

Consistent handling of collinear logarithms in the NLL Balitsky-Kovchegov equation

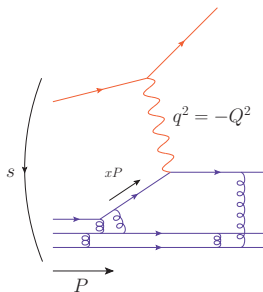
Renaud Boussarie

New Frontiers in QCD 2026

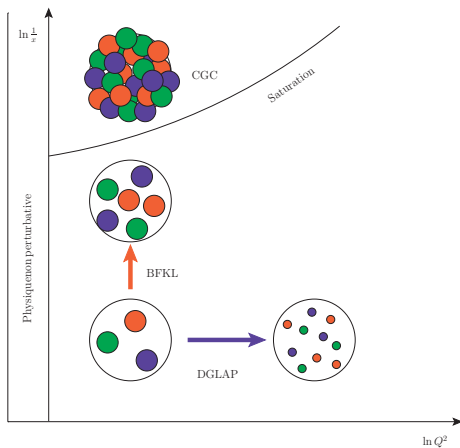


With P. Caucal, P. Korcyl, Y. Mehtar-Tani and J. Penttala

Accessing the partonic content of hadrons with an electromagnetic probe

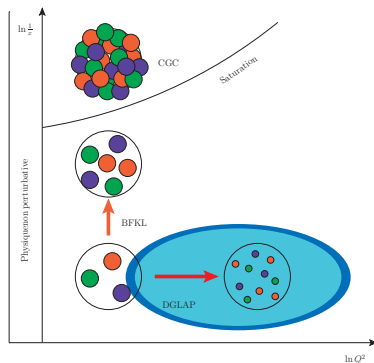


Electron-proton
collision
(parton model)



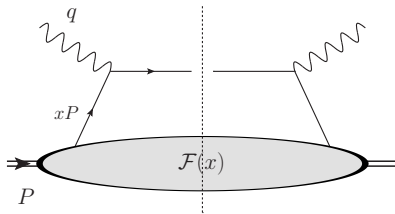
QCD at moderate $x_{Bj} \sim Q^2/s$

Bjorken limit: $Q^2 \sim s$



QCD factorization

processes with a hard scale $Q \gg \Lambda_{QCD}$



$$\sigma = \mathcal{F}(x, \mu) \otimes \mathcal{H}(x, \mu)$$

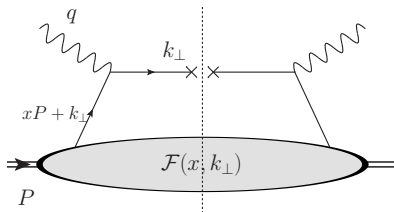
At a scale μ , the process is factorized into:

- A hard scattering subamplitude $\mathcal{H}(x, \mu)$
- A Parton Distribution Function (PDF) $\mathcal{F}(x, \mu)$

μ independence: DGLAP renormalization equation for $\mathcal{F}$

Transverse Momentum Dependent (TMD) factorization:
semi-inclusive processes with one hard and one semihard scale

$$Q \sim \sqrt{s} \gg k_{\perp}$$



$$\sigma = \mathcal{F}(x, k_{\perp}, \zeta, \mu) \otimes \mathcal{H}(\mu) \otimes \hat{\mathcal{F}}(\hat{x}, \hat{k}_{\perp}, \hat{\zeta}, \mu)$$

At a scale μ , the process is factorized into:

- A **hard scattering subamplitude** $\mathcal{H}(\mu)$
- A **TMD PDF** $\mathcal{F}(x, k_{\perp}, \zeta, \mu)$
- A **TMD FF** $\hat{\mathcal{F}}(\hat{x}, \hat{k}_{\perp}, \hat{\zeta}, \mu)$

$\mu, \zeta, \hat{\zeta}$ independence: TMD evolution for $\mathcal{F}, \hat{\mathcal{F}}$

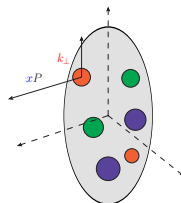
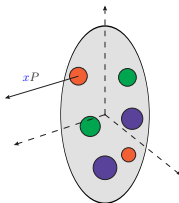
Operator definition for parton distributions

Parton distribution function

$$\mathcal{F}(x) \propto \int dz^+ e^{ixP^- z^+} \langle P | F^{-i}(z^+) [z^+, 0^+] F^{-i}(0) [0^+, z^+] | P \rangle$$

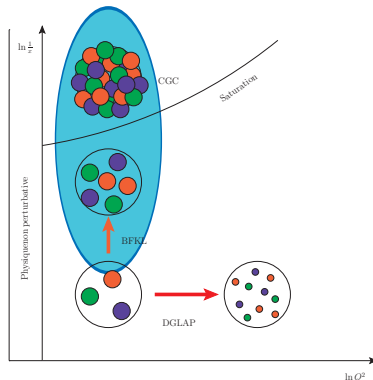
Transverse Momentum Dependent distribution

$$\mathcal{F}(x, k_\perp) \propto \int d^4 z \delta(z^-) e^{ixP^- z^+ + i(k_\perp \cdot z_\perp)} \langle P | F^{-i}(z) \mathcal{U}_{z,0} F^{-i}(0) \mathcal{U}_{0,z} | P \rangle$$



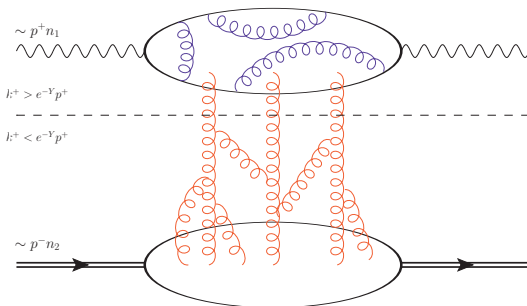
QCD at small $x_{Bj} \sim Q^2/s$

Regge limit: $Q^2 \ll s$



Rapidity separation

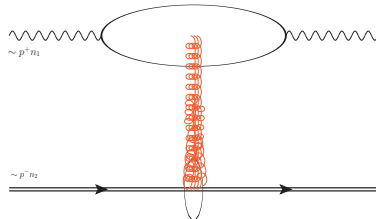
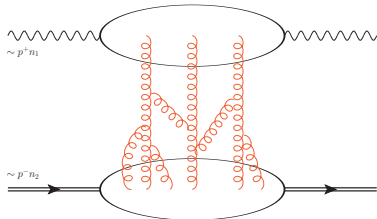
[McLerran, Venugopalan, 1994] [Balitsky, 1995]



Let us split the gluonic field between "fast" and "slow" gluons

$$\begin{aligned}
 A^{\mu a}(k^+, k^-, k) &= a^{\mu a}(|k^+| > \rho^+, k^-, k) \\
 &+ A_{\text{cl}}^{\mu a}(|k^+| < \rho^+, k^-, k)
 \end{aligned}$$

Large longitudinal boost to the projectile frame



$$A_{\text{cl}}^+(x^+, x^-, \mathbf{x})$$

$$A_{\text{cl}}^-(x^+, x^-, \mathbf{x})$$

$$A_{\text{cl}}^i(x^+, x^-, \mathbf{x})$$



$$\Lambda \sim \sqrt{\frac{s}{m_t^2}}$$

$$\frac{1}{\Lambda} A_{\text{cl}}^+(\Lambda x^+, \frac{x^-}{\Lambda}, \mathbf{x})$$

$$\Lambda A_{\text{cl}}^-(\Lambda x^+, \frac{x^-}{\Lambda}, \mathbf{x})$$

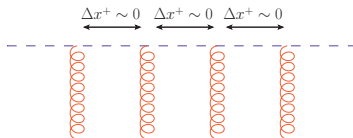
$$A_{\text{cl}}^i(\Lambda x^+, \frac{x^-}{\Lambda}, \mathbf{x})$$

$$A_{\text{cl}}^\mu(x) \rightarrow A_{\text{cl}}^-(x) n_2^\mu = \delta(x^+) \mathbf{A}(x) n_2^\mu + O(\sqrt{\frac{m_t^2}{s}})$$

Shock wave approximation

Effective Feynman rules in the slow background field

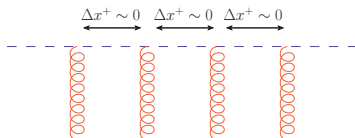
Effective fermion propagator in the external classical field



- $A_{\text{cl}}^i = 0$, $A_{\text{cl}}^+ = 0$: the Dirac structure **factorizes**
- A_{cl} does not depend on x^- : **conservation** of $+$ momentum
- A_{cl} is **peaked** around $x^+ = 0$:
 - Most external propagators get **factorized out**
 - Gaussians $\sim \delta$ functions: **conservation of transverse position**
 - Possibility to **extend Wilson lines** to infinity $[x^+, y^+]_x = [\infty^+, -\infty^+]_x \equiv U_x$

Effective Feynman rules in the slow background field

The interactions with the background field can be **exponentiated**

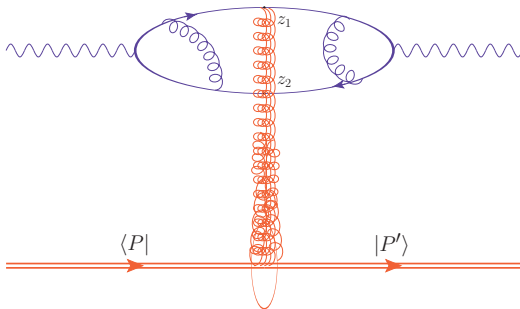


$$D_F(x_2, x_0)|_{x_2^+ > 0, x_0^+ < 0} = \int d^D x_1 \delta(x_1^+) D_0(x_2, x_1) \gamma^+ V_{\mathbf{x}_1} D_0(x_1, x_0)$$

Each fast parton is dressed by an infinite Wilson line

$$V_x \equiv \mathcal{P} \exp \left[ig \int_{-\infty}^{+\infty} dx^+ A_{\text{cl}}^-(x) \right]$$

Factorized picture



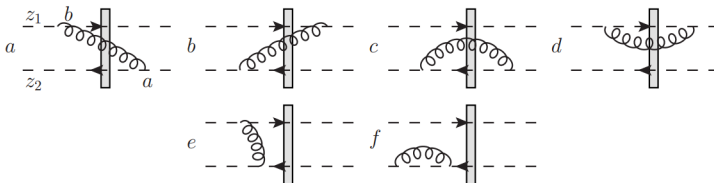
Factorized amplitude

$$\mathcal{S} = \int dx_1 dx_2 \Phi^{\rho^+}(x_1, x_2) \langle P' | \frac{1}{N_c} \text{tr}(V_{x_1}^{\rho^+} V_{x_2}^{\rho^+ \dagger}) - 1 | P \rangle$$

Written similarly for any number of Wilson lines in **any color representation!**

Y independence: **B-JIMWLK**, **BK** equations. Resums **logarithms of s**

Evolution for the dipole operator



B-JIMWLK hierarchy of equations

[Balitsky, Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner]

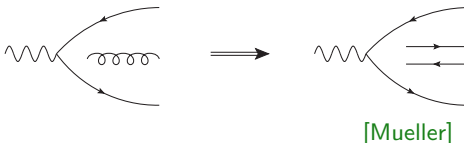
$$\frac{\partial S_{12}^{\rho+}}{\partial \ln \rho^+} = \frac{\alpha_s N_c}{2\pi^2} \int d\vec{z}_3 \frac{\vec{z}_{12}^2}{\vec{z}_{13}^2 \vec{z}_{23}^2} \left[S_{13}^{\rho+} + S_{32}^{\rho+} - S_{12}^{\rho+} + S_{13}^{\rho+} S_{32}^{\rho+} \right]$$

$$\frac{\partial S_{13}^{\rho+} S_{32}^{\rho+}}{\partial \ln \rho^+} = \dots$$

Evolves a **dipole** into a **double dipole**

The BK equation

Mean field approximation, or 't Hooft planar limit $N_c \rightarrow \infty$ in the dipole B-JIMWLK equation



⇒ BK equation [Balitsky, 1995] [Kovchegov, 1999]

$$\frac{\partial \langle S_{12}^{\rho+} \rangle}{\partial \ln \rho^+} = \frac{\alpha_s N_c}{2\pi^2} \int d\vec{z}_3 \frac{\vec{z}_{12}^2}{\vec{z}_{13}^2 \vec{z}_{23}^2} \left[\langle S_{13}^{\rho+} \rangle + \langle S_{32}^{\rho+} \rangle - \langle S_{12}^{\rho+} \rangle + \langle S_{13}^{\rho+} \rangle \langle S_{32}^{\rho+} \rangle \right]$$

BFKL/BKP part

Triple pomeron vertex

Non-linear term : one type of saturation

High energy OPE

$$\sigma = W_0^{\rho^+} \otimes \mathcal{O}_0^{\rho^+} + \alpha_s W_1^{\rho^+} \otimes \mathcal{O}_1^{\rho^+} + \alpha_s^2 W_2^{\rho^+} \otimes \mathcal{O}_2^{\rho^+} + \dots$$

Operators $\mathcal{O}_n^{\rho^+}$ are built from infinite lightlike Wilson lines.

$$\frac{\partial \mathcal{O}_n^{\rho^+}}{\partial \ln \rho^+} = \alpha_s K_1 \otimes \mathcal{O}_{n+1}^{\rho^+} + \alpha_s^2 K_2 \otimes \mathcal{O}_{n+2}^{\rho^+} \dots$$

Rapidity regulator ρ^+ : gluons with $k^+ > \rho^+$ belong to the **Wilson coefficients**, gluons with $k^+ < \rho^+$ constitute the **operators**

The seemingly incompatible nature of the distributions

Two different kinds of gluon distributions

Moderate x distributions

Low x distributions

TMD, PDF...

Dipole scattering amplitude

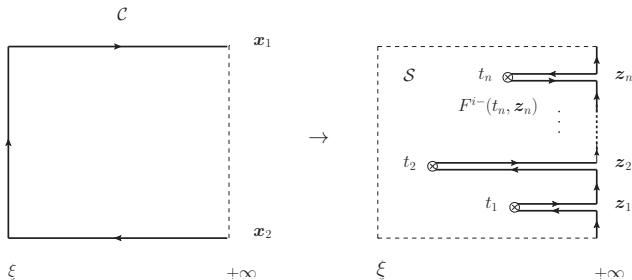
$$\langle P | F^{-i} W F^{-j} W | P \rangle$$

$$\langle P | \text{tr}(V_1 V_2^\dagger) | P \rangle$$

The Wilson line $\leftrightarrow$ parton distribution equivalence

Most general equivalence: use the **Non-Abelian Stokes theorem**

[RB, Mehtar-Tani]



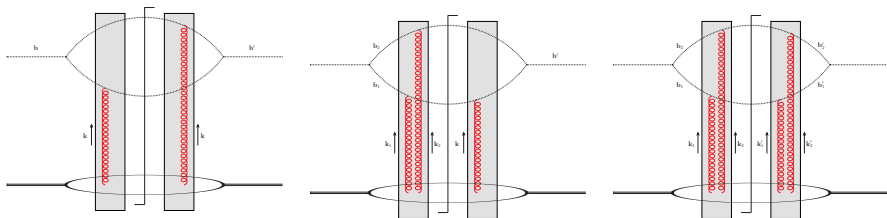
$$\mathcal{P} \exp \left[\oint_C dx_\mu A^\mu(x) \right] = \mathcal{P} \exp \left[\int_S d\sigma_{\mu\nu} WF^{\mu\nu} W^\dagger \right]$$

$$V_{x_{1\perp}} V_{x_{2\perp}}^\dagger = [\hat{x}_{1\perp}, \hat{x}_{2\perp}]$$

Inclusive low x cross sectionNon-exclusive low x cross section = TMD cross section

[Altinoluk, RB, Kotko], [Altinoluk, RB]

Generalizes [Dominguez, Marquet, Xiao, Yuan]

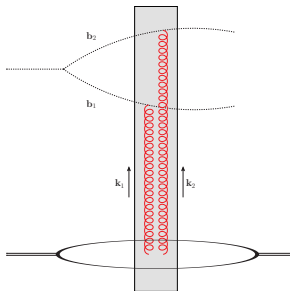


$$\begin{aligned}
 \sigma &= \mathcal{H}_2^{ij}(k) \otimes f_2^{ij}(x=0, k) \\
 &+ \mathcal{H}_3^{ijk}(k, k_1) \otimes f_3^{ijk}(x=0, x_1=0, k, k_1) \\
 &+ \mathcal{H}_4^{ijkl}(k, k_1, k'_1) \otimes f_4^{ijkl}(x=0, x_1=0, x'_1=0, k, k_1, k'_1)
 \end{aligned}$$

All distributions are evaluated in the **strict $x = 0$ limit**

Exclusive low x cross section

Exclusive low x amplitude = GTMD amplitude
 [Altinoluk, RB]

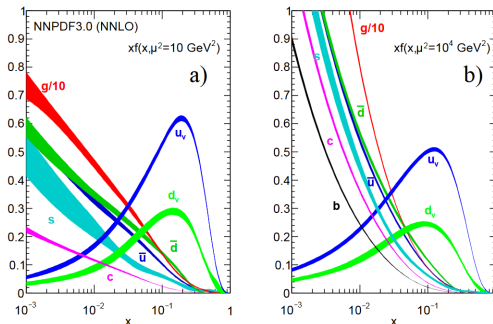


$$\mathcal{H}^{ij}(k_1, k_2) \otimes f^{ij}(x=0, \xi=0; \mathbf{k}, \Delta)$$

Every exclusive low x process probes
 a **Wigner distribution**!

All distributions are evaluated in the **strict $x=0$ limit**

All distributions are evaluated in the **strict $x = 0$ limit**



[NNLO NNPDF3.0 global analysis, taken from PDG2018]

Instabilities in the collinear corner of the phase space

All distributions are evaluated in the **strict $x = 0$ limit**

Hard part $\mathcal{H}$ and gluon distribution f for an inclusive observable:

Bjorken limit

$$s \sim Q^2$$

$$\int d\mathbf{x} f(\mathbf{x}) \mathcal{H}(\mathbf{x})$$

Leading twist of small x

$$s \gg Q^2, Q^2 \rightarrow \infty$$

$$f(0) \int d\mathbf{x} \mathcal{H}(\mathbf{x})$$

Too late to restore a dependence on x via evolution: x is already integrated over

The double log limit

Collinear logs are a known problem for small x_{Bj} physics

Proposed *ad hoc* solutions to the symptoms:

- Imposed kinematic orderings on — momenta or light cone times
[Beuf], extends [Ciafaloni, Colferai, Salam, Stasto...]
- Resummation of logarithms
[Iancu, Madrigal, Mueller, Soyez, Triantafyllopoulos], [Liu, Kang, Liu]
[Caucal, Salazar, Schenke, Venugopalan], [Taels, Altinoluk, Beuf, Marquet]
- Non-local factorization
[Iancu, Mueller, Triantafyllopoulos]
- Better choice of evolution variable
[Ducloué, Iancu, Mueller, Soyez, Triantafyllopoulos], [Altinoluk, Beuf, Penttala]

Mimic a dependence on x , and are not compatible with k^+ -space factorization hence with having Wilson line operators in the first place

First suggestion: take the small x limit properly

All distributions are evaluated in the **strict $x = 0$ limit**

Hard part $\mathcal{H}$ and gluon distribution f for an inclusive observable:

Bjorken limit

$$s \sim Q^2$$

$$\int d\mathbf{x} f(\mathbf{x}) \mathcal{H}(\mathbf{x})$$

Leading twist of small x

$$s \gg Q^2, Q^2 \rightarrow \infty$$

$$f(0) \int d\mathbf{x} \mathcal{H}(\mathbf{x})$$

Too late to restore a dependence on x via evolution: x is already integrated over

Summary so far

Distributions involved in pQCD observables

Overarching scheme?

$$f(x_1 \dots x_n; k_{\perp 1} \dots k_{\perp n})$$

Bjorken limit

$$s \sim Q^2$$

$$f(x; 0_{\perp}) + O(Q^{-2})$$

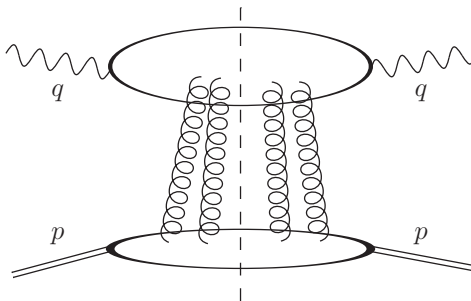
Regge limit

$$s \gg Q^2$$

$$f(0 \dots 0, k_{\perp 1} \dots k_{\perp n}) + O(x_{Bj})$$

Look for an interpolating scheme for simple observables

An interpolating scheme for collinearly factorizable observables: DIS, DVCS, TCS, DDVCS



[RB, Mehtar-Tani]

Final result

Interpolating formula for DIS from a modified effective theory

$$\begin{aligned}
 \mathcal{A} = & -i\pi g^2 \sum_f q_f^2 \int_0^1 \frac{dz}{2\pi} \int \frac{d^d \ell}{(2\pi)^d} \int d^d \mathbf{k} \\
 & \times (\partial^i \Phi)(z, \ell - \mathbf{k}/2) (\partial^j \Phi^*)(z, \ell + \mathbf{k}/2) \\
 & \times \int dx \, \mathcal{G}^{ij}(x, \mathbf{k}) \delta \left(x - x_{\text{Bj}} - \frac{\ell^2}{2z\bar{z}q^+P^-} \right)
 \end{aligned}$$

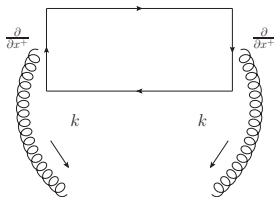
Standard wave functions Φ

x -dependent unintegrated PDF operator $\mathcal{G}^{ij}(x, \mathbf{k})$

The unintegrated PDF

uPDF as a finite Wilson loop

$$\begin{aligned}
 & \int d^2 k e^{i(k \cdot r)} r^i r^j \mathcal{G}^{ij}(x, k) \\
 &= \frac{1}{\alpha_s} \int \frac{d^4 v_1 d^4 v_2}{(2\pi)^4} \delta(v_1^-) \delta(v_2^-) e^{-ik \cdot (v_1 - v_2)} \\
 & \times \frac{\partial}{\partial v_1^+} \frac{\partial}{\partial v_2^+} \frac{\langle p | \text{tr}[v_1^+, v_2^+]_{v_1} [v_1, v_2]_{v_2^+} [v_2^+, v_1^+]_{v_2} [v_2, v_1]_{v_1^+} | p \rangle}{\langle p | p \rangle}
 \end{aligned}$$



x-dependent unintegrated PDF $\Leftrightarrow$ FT of a finite Wilson loop

Summary

Interpolating scheme for Deep Inelastic scattering

Overarching scheme

$$\int d\mathbf{x} \int d^d \mathbf{k} \mathcal{G}^{ij}(\mathbf{x}, \mathbf{k}) H^{ij}(\mathbf{x}, \mathbf{k})$$

Bjorken limit

$$\int d\mathbf{x} H^{ij}(\mathbf{x}, \mathbf{0}) \int d^d \mathbf{k} \mathcal{G}^{ij}(\mathbf{x}, \mathbf{k})$$

Regge limit

$$\int d^d \mathbf{k} \mathcal{G}^{ij}(\mathbf{0}, \mathbf{k}) \int d\mathbf{x} H^{ij}(\mathbf{x}, \mathbf{k})$$

We found an interpolating scheme

Summary

Interpolating scheme for Deep Inelastic scattering

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$$\int d\mathbf{x} H^{ij}(\mathbf{x}, \mathbf{0}) \int d^d \mathbf{k} \mathcal{G}^{ij}(\mathbf{x}, \mathbf{k})$$

Regge limit

$$\int d^d \mathbf{k} \mathcal{G}^{ij}(\mathbf{0}, \mathbf{k}) \int d\mathbf{x} H^{ij}(\mathbf{x}, \mathbf{k})$$

We found an interpolating scheme

Can we find an interpolating evolution equation?

Evolution equation

$x = 0$ limit of the interpolating evolution equation

$$\Delta S_{12} - \bar{\alpha}_s K_{BK} \otimes S_{12} \\ = -\bar{\alpha}_s \int_3 \frac{x_{12}^2}{x_{13}^2 x_{32}^2} \left(\ln \frac{|x_{13}| |x_{32}| \mu}{|x_{12}|} + \ln |x_{12}| \mu + \frac{x_{23}^2 - x_{13}^2}{x_{12}^2} \ln \frac{|x_{13}|}{|x_{32}|} \right) (S_{13} S_{32} - S_{12})$$

$$\text{where } \mu^2 = 2\rho_0^+ P^-$$

Balitsky and Chirilli's conformal dipole evolution, and an extra term

Evolution equation

$x = 0$ limit of the interpolating evolution equation

$$\Delta S_{12} - \bar{\alpha}_s K_{\text{BK}} \otimes S_{12} \\ = -\bar{\alpha}_s \int_3 \frac{x_{12}^2}{x_{13}^2 x_{32}^2} \left(\ln \frac{|x_{13}| |x_{32}| \mu}{|x_{12}|} + \ln |x_{12}| \mu + \frac{x_{23}^2 - x_{13}^2}{x_{12}^2} \ln \frac{|x_{13}|}{|x_{32}|} \right) (S_{13} S_{32} - S_{12})$$

$$\text{where } \mu^2 = 2\rho_0^+ P^-$$

Conformal dipole term: cancels double logs, but **generates instabilities**

Extra term: compensates those instabilities

Second suggestion: general solution

The double log limit

Collinear log issue as formulated at small x_{Bj}

Dipole amplitude $S_{12}(\zeta = 2\rho^+ P^-) \equiv \frac{1}{N_c} \langle P | \text{tr}(V_{x_1}^{\rho^+} V_{x_2}^{\rho^+ \dagger}) | P \rangle$
evolves wrt ζ via the BK equation

$$\begin{aligned} \frac{\partial S_{12}^\zeta}{\partial \ln \zeta} = & \frac{1}{2} \int_3 K_{123} (S_{13}^\zeta S_{32}^\zeta - S_{12}^\zeta) + \frac{n_f}{2N_c} \int_{3,4} K_f S_{32}^\zeta (S_{14}^\zeta - S_{13}^\zeta) \\ & + \frac{1}{2} \int_{3,4} K_{1234} (S_{13}^\zeta S_{34}^\zeta S_{42}^\zeta - S_{13}^\zeta S_{32}^\zeta) \end{aligned}$$

Double log in the kernel at two loops

$$K_{123} \supset \frac{x_{12}^2}{x_{13}^2 x_{32}^2} \ln \frac{x_{13}^2}{x_{12}^2} \ln \frac{x_{32}^2}{x_{12}^2}$$

Wrong resummation variable

$$\ln \zeta \rightarrow \ln s = \ln \frac{1}{x_{Bj}} + \ln Q^2$$

Generalities

Doi-Peliti notations

Observables = convolution of an **impact factor** H (bra) and **target operators** S (ket)

Basis of bras and kets as a dipole-chain Fock space

$|0\rangle, |\mathbf{x}_1\mathbf{x}_2\rangle, |\mathbf{x}_1\mathbf{x}_3\mathbf{x}_2\rangle, \dots$

$$(\mathbf{x}_1\mathbf{x}_2|S) = S_{12}$$

$$(\mathbf{x}_1\mathbf{x}_3\mathbf{x}_2|S) = S_{13}S_{32}$$

$$(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\dots\mathbf{x}_n|S) = S_{12}S_{23}\dots S_{(n-1)n}$$

Generalities

DIS cross section in Doi-Peliti notations

$$\begin{aligned}
 \sigma_{\text{DIS}} &= (H^\zeta | S^\zeta) = (H^\zeta \sum_n \int_{1\dots n} | \mathbf{x}_1 \dots \mathbf{x}_n \rangle) (\mathbf{x}_1 \dots \mathbf{x}_n | S^\zeta) \\
 &= (H^\zeta | \mathbf{x}_1 \mathbf{x}_2 \rangle) (\mathbf{x}_1 \mathbf{x}_2 | S^\zeta) + (H^\zeta | \mathbf{x}_1 \mathbf{x}_3 \mathbf{x}_2 \rangle) (\mathbf{x}_1 \mathbf{x}_3 \mathbf{x}_2 | S^\zeta) + \dots \\
 &= \int_{1,2} H_{\text{dipole}}^\zeta S_{12}^\zeta + \int_{1,2,3} H_{\text{double-dipole}}^\zeta S_{13}^\zeta S_{32}^\zeta + \dots
 \end{aligned}$$

Generalities

BK equation in Doi-Peliti notations

$$\frac{\partial |S^\zeta\rangle}{\partial \ln \zeta} = K |S^\zeta\rangle$$

with

$$\begin{aligned} (\mathbf{x}_1 \mathbf{x}_2 | K_{\text{NLL}} = & \frac{1}{2} \int_3 K_{123} [(\mathbf{x}_1 \mathbf{x}_3 \mathbf{x}_2 | - (\mathbf{x}_1 \mathbf{x}_2 |] \\ & + \frac{1}{2} \int_{3,4} K_{1234} [(\mathbf{x}_1 \mathbf{x}_3 \mathbf{x}_4 \mathbf{x}_2 | - (\mathbf{x}_1 \mathbf{x}_3 \mathbf{x}_2 |] \\ & + \frac{n_f}{2N_c} \int_{3,4} K_f(\mathbf{x}_3 \mathbf{x}_2 | \otimes [(\mathbf{x}_1 \mathbf{x}_4 | - (\mathbf{x}_1 \mathbf{x}_3 |] \end{aligned}$$

Change of variables

How to build a stable solution

- Introduce a **collinear scale** μ^2 and an **operator** $L(\mu^2)$
- Change variables

$$\bar{S}(\mu^2, \zeta) = e^{-L(\mu^2)} S^\zeta$$

leaving the observable invariant

$$\sigma = (H^\zeta | S^\zeta) = (H^\zeta | e^{L(\mu^2)} e^{-L(\mu^2)} | S^\zeta) = (\bar{H}(\zeta, \mu^2) | \bar{S}(\zeta, \mu^2))$$

- $\bar{S}$ evolves as

$$\frac{\partial \bar{S}(\zeta, \mu^2)}{\partial \ln \zeta} = \bar{K}(\mu^2) \bar{S}(\zeta, \mu^2), \quad \bar{K}(\mu^2) = e^{-L(\mu^2)} K e^{L(\mu^2)}$$

Change of variables

How to build a stable solution

Impose, by construction of $L(\mu^2)$, that

$$\frac{\partial \bar{S}(\zeta, \mu^2)}{\partial \ln \zeta} + \frac{\partial \bar{S}(\zeta, \mu^2)}{\partial \ln \mu^2} = 0 \quad (1)$$

- $\bar{S}$ can be built to be a function of $\ln(\zeta/\mu^2)$
- Then for $(\zeta, \mu^2) = (s, Q^2)$ its evolution **resums $\ln(1/x_{Bj})$**
- Condition (1) implies

$$\frac{\partial \bar{K}(\mu^2)}{\partial \ln \mu^2} = 0$$

Change of variables

Stable solutions

$$\frac{\partial \bar{S}(\zeta, \mu^2)}{\partial \ln \zeta} + \frac{\partial \bar{S}(\zeta, \mu^2)}{\partial \ln \mu^2} = 0$$

For $L(\mu^2)$ polynomial in $\ln \mu^2$ of degree $\leq$ number of loops, we find at NNLL:

$$L_{LL} = L_1 + K_{LL} \ln \mu^2$$

$$L_{NLL} = L_2 + \left\{ K_{NLL} + \frac{1}{2} [K_{LL}, L_1] \right\} \ln \mu^2$$

$$\begin{aligned} L_{NNLL} = & L_3 + \left\{ K_{NNLL} + \frac{1}{2} [K_{LL}, L_2] + \frac{1}{2} [K_{NLL}, L_1] + \frac{1}{12} [[K_{LL}, L_1], L_1] \right\} \ln \mu^2 \\ & + \frac{1}{12} [K_{LL}, [K_{LL}, L_1]] \ln^2 \mu^2 \end{aligned}$$

Fixing BK by a change of variables

Modified kernel:

$$\bar{K}_{\text{NNLL}} = K_{\text{NNLL}} + \bar{\alpha}_s[K_{\text{LL}}, L_1] + \bar{\alpha}_s^2[K_{\text{NLL}}, L_1] + \bar{\alpha}_s^2[K_{\text{LL}}, L_2] + \frac{\bar{\alpha}_s^2}{2}[[K_{\text{LL}}, L_1], L_1]$$

Educated choice for L_1, L_2 : convolution of the BK kernel with transverse logs of daughter dipole sizes

$$(x_1 x_2 | L_1 = \int_3 K_{123} \ln(|x_{13}| |x_{32}|) [(x_1 x_3 x_2 | - (x_1 x_2 |]$$

so

$$(x_1 x_2 | L_{\text{LL}} = \int_3 K_{123} \ln(\mu^2 |x_{13}| |x_{32}|) [(x_1 x_3 x_2 | - (x_1 x_2 |]$$

which means

$$\bar{S}_{12} = S_{12} - \bar{\alpha}_s \int_3 \frac{x_{12}^2}{x_{13}^2 x_{32}^2} \ln(\mu^2 |x_{13}| |x_{32}|) (S_{13} S_{32} - S_{12})$$

Result

Final result in our scheme choice for L_1 at NLL

$$\bar{S}_{12} = S_{12} - \bar{\alpha}_s \int_3 \frac{x_{12}^2}{x_{13}^2 x_{32}^2} \ln(\mu^2 |x_{13}| |x_{32}|) (S_{13} S_{32} - S_{12})$$

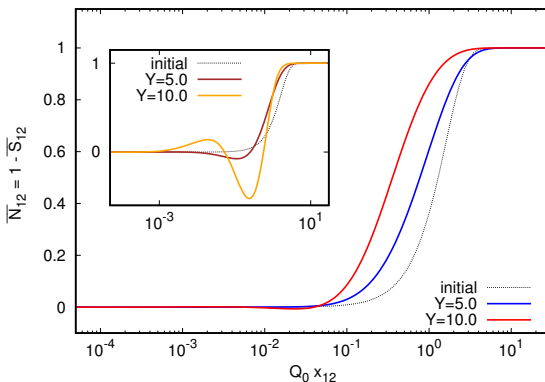
evolves as

$$\begin{aligned} \frac{\partial \bar{S}_{12}}{\partial \ln(\zeta/\mu^2)} &= \bar{\alpha}_s (x_1 x_2 |K| \bar{S}) \\ &+ \frac{\bar{\alpha}_s^2}{2} \int_3 \frac{x_{12}^2}{x_{13}^2 x_{32}^2} \ln \frac{x_{13}^2}{x_{12}^2} \ln \frac{x_{32}^2}{x_{12}^2} (\bar{S}_{13} \bar{S}_{32} - \bar{S}_{12}) \\ &+ \frac{\bar{\alpha}_s^2}{2} \int_{3,4} \frac{x_{12}^2}{x_{13}^2 x_{34}^2 x_{42}^2} \ln \left(\frac{x_{34}^2}{x_{14}^2} \right) \bar{S}_{13} (\bar{S}_{34} \bar{S}_{42} - \bar{S}_{32}) \\ &+ \frac{\bar{\alpha}_s^2}{2} \int_{3,4} \frac{x_{12}^2}{x_{14}^2 x_{43}^2 x_{32}^2} \ln \left(\frac{x_{43}^2}{x_{42}^2} \right) (\bar{S}_{14} \bar{S}_{43} - \bar{S}_{13}) \bar{S}_{32} \end{aligned}$$

Evolves with the **right variable**, and **compensates double logs** in K

Result

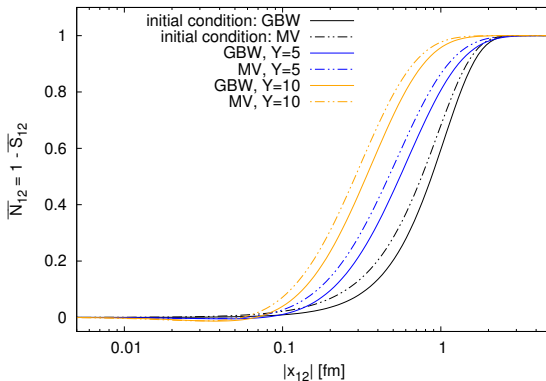
Rotated dipole at NLL



Fixed coupling result, compared to the unrotated dipole evolution

Result

Rotated dipole at NLL



Running coupling result, BLM scale fixing

Comparison to other schemes

Our scheme choice:

$$(\mathbf{x}_1 \mathbf{x}_2 | L_{LL} = \int_3 K_{123} \ln(\mu^2 |\mathbf{x}_{13}| |\mathbf{x}_{32}|) [(\mathbf{x}_1 \mathbf{x}_3 \mathbf{x}_2 | - (\mathbf{x}_1 \mathbf{x}_2 |]$$

Evolution in k^- without kinematical constraint

[Ducloué, Iancu, Mueller, Soyez, Triantafyllopoulos]

$$(\mathbf{x}_1 \mathbf{x}_2 | L_{LL} = \int_3 K_{123} \ln(\mu^2 \mathbf{x}_{12}^2) [(\mathbf{x}_1 \mathbf{x}_3 \mathbf{x}_2 | - (\mathbf{x}_1 \mathbf{x}_2 |]$$

Evolution in k^- from an impact factor regulator

[Altinoluk, Beuf, Penttala]

$$\begin{aligned} (\mathbf{x}_1 \mathbf{x}_2 | L_{LL} = \int_3 K_{123} \ln(\mu^2 |\mathbf{x}_{13}| |\mathbf{x}_{32}| e^{(\mathbf{x}_{23}^2 - \mathbf{x}_{13}^2) \ln(\mathbf{x}_{13}^2 / \mathbf{x}_{23}^2) / \mathbf{x}_{12}^2}) \\ \times [(\mathbf{x}_1 \mathbf{x}_3 \mathbf{x}_2 | - (\mathbf{x}_1 \mathbf{x}_2 |] \end{aligned}$$

We also get the **counterterms** to the other part of the OPE

The original Balitsky-Chirilli scheme

Construction of our scheme: Impose, by construction of $L(\mu^2)$, that

$$\frac{\partial \bar{S}(\zeta, \mu^2)}{\partial \ln \zeta} + \frac{\partial \bar{S}(\zeta, \mu^2)}{\partial \ln \mu^2} = 0$$

Function of $\ln(\zeta/\mu^2) \sim \ln(x_{Bj})$: forward processes

Balitsky-Chirilli scheme: Impose, by construction of $L(\mu^2)$, that

$$2 \frac{\partial \bar{S}(\zeta, \mu^2)}{\partial \ln \zeta} + \frac{\partial \bar{S}(\zeta, \mu^2)}{\partial \ln \mu^2} = 0$$

Function of $\ln(\zeta^2/\mu^2) \sim \ln(Y_{\text{projectile}}/Y_{\text{target}})$: central processes

Fixing BK by a change of variables

Summary: change of variables

We can define a composite dipole $\bar{S}$ which:

- Evolves with $\zeta/\mu^2 \rightarrow 1/x_{Bj}$ instead of projectile momentum $\sqrt{s}$
- Evolves with an evolution equation **free from collinear double logarithms**
- Is still compatible with the OPE and with impact factors computed in the standard **regularization in k_+**

Correction to standard impact factors, for cross section independence on the choice of L :

$$\bar{H} = H e^{\bar{\alpha}_s L(\mu^2)}$$

Conclusion

Summary of our findings

Bad news

- Semi-classical small x physics has, **at its core**, issues with **collinear logarithms**
- The problem can be traced down **to the very starting point**

Good news

- We now have a **minimal correction** of semi-classical small x which solves the problem **from first principles**
- Known impact factors are **compatible with our scheme**

Necessary gluon fields in the Regge limit:

$$A^\mu(x) = A^-(x^+, 0^-, \mathbf{x}) n_2^\mu$$

Necessary gluon fields in the Bjorken limit?

$$A^\mu(x) = A^-(x^+, x^-, \mathbf{x}) n_2^\mu + A^+_\perp(x^+, x^-, \mathbf{x})$$

Necessary gluon fields in the Regge limit:

$$A^\mu(x) = A^-(x^+, 0^-, \mathbf{x}) n_2^\mu$$

Necessary gluon fields in the Bjorken limit?

$$A^\mu(x) = A^-(x^+, \textcolor{red}{x}^-, x) n_2^\mu + A^+_\perp(x^+, \textcolor{red}{x}^-, x)$$

Dependence on x^- : sub-sub-leading in twist counting

Necessary gluon fields in the **Regge limit**:

$$A^\mu(x) = A^-(x^+, 0^-, \mathbf{x}) n_2^\mu$$

Necessary gluon fields in the Bjorken limit:

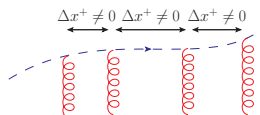
$$A^\mu(x) = A^-(x^+, 0^-, \mathbf{x}) n_2^\mu$$

Diagram illustrating the separation of scales in the collinear limit. A horizontal dashed blue line represents the collinear limit. Below it, four vertical orange chains of circles represent the collinear partons. Above the dashed line, three double-headed arrows indicate the separation of scales, each labeled $\Delta x^+ \sim 0$.

- $A_{cl}^i = 0$, $A_{cl}^+ = 0$: the Dirac structure **factorizes**
- A_{cl} does not depend on x^- : **conservation** of $+$ momentum
- A_{cl} is **peaked** around $x^+ = 0$:
 - Most external propagators get **factorized out**
 - Gaussians $\sim \delta$ functions: **conservation of transverse position**
 - Possibility to **extend Wilson lines** to infinity $[x^+, y^+]_x = [\infty^+, -\infty^+]_x \equiv U_x$

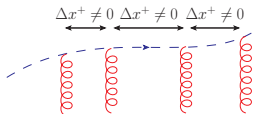
Effective Feynman rules in the slow background field

Effective fermion propagator in the external classical field



- $A_{\text{cl}}^i = 0$, $A_{\text{cl}}^+ = 0$: the Dirac structure **factorizes**
- A_{cl} does not depend on x^- : **conservation** of $+$ momentum
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Effective fermion propagator in the external classical field

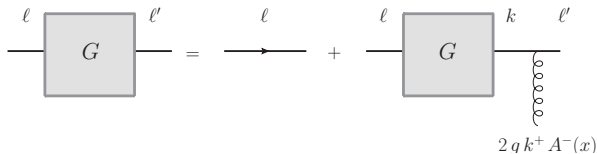


- $A_{\text{cl}}^i = 0$, $A_{\text{cl}}^+ = 0$: the Dirac structure **factorizes**
- A_{cl} does not depend on x^- : **conservation** of $+$ momentum

$$D_F(\ell', \ell) = i \frac{\gamma^+}{2\ell^+} (2\pi)^D \delta^D(\ell' - \ell) + i \frac{\ell' \gamma^+ \ell}{2\ell^+} G_{\text{scal}}(\ell', \ell)$$

Effective Feynman rules in the slow background field

Effective scalar propagator in the external classical field

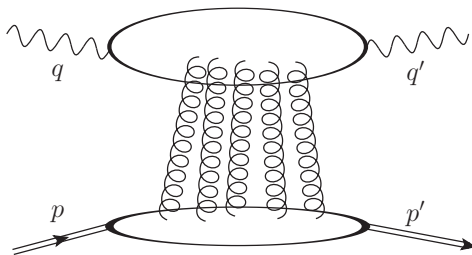


$$G_{\text{scal}}(\ell', \ell) = G_0(\ell')(2\pi)^D \delta^D(\ell' - \ell) \\ = 2g \int d^D z \int \frac{d^D k}{(2\pi)^D} e^{i(\ell' - k) \cdot z} G_0(\ell') (k \cdot A)(z) G_{\text{scal}}(k, \ell).$$

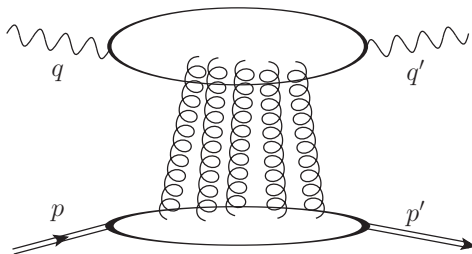
In coordinate space, it satisfies the **Klein-Gordon equation in a potential**

$$[-\square_z + 2igA(z) \cdot \partial_z] G_{\text{scal}}(z, z_0) = \delta^D(z - z_0)$$

Application to the $\gamma^{(*)}(q)P(p) \rightarrow \gamma^{(*)}(q')P(p')$
amplitude

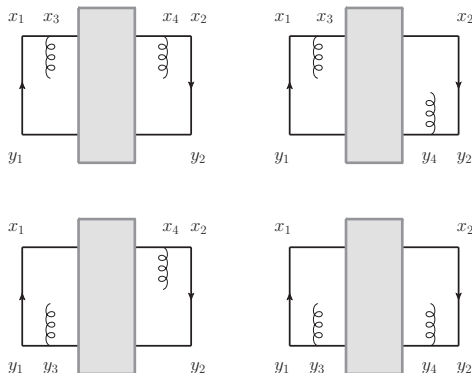


Computing the $\gamma^{(*)}(q)P(p) \rightarrow \gamma^{(*)}(q')P(p')$ amplitude



$$\mathcal{A} = \frac{e^2}{\mu^{d-2}} \varepsilon_q^\mu \varepsilon_{q'}^{\nu*} \sum_f q_f^2 \int \frac{d^D \ell}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} \\ \times \langle p' | \text{tr} [\gamma_\nu D_F(k, \ell) \gamma_\mu D_F(-q + \ell, -q' + \ell + k)] | p \rangle$$

Some operator algebra



$$\begin{aligned}
 & \text{tr} G_{\text{scal}}^R(x_2, x_1) G_{\text{scal}}^A(y_1, y_2) \\
 &= 16g^2 \int d^D x_3 \int d^D x_4 \int d^D y_3 \int d^D y_4 \delta(y_3^+ - x_3^+) \delta(x_4^+ - y_4^+) \\
 &\times (\partial_{x_3}^+ G_0^R)(x_3, x_1) (\partial_{x_4}^+ G_0^R)(x_2, x_4) (\partial_{y_3}^+ G_0^A)(y_1, y_3) (\partial_{y_4}^+ G_0^A)(y_4, y_2) \\
 &\times \text{tr} \left\{ [A^-(y_3) - A^-(x_3)] G_{\text{scal}}^A(y_3, y_4) [A^-(y_4) - A^-(x_4)] G_{\text{scal}}^R(x_4, x_3) \right\}
 \end{aligned}$$

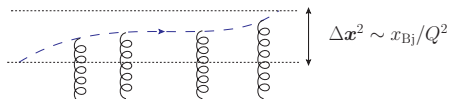
Fully general result

$$\mathcal{A} \propto \mathcal{U}^{ij}(z, \ell_1, \ell_2) \otimes_{z, \ell_1, \ell_2} (\partial^i \Phi)(z, \ell_1) (\partial^j \Phi^*)(z, \ell_2)$$

- Φ : standard wave functions
- $\mathcal{U}^{ij}$: generalization of the dipole operator

Contains unnecessary subleading powers of x_{Bj}, ξ and Q, Q'

Partial twist expansion



Typical transverse recoil of a fast parton:

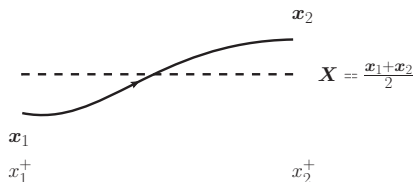
$$\Delta x^2 \sim 1/(q^+(p^- + p'^-)) \sim 1/s$$

$1/s$: eikonally suppressed in the **Regge** limit

$1/s \sim 1/Q^2$: twist suppressed in in the **Bjorken** limit.

We can **get rid of all corrections from transverse recoils** without loss of accuracy

Partial twist expansion



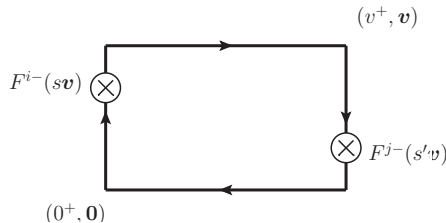
$$G_{\text{scal}}(x_1, x_2) \simeq [x_1^+, x_2^+]_{\frac{x_1 + x_2}{2}} G_{\text{scal}}^{(0)}(x_1, x_2)$$

[Altinoluk, Armesto, Beuf, Martinez, Salgado]

$$F^{i-}(x_1^+, s\mathbf{x}_1 + \bar{s}\mathbf{y}_1) \simeq F^{i-}\left(x_1^+, s\frac{\mathbf{x}_1 + \mathbf{x}_2}{2} + \bar{s}\frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right)$$

Partial twist expansion

$$\frac{\langle p' | \mathcal{U}^{ij}(z, \ell_1, \ell_2) | p \rangle}{\langle p | p \rangle} \simeq -i \frac{(2\pi)^d}{8z\bar{z}(q^+)^2} \int dx \frac{\mathcal{G}^{ij}(x, \ell_2 - \ell_1)}{x - x_{Bj} - \frac{(\frac{\ell_1 + \ell_2}{2})^2}{z\bar{z}q^+(p^- + p'^-) + i0}},$$



x-dependent unintegrated GPD

$$\mathcal{G}^{ij}(x, \xi, \mathbf{k}, \mathbf{\Delta}) \equiv \frac{2}{p^- + p'^-} \int \frac{dv^+}{2\pi} e^{ix \frac{p^- + p'^-}{2} v^+} \int \frac{d^d \mathbf{v}}{(2\pi)^d} e^{-i(\mathbf{k} \cdot \mathbf{v})} \int_0^1 ds ds' \\ \times \langle p' | \text{tr}_c \{ [v^+, 0^+]_0 F^{i-}(0^+, sv) [0^+, v^+]_v F^{j-}(v^+, s'v) \} | p \rangle$$

Final expression for the DDVCS amplitude

$$\begin{aligned} \mathcal{A} = & g^2 \sum_f q_f^2 \int_0^1 \frac{dz}{2\pi} \int \frac{d^d \ell}{(2\pi)^d} \int d^d \mathbf{k} \\ & \times (\partial^i \Phi)(z, \ell - \mathbf{k}/2) (\partial^j \Phi^*)(z, \ell + \mathbf{k}/2) \\ & \times \int dx \frac{\mathcal{G}^{ij}(x, \xi, \mathbf{k}, \Delta)}{x - x_{Bj} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0} \end{aligned}$$

Standard wave functions Φ

x -dependent unintegrated GPD operator $\mathcal{G}^{ij}(x, \xi, \mathbf{k}, \Delta)$ (includes polarized terms)

Bjorken limit and Regge limit

Recovering the Bjorken limit

The Bjorken limit is reached by **neglecting transverse momentum transfert** from the target:

$$|\ell| \sim Q \gg |k|$$

Key observation: $\mathcal{G}^{ij}$ integrates into GPDs

$$\begin{aligned} & \int d^d k (\partial^i \phi)(z, \ell - k/2) (\partial^j \phi^*)(z, \ell + k/2) \mathcal{G}^{ij}(x, k) \\ & \simeq (\partial^i \phi)(z, \ell) (\partial^j \phi^*)(z, \ell) \int d^d k \mathcal{G}^{ij}(x, k) \\ & \simeq (\partial^i \phi)(z, \ell) (\partial^j \phi^*)(z, \ell) G^{ij}(x) \end{aligned}$$

We fully recover the well-known one-loop exclusive Compton scattering amplitudes

Unpolarized contribution

$$2\alpha_{\text{em}}\alpha_s \sum_f q_f^2 \int dx \frac{(\epsilon_q \cdot \epsilon_{q'}^*) G(x, \xi, \mathbf{\Delta})}{(x + \xi - i0x_{\text{Bj}})^2 (x - \xi + i0x_{\text{Bj}})^2} \\ \times \frac{1}{2\xi} \left\{ \left[(x_{\text{Bj}} + \xi)(x^2 - \xi^2 + 4\xi x_{\text{Bj}} + 4\xi x) \ln \left(\frac{x_{\text{Bj}} + x - i0}{x_{\text{Bj}} + \xi - i0} \right) \right. \right. \\ - \frac{x_{\text{Bj}} + \xi}{2} (x^2 - \xi^2 + 2\xi x_{\text{Bj}} + 2\xi x) \left[\ln^2 \left(\frac{x_{\text{Bj}} + x - i0}{\xi} \right) - \ln^2 \left(\frac{x_{\text{Bj}} + \xi - i0}{\xi} \right) \right] \\ + \frac{x_{\text{Bj}} - \xi}{2} (x^2 - \xi^2 - 2\xi x_{\text{Bj}} - 2\xi x) \left[\ln^2 \left(\frac{x_{\text{Bj}} + x - i0}{\xi} \right) - \ln^2 \left(\frac{x_{\text{Bj}} - \xi - i0}{\xi} \right) \right] \\ \left. \left. - (x_{\text{Bj}} - \xi)(x^2 - \xi^2 - 4\xi x_{\text{Bj}} + 4\xi) \ln \left(\frac{x_{\text{Bj}} + x - i0}{x_{\text{Bj}} - \xi - i0} \right) \right] + (x \rightarrow -x) \right\}$$

Polarized contribution

$$\begin{aligned}
& 2\alpha_{\text{em}}\alpha_s \sum_f q_f^2 \int d\mathbf{x} \frac{\epsilon^{mn} \mathbf{e}_h^m \mathbf{e}_{h'}^{n*} \tilde{G}(\mathbf{x}, \xi, \mathbf{\Delta})}{(\mathbf{x} + \xi - i0\mathbf{x}_{\text{Bj}})^2 (\mathbf{x} - \xi + i0\mathbf{x}_{\text{Bj}})^2} \\
& \times \left\{ \left[2(\mathbf{x}^2 + \xi^2) \ln \left(\frac{\mathbf{x}_{\text{Bj}} + \mathbf{x} - i0}{\xi} \right) \right. \right. \\
& + 3\mathbf{x}(\mathbf{x}_{\text{Bj}} + \xi) \ln \left(\frac{\mathbf{x}_{\text{Bj}} + \mathbf{x} - i0}{\mathbf{x}_{\text{Bj}} + \xi - i0} \right) + 3\mathbf{x}(\mathbf{x}_{\text{Bj}} - \xi) \ln \left(\frac{\mathbf{x}_{\text{Bj}} + \mathbf{x} - i0}{\mathbf{x}_{\text{Bj}} - \xi - i0} \right) \\
& - \frac{1}{2}\mathbf{x}(\mathbf{x}_{\text{Bj}} + \xi) \left[\ln^2 \left(\frac{\mathbf{x}_{\text{Bj}} + \mathbf{x} - i0}{\xi} \right) - \ln^2 \left(\frac{\mathbf{x}_{\text{Bj}} + \xi - i0}{\xi} \right) \right] \\
& - \frac{1}{2}\mathbf{x}(\mathbf{x}_{\text{Bj}} - \xi) \left[\ln^2 \left(\frac{\mathbf{x}_{\text{Bj}} + \mathbf{x} - i0}{\xi} \right) - \ln^2 \left(\frac{\mathbf{x}_{\text{Bj}} - \xi - i0}{\xi} \right) \right] \\
& \left. \left. - \frac{1}{2}(\mathbf{x}^2 + \xi^2) \ln^2 \left(\frac{\mathbf{x}_{\text{Bj}} + \mathbf{x} - i0}{\xi} \right) \right] - (\mathbf{x} \rightarrow -\mathbf{x}) \right\}
\end{aligned}$$

Transversity contribution

$$2\alpha_{\text{em}}\alpha_s \sum_f q_f^2 \tau^{mn,ij} \mathbf{e}_h^m \mathbf{e}_{h'}^{n*} \int dx \frac{G_T^{ij}(x, \xi, \mathbf{\Delta})}{(x - \xi + i0x_{\text{Bj}})^2 (x + \xi - i0x_{\text{Bj}})^2} \\ \times \left[(x^2 - \xi^2) + (x_{\text{Bj}}^2 - \xi^2) \ln \frac{(x_{\text{Bj}} - x - i0)(x_{\text{Bj}} + x - i0)}{(x_{\text{Bj}} - \xi - i0)(x_{\text{Bj}} + \xi - i0)} \right]$$

Recovering the Regge limit? What is x ?

Naive argument

- In the Regge limit, the amplitude is dominated by its **imaginary part**
- Leading order amplitude:

$$\text{Im}\mathcal{A}_{LO} \propto \text{Im} \int dx H^q(x, \xi, t) \frac{1}{x - x_{Bj} + i\epsilon} = -\pi H^q(x_{Bj}, \xi, t)$$

- Hence **take** $x = x_{Bj}$

Problems

- At NLO, the **x cut** is way more complicated
- For DDVCS and for TCS, **s -channel cuts** also contribute to the imaginary part

Recovering the Regge limit

The Regge limit is reached by neglecting x_{Bj} and setting $\frac{\ell^2}{z\bar{z}} \ll q \cdot P$, then taking the x cut:

$$\frac{1}{x - x_{\text{Bj}} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0} \rightarrow \frac{1}{x + i0} \rightarrow -i\pi\delta(x),$$

then taking $x_{Bj}, \xi \ll 1$.

Key observation:

$$\begin{aligned} & \int \frac{d^d \ell_1}{(2\pi)^d} \int \frac{d^d \ell_2}{(2\pi)^d} e^{-i(\ell_1 \cdot \mathbf{r}_1) + i(\ell_2 \cdot \mathbf{r}_2)} \mathbf{r}_1^i \mathbf{r}_2^j [x G^{ij}(x, \ell_2 - \ell_1)]_{x=0} \\ &= \frac{N_c}{2\pi^2 \alpha_s} \delta^d(\mathbf{r}_1 - \mathbf{r}_2) \int \frac{d^d \mathbf{v}_2}{(2\pi)^d} \text{Re} \frac{\langle P | 1 - \frac{1}{N_c} \text{tr}_c (U_{\mathbf{v}_2 + \mathbf{r}_1} U_{\mathbf{v}_2}^\dagger) | P \rangle}{\langle P | P \rangle} \end{aligned}$$

Recovering the Regge limit

$$\begin{aligned}
 & (\partial^i \Phi)(z, \ell - \frac{k}{2})(\partial^j \Phi^*)(z, \ell + \frac{k}{2}) \otimes_{\ell, k} x G^{ij}(x, k) \delta(x) \\
 & \rightarrow \Psi(z, r_1) \Psi^*(z, r_2) \otimes_{r_1, r_2} r_1^i r_2^j [x G^{ij}(x, k)]_{x=0} \\
 & \rightarrow \Psi(z, r_1) \Psi^*(z, r_2) \otimes_{r_1, r_2} \delta^d(r_1 - r_2) U U \\
 & \rightarrow |\Psi(z, r)|^2 \otimes_r D(r)
 \end{aligned}$$

We fully recover the small- x description of exclusive Compton scattering e.g. [Hatta, Xiao, Yuan].

Rq: $x = 0$ is the reason why wave functions involve the same dipole size in the wave functions

Non-commutativity of the limits

Do the two limits commute?

Leading twist limit of the Regge limit

$$\lim_{Q^2+Q'^2\rightarrow\infty}\mathcal{A}_{\text{Regge}}=g^2\sum_fq_f^2\int_0^1\frac{\mathrm{d}z}{2\pi}\int\frac{\mathrm{d}^d\ell}{(2\pi)^d}$$

$$\times(-i\pi)\textcolor{red}{G}^{ij}(0,\xi,\textcolor{red}{t})(\partial^i\Phi)(z,\ell)(\partial^j\Phi^*)(z,\ell)$$

Eikonal limit of the Bjorken limit

$$\lim_{x_{\text{Bj}}, \xi \rightarrow 0} \mathcal{A}_{\text{Bjorken}} = g^2 \sum_f q_f^2 \int_0^1 \frac{dz}{2\pi} \int \frac{d^d \ell}{(2\pi)^d} \\ \times \lim_{x_{\text{Bj}}, \xi \rightarrow 0} \int dx \frac{G^{ij}(x, \xi, t)(\partial^i \Phi)(z, \ell)(\partial^j \Phi^*)(z, \ell)}{x - x_{\text{Bj}} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0}$$

Do the two limits commute?

If $G^{ij}(x, \xi, t)$ is a constant at $x = 0$:

$$\begin{aligned} & \int d\mathbf{x} \frac{G^{ij}(\mathbf{x}, \xi, t)(\partial^i \Phi)(z, \ell)(\partial^j \Phi^*)(z, \ell)}{x - x_{\text{Bj}} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0} \\ & \simeq G^{ij}(0, \xi, t) \int d\mathbf{x} \frac{(\partial^i \Phi)(z, \ell)(\partial^j \Phi^*)(z, \ell)}{x - x_{\text{Bj}} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0} \\ & = G^{ij}(0, \xi, t)(\partial^i \Phi)(z, \ell)(\partial^j \Phi^*)(z, \ell) \\ & \times \ln \left(\frac{1 - x_{\text{Bj}} - \frac{\ell^2}{z\bar{z} \frac{Q^2 + Q'^2}{2}} \xi + i0}{-1 - x_{\text{Bj}} - \frac{\ell^2}{z\bar{z} \frac{Q^2 + Q'^2}{2}} \xi + i0} \right) \end{aligned}$$

and thus

$$\lim_{x_{\text{Bj}}, \xi \rightarrow 0} \int dx \frac{G^{ij}(x, \xi, t) (\partial^i \Phi)(z, \ell) (\partial^j \Phi^*)(z, \ell)}{x - x_{\text{Bj}} - \frac{\ell^2}{2z\bar{z}q^+p^-} + i0}$$

Do the two limits commute?

Leading twist limit of the Regge limit

$$\lim_{Q^2+Q'^2 \rightarrow \infty} \mathcal{A}_{\text{Regge}} = g^2 \sum_f q_f^2 \int_0^1 \frac{dz}{2\pi} \int \frac{d^d \ell}{(2\pi)^d} \\ \times (-i\pi) \textcolor{red}{G}^{ij}(0, \xi, t) (\partial^i \Phi)(z, \ell) (\partial^j \Phi^*)(z, \ell)$$

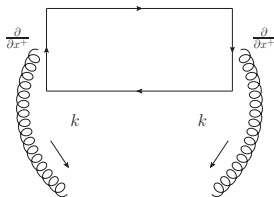
Eikonal limit of the Bjorken limit **provided the GPDs are constant at $x = 0$**

$$\lim_{x_{\text{Bj}}, \xi \rightarrow 0} \mathcal{A}_{\text{Bjorken}} = g^2 \sum_f q_f^2 \int_0^1 \frac{dz}{2\pi} \int \frac{d^d \ell}{(2\pi)^d} \times (-i\pi) \textcolor{red}{G}^{ij}(0, \xi, t) (\partial^i \Phi)(z, \ell) (\partial^j \Phi^*)(z, \ell)$$

Checked with explicit final expressions for both double limits

uGPD as a finite Wilson loop

$$\begin{aligned} & \int d^2 k e^{i(k \cdot r)} r^i r^j \mathcal{G}^{ij}(x, \xi, k, \Delta) \\ &= \frac{1}{\alpha_s} \int \frac{d^4 v_1 d^4 v_2}{(2\pi)^4} \delta(v_1^-) \delta(v_2^-) e^{-i(k - \frac{\Delta}{2}) \cdot v_1 + i(k + \frac{\Delta}{2}) \cdot v_2} \\ & \times \frac{\partial}{\partial v_1^+} \frac{\partial}{\partial v_2^+} \frac{\langle p' | \text{tr}[v_1^+, v_2^+] v_1 [\mathbf{v}_1, \mathbf{v}_2]_{v_2^+} [v_2^+, v_1^+]_{v_2} [\mathbf{v}_2, \mathbf{v}_1]_{v_1^+} | p \rangle}{\langle p | p \rangle} \end{aligned}$$



x-dependent unintegrated GPD $\Leftrightarrow$ FT of a finite Wilson loop

Interpolating scheme for exclusive Compton scattering

Overarching scheme

$$\int d\mathbf{x} \int d^d \mathbf{k} \mathcal{G}^{ij}(\mathbf{x}, \xi, \mathbf{k}, \mathbf{\Delta}) H^{ij}(\mathbf{x}, \xi, \mathbf{k}, \mathbf{\Delta})$$

Bjorken limit

$$\int d\mathbf{x} H^{ij}(\mathbf{x}, \xi, \mathbf{0}, \mathbf{\Delta}) \times [\int d^d \mathbf{k} \mathcal{G}^{ij}(\mathbf{x}, \xi, \mathbf{k}, \mathbf{\Delta})]$$

Regge limit

$$\int d^d \mathbf{k} \mathcal{G}^{ij}(\mathbf{0}, \xi, \mathbf{k}, \mathbf{\Delta})$$

We found an interpolating scheme

Interpolating scheme for exclusive Compton scattering

Overarching scheme

$$\int d\mathbf{x} \int d^d \mathbf{k} \mathcal{G}^{ij}(\mathbf{x}, \xi, \mathbf{k}, \mathbf{\Delta}) H^{ij}(\mathbf{x}, \xi, \mathbf{k}, \mathbf{\Delta})$$

Bjorken limit

$$\int d\mathbf{x} H^{ij}(\mathbf{x}, \xi, \mathbf{0}, \mathbf{\Delta}) \times [\int d^d \mathbf{k} \mathcal{G}^{ij}(\mathbf{x}, \xi, \mathbf{k}, \mathbf{\Delta})]$$

Regge limit

$$\int d^d \mathbf{k} \mathcal{G}^{ij}(\mathbf{0}, \xi, \mathbf{k}, \mathbf{\Delta})$$

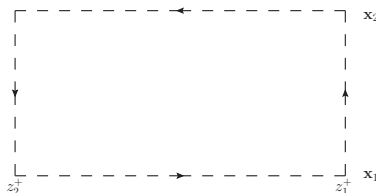
We found an interpolating scheme

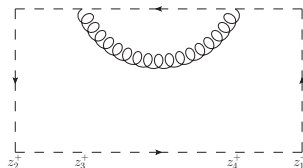
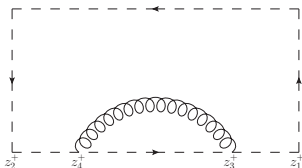
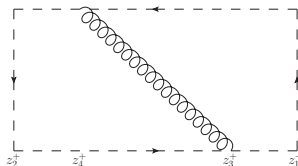
Can we find an interpolating evolution equation?

Does it correct the collinear structure of BK?

Evolution equation for the x -dependent dipole operator

$$S^{(2)}(x, \mathbf{x}_1, \mathbf{x}_2) \equiv \frac{1}{N_c} \int d\mathbf{z}_1^+ \int_{-\infty}^{z_1^+} d\mathbf{z}_2^+ e^{ixP^- z_{12}^+} \frac{\partial^2}{\partial z_1^+ \partial z_2^+} \text{tr}[\mathbf{z}_1^+, \mathbf{z}_2^+]_{\mathbf{x}_1} [\mathbf{z}_2^+, \mathbf{z}_1^+]_{\mathbf{x}_2}$$



$$A^\mu = A^\mu_{\text{cl}} + a^\mu$$


Every line is dressed with classical gluon fields

The second line is power suppressed in the Regge limit

Deriving the evolution equation

Result, up to subeikonal terms

$$\begin{aligned} \Delta S^{(2)}(x, x_1, x_2) &= -\frac{2\alpha_s}{N_c} \int \frac{dk^+}{k^+} \theta(k^+) \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \int d^d x_3 \\ &\times \frac{(k_1 \cdot k_2) \left(e^{i(k_1 \cdot x_{13})} - e^{i(k_1 \cdot x_{23})} \right) \left(e^{-i(k_2 \cdot x_{13})} - e^{-i(k_2 \cdot x_{23})} \right)}{(k_1^2 - 2xP^- k^+ - i0)(k_2^2 - 2xP^- k^+ - i0)} \\ &\times \int dz_1^+ \int_{-\infty}^{z_1^+} dz_2^+ e^{i\left(xP^- - \frac{k_1^2 - i0}{2k^+}\right)z_{12}^+} \\ &\times \partial_{z_1^+} \partial_{z_2^+} \left\{ \text{tr}[z_1^+, z_2^+]_{x_1} [z_2^+, z_1^+]_{x_3} \text{tr}[z_1^+, z_2^+]_{x_3} [z_2^+, z_1^+]_{x_2} - N_c \text{tr}[z_1^+, z_2^+]_{x_1} [z_2^+, z_1^+]_{x_2} \right\} \end{aligned}$$

x -dependent BK kernel, x -dependent double dipole

Deriving the evolution equation

Result, at $x = 0$

$$\begin{aligned} \Delta S^{(2)}(x=0, \mathbf{x}_1, \mathbf{x}_2) &= -\frac{2\alpha_s}{N_c} \int \frac{d\mathbf{k}^+}{k^+} \theta(k^+) \int \frac{d^d \mathbf{k}_1}{(2\pi)^d} \frac{d^d \mathbf{k}_2}{(2\pi)^d} \int d^d \mathbf{x}_3 \\ &\times \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2) \left(e^{i(\mathbf{k}_1 \cdot \mathbf{x}_{13})} - e^{i(\mathbf{k}_1 \cdot \mathbf{x}_{23})} \right) \left(e^{-i(\mathbf{k}_2 \cdot \mathbf{x}_{13})} - e^{-i(\mathbf{k}_2 \cdot \mathbf{x}_{23})} \right)}{k_1^2 k_2^2} \\ &\times \int d\mathbf{z}_1^+ \int_{-\infty}^{z_1^+} d\mathbf{z}_2^+ e^{-i \frac{k_1^2 - i0}{2k^+} z_{12}^+} \\ &\times \partial_{z_1^+} \partial_{z_2^+} \left\{ \text{tr}[z_1^+, z_2^+]_{x_1} [z_2^+, z_1^+]_{x_3} \text{tr}[z_1^+, z_2^+]_{x_3} [z_2^+, z_1^+]_{x_2} - N_c \text{tr}[z_1^+, z_2^+]_{x_1} [z_2^+, z_1^+]_{x_2} \right\} \end{aligned}$$

Non-zero phase: not BK!

Deriving the evolution equation

Result, at $x = 0$

Standard BK:

$$\int \frac{dk^+}{k^+} \theta(k^+) e^{-i \frac{k_1^2 - i0}{2k^+} z_{12}^+} \rightarrow \int_{\rho_0^+}^{\rho_0^+ + \delta \rho_0^+} \frac{dk^+}{k^+} \theta(k^+) = \delta \ln \rho_0^+$$

Here:

$$\int \frac{dk^+}{k^+} \theta(k^+) e^{-i \frac{k_1^2 - i0}{2k^+} z_{12}^+} \rightarrow (\rho^+)^{\eta} \int \frac{dk^+}{(k^+)^{1+\eta}} e^{-i \frac{k_1^2 - i0}{2k^+} z_{12}^+} \theta(k^+)$$

Then,

$$\Delta S^{(2)}(x=0, \mathbf{x}_1, \mathbf{x}_2) = \Delta S_{\text{BK}}^{(2)}(x=0, \mathbf{x}_1, \mathbf{x}_2) - \ln \frac{k_1^2}{2\rho_0^+ P^-} \otimes S^{(3)}(x=0) + \text{const.}$$