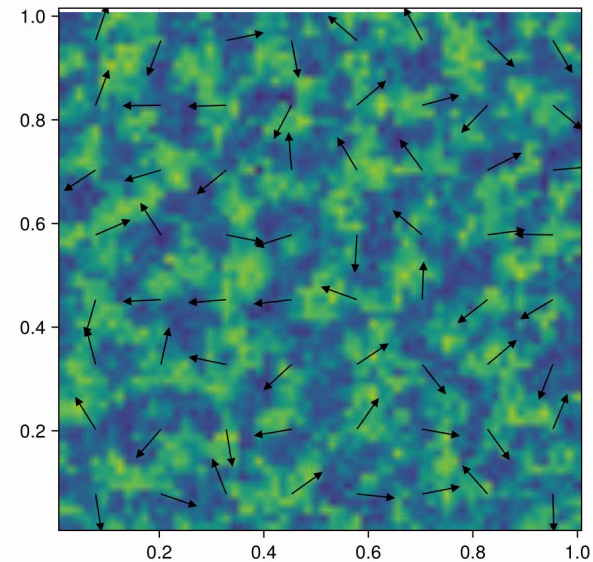
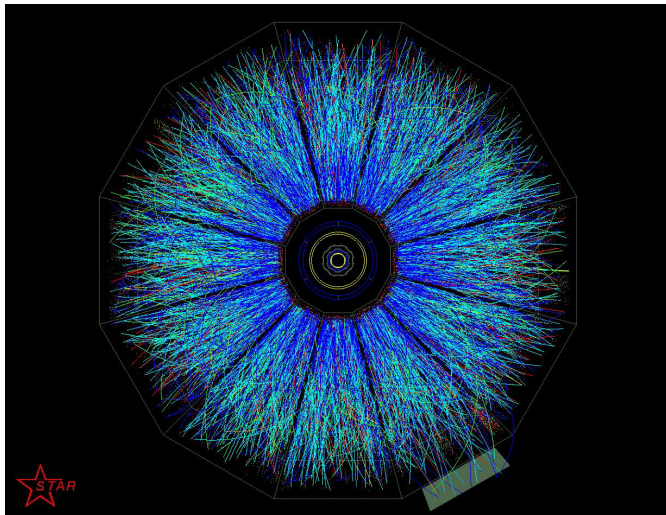


# Real-time stochastic dynamics near a QCD critical point

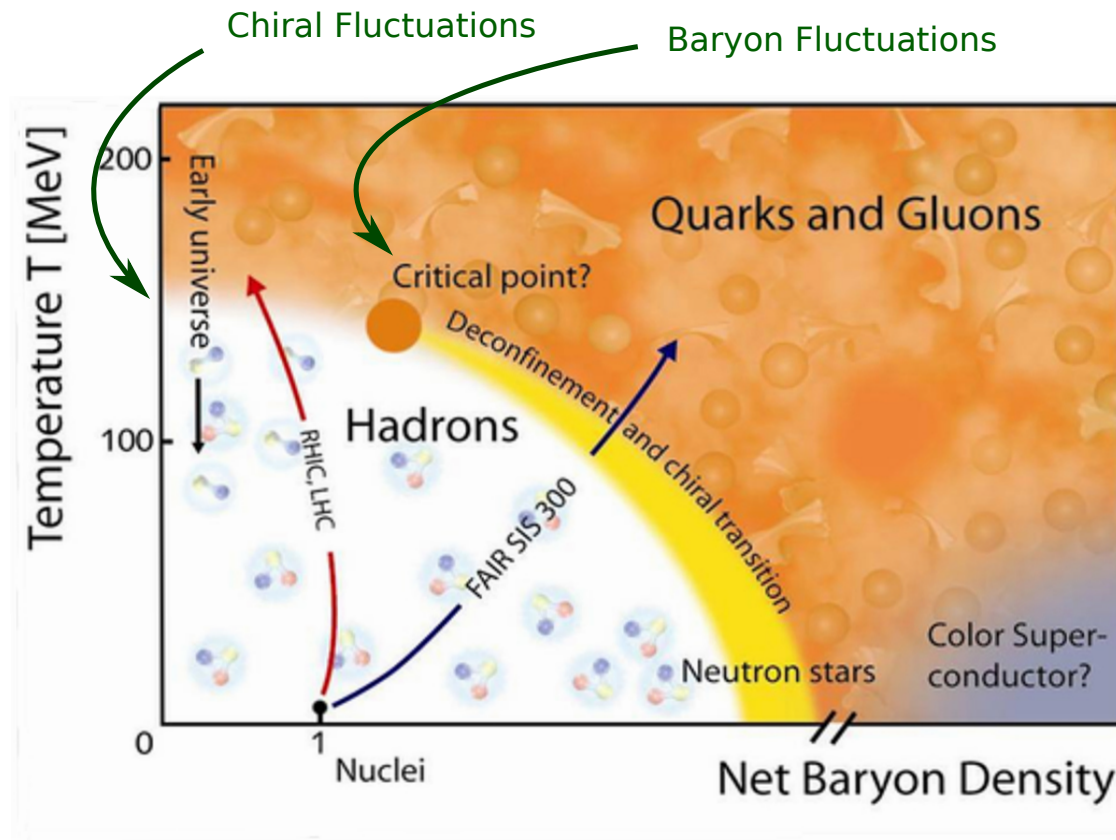
Thomas Schäfer, North Carolina State University



With C. Chattopadhyay, J. Ott, V. Skokov

References: 2403.10608, 2411.15994, 2510.12557, 2603.21479

# Motivation



Can we locate the chiral phase transition, or the endpoint of a first-order QGP-hadron gas transition?

# Dynamical Theory

What is the dynamical theory near the critical point?

The basic logic of fluid dynamics still applies. Important modifications:

- Critical equation of state.
- Stochastic fluxes, fluctuation-dissipation relations.
- Possible Goldstone modes (chiral field in QCD)

Chiral phase transition: Model G (Rajagopal & Wilzcek)

Possible critical endpoint: Model H (Son & Stephanov)

## Digression: Diffusion

Consider a Brownian particle

$$\dot{p}(t) = -\gamma_D p(t) + \zeta(t)$$

drag (dissipation)

$$\langle \zeta(t) \zeta(t') \rangle = \kappa \delta(t - t')$$

white noise (fluctuations)

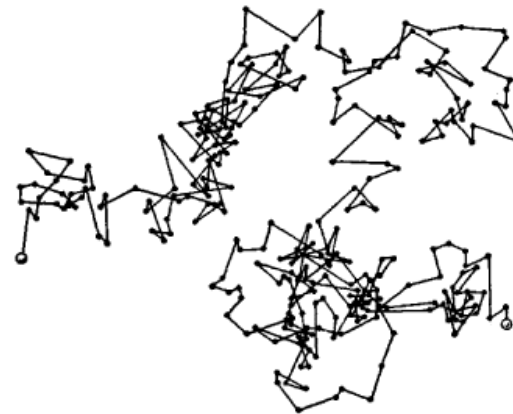
For the particle to eventually thermalize

$$\langle p^2 \rangle = 2mT$$

drag and noise must be related

$$\kappa = \frac{mT}{\gamma_D}$$

Einstein (Fluctuation-Dissipation)



## Relation to Effective Field Theory

This theory is equivalent to a 0+1 EFT, the KMS (or MSR) theory

$$Z = \int \mathcal{D}x_a \mathcal{D}x_r e^{i \int dt L} \quad L = M \dot{x}_r \dot{x}_a - c x_a x_r - \nu x_a \dot{x}_r + \frac{i}{2} \sigma x_a^2$$

with  $2\nu/\sigma = 1/T$ . Follows from

$$Z = \int \mathcal{D}x_r \mathcal{D}\xi \delta(M \ddot{x}_r - \nu \dot{x}_r + c x_r - \xi) e^{-\int dt \xi^2 / (2\sigma)}$$

Fluctuation-Dissipation relation encoded in KMS symmetry

$$x_r(t) \rightarrow x_r(-t)$$

$$x_a(t) \rightarrow x_a(-t) + i\beta \partial_t x_r(-t)$$

## Hydrodynamic equation for critical mode

Equation of motion for critical mode  $\phi$  (“model H”)

$$\frac{\partial \phi}{\partial t} = \kappa \nabla^2 \frac{\delta \mathcal{F}}{\delta \phi} - g \left( \vec{\nabla} \phi \right) \cdot \frac{\delta \mathcal{F}}{\delta \vec{\pi}^T} + \zeta \quad (g = 1)$$

Diffusion      Advection      Noise

Equation of motion for momentum density  $\pi$

$$\frac{\partial \vec{\pi}^T}{\partial t} = \eta \nabla^2 \frac{\delta \mathcal{F}}{\delta \vec{\pi}^T} + g \left( \vec{\nabla} \phi \right) \cdot \frac{\delta \mathcal{F}}{\delta \phi} - g \left( \frac{\delta \mathcal{F}}{\delta \vec{\pi}^T} \cdot \vec{\nabla} \right) \vec{\pi}^T + \vec{\xi}$$

Free energy functional: Order parameter  $\phi$ , momentum density  $\vec{\pi} = w\vec{v}$

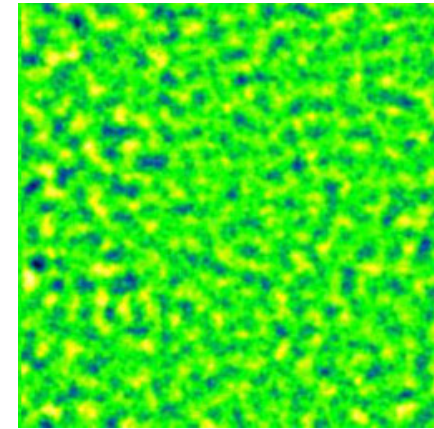
$$\mathcal{F} = \int d^3x \left[ \frac{1}{2w} \vec{\pi}^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{m^2}{2} \phi^2 + \lambda \phi^4 \right] \quad D = m^2 \kappa$$

## Fluctuation-Dissipation relation

$$\langle \zeta(x, t) \zeta(x', t') \rangle = -2\kappa T \nabla^2 \delta(x - x') \delta(t - t')$$

$$\langle \xi_i(x, t) \xi_j(x', t') \rangle = -2\eta T \nabla^2 P_{ij}^T \delta(x - x') \delta(t - t')$$

$$\text{ensures } P[\phi, \vec{\pi}] \sim \exp(-\mathcal{F}[\phi, \vec{\pi}]/T)$$



Mode couplings governed by Poisson brackets:  $\psi^a = (\phi, \vec{\pi})$

$$\partial_t \psi^a = \{\mathcal{H}, \psi^a\} = - \int d^3x \{\psi^a, \psi^b\} \frac{\delta \mathcal{H}}{\delta \psi^b} = - \int d^3x Q^{ab} \psi^b$$

Provides mutual coupling between  $\phi$  and  $\vec{\pi}$

$$\frac{\partial \phi}{\partial t} = - \left( \vec{\nabla} \phi \right) \cdot \frac{\delta \mathcal{F}}{\delta \vec{\pi}^T} \qquad \frac{\partial \vec{\pi}^T}{\partial t} = + \left( \vec{\nabla} \phi \right) \cdot \frac{\delta \mathcal{F}}{\delta \phi}$$

as well as self coupling (self-advection) of  $\vec{\pi}$

$$\frac{\partial \vec{\pi}^T}{\partial t} = - \left( \frac{\delta \mathcal{F}}{\delta \vec{\pi}^T} \cdot \vec{\nabla} \right) \vec{\pi}^T$$

Note: There is a consistent truncation (“model H0”) in which the self-coupling of  $\vec{\pi}$  is dropped. This model is claimed to be in the same dynamical universality class as model H.

Hohenberg, Halperin, RMP (1977)

And: There is a generalization (“compressible model H”) in which  $\vec{\pi}^L$  is retained. This theory is needed to understand the critical behavior of the bulk viscosity.

M. Martinez, T.S., V.S, arXiv:1906.11306

## Why direct numerical simulation?

Critical dynamics is non-perturbative.

Need non-perturbative approach capable of dealing with non-trivial background.

## Numerical realization

Stochastic relaxation equation (“model A”)

$$\partial_t \psi = -\Gamma \frac{\delta \mathcal{F}}{\delta \psi} + \zeta \quad \langle \zeta(x, t) \zeta(x', t') \rangle = \Gamma T \delta(x - x') \delta(t - t')$$

Naive discretization

$$\psi(t + \Delta t) = \psi(t) + (\Delta t) \left[ -\Gamma \frac{\delta \mathcal{F}}{\delta \psi} + \sqrt{\frac{\Gamma T}{(\Delta t) a^3}} \theta \right] \quad \langle \theta^2 \rangle = 1$$

Noise dominates as  $\Delta t \rightarrow 0$ , leads to discretization ambiguities in the equilibrium distribution.

Idea: Use Metropolis update

$$\psi(t + \Delta t) = \psi(t) + \sqrt{2\Gamma(\Delta t)} \theta \quad p = \min(1, e^{-\beta \Delta \mathcal{F}})$$

## Numerical realization

Central observation

$$\begin{aligned}\langle \psi(t + \Delta t, \vec{x}) - \psi(t, \vec{x}) \rangle &= -(\Delta t) \Gamma \frac{\delta \mathcal{F}}{\delta \psi} + O((\Delta t)^2) \\ \langle [\psi(t + \Delta t, \vec{x}) - \psi(t, \vec{x})]^2 \rangle &= 2(\Delta t) \Gamma T + O((\Delta t)^2) .\end{aligned}$$

Metropolis realizes both diffusive and stochastic step. Also

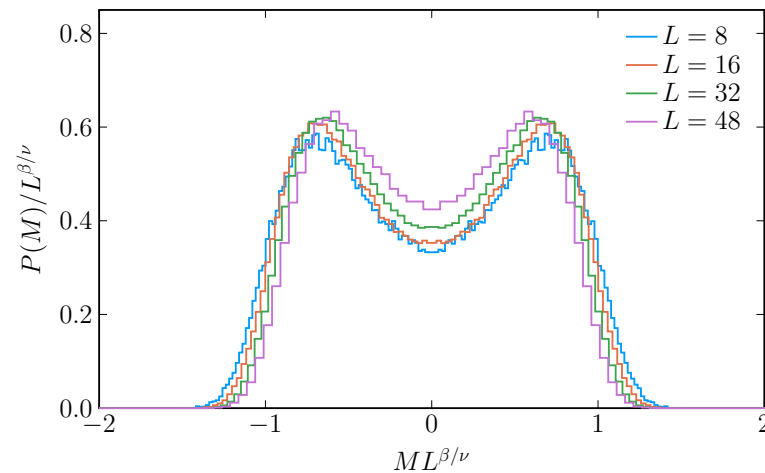
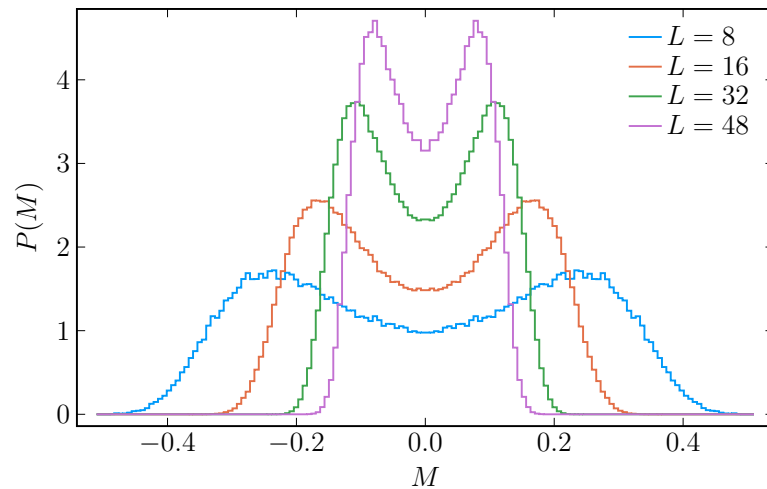
$$P[\psi] \sim \exp(-\beta \mathcal{F}[\psi])$$

Note: Still have short distance noise; need to adjust bare parameters such as  $\Gamma, m^2, \lambda$  to reproduce physical quantities.

## Model A: Static Distribution

Tune  $m^2$  to critical point  $m^2 = m_c^2$  (Ising critical point)

Check finite size scaling

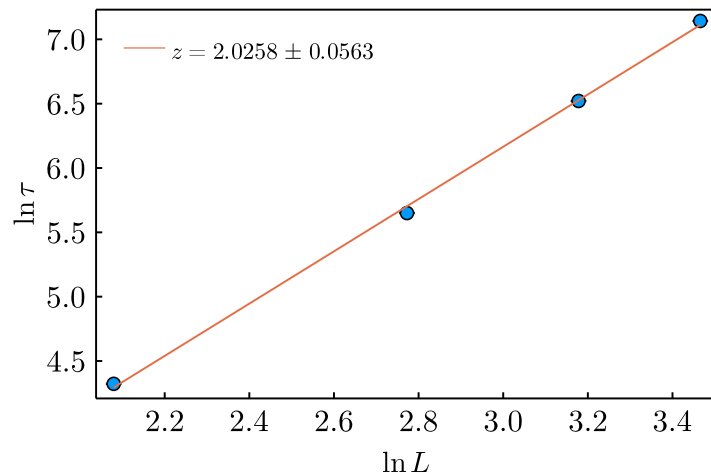


Critical exponents  $\beta = 0.326$  and  $\nu = 0.630$ .

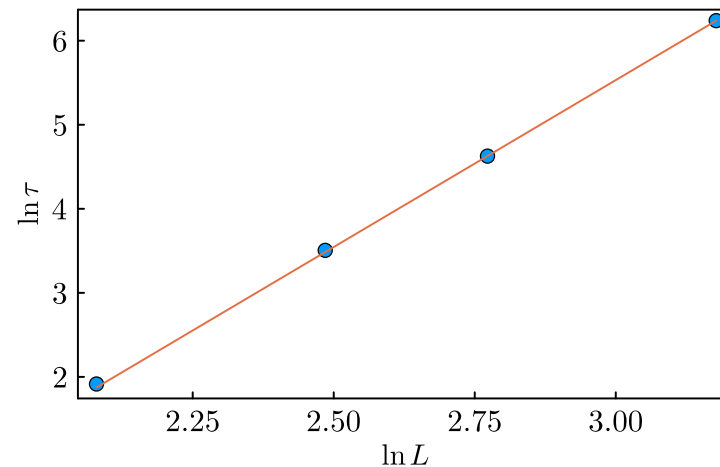
## Model A/B: Dynamical Scaling

Finite size scaling of dynamic order parameter correlation function

$$z_A \simeq 2.026$$



$$z_B \simeq 3.906$$



$$G(t, k) = \int d^3x e^{i\vec{k} \cdot \vec{x}} \langle \psi(0, 0) \psi(\vec{x}, t) \rangle$$

Look for dynamic scaling  $G(t, k, L) = \tilde{G}(t/L^z, kL)$

## Numerical realization: Model H

Model H: Conserving update

$$\begin{aligned}\pi_{\nu}^{trial}(\vec{x}, t + \Delta t) &= \pi_{\nu}(\vec{x}, t) + r_{\nu\mu}, \\ \pi_{\nu}^{trial}(\vec{x} + \hat{\mu}, t + \Delta t) &= \pi_{\nu}(\vec{x} + \hat{\mu}, t) - r_{\nu\mu},\end{aligned}\quad r_{\nu\mu} = \sqrt{2\eta T(\Delta t)} \zeta_{\nu}.$$

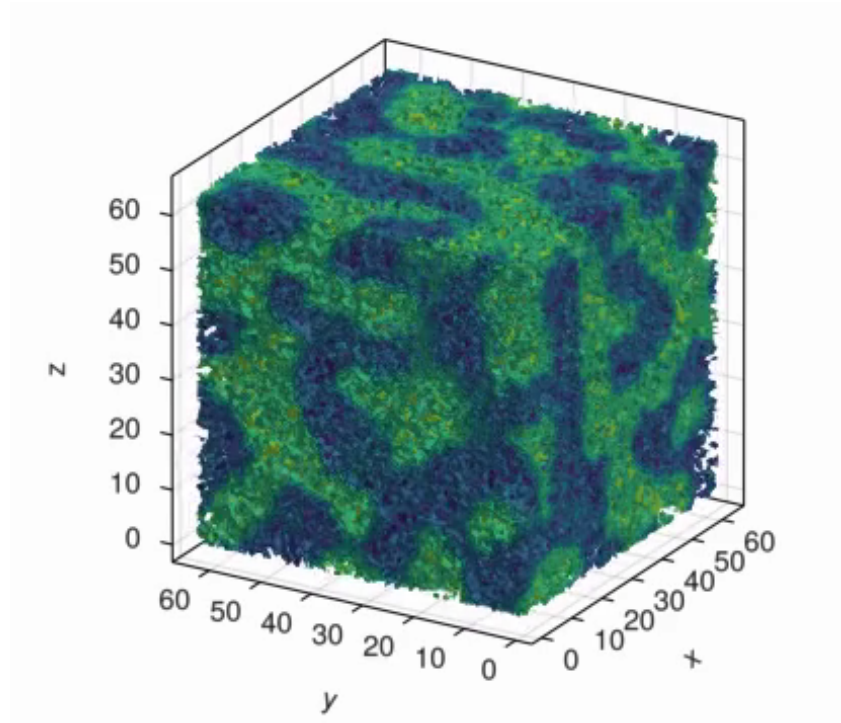
Advection (PB terms) conserves  $\mathcal{H}$ . On the lattice use “skew” discretized derivatives

$$\begin{aligned}\dot{\phi} &= -\frac{1}{\rho} \pi_{\mu}^T \nabla_{\mu}^c \phi, \\ \dot{\pi}_{\mu}^T &= -\left[ \frac{1}{2} \nabla_{\nu}^c \left( \frac{1}{\rho} \pi_{\nu}^T \pi_{\mu}^T \right) + \frac{1}{2\rho} \pi_{\nu}^T \nabla_{\nu}^c \pi_{\mu}^T + (\nabla_{\mu}^c \phi) (\nabla_{\nu}^c \nabla_{\nu}^c \phi) \right],\end{aligned}$$

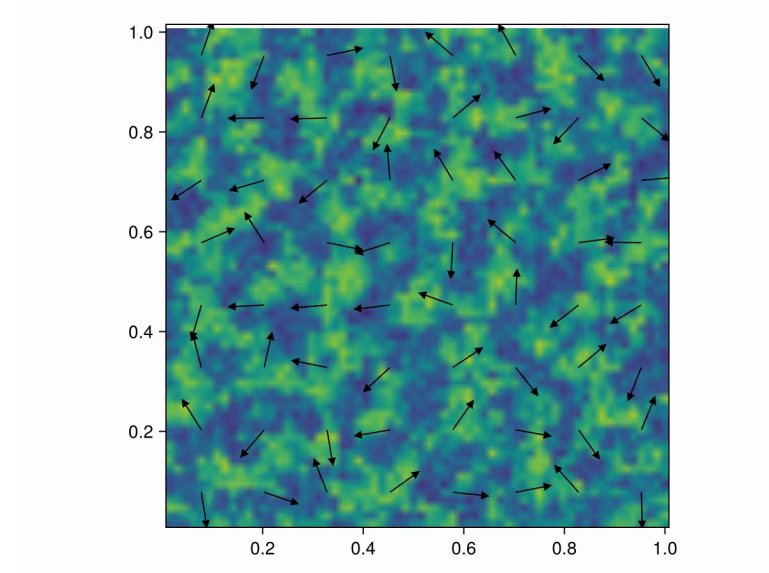
and project on  $\pi_{\mu}^T$  using Fourier transforms.

## Numerical results (critical Navier-Stokes)

Order parameter (3d)



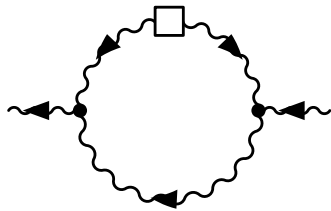
Order parameter/velocity field (2d)



## Renormalized viscosity

Renormalization of  $\eta$

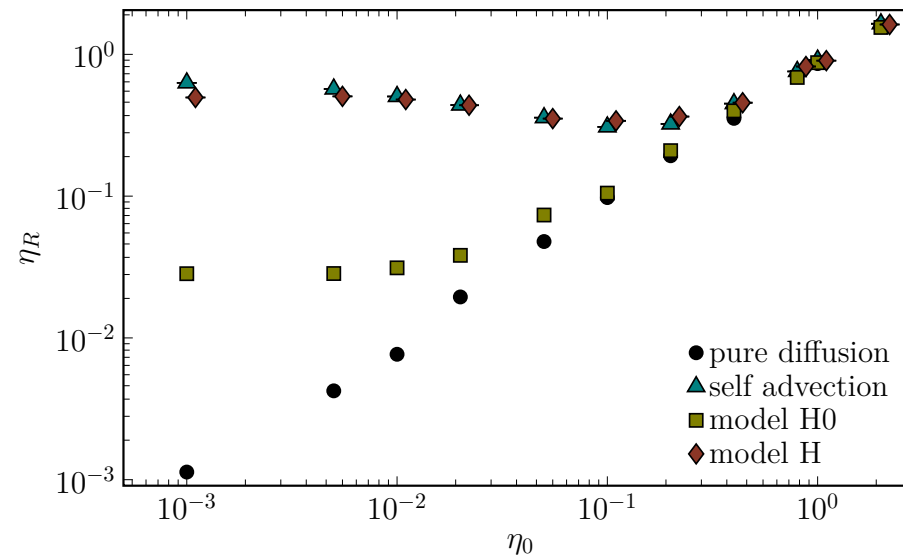
“Stickiness of shear waves”



$$\eta_R = \eta + \frac{7}{60\pi^2} \frac{\rho T \Lambda}{\eta}$$

Leads to minimum viscosity

Top: Model H

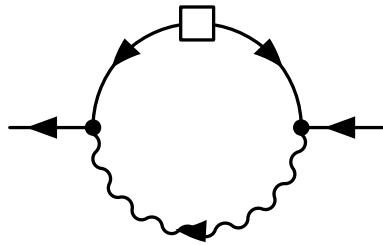


Middle: No self-advection

Bottom: No advection

## Relaxation Rate

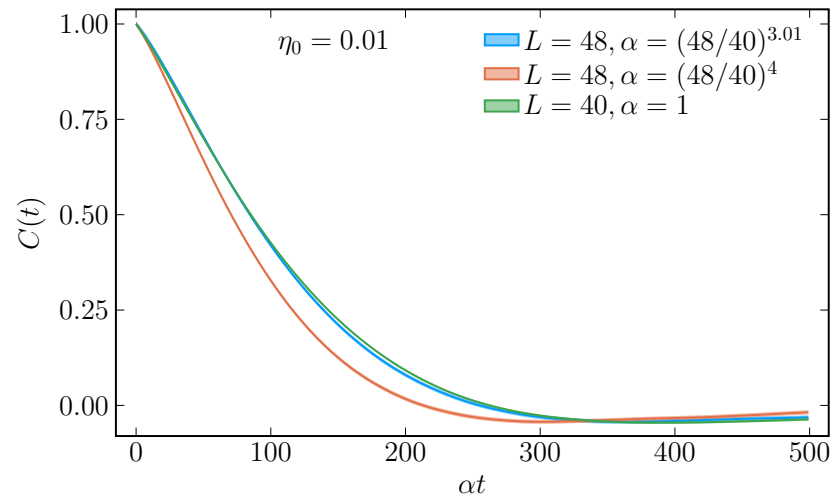
Order parameter relaxation rate



$$\Gamma_k = \frac{\Gamma}{\xi^4} (k\xi)^2 (1 + (k\xi)^2) + \frac{T}{6\pi\eta_R\xi^3} K(k\xi)$$

Crossover from  $\tau_R \sim \xi^4$  at large  $\eta_R$   
to  $\tau_R \sim \xi^3$  for small  $\eta_R$

$$C(t) = \langle \phi_k(0) \phi_{-k}(t) \rangle$$

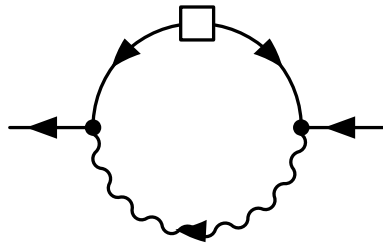


Dynamic Scaling:

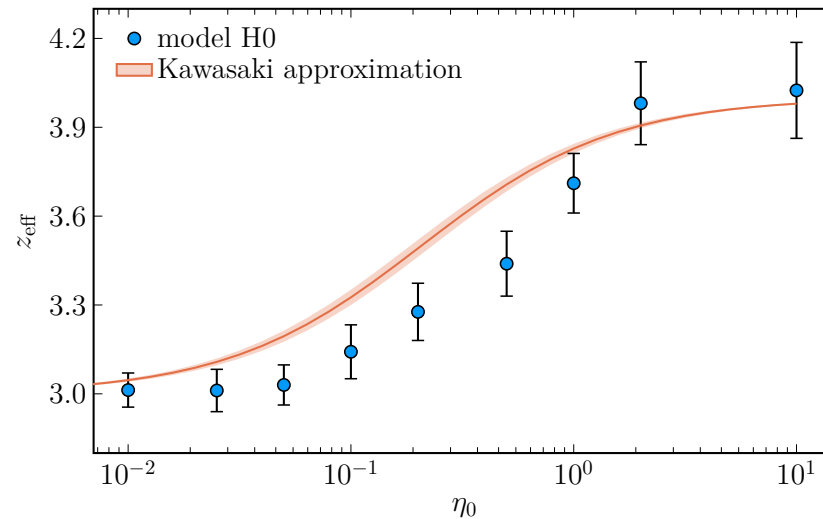
$$z(\eta=0.01) = 3.07$$

## Relaxation Rate

Order parameter relaxation rate



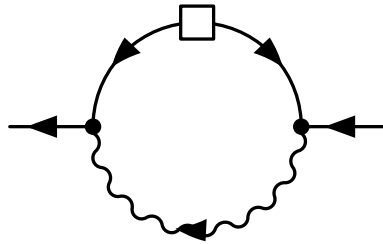
$$\Gamma_k = \frac{\Gamma}{\xi^4} (k\xi)^2 (1 + (k\xi)^2) + \frac{T}{6\pi\eta_R\xi^3} K(k\xi)$$



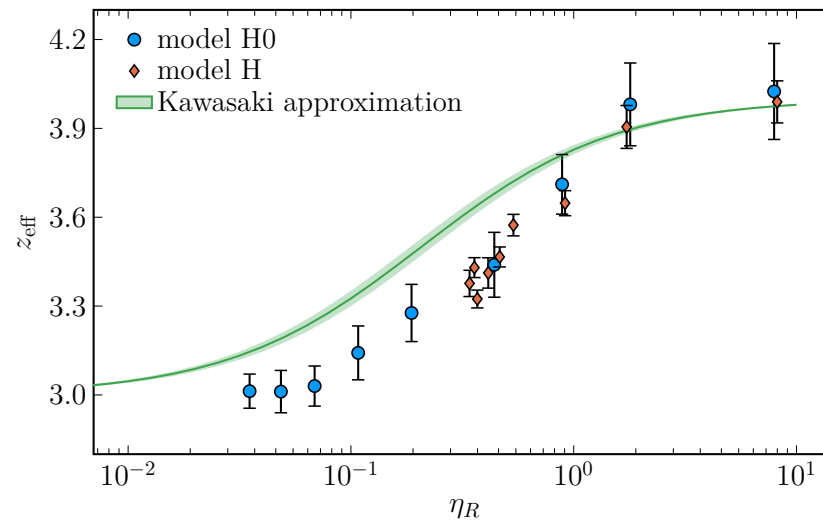
Crossover from  $\tau_R \sim \xi^4$  at large  $\eta_R$   
to  $\tau_R \sim \xi^3$  for small  $\eta_R$

## Universality: model H/H0

Order parameter relaxation rate



$$\Gamma_k = \frac{\Gamma}{\xi^4} (k\xi)^2 (1 + (k\xi)^2) + \frac{T}{6\pi\eta_R\xi^3} K(k\xi)$$

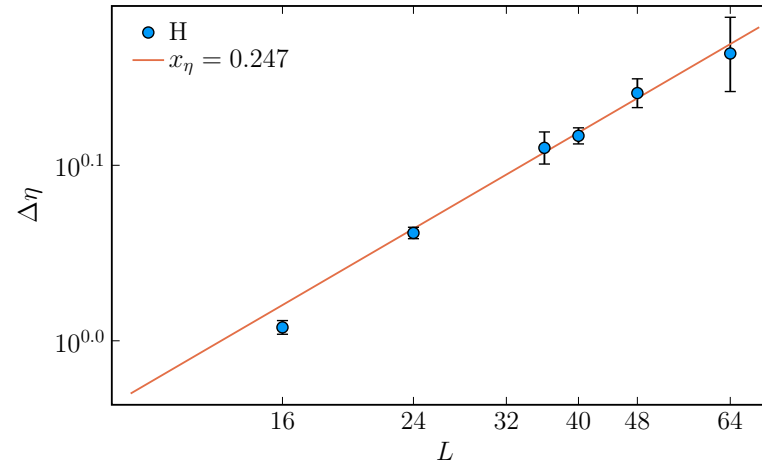
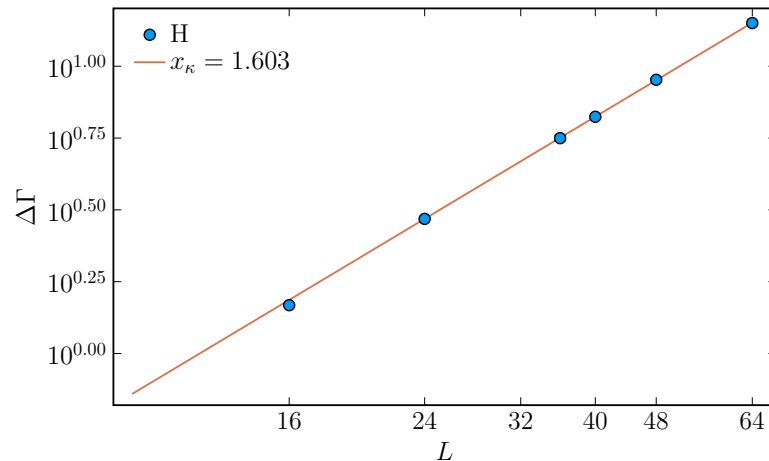


Crossover from  $\tau_R \sim \xi^4$  at large  $\eta_R$   
to  $\tau_R \sim \xi^3$  for small  $\eta_R$

## Critical behavior of transport coefficients (Here: 2d)

$$\kappa = \frac{1}{3VT} \int dt d^3x_{1,2} \langle \vec{j}(0, x_1) \vec{j}(t, x_2) \rangle$$

$$\eta = \frac{1}{VT} \int dt d^3x_{1,2} \langle \Pi_{xy}(0, x_1) \Pi_{xy}(t, x_2) \rangle$$



Consistent with  $z = 4 - x_\kappa - \eta^*$  ( $\eta^* = 0.25$  correlation function exponent):

$$z = 2.11$$

$$4 - x_\kappa - \eta^* = 2.15$$

## Relativistic Fluid Dynamics in the Density Frame

Idea: Give up manifest covariance, define (using lab-frame densities)

$$T^{0\mu} = (\mathcal{E}, M^i) \equiv \mathcal{T}^{0\mu}(\beta_\mu) \quad \mathcal{T}^{\mu\nu} = [e(\beta) + P(\beta)] u^\mu u^\nu + P(\beta) \eta^{\mu\nu}$$

Ideal equations of motion

$$\partial_t \mathcal{E} + \partial_i M^i = 0$$

$$\partial_t M^i + \partial_j \mathcal{T}^{ij} = 0$$

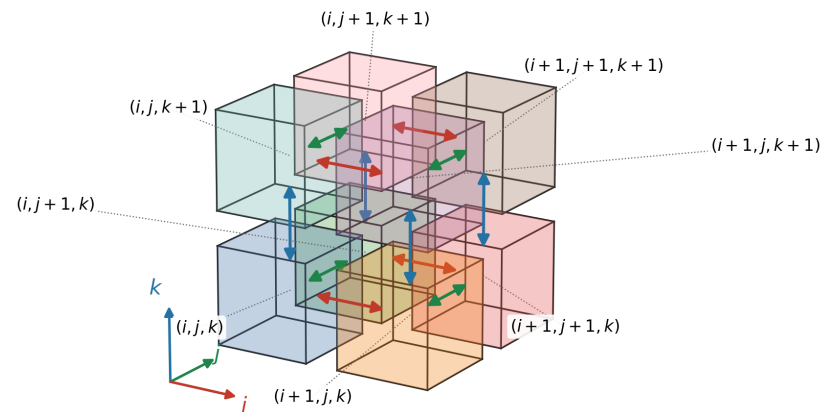
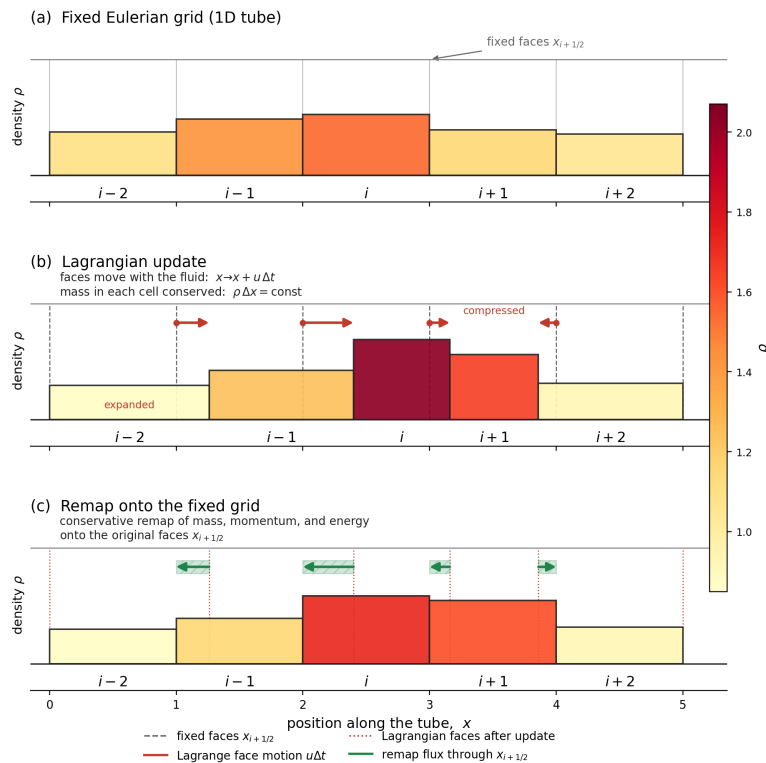
Match dissipative stresses to the Landau frame

$$T^{ij} = \mathcal{T}^{ij} + \Pi^{ij} \quad \Pi^{ij} = -T \kappa^{ijkl} \partial_{(k} \beta_{l)}$$

Can be sampled via Metropolis.

# Warmup: Stochastic compressible Navier-Stokes

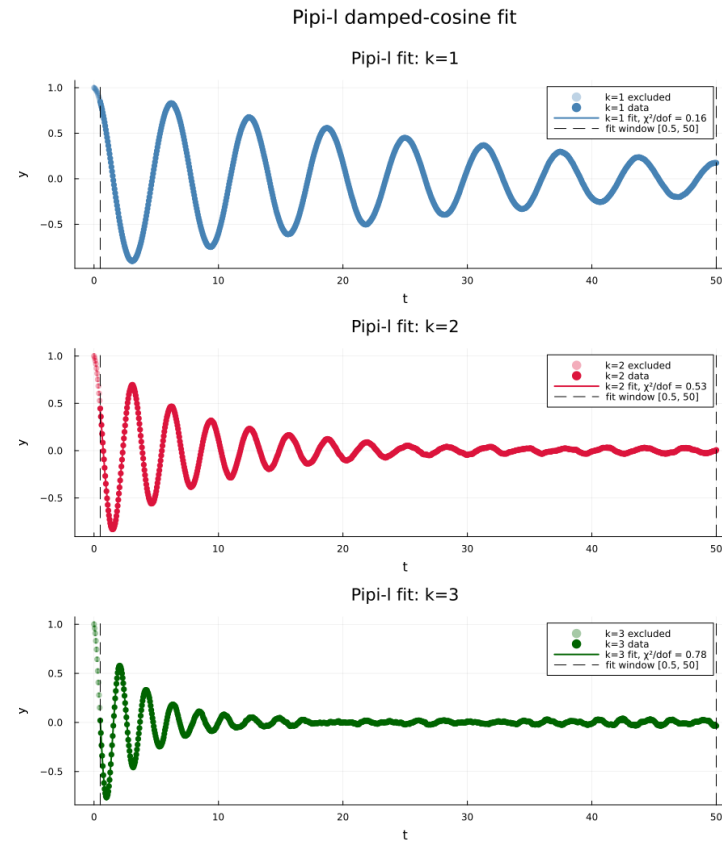
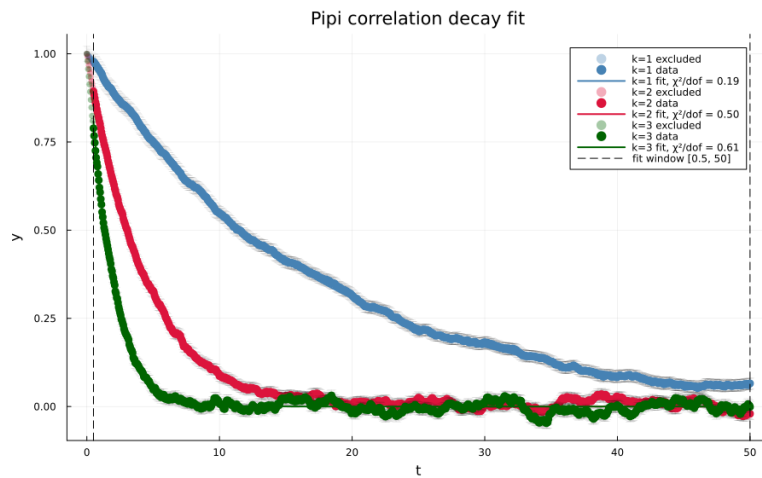
Combine Lagrange-step  $\Phi^I(x(x'), t) = x'$  with Metropolis.



# Stochastic compressible Navier-Stokes

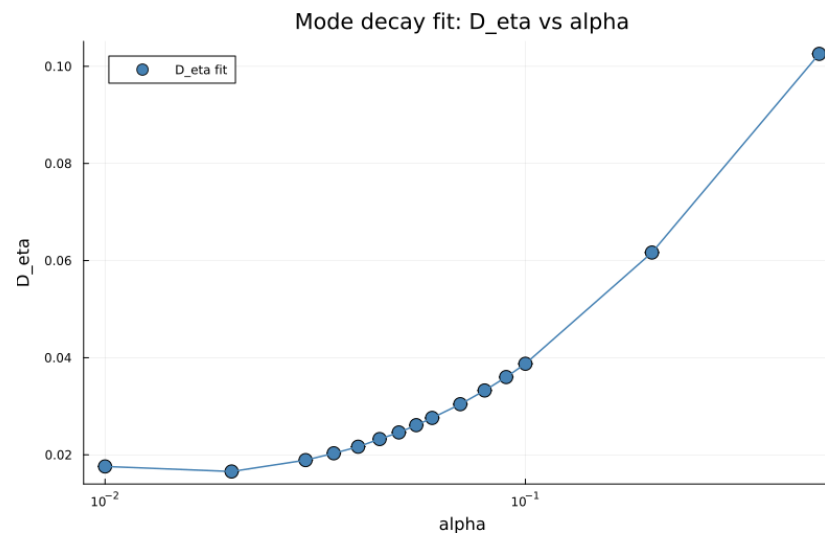
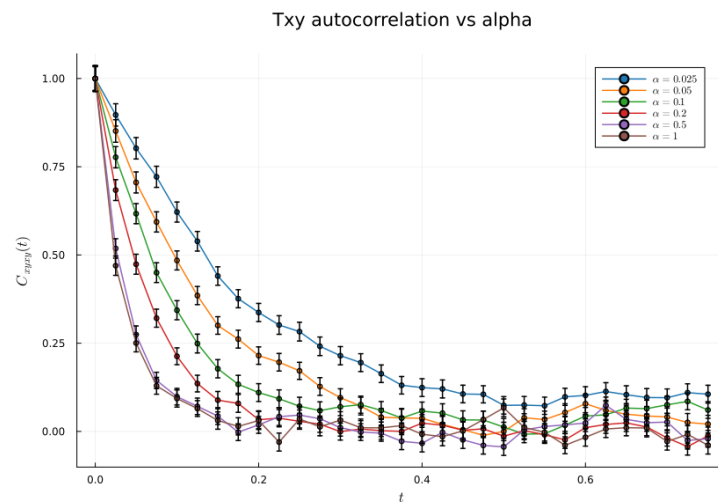
shear:  $\langle \pi_x(k_y, 0) \pi_x(-k_y, t) \rangle$

sound:  $\langle \pi_x(k_x, 0) \pi_x(-k_x, t) \rangle$



Find hydrodynamic scaling  $D \sim k^2$  and  $\Gamma_s = \frac{4}{3}\Gamma_\eta$ .

# Stochastic compressible Navier-Stokes



Stress tensor correlation function

$$\langle \Pi_{xy}(0) \Pi_{xy}(t) \rangle \sim 1/(D_{\eta} t^{3/2})$$

Renormalization of shear mode

decay rate  $D_{\eta}$ .

## Summary and Outlook

Numerical simulation of stochastic fluid dynamics, observed renormalization of shear viscosity, dynamical scaling, and critical transport.

Other models: Chiral dynamics (model G), see Derek, Aleksas, and friends. Superfluid dynamics (model F).

Compressible (full Navier-Stokes theory) and relativistic fluid dynamics in progress. Density frame approach seems promising. Goal is to do full dynamics of critical and non-critical fluctuations in a heavy ion collision.