}\right|_{h=0} = -\frac{1}{2\pi i}\int dE\, e^{-\beta E}\, \mathrm{Tr}_c\left[S^{-1} \otimes F_0\right]$$

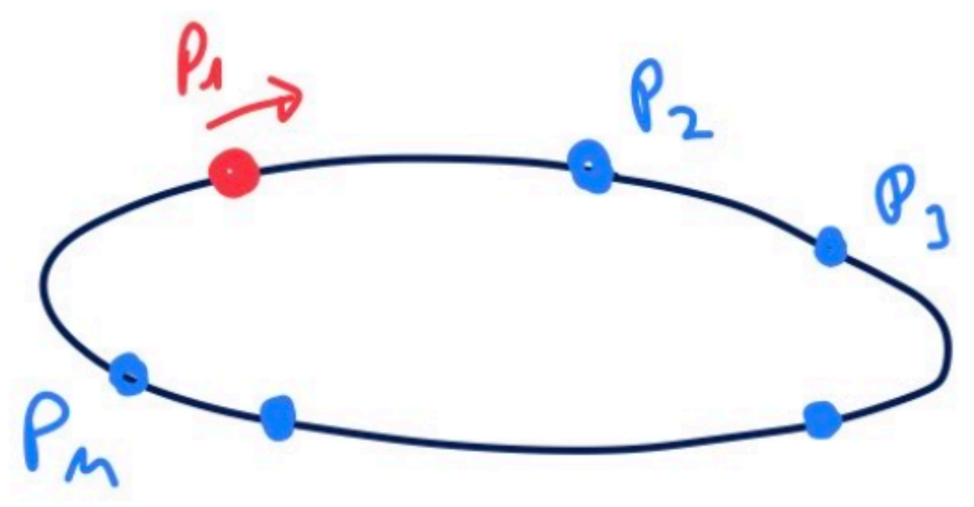
2 - Inverse problem: S-matrix in terms finite volume states. Lüscher, Sharpe, Hansen, ...

- It will be fun to continue pursuing a revision of the pQCD equation of state in view of this formalism.

Speculation: $k \sim g^2 T$ divs. or "Linde problem" may be treated in a collider-inspired approach.



# Review of Asymptotic Bethe Ansatz




Two particle state in infinite volume

$$\psi(x_1, x_2) = \langle 0 | \phi(x_1) \phi(x_2) | p_1, p_2 \rangle$$

Consider $x_1 \ll x_2$ (mutatis mutandis $x_1 \gg x_2$)

$$\psi(x_1 \gg x_2) = e^{i p_1 x_1 + i p_2 x_2} + e^{i p_1 x_2 + i p_2 x_1} S_{12}(p_1, p_2)$$

Place the theory at finite volume w/ $x_1 \ll x_2 \ll x_1 + R$,

$$\psi(x_1, x_2) = \psi(x_1 + R, x_2) \implies e^{i p_1 R} S_{12}(p_1 p_2) = 1$$

Same logic for multiple rescatterings. all in all

$$p_i R + \sum_j 2\,\delta_{ij}(p_i, p_j) = 2\pi N_i$$

Winding corrections exponentially suppressed for massive theories.