Recent progress on dense QC_2D

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featuring
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Outline

Background
- QC_2D vs QCD
- Lattice formulation

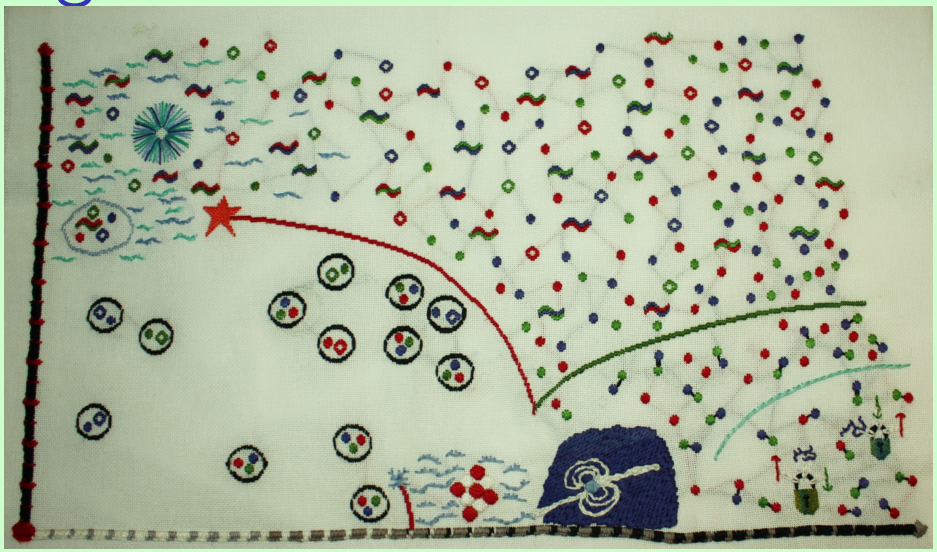
Thermodynamics results
- Phase transitions
- Bulk thermodynamics
- Improved action

Gluodynamics
- Ground-state field structure

Heavy quarkonium

Summary

Background



- A plethora of phases at high μ, low T
- Based on models and perturbation theory

Indirect approach

Study QCD-like theories without a sign problem

- Generic features of strongly interacting systems at $\mu \neq 0$
- Check on model calculations, functional methods

QC_2D vs QCD

- Baryons are bosons (diquarks); superfluid 'nuclear matter'
- Scalar diquark is pseudo-Goldstone (degenerate with pion)
- Onset transition at $\mu_q = m_\pi/2$, not $m_N/3$

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Phase diagram

- Superfluid phase for $\mu > m_\pi/2$: BEC $\longrightarrow$ BCS?
- Exotic phases: quarkyonic, spatially varying?
- Deconfinement at high density, shape of deconfinement line?

Global symmetries of QC_2D

Quarks and antiquarks are in the same representation

Anti-unitary symmetry: $KMK^{-1} = \mathcal{M}^*$ with $K \equiv C\gamma_5\tau_2$

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$m = \mu = 0$:
global SU($2N_f$) symmetry $\longrightarrow$ Sp($2N_f$) by $\langle\overline{\psi}\psi\rangle \neq 0$.

$\Rightarrow N_f(2N_f - 1) - 1$Goldstone modes

$N_f = 2$: 5 modes

$\overline{\psi}\vec{\sigma}\gamma_5\psi$ pion $\quad \psi^T\epsilon\tau_2 C\gamma_5\psi, \quad \overline{\psi}\epsilon\tau_2 C\gamma_5\overline{\psi}^T$ scalar diquark

Diquark condensation

Diquarks are colour singlets in QC_2D

$\rightarrow$ superfluidity rather than colour superconductivity

$\rightarrow$ exact Goldstone mode from breaking of $U(1)_B$ symmetry

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Bose–Einstein Condensation:

Condensation of tightly bound diquarks (Goldstone baryons)

$\leftrightarrow$ Chiral perturbation theory

$$\langle \psi\psi \rangle \propto \sqrt{1-(\mu/\mu_o)^4}$$

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Bardeen–Cooper–Schrieffer:

Pairing of quarks near the Fermi surface

$$\langle \psi\psi \rangle \propto \Delta\mu^2$$

Bulk thermodynamics expectations

Chiral perturbation theory

- Pseudo-Goldstone bosons (diquarks)
- Separation of scales $m_\pi \ll m_\rho$
- Corresponds to BEC

$$n_q^{\chi PT} = 8\mu N_f F^2 \left[1 - \left(\frac{\mu_o}{\mu}\right)^2\right]$$

Free fermions

- Fermi sphere of weakly interacting quarks
- Corresponds to BCS

$$n_q^{SB} = \frac{N_f N_c}{3}\left(\mu T^2 + \frac{\mu^3}{\pi^2}\right)$$

Gluodynamics

Gluodynamics — SU(2) and SU(3) very similar?

- Impact of fermion density on topology?
- Impact on gauge field strengths, vacuum expectation values?
- Effects of deconfinement on gluon propagation?
- Gap equation with effective or one-gluon interaction used to determine superconducting gap $\rightarrow$ more realistic input?

Lattice formulation

We use Wilson fermions:

- Correct symmetry breaking pattern, Goldstone spectrum
- $N_f < 4$ needed to guarantee continuum limit
- No problems with locality, fourth root trick
- Chiral symmetry buried at bottom of Fermi sea

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$$S = \overline{\psi}_1 M(\mu)\psi_1 + \overline{\psi}_2 M(\mu)\psi_2 - J\overline{\psi}_1 (C\gamma_5)\tau_2 \overline{\psi}_2^T + \overline{J}\psi_2^T (C\gamma_5)\tau_2 \psi_1$$

$$\gamma_5 M(\mu)\gamma_5 = M^\dagger(-\mu), \quad C\gamma_5\tau_2 M(\mu) C\gamma_5\tau_2 = -M^*(\mu)$$

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Diquark source $J \equiv \kappa j$ introduced to

- lift low-lying eigenmodes in the superfluid phase
- study diquark condensation without uncontrolled approximations

Simulation parameters

Name	β	κ	a	am_π	m_π/m_ρ
Coarse	1.9	0.1680	0.18fm	0.65	0.80
Light	1.7	0.1810	0.19fm	0.44	0.61
Fine	2.1	0.1577	0.14fm	0.45	0.81

Ensemble	N_s	N_τ	T (MeV)	μa	ja
Coarse	12	24	47	0.25–1.10	0.02, 0.04 (0.03)
	16	24	47	0.30–0.90	0.04
	12	16	70	0.30–0.90	0.04
	16	12	94	0.20–0.90	0.02, 0.04
	16	8	141	0.10–0.90	0.02, 0.04
Fine	16	32	45	0.15–0.80	0.005, 0.01, 0.02, 0.015, 0.03
	16	20	71	0.20–0.60	0.02, 0.03
	16	16	89	0.20–0.60	0.02, 0.03
	16	12	119	0.20–0.60	0.02, 0.03
Light	12	24	43	0.10–0.80	0.02, 0.04 (0.03)
	16	12	87	0.10–0.80	0.02, 0.04
	16	8	130	0.10–0.70	0.02, 0.04

T-scans at fixed μ

Coarse lattice: $\mu a = 0.0, 0.35, 0.40, 0.50, 0.60$

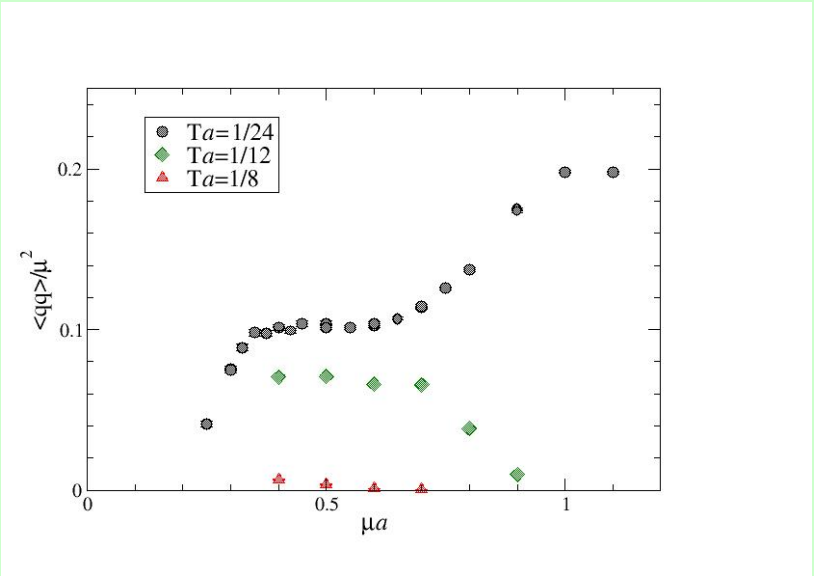
Fine lattice: $\mu a = 0.0, 0.20, 0.30, 0.40, 0.50$

Diquark condensate — μ-scan

Coarse ensemble

Results shown are for linear extrapolation

Power law $\langle qq \rangle = Aj^{\alpha}$ works for $\mu a \lesssim 0.4$, with $\alpha = 0.85 - 0.5$.
[Effective field theory predicts $\alpha = \frac{1}{3}$ near onset]



- BCS scaling $\langle qq \rangle \sim \mu^2$ for $0.35 \lesssim \mu a \lesssim 0.7$
- Melted at $T = 141$MeV ($N_\tau = 8$)

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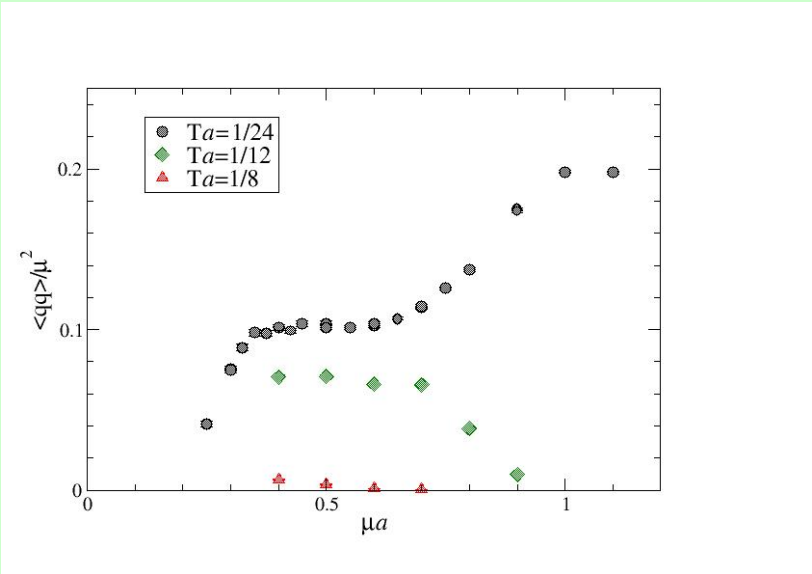


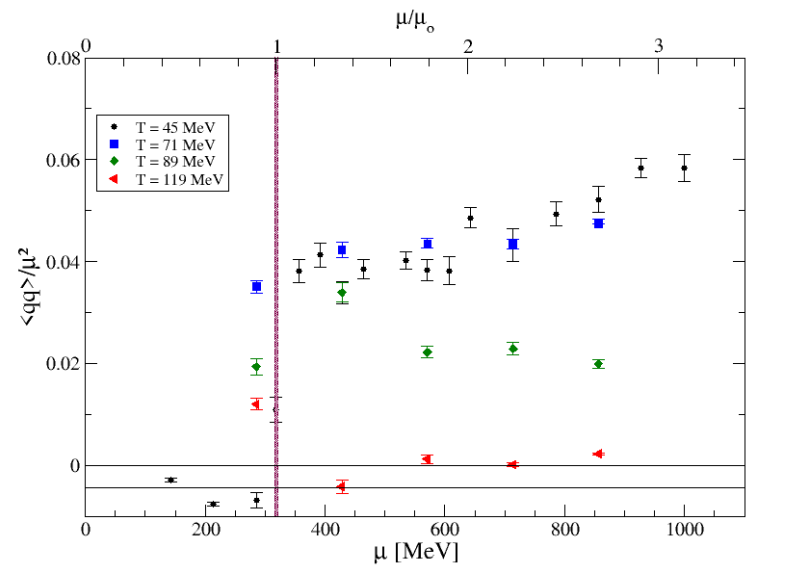
- BCS scaling $\langle qq \rangle \sim \mu^2$ for $0.35 \lesssim \mu a \lesssim 0.7$
- Melted at $T = 141$MeV ($N_\tau = 8$)
- New transition for $\mu a \gtrsim 0.7$?
- Melting for $N_\tau = 12, \mu a \gtrsim 0.7$?

$N_\tau = 16$ results are very close to $N_\tau = 24$ results.

Diquark condensate — μ-scan

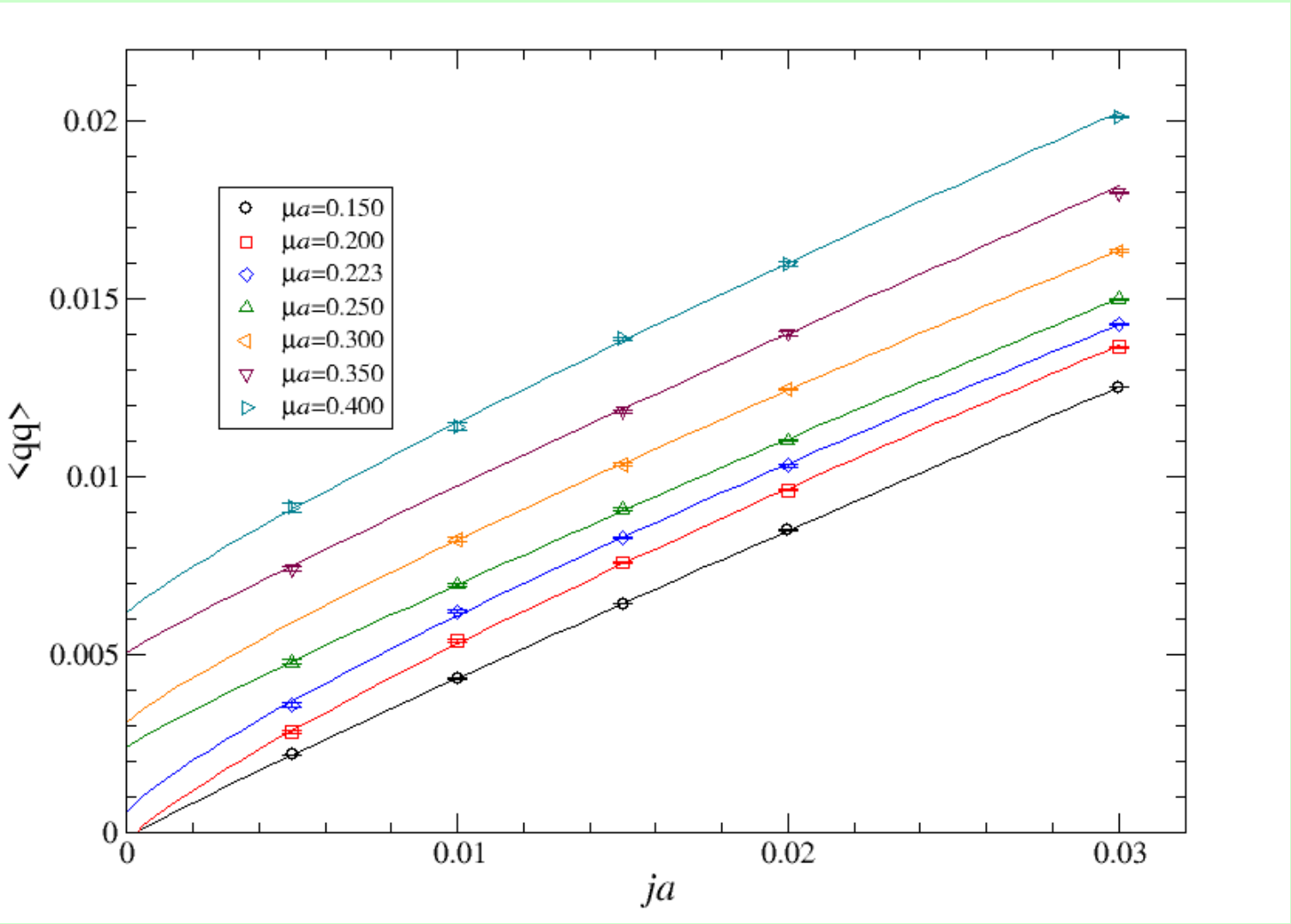
Fine ensemble

$\langle qq \rangle = \Delta + Aj^{\alpha}$ extrapolation for lowest T
Linear extrapolation for higher T



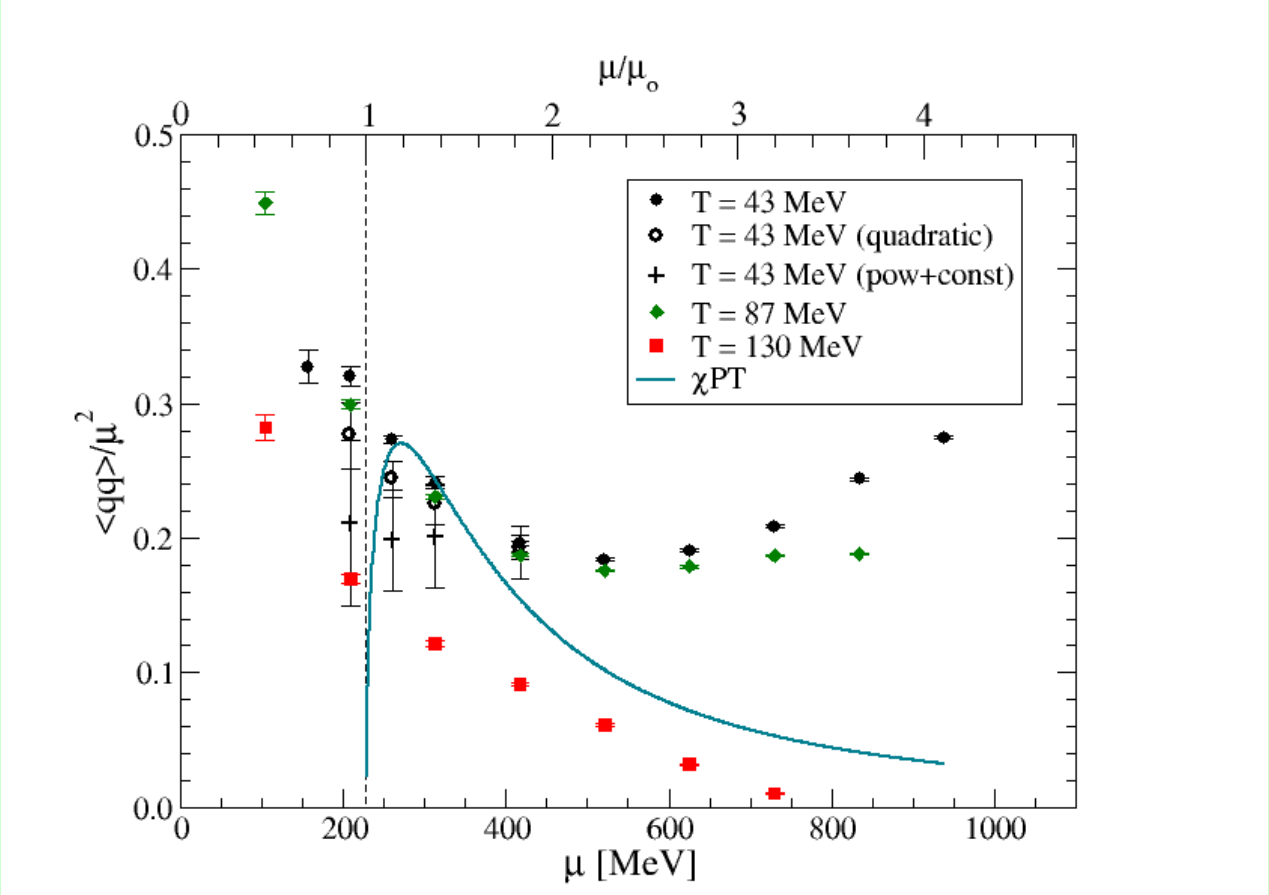
- BCS scaling $\langle qq \rangle \sim \mu^2$ confirmed
- Melted at $T \approx 130$MeV ($N_\tau = 12$)
- No sign of second transition

Diquark source extrapolation (fine ensemble)



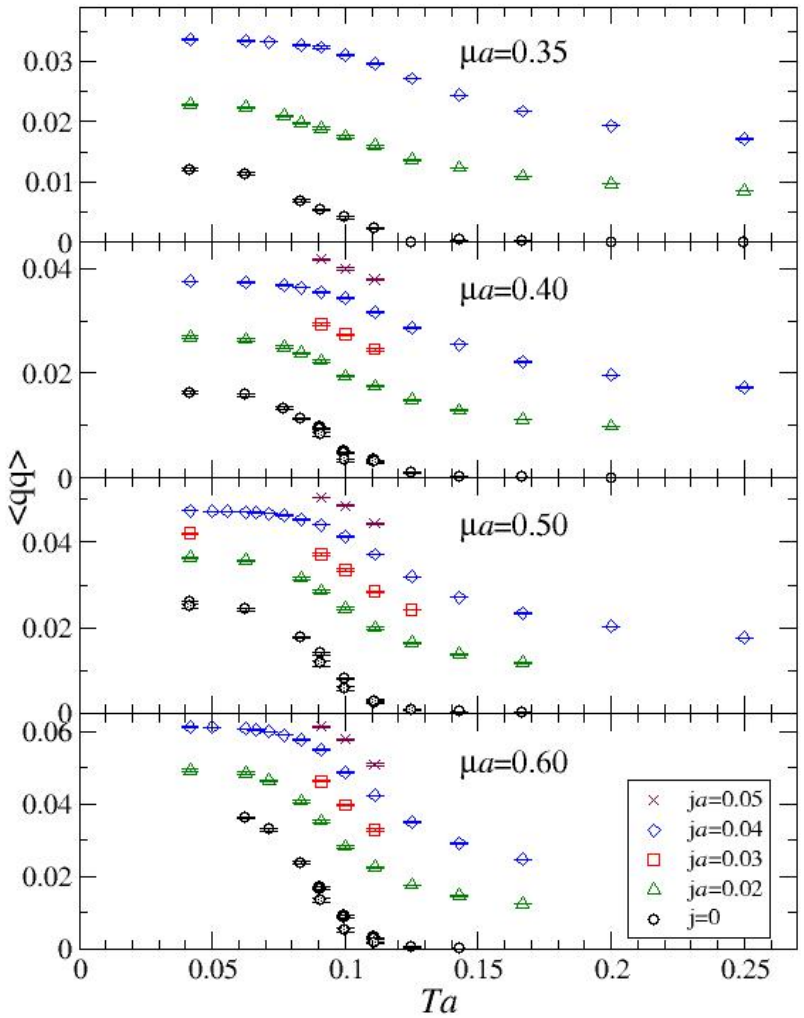
Diquark condensate — μ-scan

Light ensemble



Signs of a BEC window?

Superfluid to normal transition — coarse ensemble



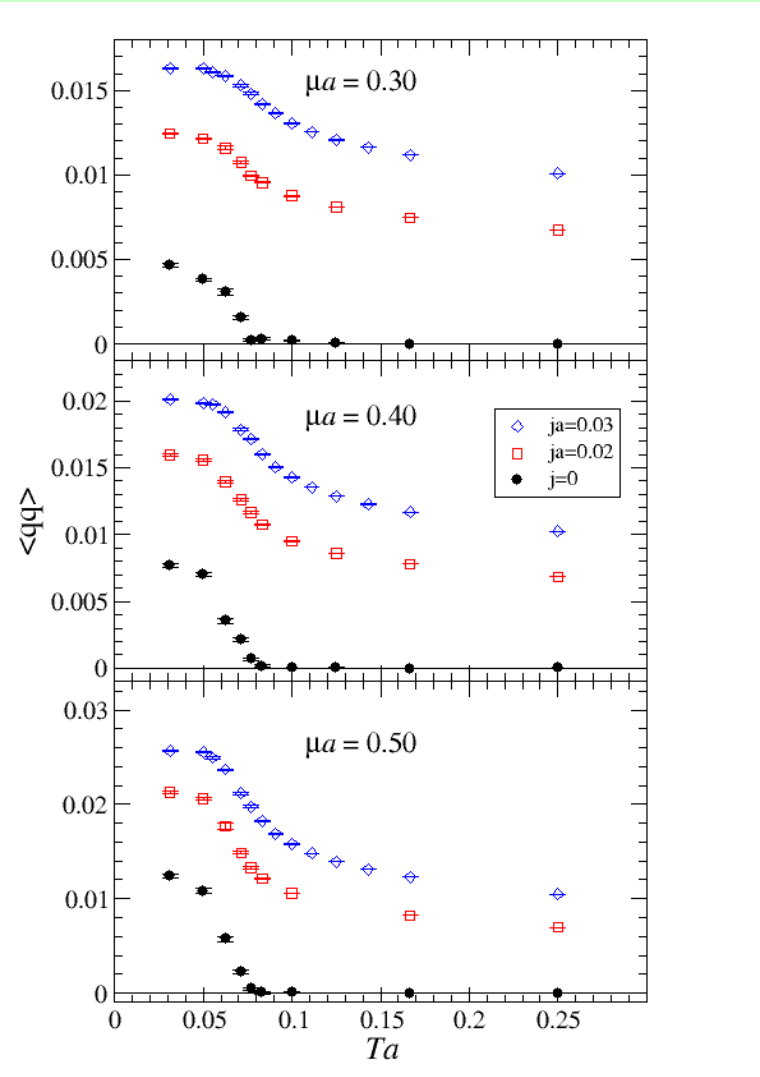
$j \to 0$ extrapolation not fully under control
Linear form used here

Transition temperatures from inflection points (and $j \to 0$)

$a\mu$	T_s (MeV)
0.35	82(27)
0.40	94(9)
0.50	93(6)
0.60	93(7)

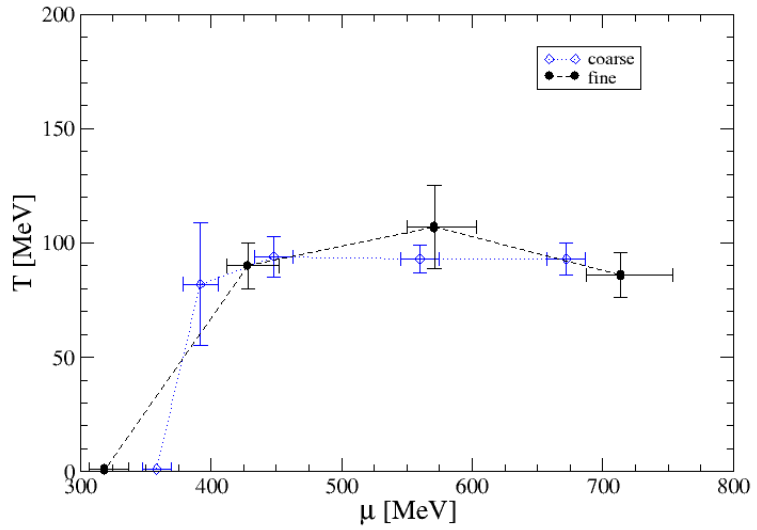
Remarkably constant!

Fine ensemble



Transition temperatures

$a\mu$	T_s (MeV)
0.30	90(10)
0.40	107(18)
0.50	86(10)



Agreement coarse–fine

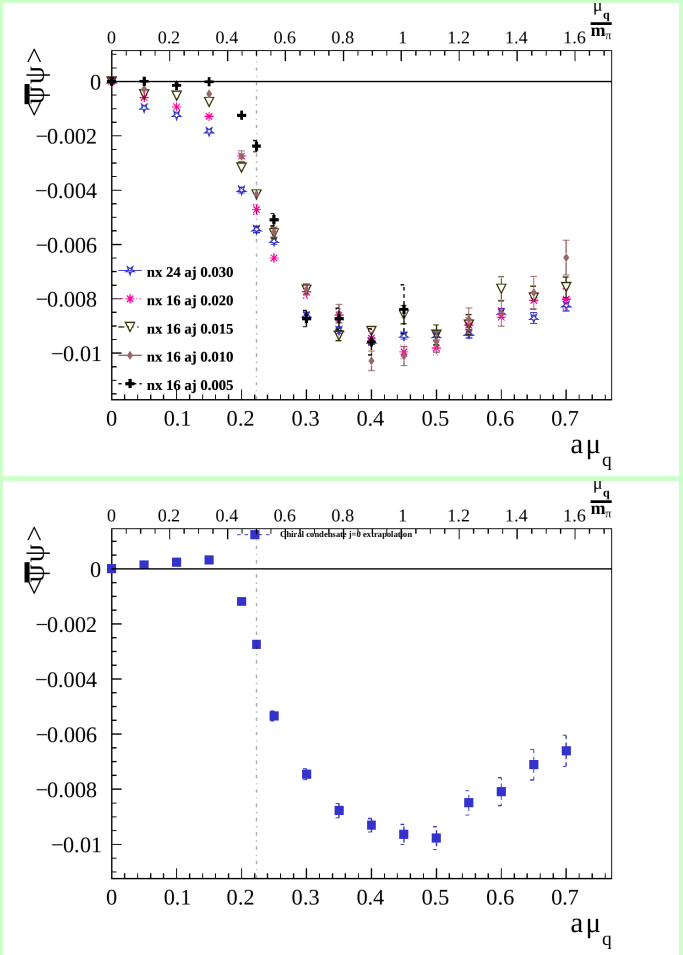
Chiral transition

Subtracted chiral condensate
$\langle\overline{\psi}\psi\rangle - \langle\overline{\psi}\psi\rangle_0$

Renormalisation = constant factor
Does not affect shape

Chiral transition coincides with onset

Second transition at $\mu \simeq m_\pi$?



Deconfinement transition

Polyakov loop L requires renormalisation,

$$L_R = e^{-F_q/T} = e^{-(F_0+\Delta F)/T} = Z_L^{N_\tau} L_0$$

We use two schemes to determine $Z_L = \exp(-a\Delta F)$,

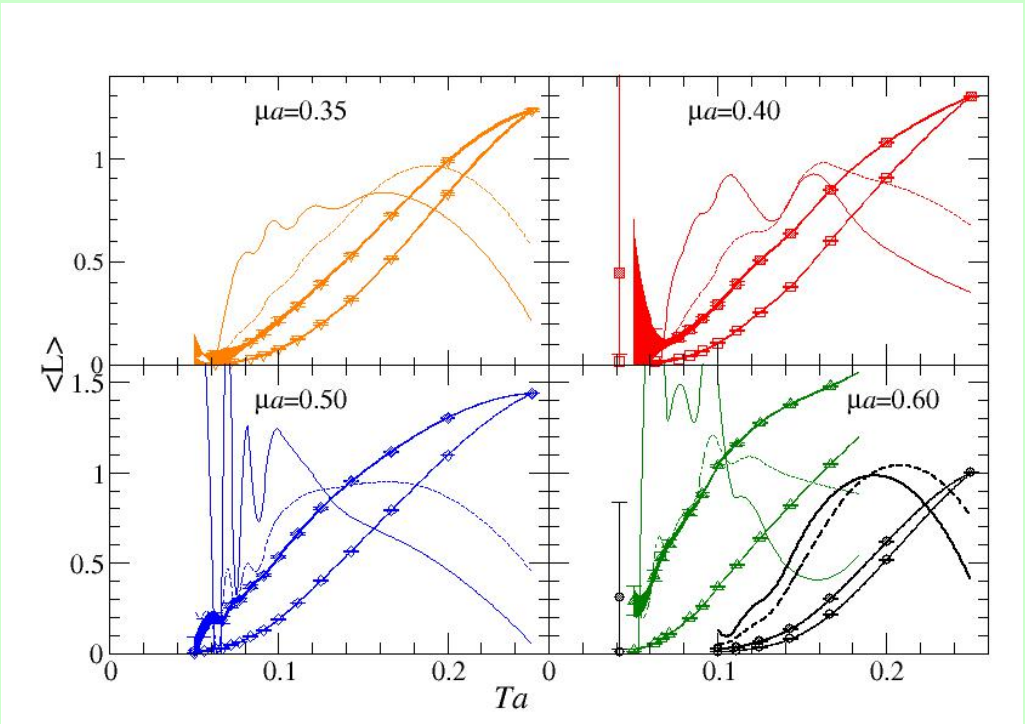
Scheme A $$L_R(T = \frac{1}{4a}, \mu = 0) = 1\,,$$

Scheme B $$L_R(T = \frac{1}{4a}, \mu = 0) = 0.5\,.$$

We determine the deconfinement temperature (crossover region) from the inflection point (linear region) of L_R

Deconfinement transition

Coarse ensemble

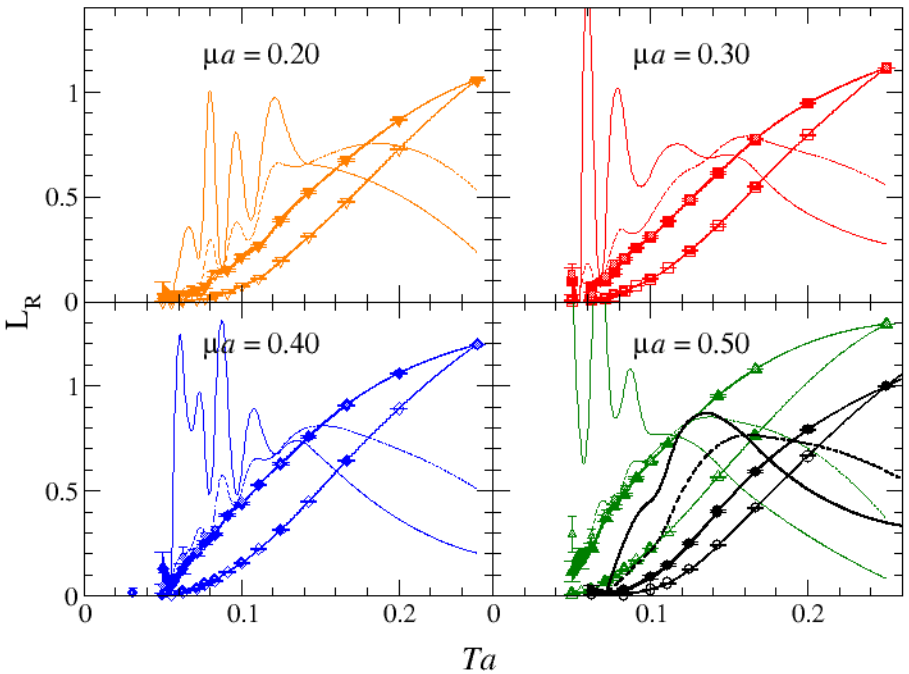


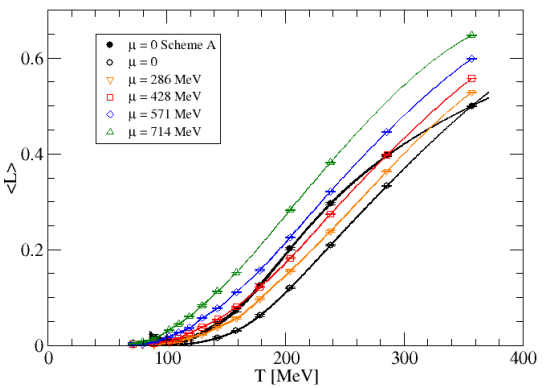
Estimates from Scheme B, encompassing Scheme A

μa	$T_d a$	T_d (MeV)
0.0	0.193(20)	217(23)
0.35	0.140–0.220	157–247
0.40	0.108–0.200	121–225
0.50	0.080–0.200	90–225
0.60	0.060–0.135	67–152

Scheme dependence $\longleftrightarrow$
broad crossover?

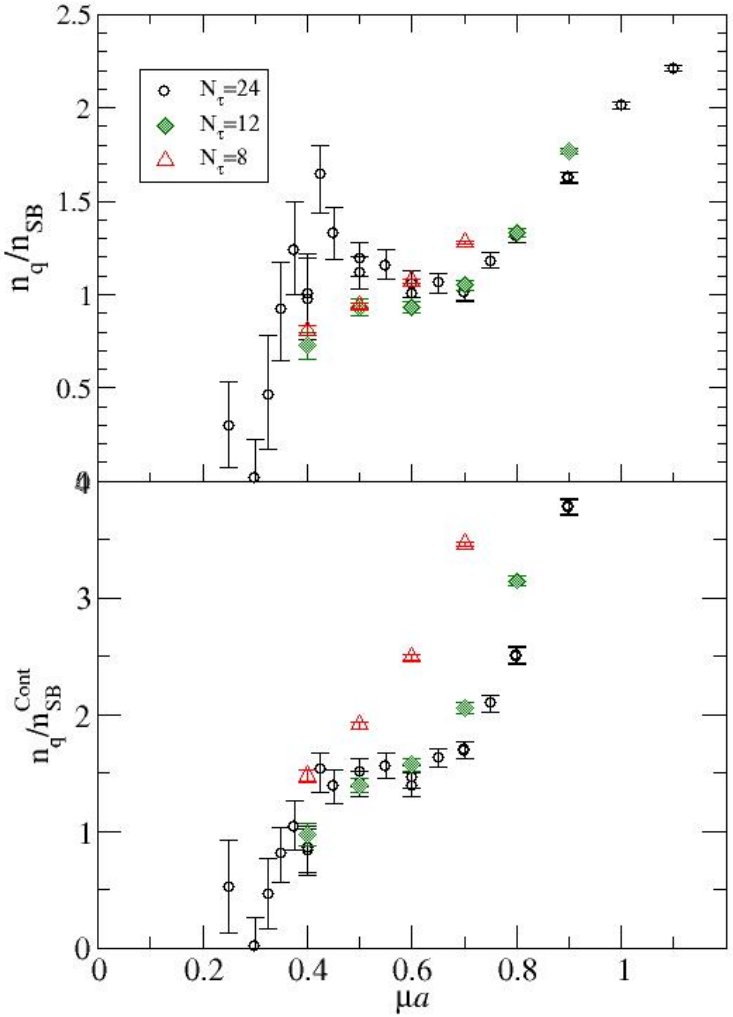
Fine ensemble





Qualitative features as for coarse ensemble

Number density: Coarse ensemble

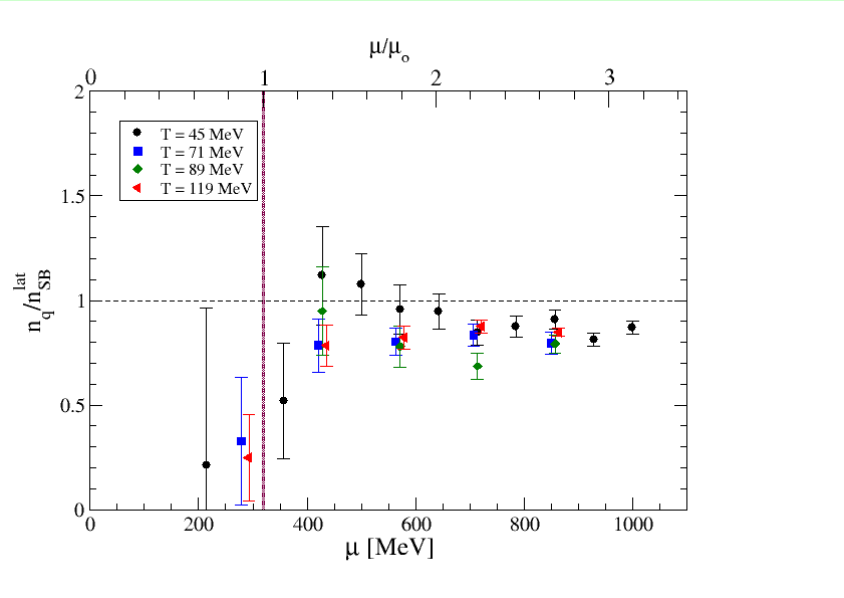


- Normalised to lattice SB or

$$n_{SB}^{\text{Cont}} = \frac{4\mu T^2}{3} + \frac{4\mu^3}{3\pi^2}$$

- Close to SB scaling for $0.4 \lesssim \mu a \lesssim 0.7$, $N_\tau = 24, 12$.
- Little T-dependence for $T \lesssim 100$MeV.
- Strong thermal effects for $N_\tau = 8$
- $N_\tau = 16, 24$ almost identical

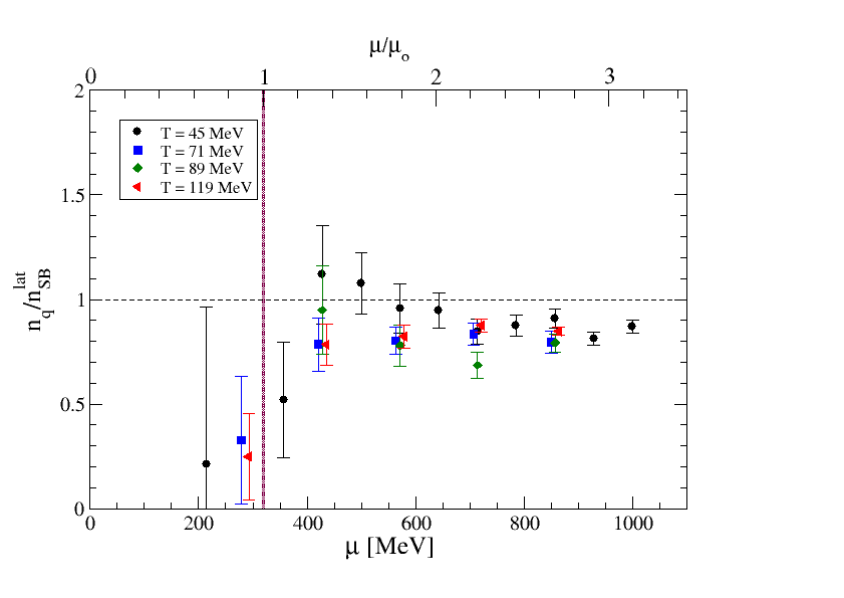
Number density on fine and light ensemble

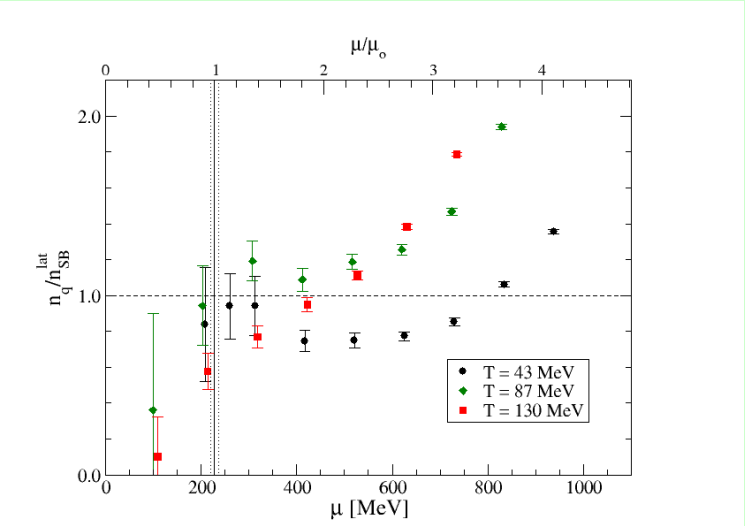


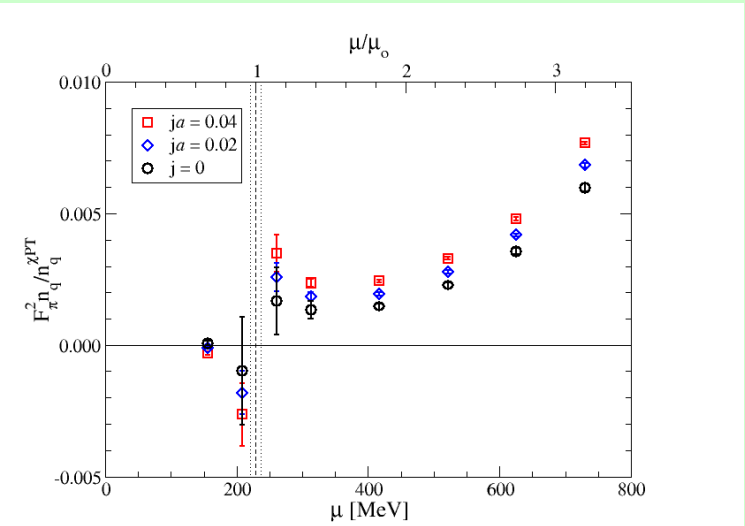
Fine ensemble:

Same qualitative features as coarse!

Number density on fine and light ensemble







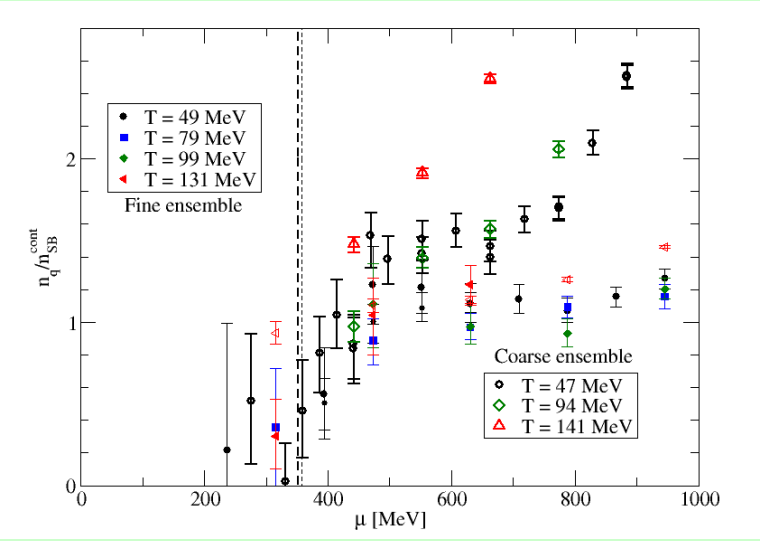
Fine ensemble:

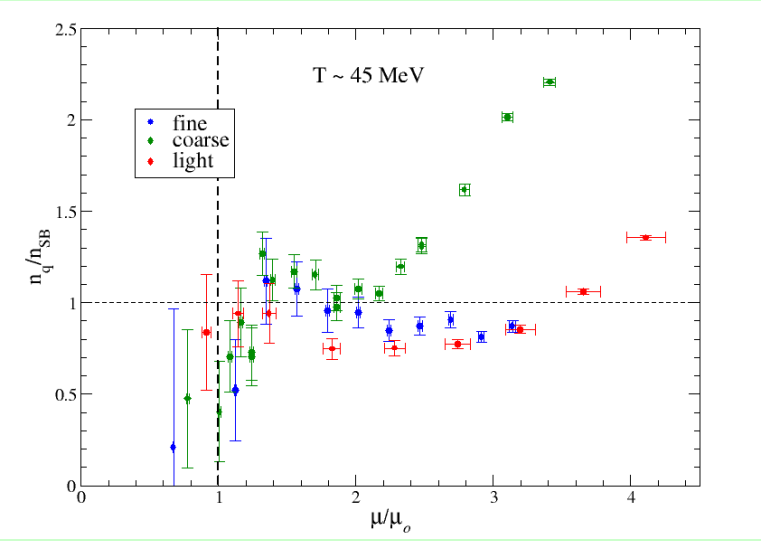
Same qualitative features as coarse!

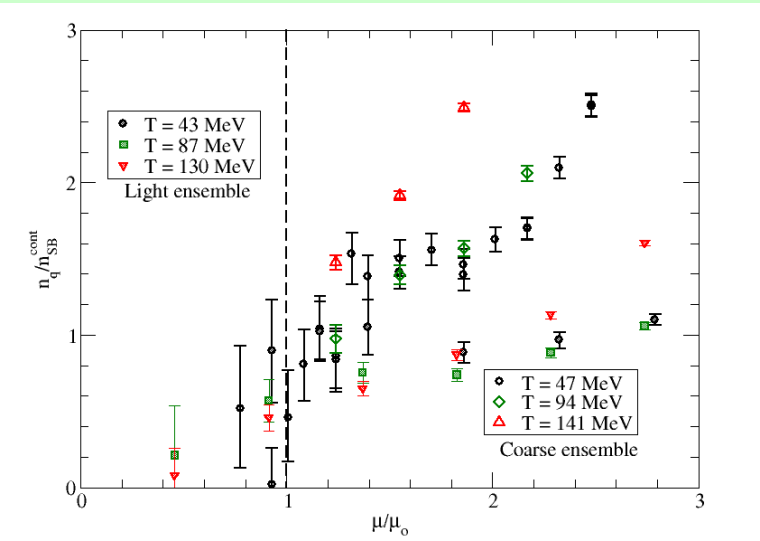
Light ensemble:

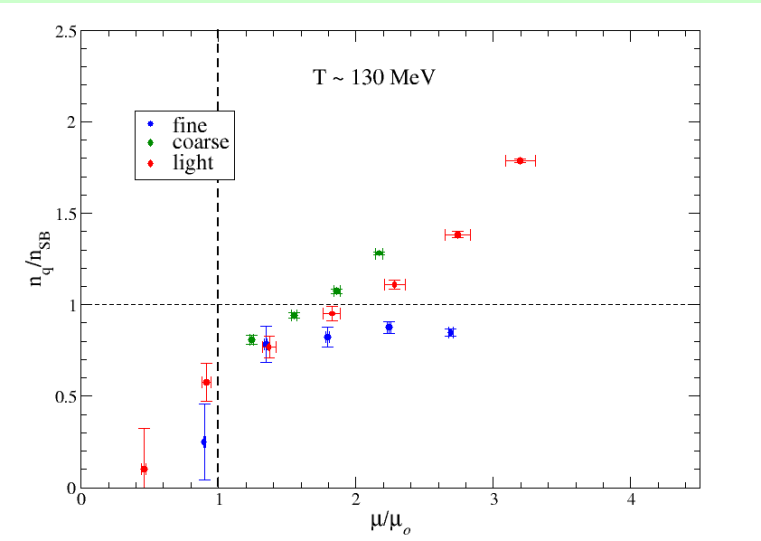
Deviation from BCS scaling — BEC window?

Number density: comparison









Equation of state

Pressure
$$p(\mu) = \int_0^\mu n_q(\mu')d\mu'$$

Trace anomaly
$$T(\mu) = \varepsilon(\mu) - 3p(\mu) = T^g + T^q$$
$$T^g = -\frac{3a}{N_c}\frac{\partial\beta}{\partial a}\,\mathrm{Re}(\mathrm{Tr}\,U_{ss} + \mathrm{Tr}\,U_{st})$$
$$T_q = -a\frac{\partial\kappa}{\partial a}\kappa^{-1}(4N_f N_c - \langle\overline{\psi}\psi\rangle)$$

Speed of sound
$$c_s^2 = \frac{\partial p}{\partial\varepsilon}$$

Speed of sound: $2\frac{1}{2}$ different ways

Pressure from $p = \int n_q d\mu$, with correction for lattice artefacts

$$\left(\frac{p}{p_{SB}}\right)''(\mu) = \frac{1}{p_{SB}^{\text{Cont}}(\mu)} \int_0^\mu \frac{n_{SB}^{\text{Cont}}}{n_{SB}^{\text{Lat}}}(\mu') n_q(\mu') d\mu'$$

Three ways of determining energy density ε:

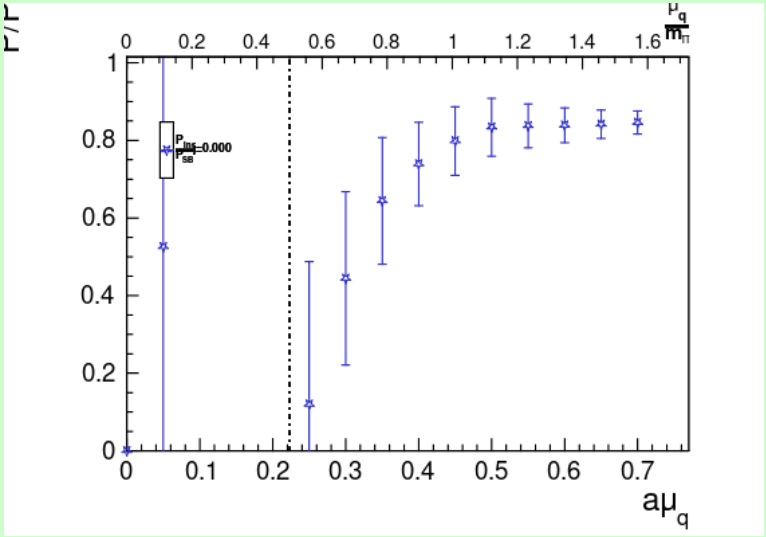
Method 1: $\varepsilon(\mu) = T_{\mu\mu} + 3p$

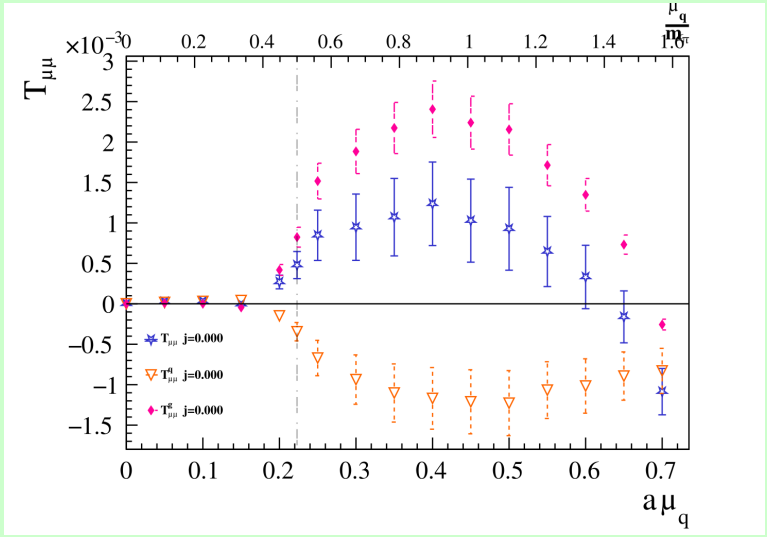
Method 2: $\varepsilon(\mu) = \frac{n_q}{\mu}\frac{d\mu}{dn_q} = \frac{n_q}{\mu dn_q/d\mu}$ using finite differences

Method 3: $\varepsilon(\mu) = \frac{n_q}{\mu\chi_q}$ direct evaluation of χ_q

$dp/d\varepsilon$ evaluated by interpolation + numerical derivatives.
Method 3 not yet implemented.

Bulk thermodynamics





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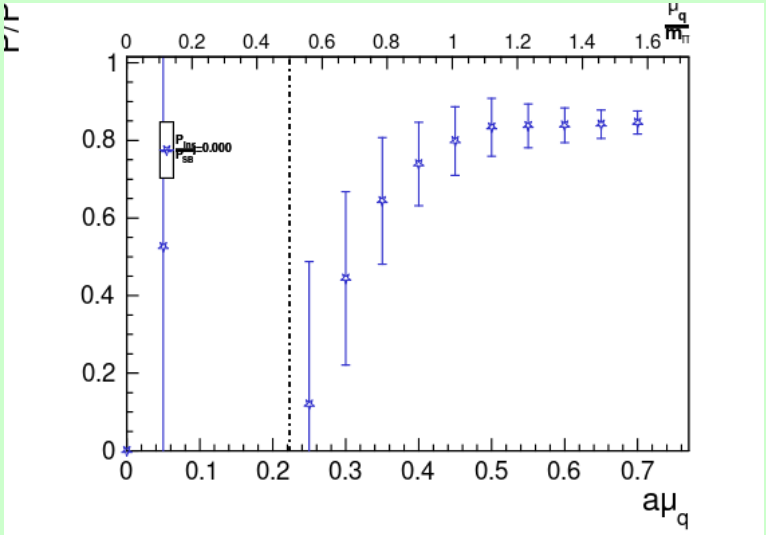
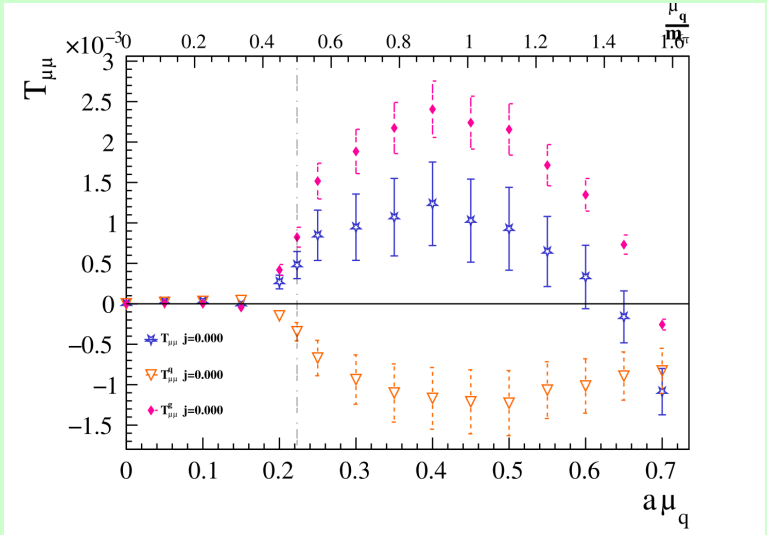
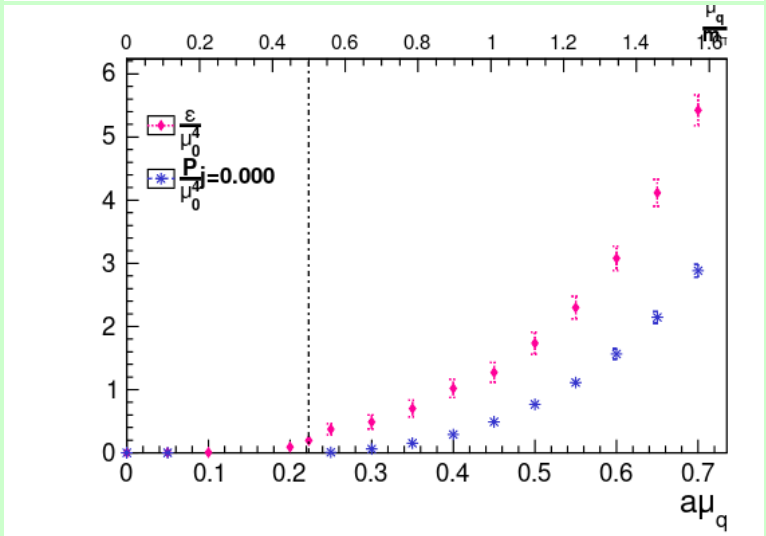
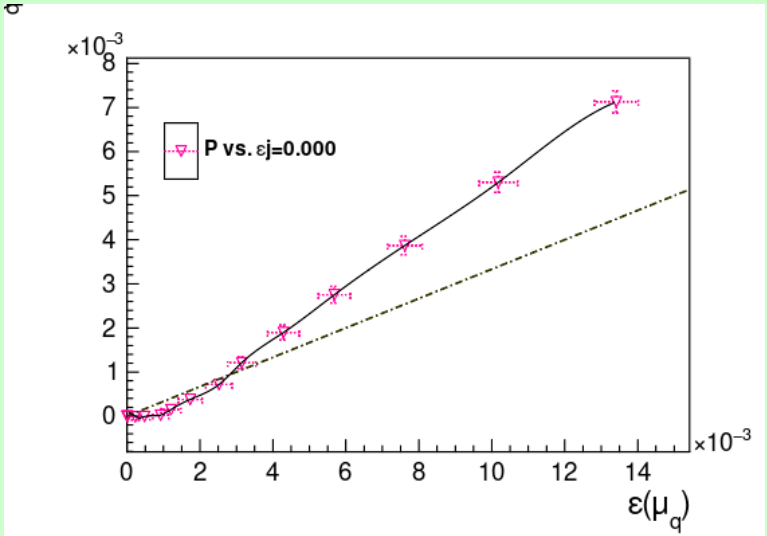
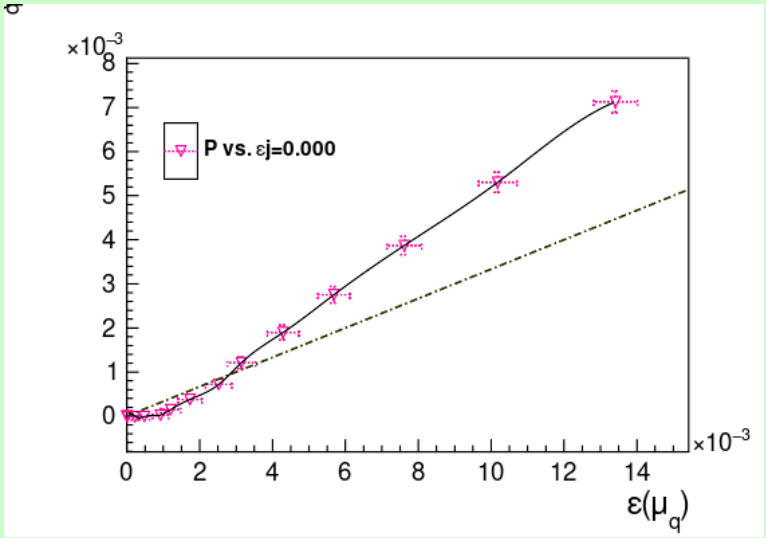


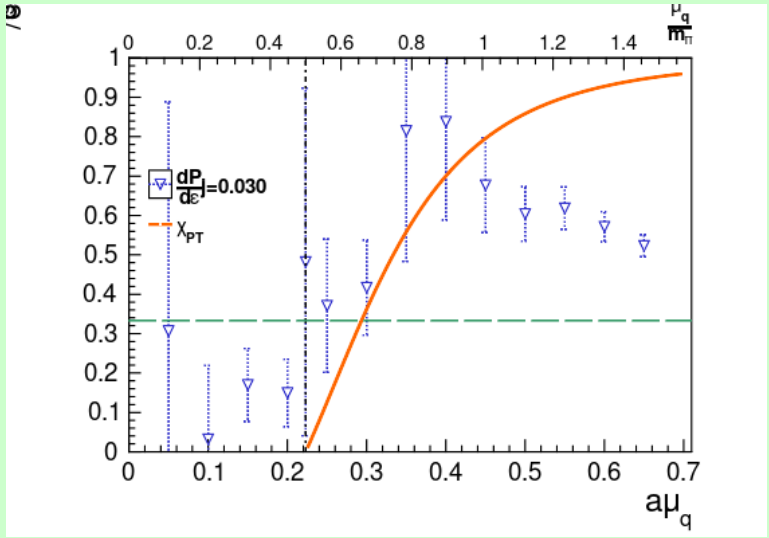






Speed of sound





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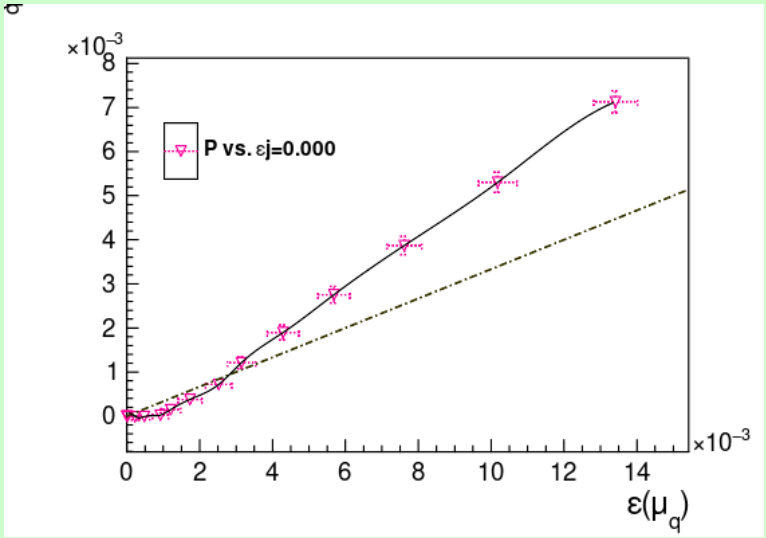
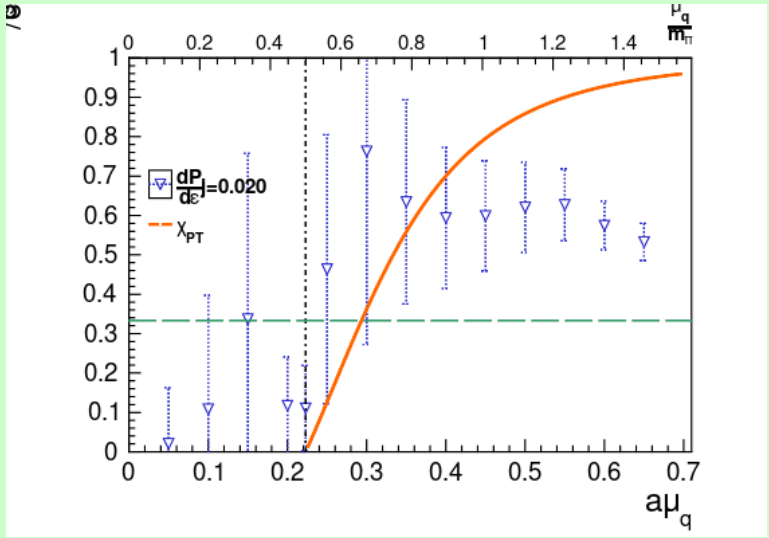
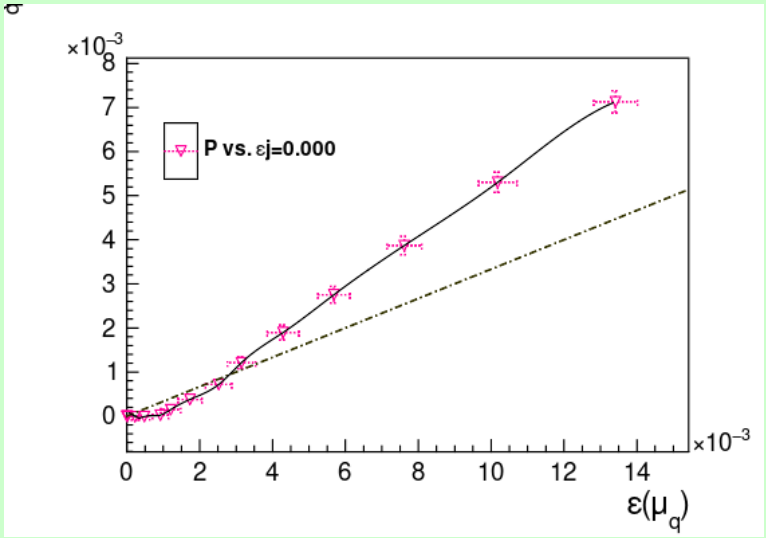
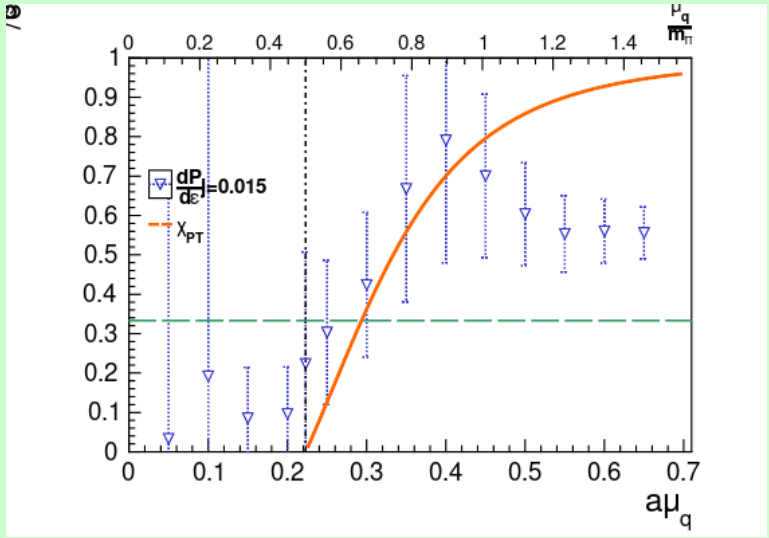


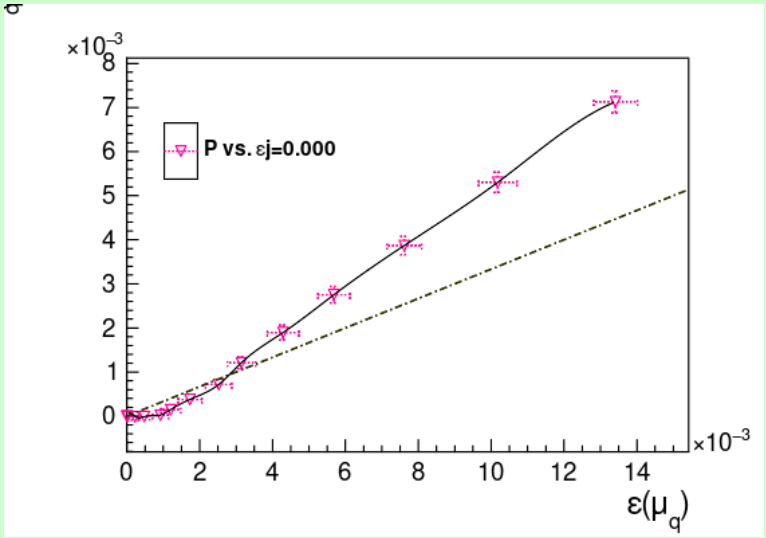


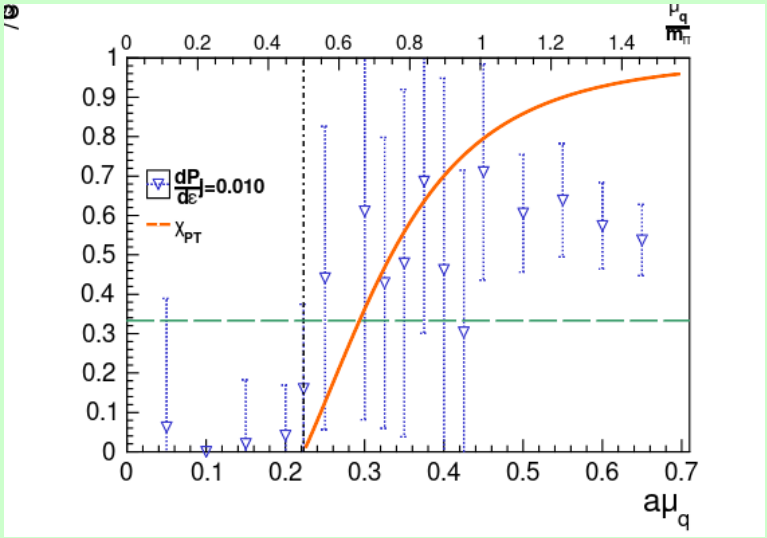
Speed of sound



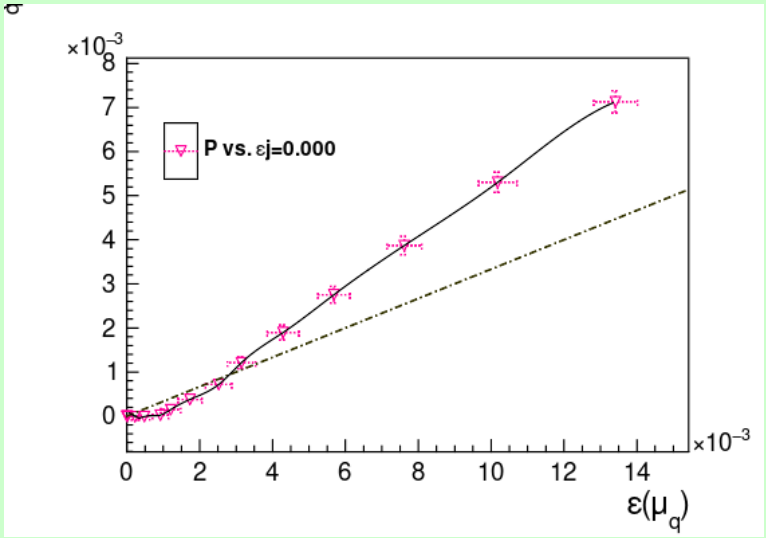


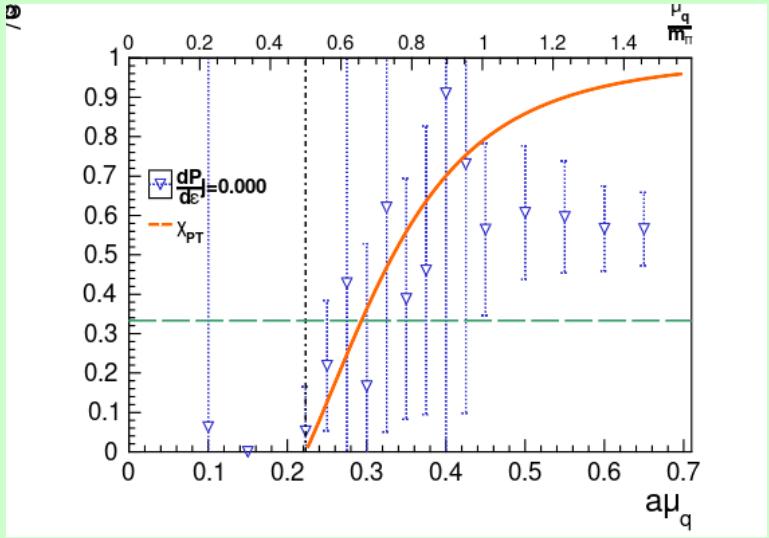
Speed of sound



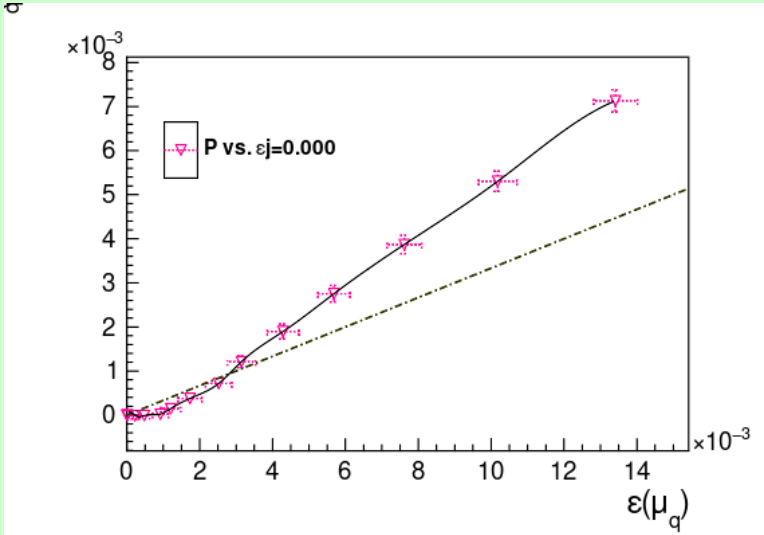


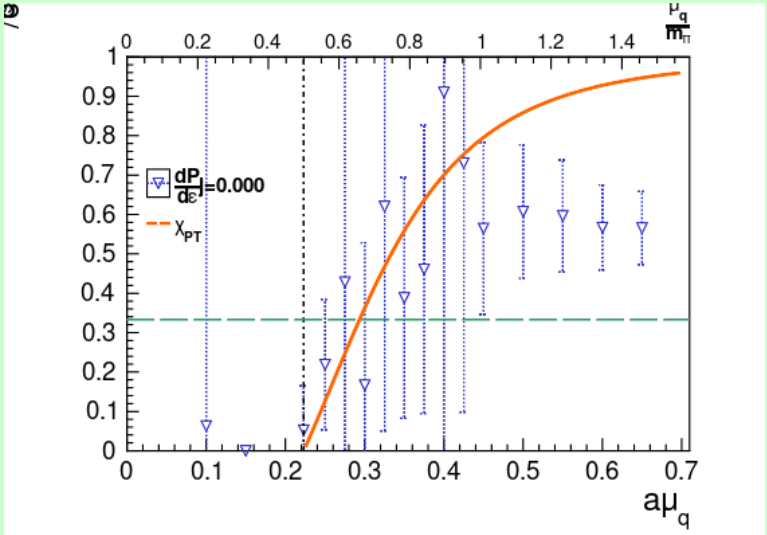
Speed of sound

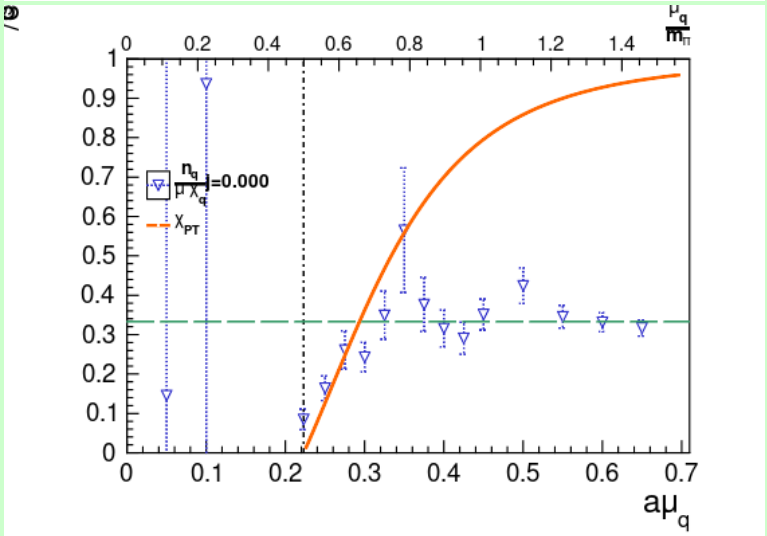




Speed of sound





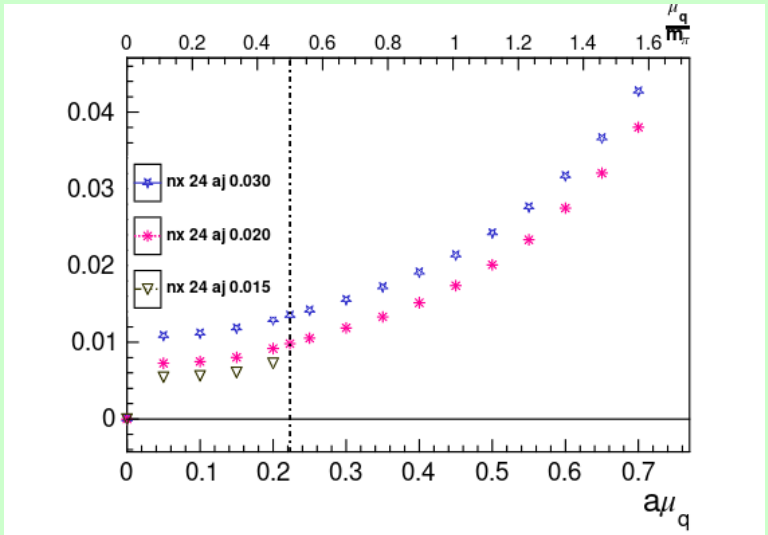


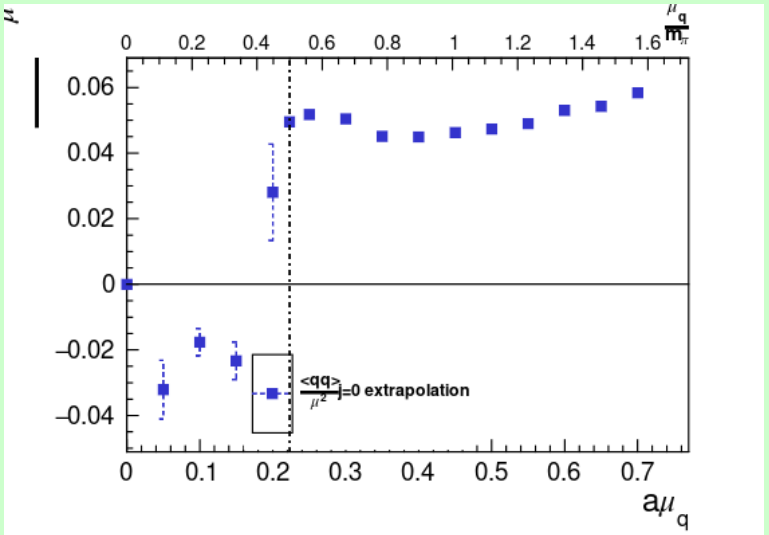
Implementing clover fermions

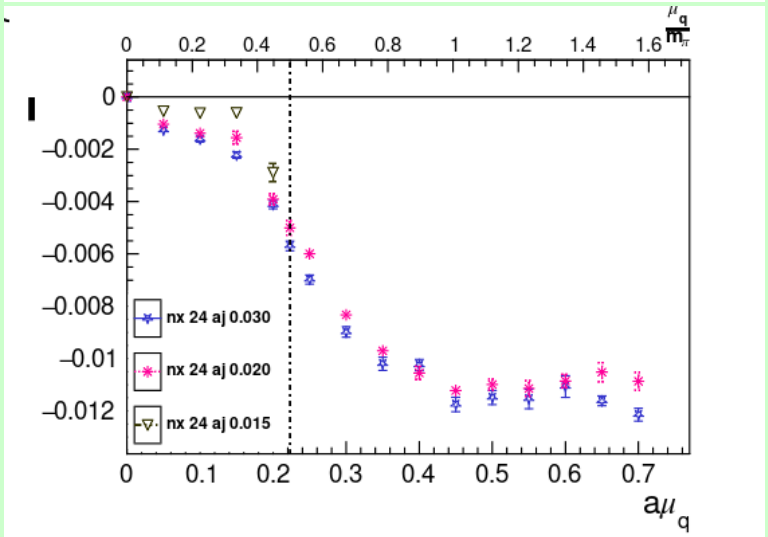
Tree-level $\mathcal{O}(a)$-improved action implemented ($c_{SW} = 1$).
Simulation parameters chosen to match fine Wilson lattice

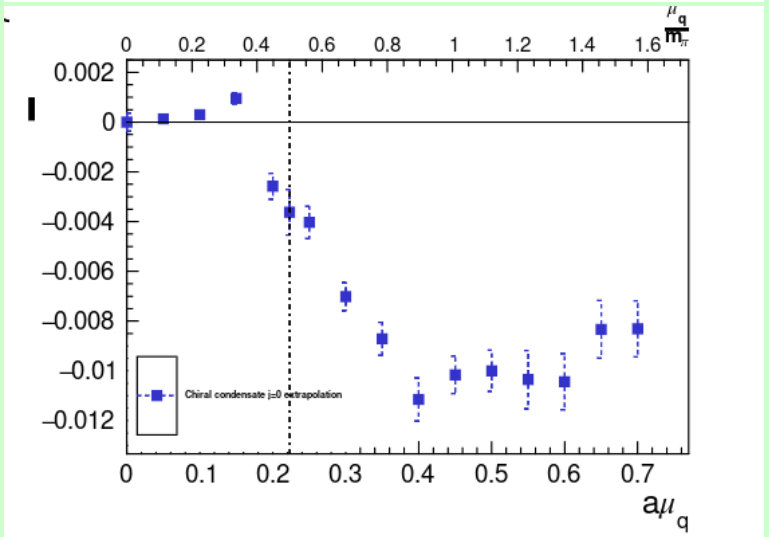
β	κ	a (fm)	am_π	m_π/m_ρ
1.95	0.1586	0.1835(31)	0.800(8)	0.632
2.00	0.1560	0.1687(15)	0.799(41)	0.615
2.05	0.1540	0.1448(22)	0.791(8)	0.528
2.10	0.1520	0.1215(20)	0.796(2)	0.446
2.15	0.1501	0.1042(19)	0.804(18)	0.390
2.20	0.1482	0.0886(19)	0.794(08)	0.360

Clover results: diquark and chiral condensates

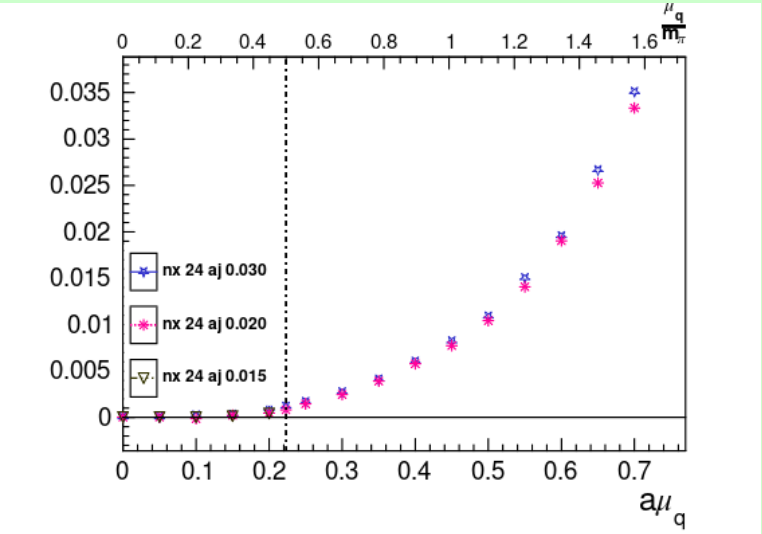


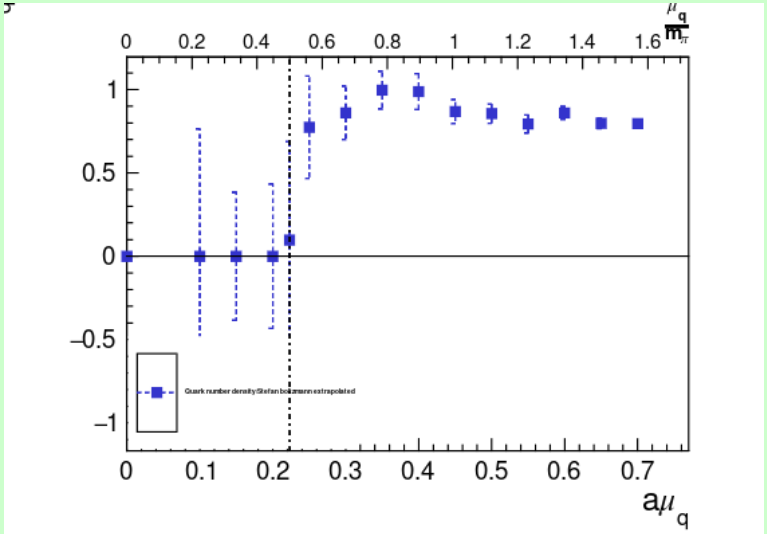


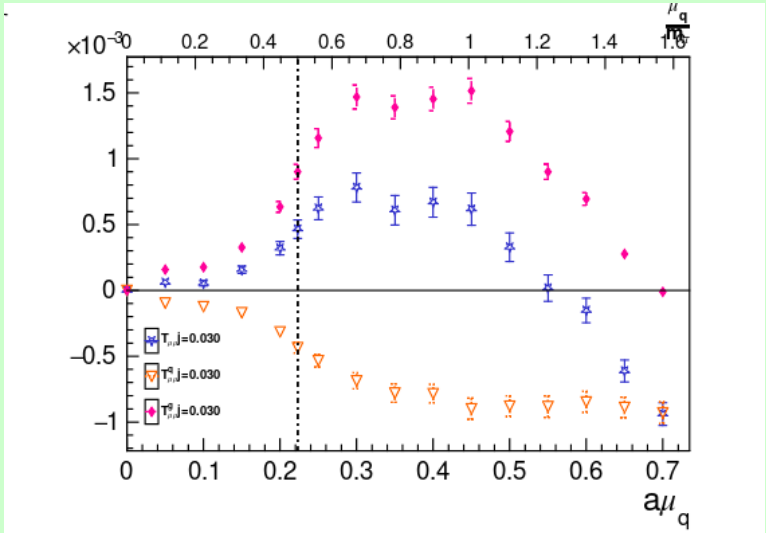


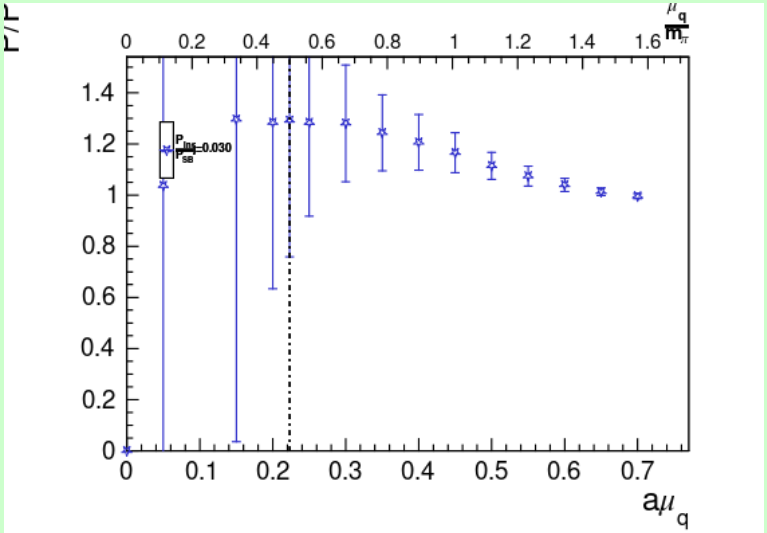


Clover results: bulk thermodynamics

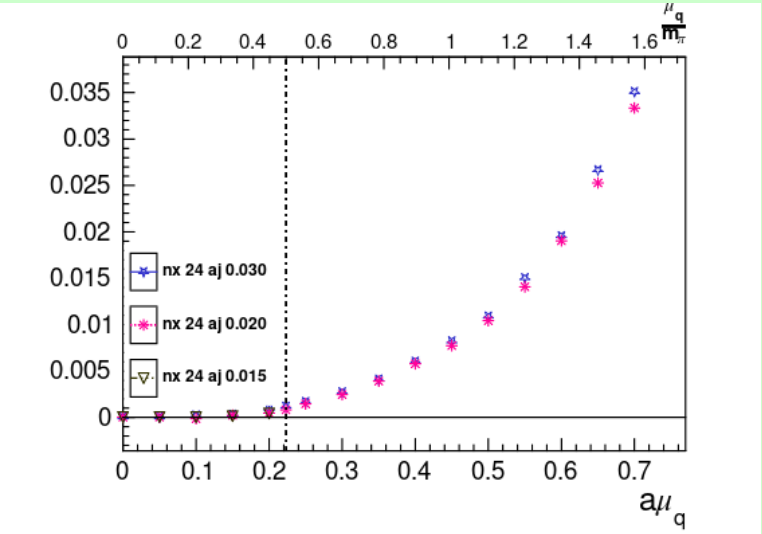


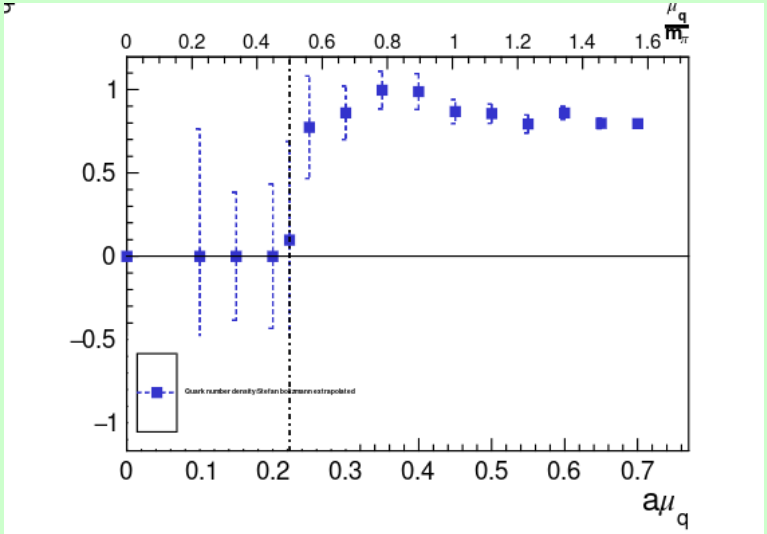


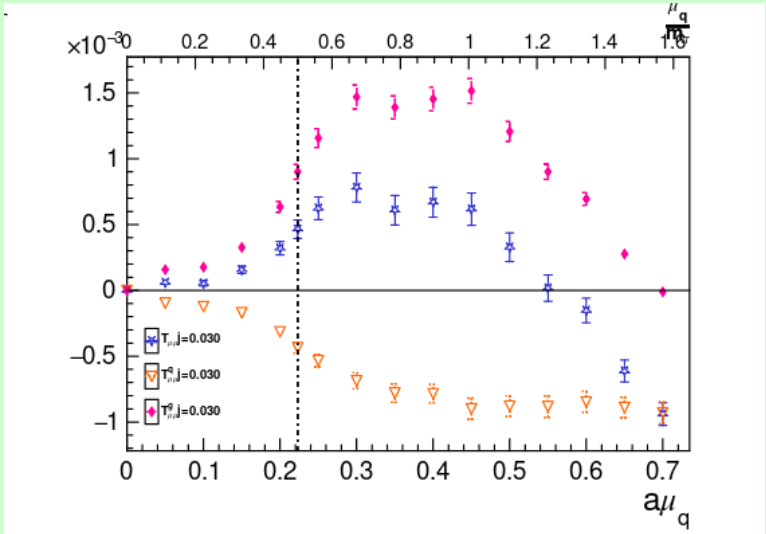


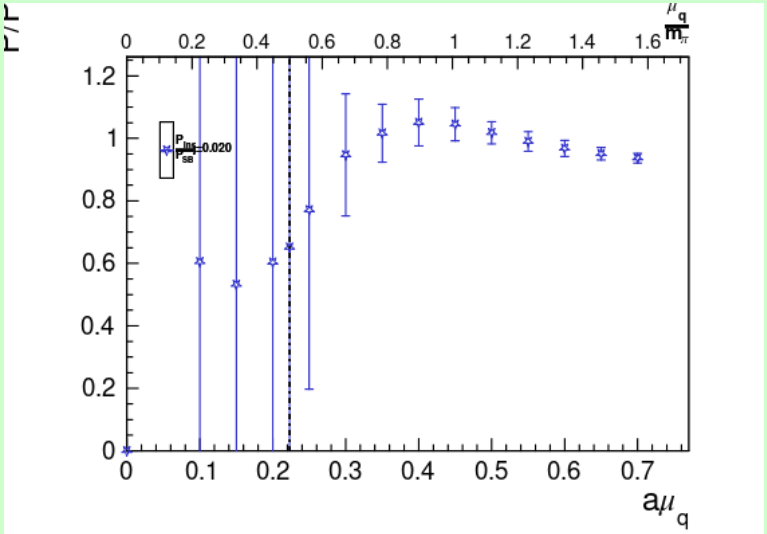


Clover results: bulk thermodynamics

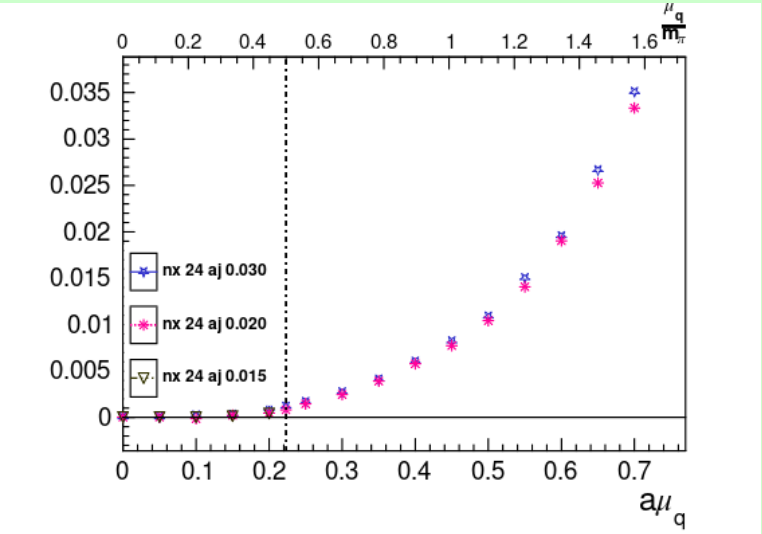


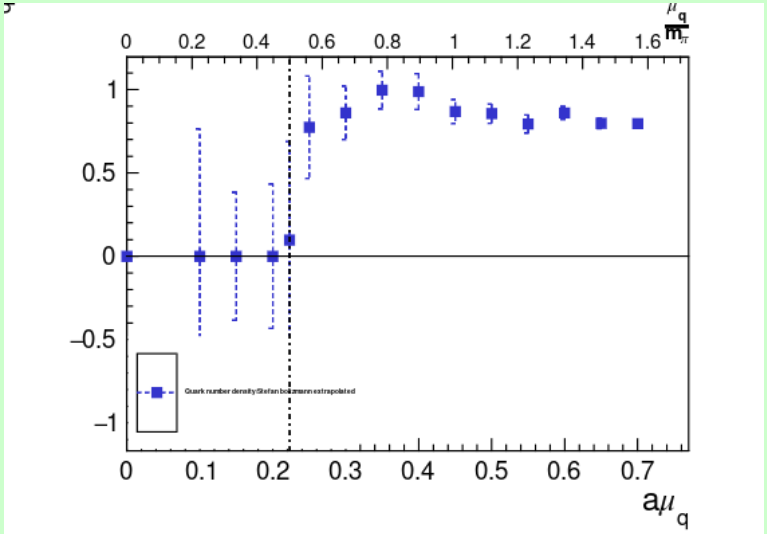


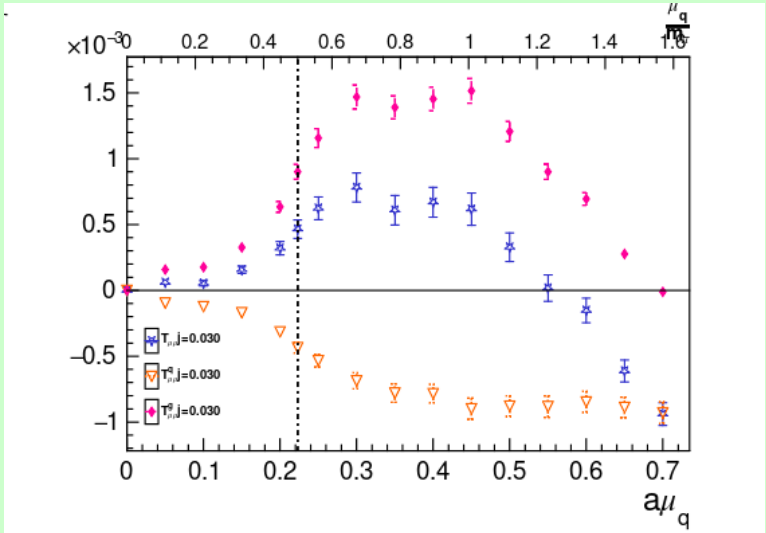


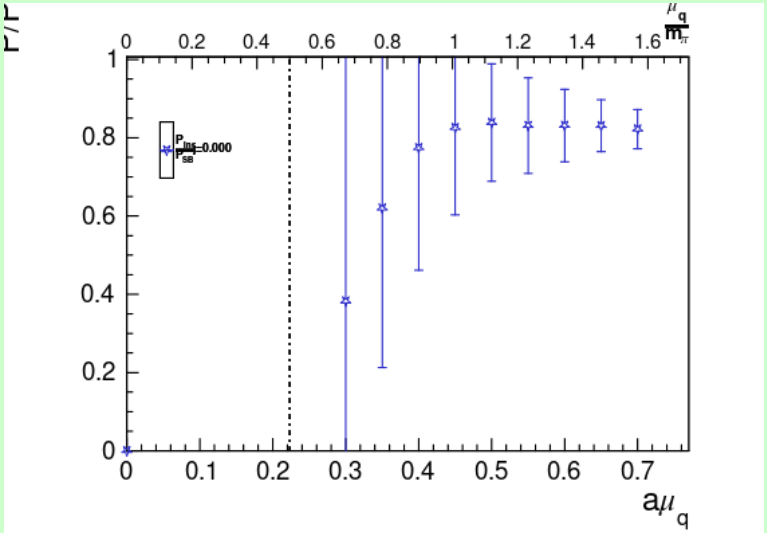


Clover results: bulk thermodynamics

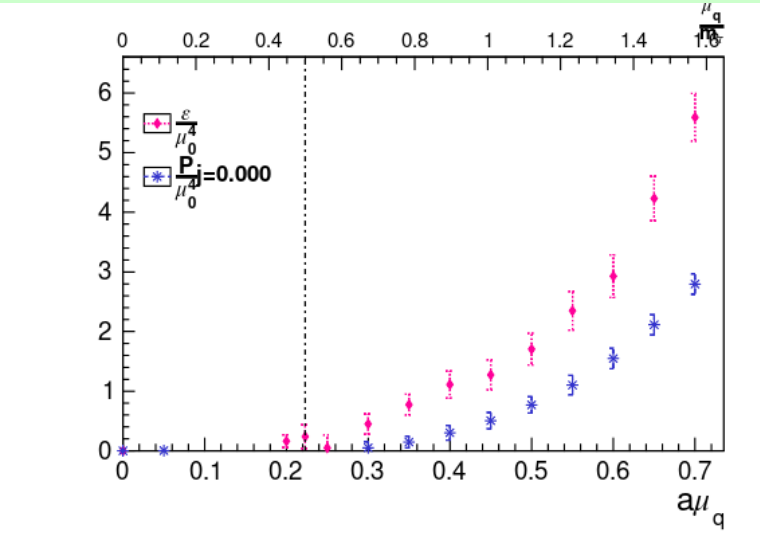


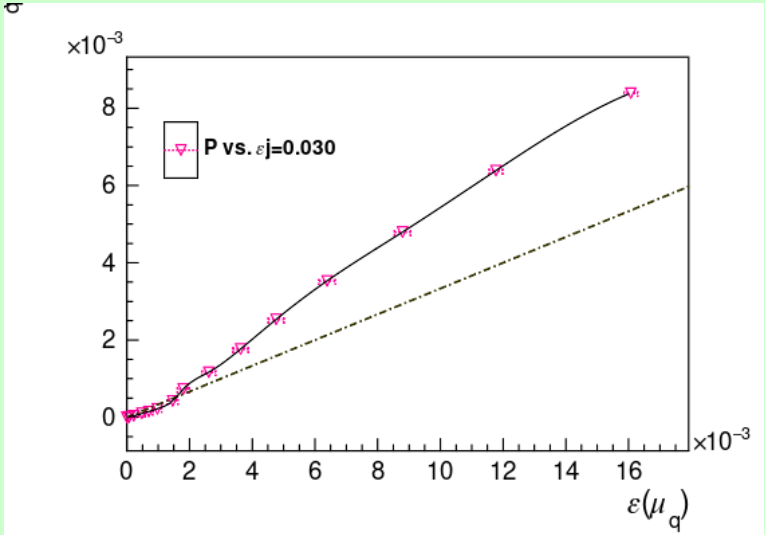


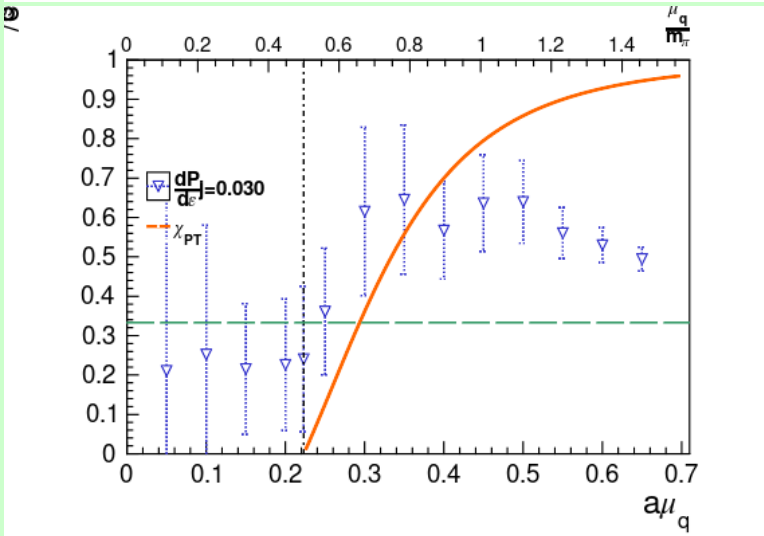




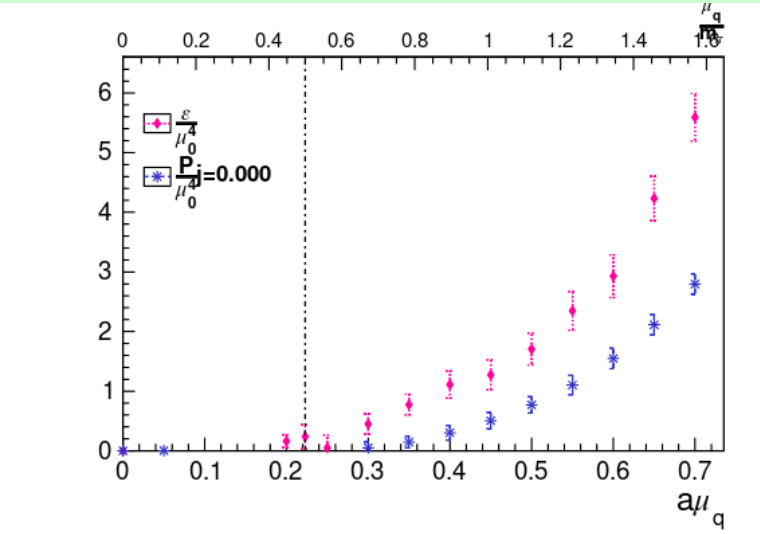
Clover results: speed of sound

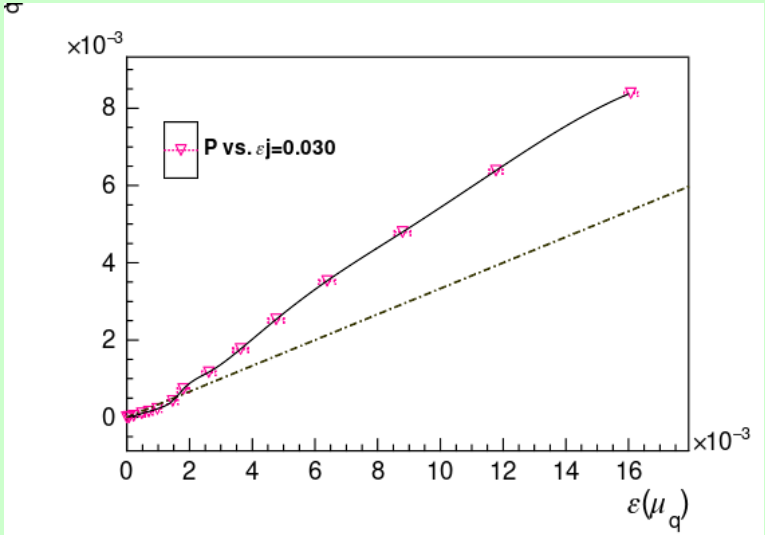


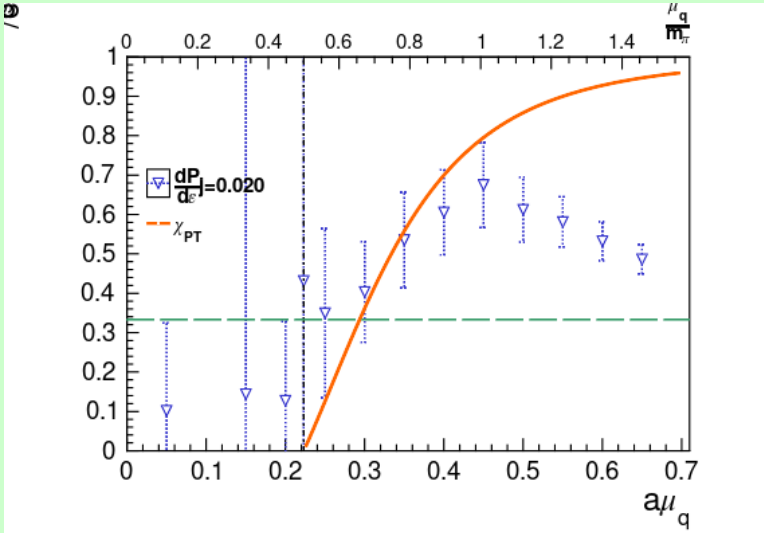




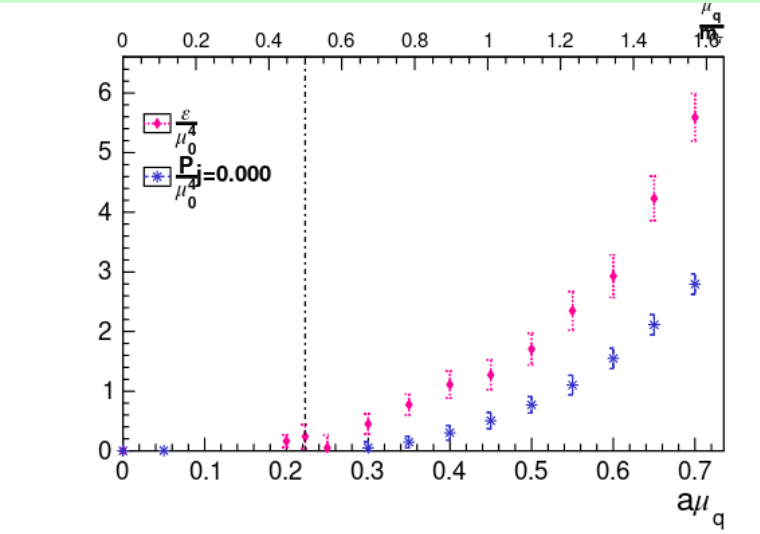
Clover results: speed of sound

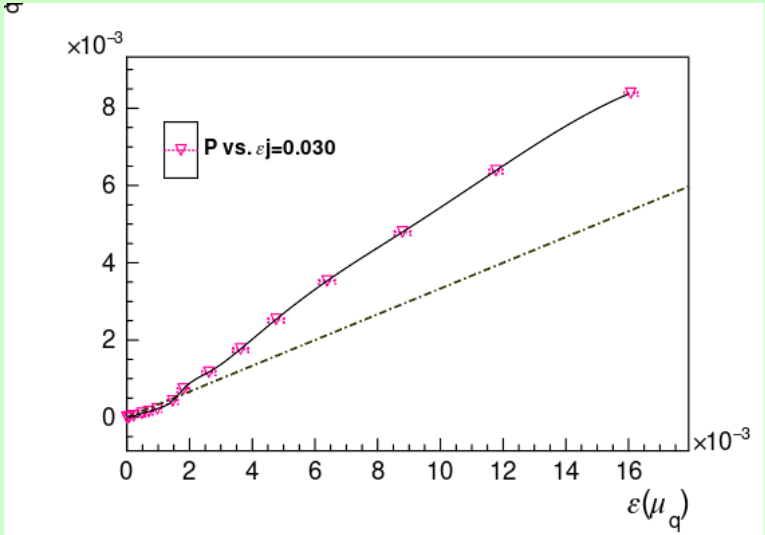


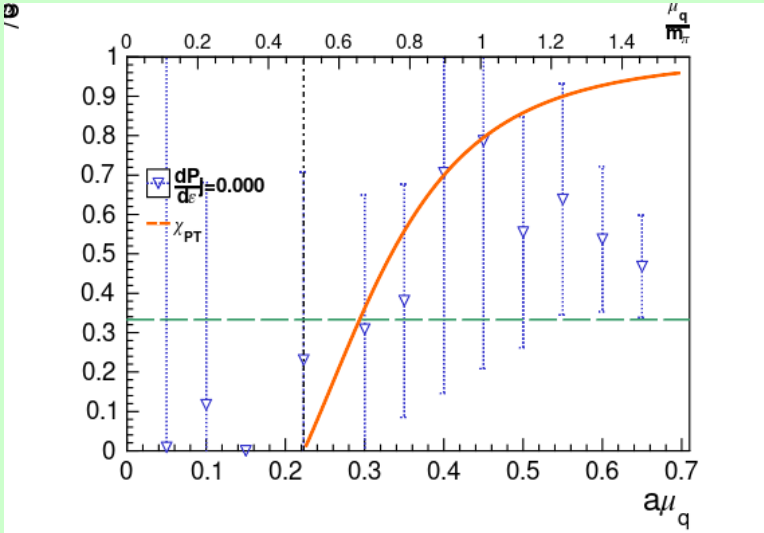




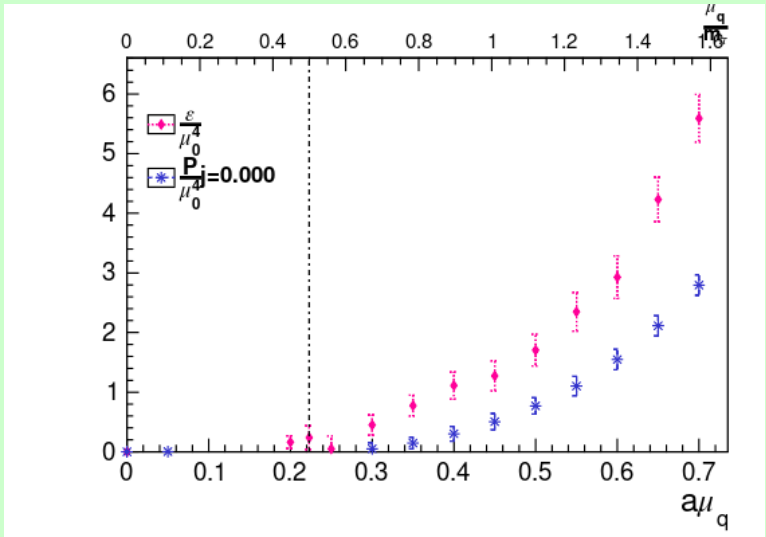
Clover results: speed of sound

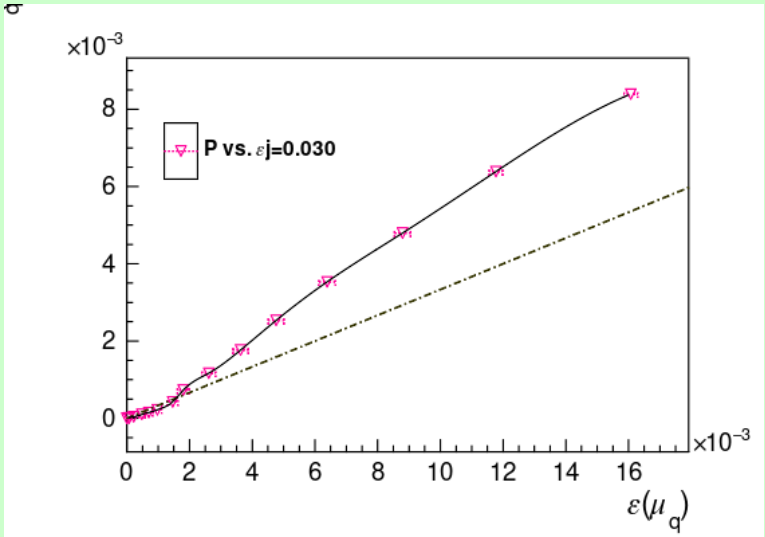


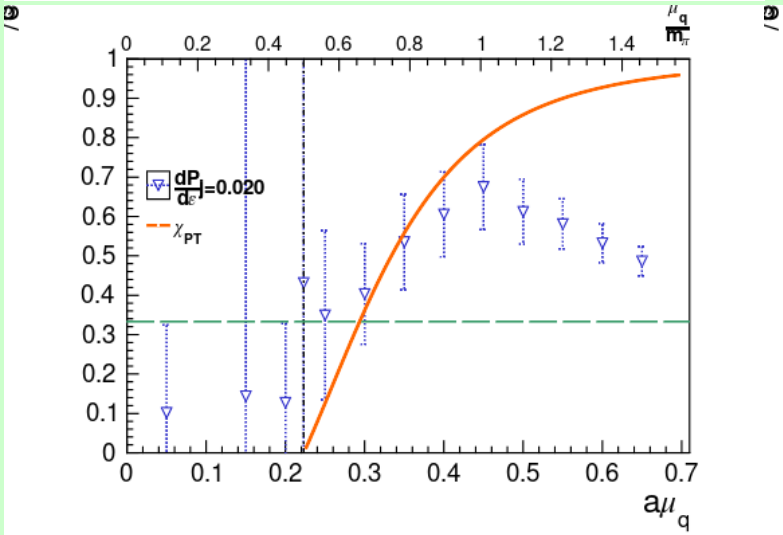


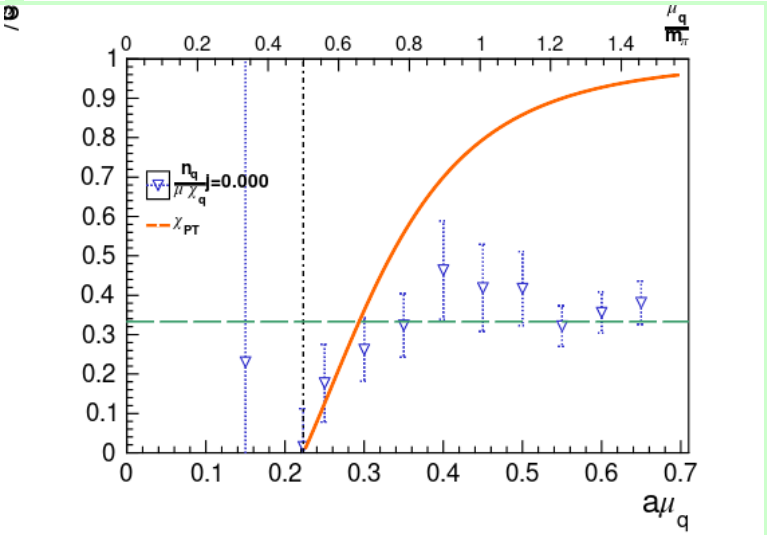


Clover results: speed of sound

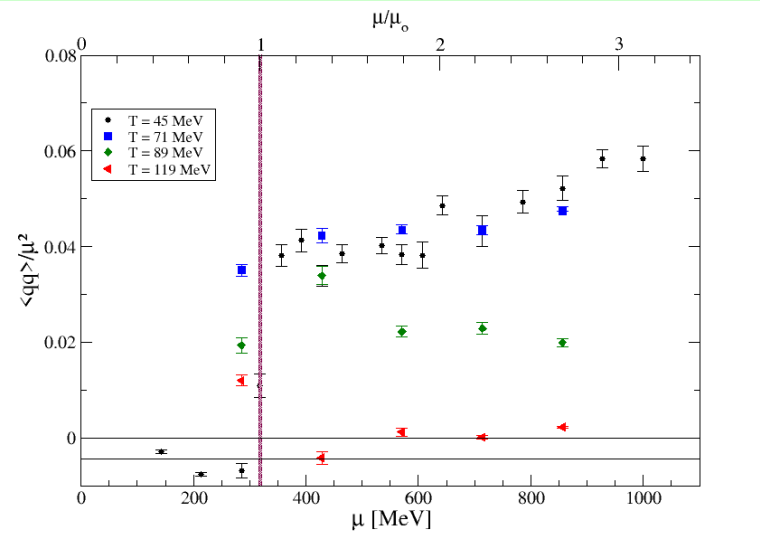


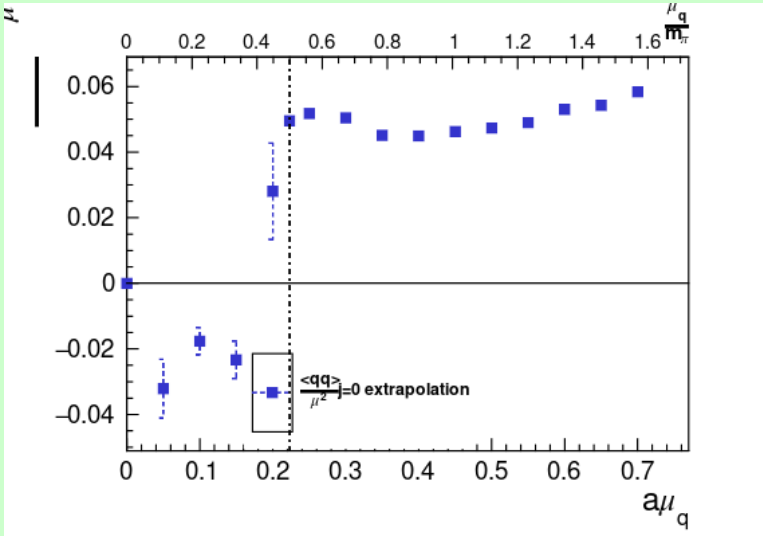


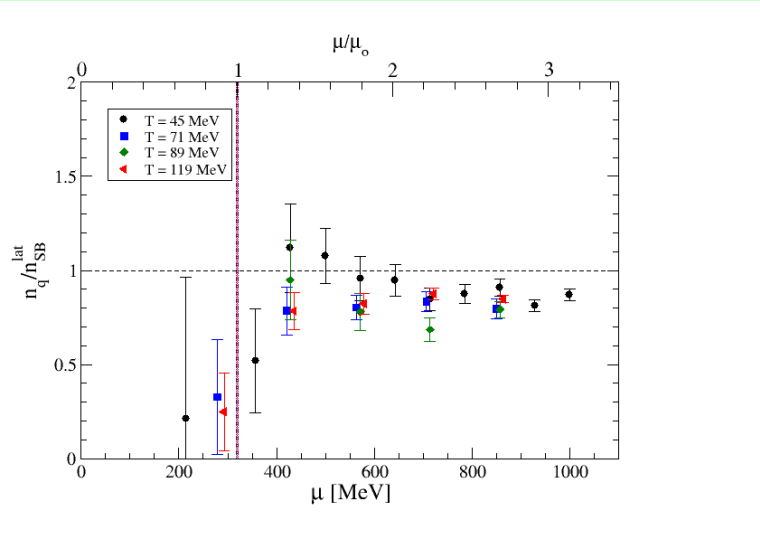


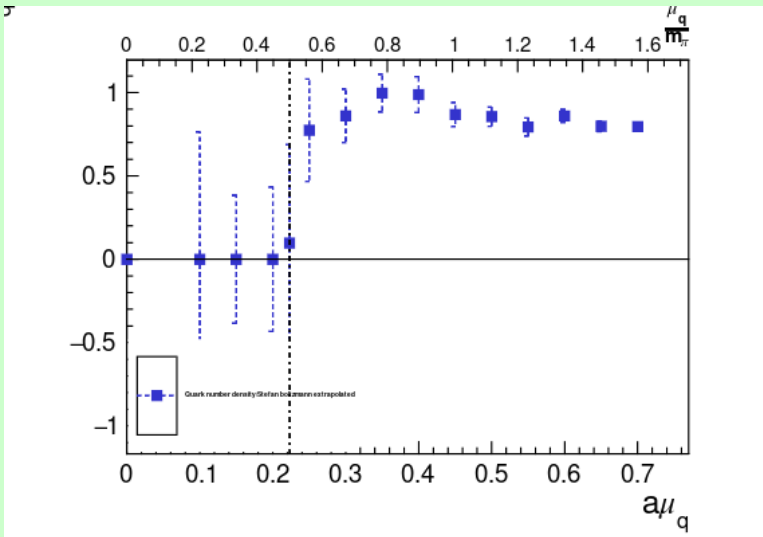


Wilson vs clover (1)

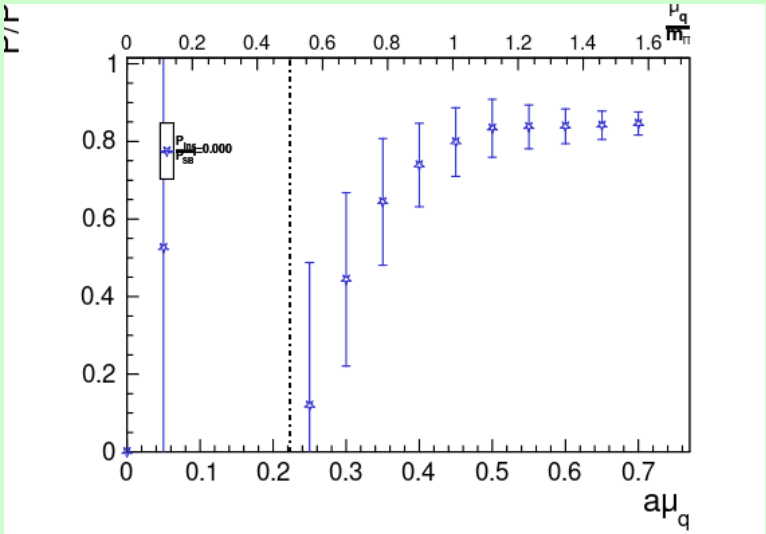


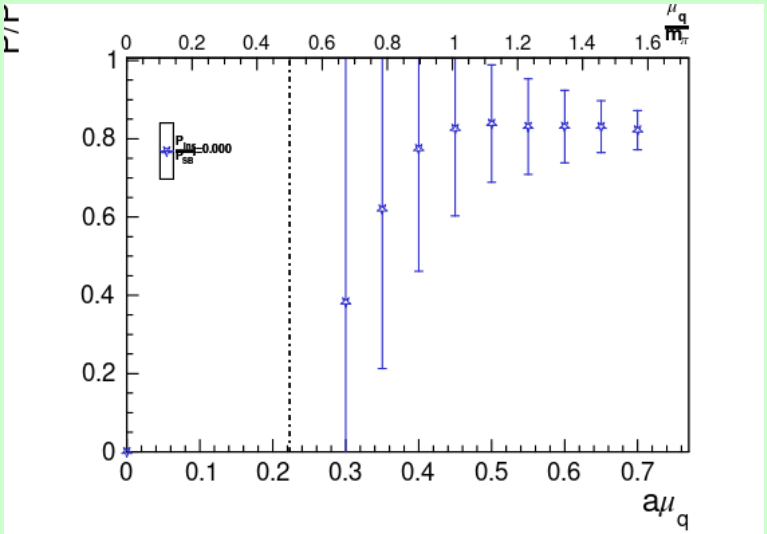


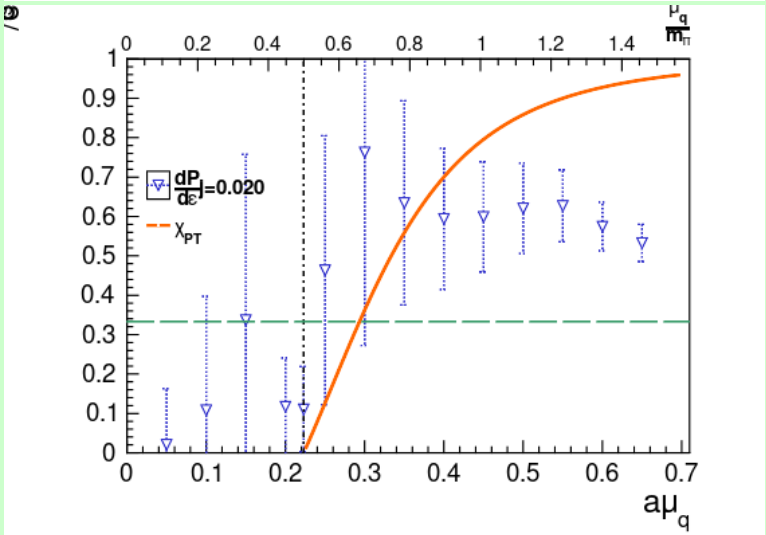


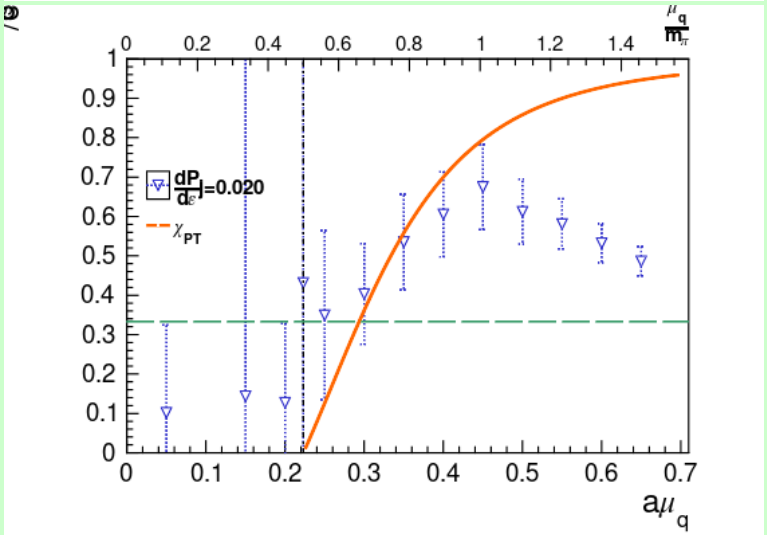


Wilson vs clover (2)









Topology and ground-state fields

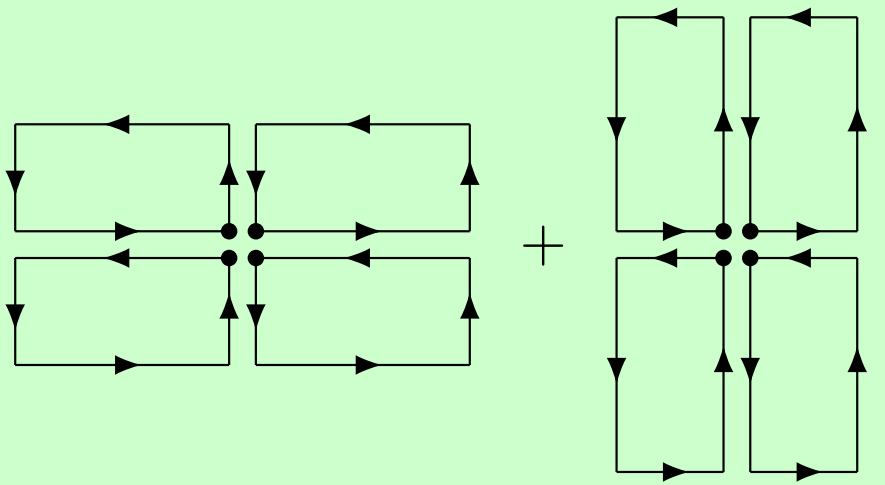
Topological charge density

$$q(x) = \frac{g^2}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\left(F_{\mu\nu}(x)F_{\rho\sigma}(x)\right)$$

Using improved definition

$$ga^2 F_{\mu\nu}(x) = \frac{1}{8}\sum_{m,n} k^{(m\times n)}\,\mathrm{Im}\left(\mathcal{C}_{\mu\nu}^{(m\times n)}(x)\right)$$

$$\mathcal{C}_{\mu\nu}^{(m\times n)} =$$



Using (n, m) up to (3,3) to cancel $\mathcal{O}(a^2), \mathcal{O}(a^4)$ errors.

Gradient flow

Gradient flow removes UV fluctuations

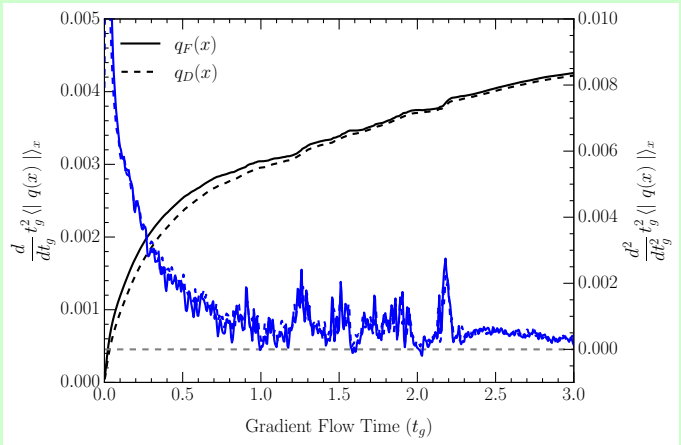
$$\frac{d}{dt_g} U_\mu(x;t_g) = -ig^2 t^a \nabla^a_{x,\mu} S_\epsilon[U] U_\mu(x;t_g)$$

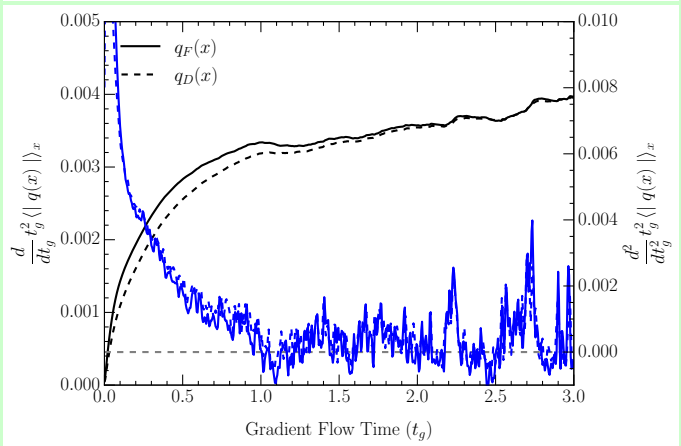
with gradient flow action

$$S_\epsilon[U] = \beta \sum_x \sum_{\mu>\nu} \left[\frac{5-2\epsilon}{3}(1 - P_{\mu\nu}(x)) - \frac{1-\epsilon}{12}\left((1 - R_{\mu\nu}(x)) + (1 - R_{\nu\mu}(x))\right) \right]$$

We use $\epsilon = -0.25$ (Moran),
$\epsilon = -2.972$ (Iwasaki)

Iwasaki





Moran

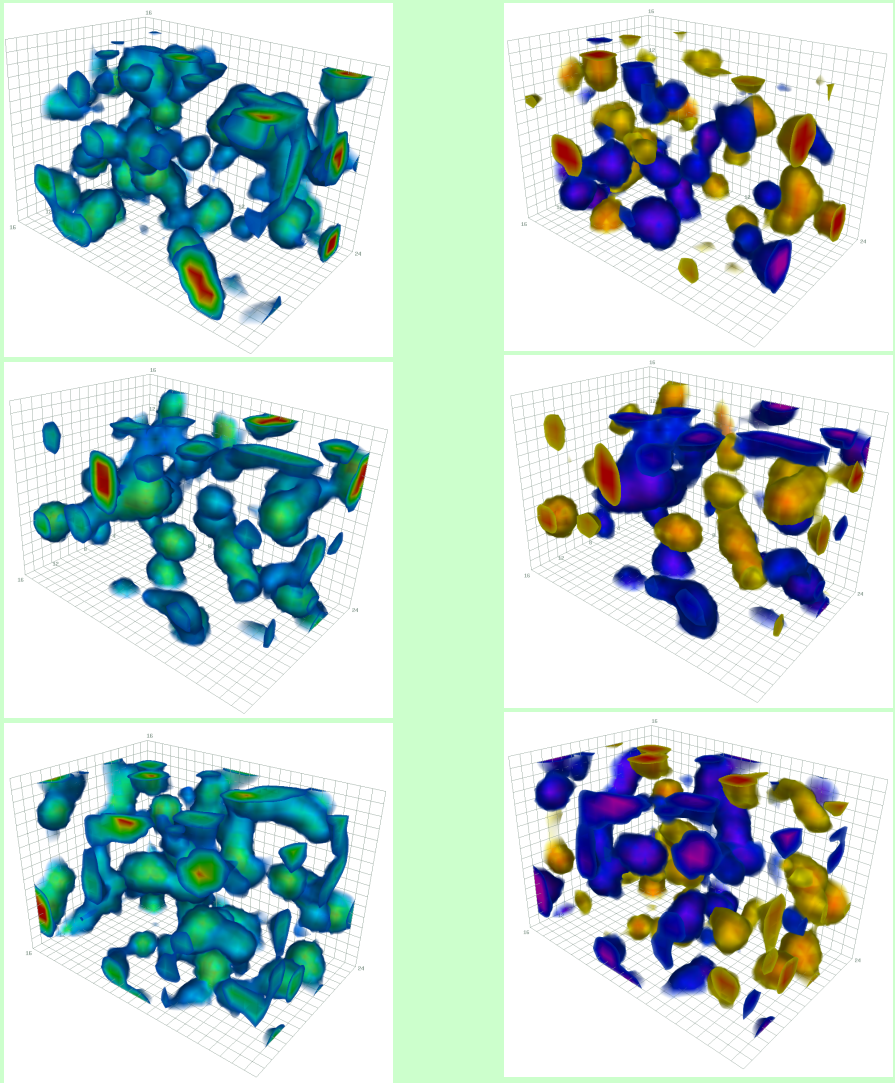
Action and topological charge density

$\mu = 0$

$a\mu = 0.4$

$a\mu = 0.7$

No visible change

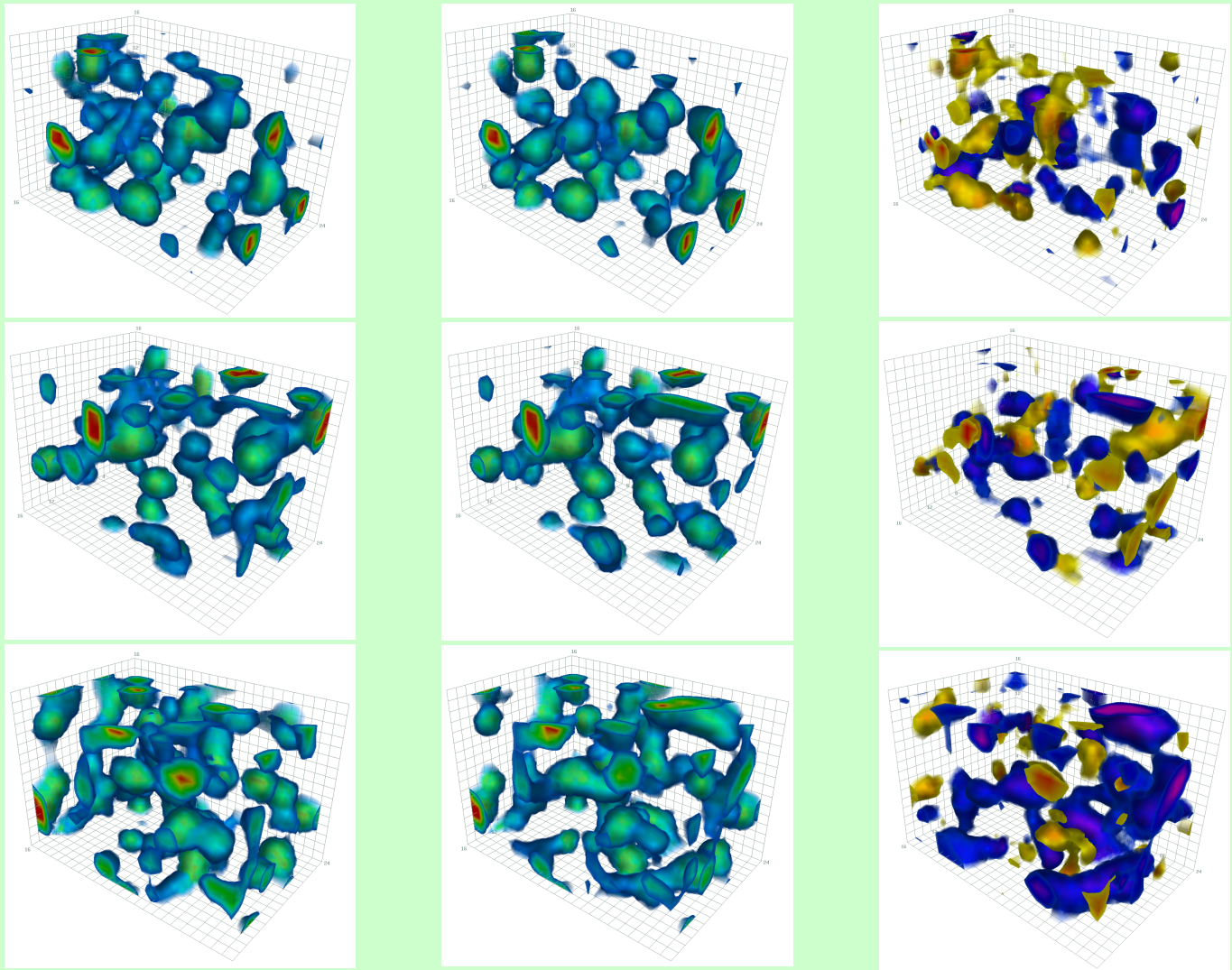


Chromoelectric, chromomagnetic field strength

$\mu = 0$

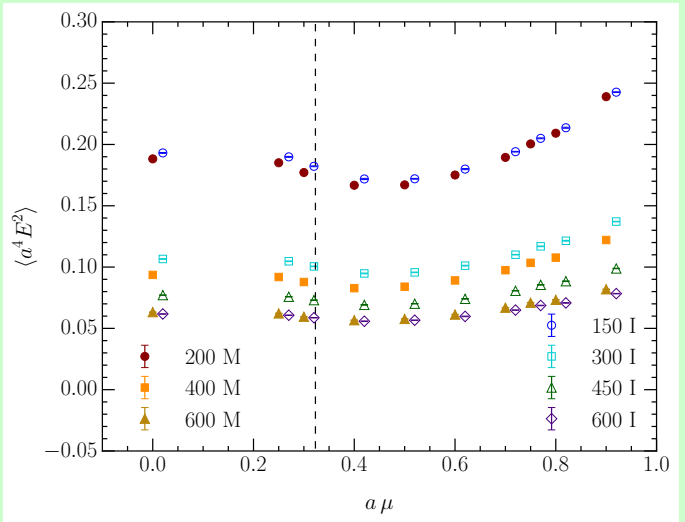
$a\mu = 0.4$

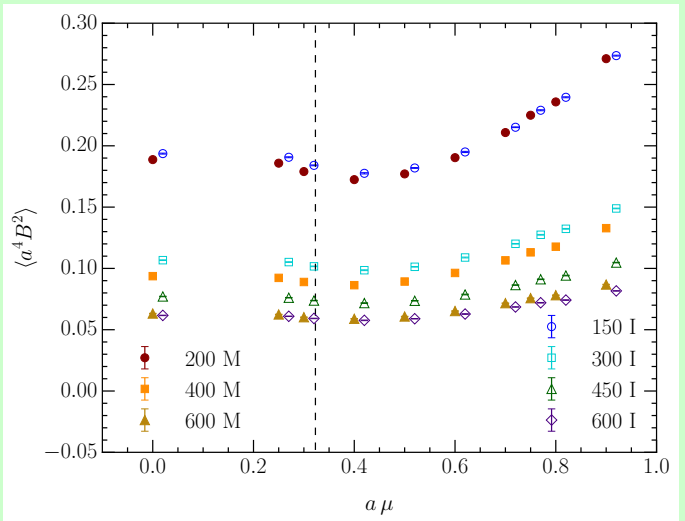
$a\mu = 0.7$



Field strengths as function of flow and μ

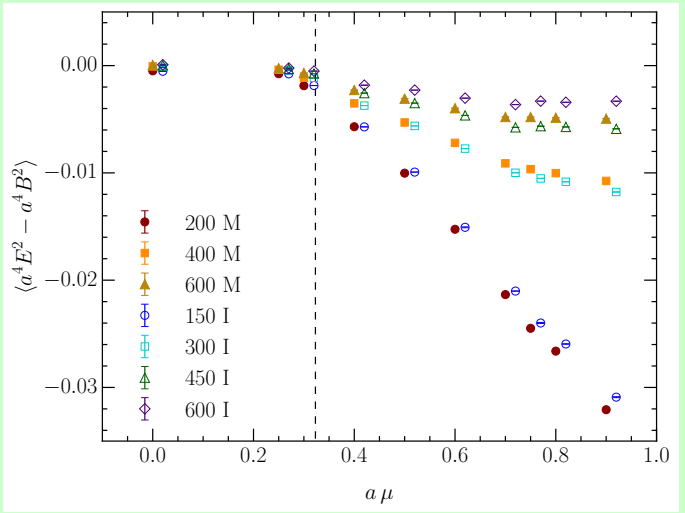
- Non-monotonic E^2, B^2 above μ_o

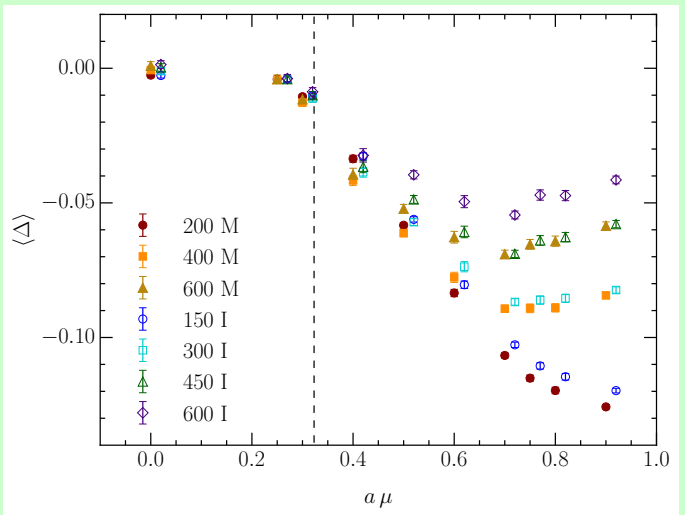




Field strengths as function of flow and μ

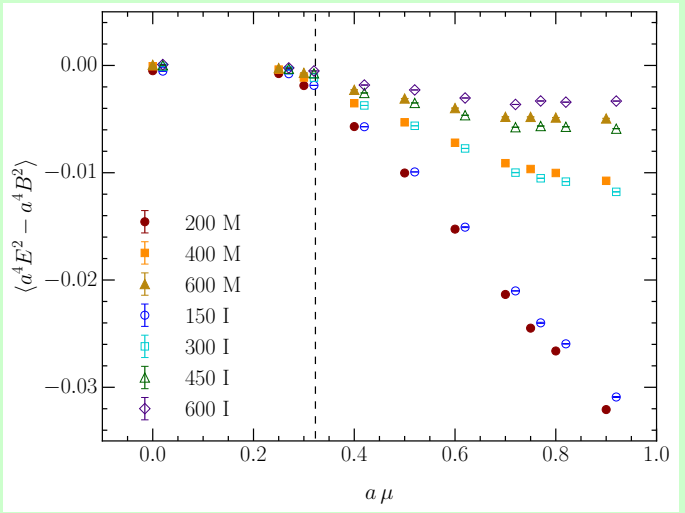
- Non-monotonic E^2, B^2 above μ_o
- $B^2 - E^2$ increasing with μ

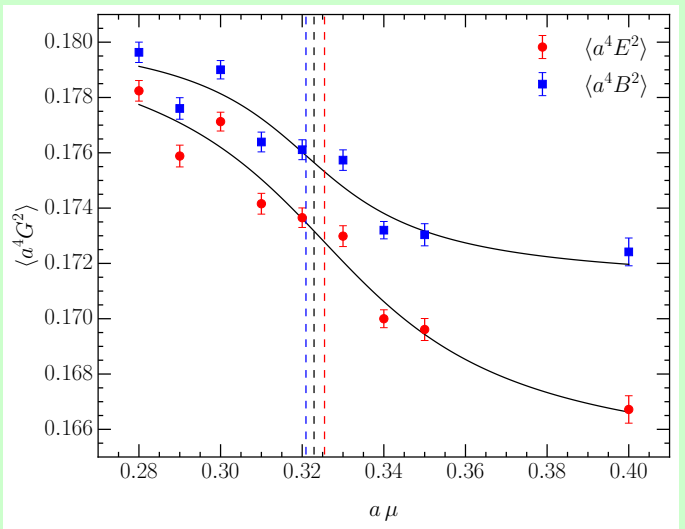




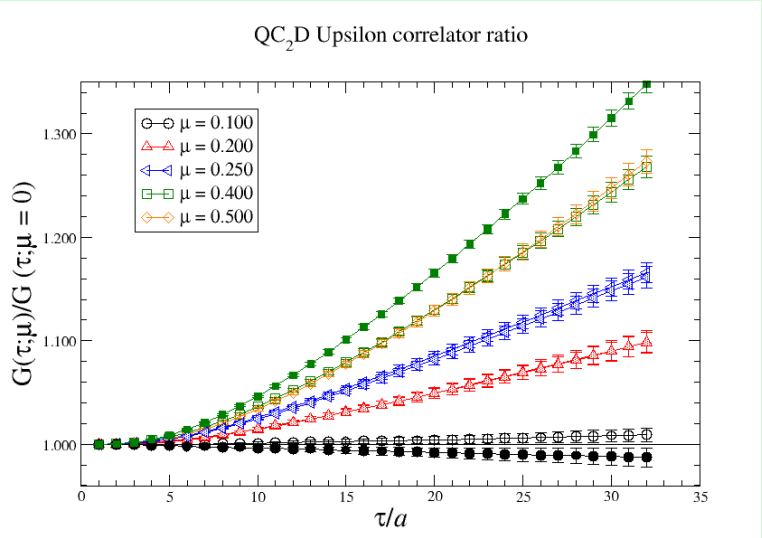
Field strengths as function of flow and μ

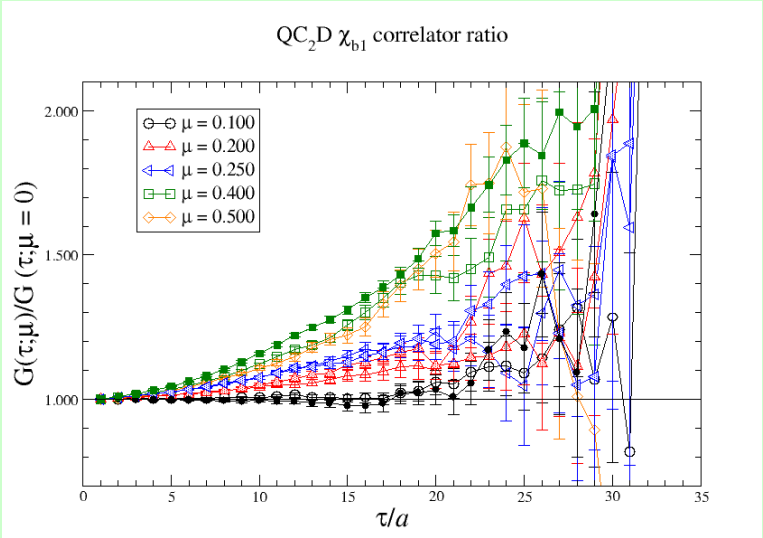
- Non-monotonic E^2, B^2 above μ_o
- $B^2 - E^2$ increasing with μ
- E^2 and B^2 have inflection at μ_o





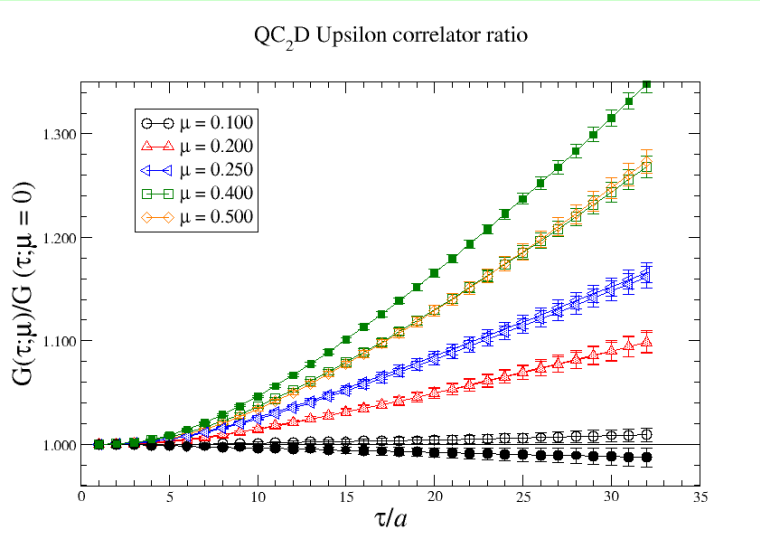
Heavy quarkonium

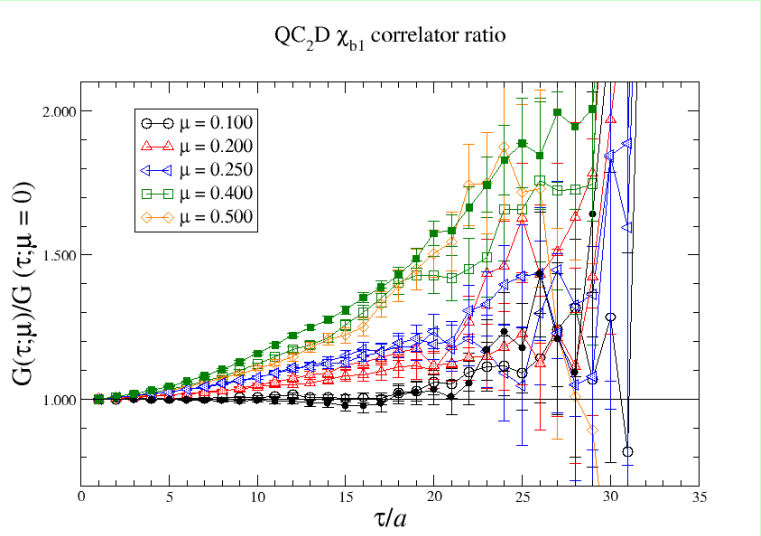


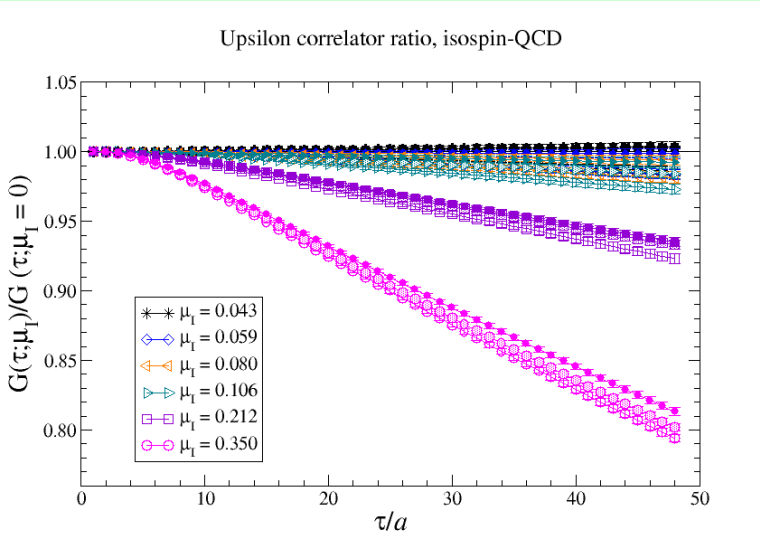


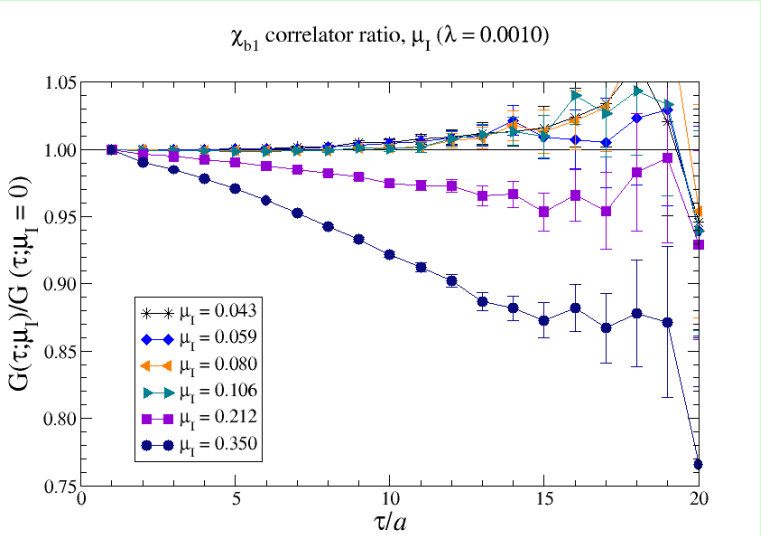
- Simulations on fine lattice (ongoing)
- Increasing correlator ratio for both Υ and χ_{b1}
  $\longrightarrow$ decreasing mass
- Opposite effect to isospin-QCD!

Heavy quarkonium









Summary

Evidence for three phases/regions

- Vacuum/hadronic phase below $\mu_o = m_\pi/2$, low T
- BCS/quarkyonic for intermediate μ, low T
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- Chromoelectric/magnetic fields sensitive to onset transition
- Results from improved (clover) action consistent with Wilson

Outlook

- Simulations with mean-field improved clover action
  - $\rightarrow$ full (T, μ) scan
  - $\rightarrow$ continuum estimate?
  - $\rightarrow$ smaller quark masses?
- Full gluodynamics analysis on fine lattices
- Matching lattice data to perturbation theory
- TOV equation — 2-colour stars?

BACKUP SLIDES

Determining β-functions

β	κ	n_{cfg}	a (fm)	$\frac{m_\pi}{m_\rho}$	am_π
1.7	*0.1780*		*0.223(4)*	*0.779(4)*	*0.79*
1.9	*0.1680*		*0.187(4)*	*0.805(9)*	*0.645*
2.1	*0.1577*		0.130(4)	*0.810(4)*	*0.446*
1.80	0.1730	1019	0.2125(19)	0.811(8)	0.731
1.85	0.1707	277	0.1846(15)	0.828(21)	0.684
1.97	0.1646	175	0.1617(8)	0.829(27)	0.546
2.00	0.1626	268	0.1528(6)	0.799(22)	0.562
2.04	0.1606	293	0.1437(4)	0.799(18)	0.520
2.14	0.1558	106	0.1004(7)	0.801(45)	0.410
2.17	0.1544	100	0.0741(16)	0.813(9)	0.381

Simulation parameters

T-scans, fixed μ (coarse lattice)

All simulations done on $16^3 \times N_\tau$ lattices

μa	ja	N_τ
0.0	0.0	4–10
0.35	0.02	4–13, 16
	0.04	4–12, 14, 16
0.40	0.02	5–13, 16
	0.04	4–13
0.50	0.02	6–12, 16
	0.04	4–16, 18, 20
0.60	0.02	6–12, 14, 16
	0.04	6–16, 20

In addition, 300 trajectories were generated at $ja = 0.03, 0.05$ for $N_\tau = 9, 10, 11$ at all $\mu > 0$.

Simulation parameters

T-scans, fixed μ (fine lattice)

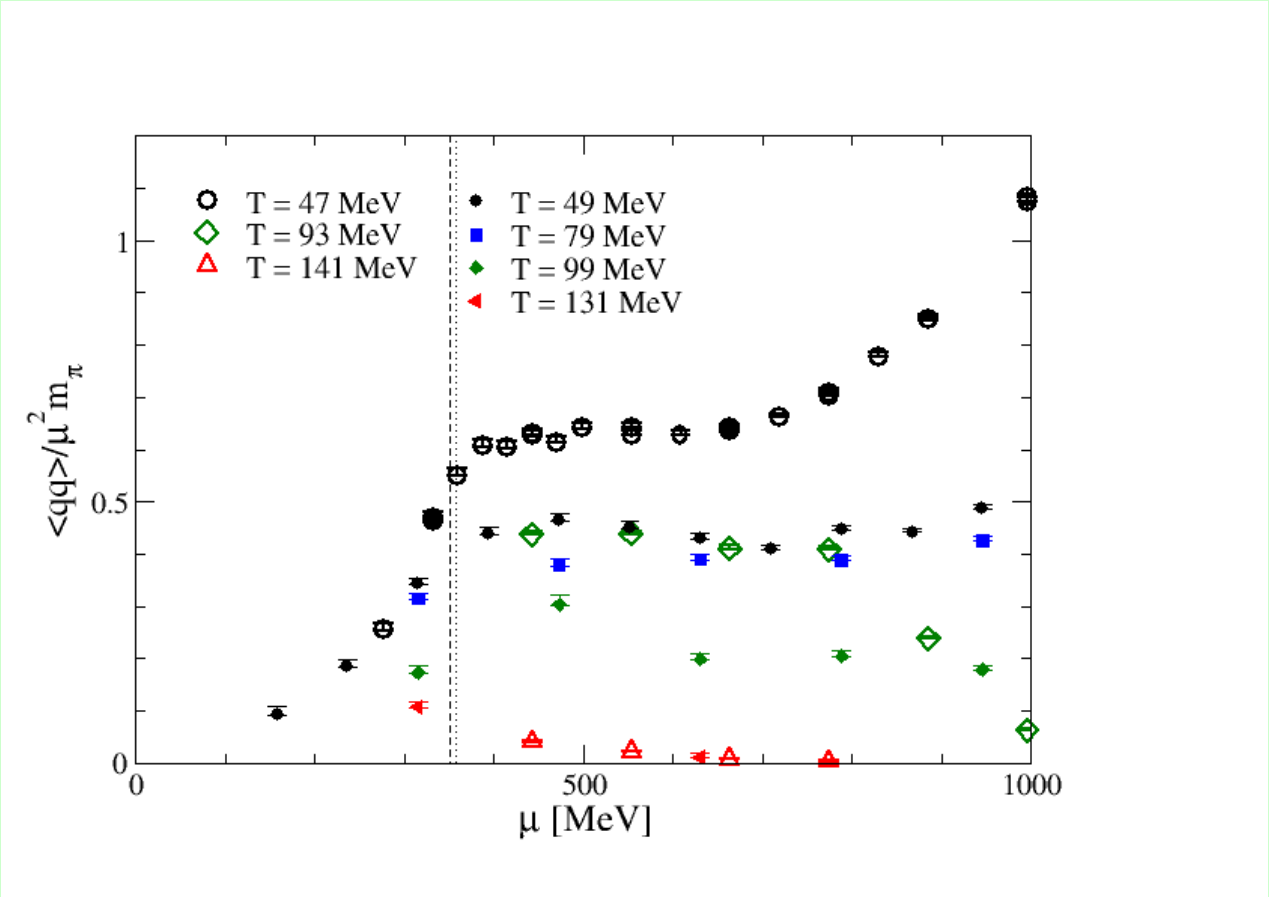
All simulations done on $16^3 \times N_\tau$ lattices

μa	ja	N_τ
0.0	0.0	4–10, 12, 16
0.2	0.03	4–14, 16, 18
0.3	0.02	4, 6, 8, 10, 12–14, 16
	0.03	4–14, 16, 18
0.4	0.02	4, 6, 8, 10, 12–14, 16
	0.03	4–14, 16, 18, 20
0.5	0.02	4, 6, 10, 12–14, 16
	0.03	4, 6–14, 16, 18

Diquark source extrapolation

$a\mu_q$	C_0	C_1	C_2	χ^2/N_{df}
0.050	-1.68(19)e-5	0.37569(56)	0.99381(45)	10.43/2
0.100	-2.97(35)e-5	0.37910(92)	0.98800(74)	58.78/2
0.150	-1.730(88)e-4	0.3495(18)	0.9459(17)	27.14/2
0.200	-2.76(61)e-4	0.2621(52)	0.8368(67)	48.31/2
0.223	5.4(12)e-4	0.2483(81)	0.826(12)	39.65/2
0.250	2.38(14)e-3	0.323(16)	0.925(17)	6.28/2
0.275	3.12(18)e-3	0.368(24)	0.963(22)	9.95/2
0.300	3.05(20)e-3	0.275(16)	0.864(21)	1.70/2
0.325	4.07(20)e-3	0.297(18)	0.889(21)	5.61/2
0.350	5.04(25)e-3	0.353(29)	0.939(28)	7.34/2
0.375	5.66(24)e-3	0.323(22)	0.908(24)	1.31/2
0.400	6.14(33)e-3	0.306(25)	0.878(30)	5.41/2
0.425	6.90(50)e-3	0.285(32)	0.851(41)	1.40/1
0.450	9.83(40)e-3	0.501(75)	1.044(51)	0.84/2
0.500	1.081(80)e-2	0.325(61)	0.879(69)	0.36/1
0.550	1.492(72)e-2	0.454(93)	0.983(72)	0.06/1
0.600	1.878(90)e-2	0.45(10)	0.964(83)	1.77/1
0.650	2.467(79)e-2	0.62(15)	1.061(83)	2.83/1
0.700	2.86(13)e-2	0.43(10)	0.902(89)	0.34/1

Coarse and fine ensemble



Smaller magnitude on fine lattice: renormalisation?