Bootstrapping string amplitudes

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Progress of Theoretical Bootstrap
YITP, Kyoto

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2401.05733, **APS, Aninda Sinha**

2409.18259, **Faizan Bhat, Debapriyo Chowdhury, APS, Aninda Sinha**

2506.03862, **Faizan Bhat, APS, Aninda Sinha**

Introduction

Scattering amplitudes in QFT satisfy some properties:

- Locality: Interactions are localized in space and time.
- Analyticity: 4-point amplitudes are analytic function of s, t and u.
- High energy behaviour: Regge bounds, Froissart bounds, etc.
- Crossing symmetry: $\mathcal{M}(s,t,u)$ when external states are identical.

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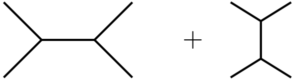
$\mathcal{M}(s,t) =$ 

Example: $\frac{1}{s-m^2} + \frac{1}{t-m^2}$ (amplitude in $\mathrm{tr}\left(\Phi^3\right)$ theory)

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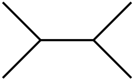
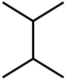
$\mathcal{M}(s,t) =$  $+$ 

Example: $\frac{1}{s-m^2} + \frac{1}{t-m^2}$ (amplitude in $\mathrm{tr}\left(\Phi^3\right)$ theory)

Can we find tree-level scattering amplitudes which are given by single channel expansion?

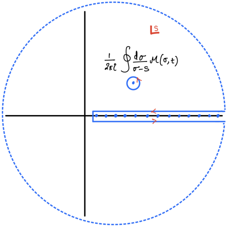
$$\text{Sum over } s\text{-channel diagrams} \stackrel{?}{=} \text{sum over } t\text{-channel diagrams}$$

Dual resonance models

$$\int_0^1 \mathrm{d}z\; z^{-s-1}\,(1-z)^{-t-1} = \frac{\boldsymbol{\Gamma}(-\mathbf{s})\,\boldsymbol{\Gamma}(-\mathbf{t})}{\boldsymbol{\Gamma}(-\mathbf{s}-\mathbf{t})}$$

String amplitudes: infinite tower of massive higher spin resonances

$$\mathcal{M}(s,t) = \frac{1}{\pi}\int_{\mu_0}^{\infty} \frac{\mathrm{d}\sigma}{\sigma - s}\,\mathrm{Disc}\left[\mathcal{M}(\sigma,t)\right] \quad t<0$$



provided the boundary contribution vanishes.

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$$\int_0^1 \mathrm{d}z\, z^{-s-1}(1-z)^{-t-1} = \frac{\boldsymbol{\Gamma(-s)\,\Gamma(-t)}}{\boldsymbol{\Gamma(-s-t)}}$$

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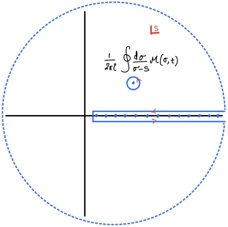

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Contrary to QFT amplitudes, string amplitudes decay with increasing energy

$$\frac{\Gamma(-s)\,\Gamma(-t)}{\Gamma(-s-t)} \sim s^{t}\,\Gamma(-t)\,\frac{\sin(\pi(s+t))}{\sin(\pi s)}, \quad s\to\infty, \quad t \text{ fixed}$$

$$\frac{\Gamma(-s)\,\Gamma(-t)}{\Gamma(-s-t)} = \begin{cases} -\sum_{n=0}^{\infty}\frac{1}{n!}\frac{(1+t)_n}{s-n}, & t<0 \\ -\sum_{n=0}^{\infty}\frac{1}{n!}\frac{(1+s)_n}{t-n}, & s<0 \end{cases}$$

Series expansions are not crossing-symmetric.

Fixed-angle limit of the amplitude can not be obtained from these representations.

Objectives

Can we obtain a field-theoretic representations of tree-level string amplitudes? These representations satisfy the following properties:

- Crossing symmetry is manifest. Sum over poles in all the channels should be included.
- Except the physical singularities, amplitude should be analytic and convergent for all values of kinematic variables.
- Regular terms in the representations, referred to as contact terms, are finite.
- For certain range of kinematic variables, say $|s_i| < \mu$, if we truncate the infinite series representation to reasonably finite number of terms, $N \sim \mu$, then the truncated sum will exhibit important features of the actual representation. The truncated sum looses its validity beyond the cut-off scale μ.
- Use the crossing-symmetric framework for S-matrix bootstrap.

Outline

- Open string amplitude
- Parametric dispersion relations
- S-matrix bootstrap
- Conclusion

Field theory expansion of string amplitude

$$\frac{\Gamma(-s_1)\Gamma(-s_2)}{\Gamma(1-s_1-s_2)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{s_1-n}+\frac{1}{s_2-n}+\frac{1}{\lambda+n}\right)\left(1-\lambda+\frac{(s_1+\lambda)(s_2+\lambda)}{\lambda+n}\right)_{n-1}$$

APS, Aninda Sinha, 2401.05733

Four-gluon amplitude is obtained with the prefactor $\mathrm{tr}\mathcal{F}^4$.

- Series has large domain of convergence in s_1 and s_2 variables.
- λ is attributed to field redefinition ambiguities of the Lagrangian.
- $\mathrm{Re}(\lambda) > -1$ is required for convergence of the series.
- λ improves convergence of the series $\left(n^{-2-\lambda} \text{ for large } n\right)$.
- Residues are independent of λ.
- Sum of the contact terms is convergent.

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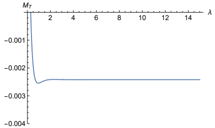

$M_T\,(s_1, s_2)$ with $N = 20$ plotted against λ for $s_1 = 7.5$, $s_2 = -2.2$.

Truncated amplitude depends on field redefinition ambiguity. But plateau indicates that $\partial_\lambda M_T \approx 0$. This demonstrates approximate QFT representation.

Mass-levels

$$t_0 = \frac{1}{s_1 s_2}$$

$$t_1 = \left\{ \frac{1}{s_1 - 1} + \frac{1}{s_2 - 1} \right\} + \frac{1}{1+\lambda}$$

$$t_2 = \frac{1}{2} \left\{ \frac{1+s_2}{s_1 - 2} + \frac{1+s_1}{s_2 - 2} \right\} + \frac{2\lambda^2 + 3\lambda + 2 + 2(\lambda+1)\,x + y}{2(\lambda+2)^2}$$

$$t_3 = \frac{1}{6} \left\{ \frac{10-d}{4\,(d-1)} \left(\frac{1}{s_1 - 3} + \frac{1}{s_2 - 3} \right) + \frac{\frac{9(d-2)}{4(d-1)} + 3s_2 + s_2^2}{s_1 - 3} + \frac{\frac{9(d-2)}{4(d-1)} + 3s_1 + s_1^2}{s_2 - 3} \right\} + \frac{1}{6} \left(3x + \frac{3x^2 - 18x + 20}{\lambda + 3} + \frac{3\left(-3x^2 + xy + 18x - 3y - 27\right)}{(\lambda+3)^2} + \frac{9x^2 + y^2 - 6xy - 54x + 18y + 81}{(\lambda+3)^3} \right)$$

Unitarity holds for $d \leq 10$

Here $x = s_1 + s_2$ and $y = s_1 s_2$.

Crossing-symmetric dispersion relations

Series representation is obtained from CSDR.

Auberson, Khuri, 1973; Sinha, Zahed, 2012.04877; Raman, Sinha, 2107.06559; Song, 2305.03669

We keep s_3 fixed. In terms s_1 and s_2, the dispersion relation is

$$\mathcal{M}(s_1, s_2) = \mathcal{M}(0,0) + \frac{1}{\pi}\int_a^\infty \frac{\mathrm{d}\sigma}{\sigma}\mathcal{A}^{(s_1)}\left(\sigma, \frac{a\sigma}{\sigma - a}\right)\left[\frac{s_1}{\sigma - s_1} + \frac{s_2}{\sigma - s_2}\right].$$

$\mathcal{A}^{(s_1)}$ is the s_1-channel discontinuity.

Define: $x = s_1 + s_2$ and $y = s_1 s_2$. This implies $a = y/x$.

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The above dispersion relation contains negative powers of x and have non local terms. Spurious singularities cancel after adding all the terms.

- Local dispersion relation for $\mathcal{M}(s_1, s_2)$: $\mathcal{M} = \sum_{p,q\geq 0} W_{pq} x^p y^q$

$$\mathcal{M}(0,0) + \frac{1}{\pi}\int_a^\infty \left\{\mathrm{d}\sigma\left[\frac{1}{\sigma - s_1} + \frac{1}{\sigma - s_2} - \frac{1}{\sigma}\right]\mathcal{A}^{(s_1)}\left(\sigma, \frac{s_1 s_2}{\sigma}\right) - \frac{\mathrm{d}\sigma}{\sigma}\mathcal{A}^{(s_1)}(\sigma, 0)\right\}$$

APS, Aninda Sinha; 2401.05733

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APS, Aninda Sinha; 2401.05733

Shift variables: $x \to x + 2\lambda$ and $y \to y + \lambda x + \lambda^2$

$$\mathcal{M}(-\lambda, -\lambda) - \frac{1}{\pi}\int_{s_0}^\infty \mathrm{d}\sigma\left\{\left(\frac{1}{s_1 - \sigma} + \frac{1}{s_2 - \sigma} + \frac{1}{\sigma + \lambda}\right)\mathcal{A}^{(s_1)}\left(\sigma, \frac{(s_1 + \lambda)(s_2 + \lambda)}{\sigma + \lambda} - \lambda\right) + \frac{1}{\sigma + \lambda}\mathcal{A}^{(s_1)}(\sigma, -\lambda)\right\}$$

Parametric dispersion relation

$$\mathcal{M} \quad = \quad -\frac{1}{\pi}\int_{s_0}^{\infty} \mathrm{d}\sigma \left(\frac{1}{s_1 - \sigma} + \frac{1}{s_2 - \sigma} + \frac{1}{\sigma + \lambda} \right) \mathcal{A}^{(s_1)} \left(\sigma, \frac{(s_1 + \lambda)(s_2 + \lambda)}{\sigma + \lambda} - \lambda \right)$$

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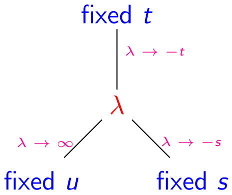
$$\Big\downarrow \quad \lambda = -s_2$$

$$\frac{1}{\pi}\int_{s_0}^{\infty} d\sigma \frac{1}{\sigma-s_1}\mathcal{A}^{(s_1)}(\sigma, s_2)$$

we retrieve fixed t dispersion relation.

Bhat, Chowdhury, APS, Sinha, 2409.18259

Dispersion relations, which have different domains of analyticity, are interpolated among each other using λ.



Subtracted dispersion relations

$$\oint \mathrm{d}\sigma \left[\frac{1}{s_1-\sigma}+\frac{1}{s_2-\sigma}+\frac{1}{\sigma+\lambda}\right]\mathcal{M}\left(\sigma, s_2'(\sigma)\right)=0$$

$\mathcal{M}(s_1, s_2) \lesssim \mathcal{O}\left(s_1^{-1}\right)$
at large s_1

Crossing symmetry implies integrations along s_1-channel and s_2-channel cut are equal. This requirement leads to

$$s_2'(\sigma)=\frac{(s_1+\lambda)(s_2+\lambda)}{\sigma+\lambda}-\lambda$$

Bhat, APS, Sinha; 2506.03862

UV-IR behaviour: $s_2'(-\lambda)=\infty$ and $s_2'(\infty)=-\lambda$

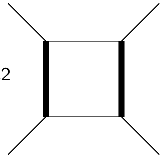
If $\mathcal{M}(s_1, s_2)$ grows faster than $\mathcal{O}\left(s_1^{-1}\right)$ at large s_1, then we can add subtractions to the dispersion relation in the form Taylor expansions around $s_1=s_2=-\lambda$. This is equivalent to adding counterterms such that the arc at infinity does not contribute in the contour integration.

$s_2'(\sigma)$ is a monotonically decreasing function for $\sigma>0$. For amplitudes having Regge behaviour (e.g. $\mathcal{M}\sim s_1^{2+s_2}$), even the un-subtracted dispersion relation works if λ is chosen sufficiently large.

Examples

- This dispersion relation works for amplitudes containing massless poles. Examples: $\frac{1}{st}$, $\frac{1}{s}+\frac{1}{t}$. Here λ acts as IR regulator.

- Dilaton scattering in $\mathcal{N}=4$ SYM on the Coulomb branch

$s^2 t^2$ 

$$\mathcal{I}_{\square}(s,t) = \frac{1}{t\sqrt{s(s-4)}}\left(\frac{1}{\epsilon}-\log(-t)\right) \log\left(1-\frac{s-\sqrt{s(s-4)}}{2}\right) + (s\leftrightarrow t)+\mathcal{O}(\epsilon)$$

- Parametric representation from full three-channel crossing symmetric dispersion relation

$$\frac{\Gamma(-s_1)\Gamma(-s_2)\Gamma(-s_3)}{\Gamma(1+s_1)\Gamma(1+s_2)\Gamma(1+s_3)} = -\frac{1}{s_1 s_2 s_3}+\sum_{n=1}^{\infty}\frac{1}{(n!)^2}\left[\frac{1}{s_1-n}+\frac{1}{s_2-n}+\frac{1}{s_3-n}+\frac{1}{\lambda+n}\right] \left(1-\frac{n}{2}+\frac{n-2\lambda}{2}\sqrt{1-\frac{4(s_1+\lambda)(s_2+\lambda)(s_3+\lambda)}{(n+\lambda)(n-2\lambda)^2}}\right)^2_{n-1}$$

Rosengren, 2409.06658; Bhat, Chowdhury, APS, Sinha, 2409.18259

Bootstrap set up

$2 \to 2$ massless scalar scattering with massive higher spin exchanges

$$\mathcal{M}(s_1, s_2) = \sum_{\ell} f_\ell(s_1)\, \mathcal{G}_\ell^{\left(\frac{D-3}{2}\right)}\left(1 + \frac{2s_2}{s_1}\right)$$

Assumptions at tree-level

$$\mathrm{Disc}_s\left[f_\ell(s)\right] = -\pi \sum_{n=1}^{\infty} c_\ell^{(n)} \delta(s-n)$$

- Unitarity implies $c_\ell^{(n)} > 0$
- Spins of the exchanges are bounded from above

$$\mathcal{M}(s_1, s_2) = \text{Massless pole} + W_{00} + \sum_{n=1}^{\infty}\sum_{\ell} c_\ell^{(n)} \Bigg\{ \left(\frac{2}{n} - \frac{1}{\lambda+n}\right) \mathcal{G}_\ell^{\left(\frac{D-3}{2}\right)}\left(\frac{n-\lambda}{n+\lambda}\right) + \left[\frac{1}{s_1-n} + \frac{1}{s_2-n} + \frac{1}{\lambda+n}\right] \mathcal{G}_\ell^{\left(\frac{D-3}{2}\right)}\left[1 + \frac{2}{n}\left(\frac{(s_1+\lambda)(s_2+\lambda)}{\lambda+n} - \lambda\right)\right] \Bigg\}$$

This one-parameter family of representations holds for amplitudes

- which have any number of only scalar exchanges,
- if exchanges of spinning states are allowed then there should be infinite of higher spin states.

Constraints for bootstrap

- $W_{00} = -\sum_{n,\ell} \frac{(D-3)_n}{n\ n!} c_\ell^{(n)}$
This form follows from open string amplitude
- Maximization of W_{00} is related to minimization of linear entropy $\mathcal{E}[\Omega] = 1 - \mathrm{tr}_A\left[\rho_A^2\right]$
- Impose $\frac{\partial^k}{\partial \lambda^k}\mathcal{M}(s_1, s_2) = 0$ for higher values of k

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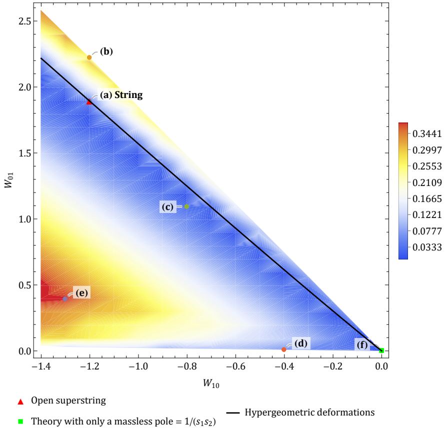
Space of 4-point amplitudes which admit fixed-t dispersion relation near forward limit. Linear Regge trajectories are present on the black curve.



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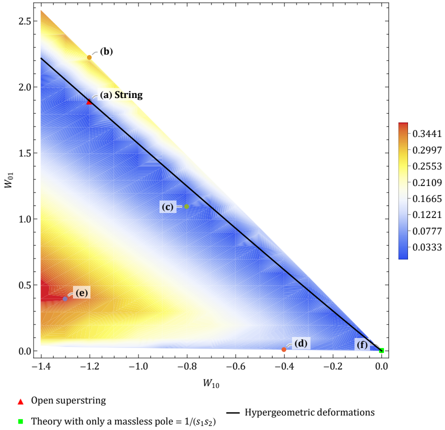

$$\sum_{n,\ell} c_\ell^{(n)} \left\{ \frac{(D-3)_n}{n\, n!} + \left(-\frac{2}{n} + \frac{1}{\lambda + n} \right) \mathcal{G}_\ell^{\left(\frac{D-3}{2}\right)} \left(\frac{n-\lambda}{n+\lambda} \right) \right\} = 0$$

condition for single channel expansion

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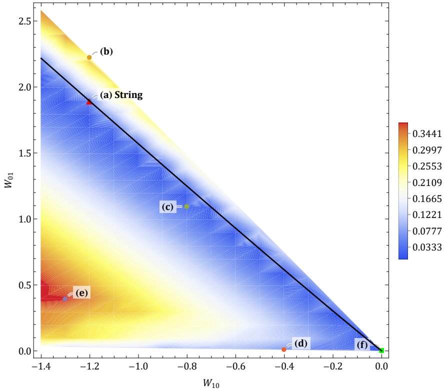

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condition for single channel expansion

$$\frac{1}{s_1 s_2} - \frac{\Gamma(1-s_1)\Gamma(1-s_2)}{(r+1)\Gamma(-s_1-s_2+2)}\, {}_3F_2\left(r+1, 1-s_1, 1-s_2; r+2, -s_1-s_2+2; 1\right)$$

Mansfield, Spradlin, 2409.09561; Bhat, Chowdhury, APS, Sinha, 2409.18259

Three-channel symmetric dispersion relation

$$\mathcal{M}(s_1, s_2) = -\frac{1}{\pi}\int_{\mu_0}^{\infty} \mathrm{d}\sigma \left[\frac{1}{s_1-\sigma} + \frac{1}{s_2-\sigma} + \frac{1}{s_3-\sigma} + \frac{1}{\lambda+\sigma}\right] \mathcal{A}^{(s_1)}\left(\sigma, s_2^{\pm}(\sigma) - \lambda\right)$$

$$s_2^{\pm}(\sigma) = -\frac{\sigma - 2\lambda}{2}\left[1 \pm \sqrt{1 + \frac{4\left(y + \lambda x - \lambda^3\right)}{(\sigma+\lambda)(\sigma - 2\lambda)^2}}\right]$$

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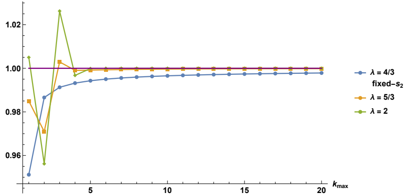
Bhat, Chowdhury, APS, Sinha, 2409.18259; Bhat, APS, Sinha, 2506.03862

Tree-level dilaton scattering: $\lim_{|s|\to\infty}\left|\frac{\mathcal{M}(s,t)}{s^2}\right| \to 0,\ t<0$

Ratio of truncated sum to the actual amplitude at $s_1 = 1/2, s_2 = -4/3$.



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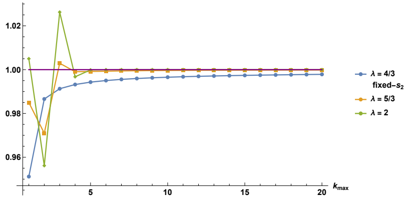
$$y = -s_1 s_2 s_3,$$
$$s_1 + s_2 + s_3 = 0$$

Bhat, Chowdhury, APS, Sinha, 2409.18259; Bhat, APS, Sinha, 2506.03862

Tree-level dilaton scattering: $\lim_{|s|\to\infty}\left|\frac{\mathcal{M}(s,t)}{s^2}\right| \to 0,\ t<0$

Ratio of truncated sum to the actual amplitude at $s_1 = 1/2, s_2 = -4/3$.



Dispersive representations for Wilson coefficients can be used for bootstrap

$$W_{1,0} = \frac{2}{\lambda} - \sum_{k=1}^{\infty}\sum_{\ell=0}^{2k+2} \mathrm{Im} f_\ell(k)\frac{2}{k^3}\left[\mathcal{G}_\ell^{\left(\frac{d-3}{2}\right)}\left(\frac{\sqrt{k-3\lambda}}{\sqrt{k+\lambda}}\right) + \frac{(d-3)\lambda(2k+3\lambda)}{\sqrt{k-3\lambda}(k+\lambda)^{3/2}}\mathcal{G}_{\ell-1}^{\left(\frac{d-1}{2}\right)}\left(\frac{\sqrt{k-3\lambda}}{\sqrt{k+\lambda}}\right)\right]$$

$$W_{0,1} = \frac{1}{\lambda^2} + \sum_{k=1}^{\infty}\sum_{\ell=0}^{2k+2} \mathrm{Im} f_\ell(k)\frac{1}{k^4}\left[3\mathcal{G}_\ell^{\left(\frac{d-3}{2}\right)}\left(\frac{\sqrt{k-3\lambda}}{\sqrt{k+\lambda}}\right) - \frac{2(d-3)k(2k+3\lambda)}{\sqrt{k-3\lambda}(k+\lambda)^{3/2}}\mathcal{G}_{\ell-1}^{\left(\frac{d-1}{2}\right)}\left(\frac{\sqrt{k-3\lambda}}{\sqrt{k+\lambda}}\right)\right]$$

Double copy

At a massive pole residues exhibit double copy relation

$$\left(1-\frac{n}{2}+\frac{n-2\lambda}{2}\sqrt{1-\frac{4\left(s_1+\lambda\right)\left(s_2+\lambda\right)\left(s_3+\lambda\right)}{\left(n+\lambda\right)\left(n-2\lambda\right)^2}}\right)^2_{n-1} \xrightarrow{s_1=n} \left(1+s_2\right)^2_{n-1}$$

From the series representation

$$\mathcal{M}_{\text{cl}}\left(s_1,s_2,s_3\right) = \mathcal{M}_{\text{op}}\left(s_1,s_2\right)\frac{\sin\left(\pi s_1\right)\sin\left(\pi s_2\right)}{\pi\sin\left(\pi\left(s_1+s_2\right)\right)}\mathcal{M}_{\text{op}}\left(s_1,s_2\right)$$

$$\Big\downarrow \; s_1,\; s_2 \sim 0$$

$$= -\frac{1}{s_1 s_2 s_3}+\frac{2}{\left(s_1+s_2\right)}\left[\sum_n \frac{\left(2\lambda+n\right)}{\lambda n^2 n!}\left(-\frac{n\lambda}{n+\lambda}\right)_n+\zeta\left(2\right)\right]+\mathcal{O}\left(1\right)$$

Sum over infinite terms required to cancel the crossing symmetry violating term.

Double copy relations can be used to constrain Wilson coefficients.

Crossing symmetry beyond 4-point

Planar-ordered cubic scalar amplitude

$$\mathcal{M}_5^{\phi^3} = \frac{1}{(s_{12}-m^2)(s_{34}-m^2)} + \frac{1}{(s_{23}-m^2)(s_{45}-m^2)} + \frac{1}{(s_{34}-m^2)(s_{51}-m^2)} + \frac{1}{(s_{45}-m^2)(s_{12}-m^2)} + \frac{1}{(s_{51}-m^2)(s_{23}-m^2)}.$$

The amplitude is invariant under the simultaneous exchanges

Ongoing work with **Debapriyo Chowdhury**, **Soumen Saren** and **Aninda Sinha**

$$\begin{aligned} s_{12} &\leftrightarrow s_{34} & s_{51} &\leftrightarrow s_{45}, \\ s_{12} &\leftrightarrow s_{23} & s_{34} &\leftrightarrow s_{51}, \\ s_{12} &\leftrightarrow s_{45} & s_{23} &\leftrightarrow s_{34}, \\ s_{12} &\leftrightarrow s_{51} & s_{23} &\leftrightarrow s_{45}, \\ s_{23} &\leftrightarrow s_{51} & s_{34} &\leftrightarrow s_{45}. \end{aligned}$$

Symmetry is satisfied by tachyonic amplitude

$$\mathcal{M}_5^{\text{string}} = \frac{\Gamma(-1-s_{12})\,\Gamma(-1-s_{23})\,\Gamma(-1-s_{34})\,\Gamma(-1-s_{45})\,\Gamma(-3-s_{23}-s_{34}-s_{45})\,\Gamma(-1-s_{51})}{\Gamma(-2-s_{12}-s_{23})\,\Gamma(-2-s_{23}-s_{34})\,\Gamma(-2-s_{34}-s_{45})\,\Gamma(-2-s_{45}-s_{51})}$$

$$\times\, {}_6F_5\left(\begin{matrix} -1-s_{23},\ -1-s_{45},\ -1+s_{12}-s_{34}-s_{45}, \\ -4-s_{23}-s_{34}-s_{45},\ -1-\frac{s_{23}}{2}-\frac{s_{34}}{2}-\frac{s_{45}}{2},\ -1-s_{23}-s_{34}+s_{51} \\ -2-s_{12}-s_{23},\ -2-s_{23}-s_{34},\ -2-s_{34}-s_{45},\ -2-s_{45}-s_{51} \end{matrix}\ ;-1\right).$$

Arkani-Hamed, Figueiredo, Remmen, 2412.20639

Summary & future works

- We have obtained a parametric crossing symmetric dispersive representation. By tuning the free parameter we can recover the known dispersion relations.
- This provides a suitable framework for bootstrap analysis for amplitudes with massive higher spin exchanges.
- We obtained a class of solutions which include string amplitude. More constraints are required to uniquely fix string amplitudes. We need to extend the analysis beyond four-point tree-level amplitudes.
- Constrain the non-analytic structures by including loops.
- What is the domain of analyticity of the dispersion relation?
- Study of Regge analysis using crossing symmetric dispersion relation.
- Crossing-symmetric dispersion relation for higher point amplitudes?

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  Suitable basis for partial wave expansion for tree-level five-point Veneziano amplitude? work in progress

$$\text{Res} = \sum_{j,l=0}^{\infty} \sum_{n=0}^{\min\{|j|,|l|\}} a_{jln} \cos(\omega n) \left(1 - x^2\right)^{\frac{|n|}{2}} \mathcal{G}_{j-|n|}^{m+|n|}(x) \left(1 - y^2\right)^{\frac{|n|}{2}} \mathcal{G}_{l-|n|}^{m+|n|}(y)$$

Thank you