

QCD equation of state and proton number cumulants

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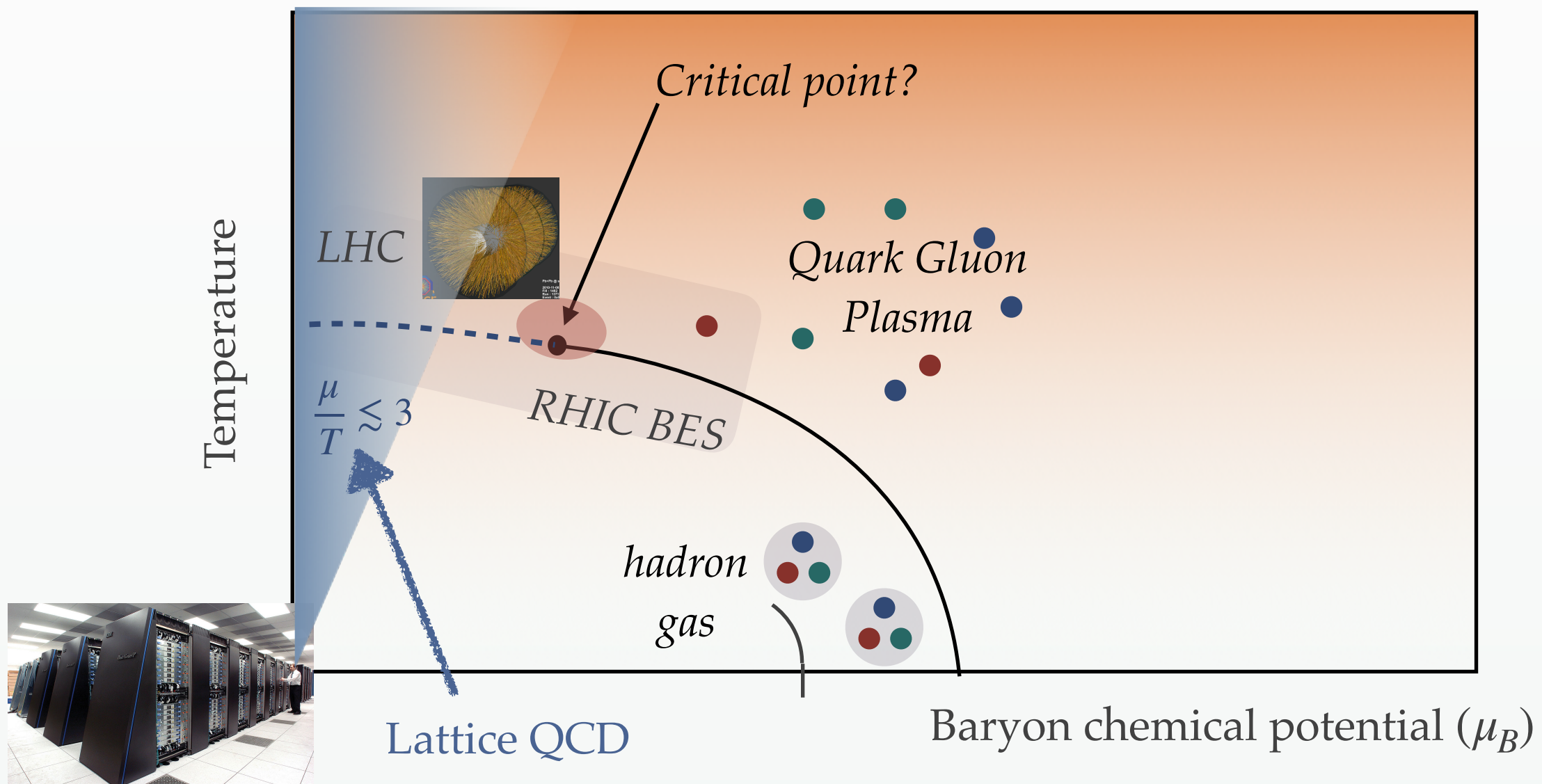
Based on:

GB, M. Pradeep, M. Stephanov, *Phys.Rev.D* 114 (2026) 5, 054048 [2603.23635](#)

+ work in progress

Motivation

To understand the QCD phase diagram via heavy-ion collisions



Motivations

Lattice QCD shows a smooth transition between hadron gas and quark gluon plasma

This is observed in thermodynamic quantities such as pressure and susceptibilities

Susceptibilities are related to *fluctuations* which are measured in experiments
(will focus on proton number fluctuations here)

The susceptibilities are naturally identified with conserved charges
(baryon number, charge, strangeness...)

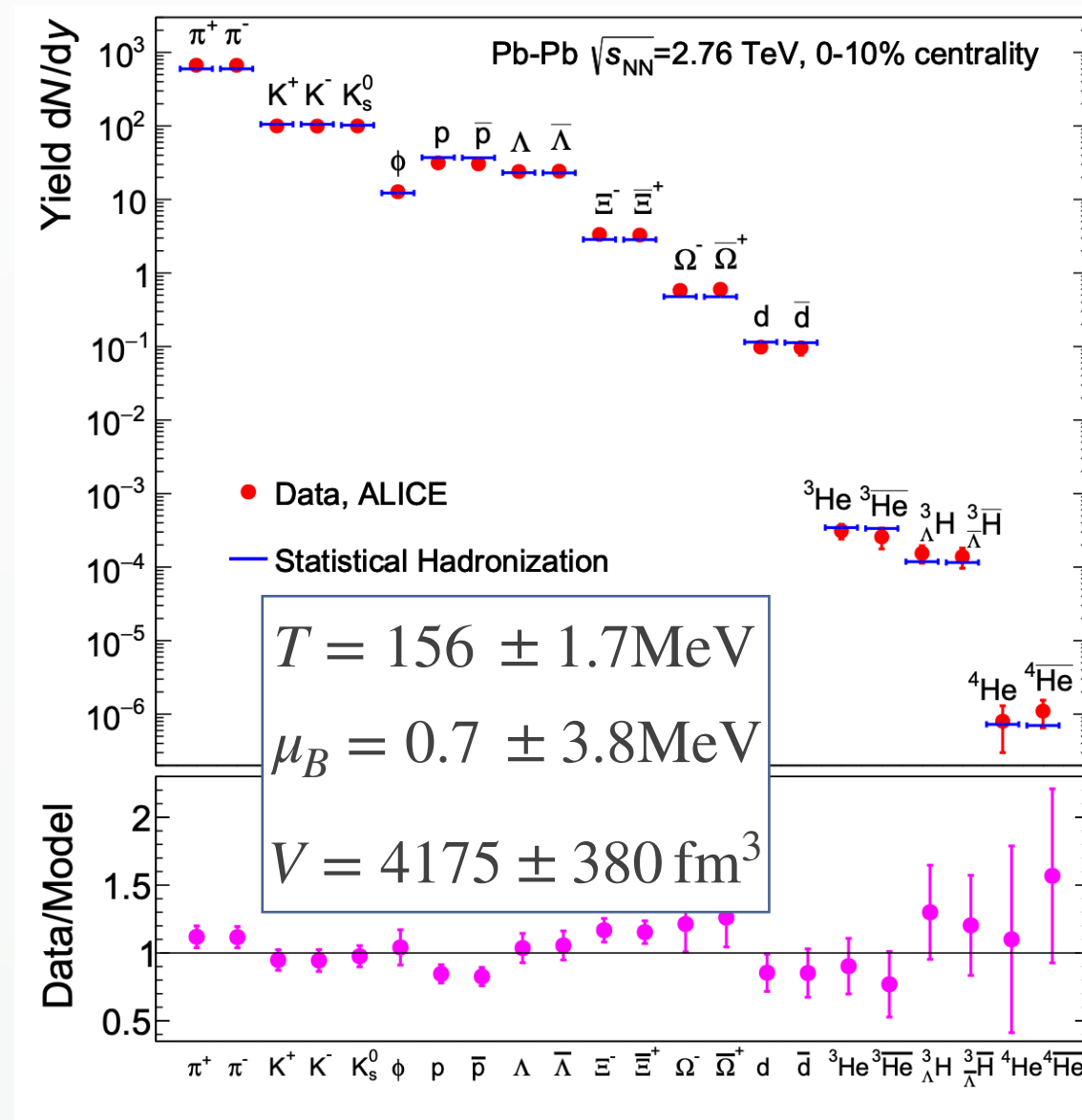
But baryon number is not proton number
(e.g. protons carry energy, electric charge whose fluctuations affect proton number fluctuations)

Relation between susceptibilities of conserved charges and proton number fluctuations ?

Motivations

Near the crossover region QCD EoS is very well described by the EoS of ideal gas of resonances

[Dashen et al '69...]

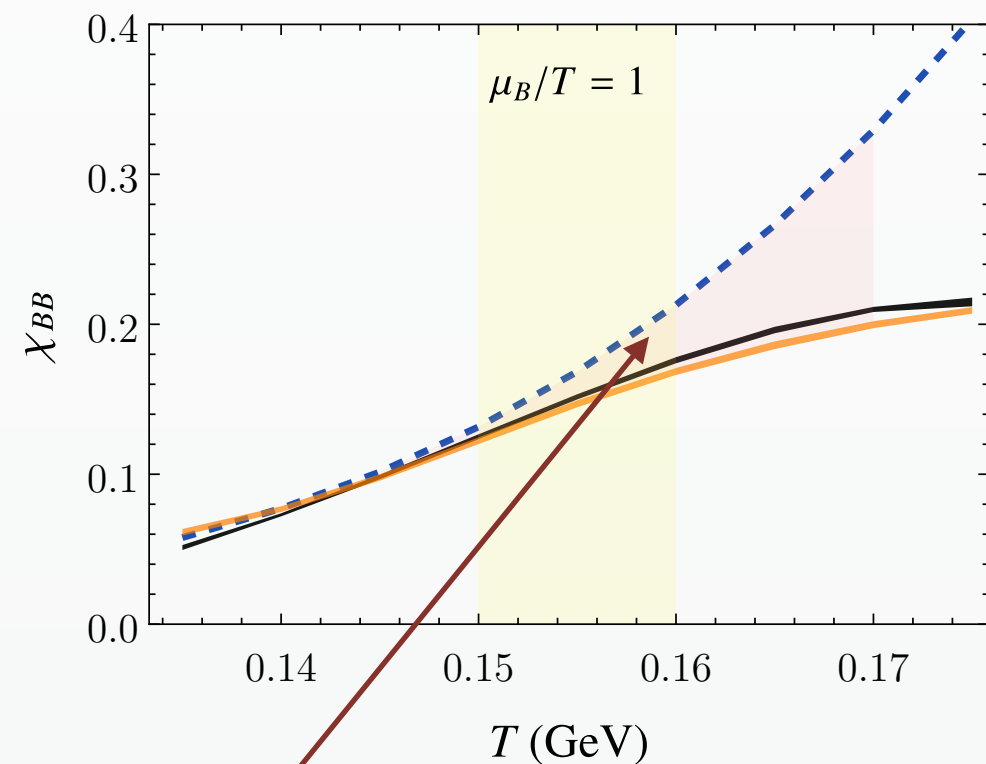
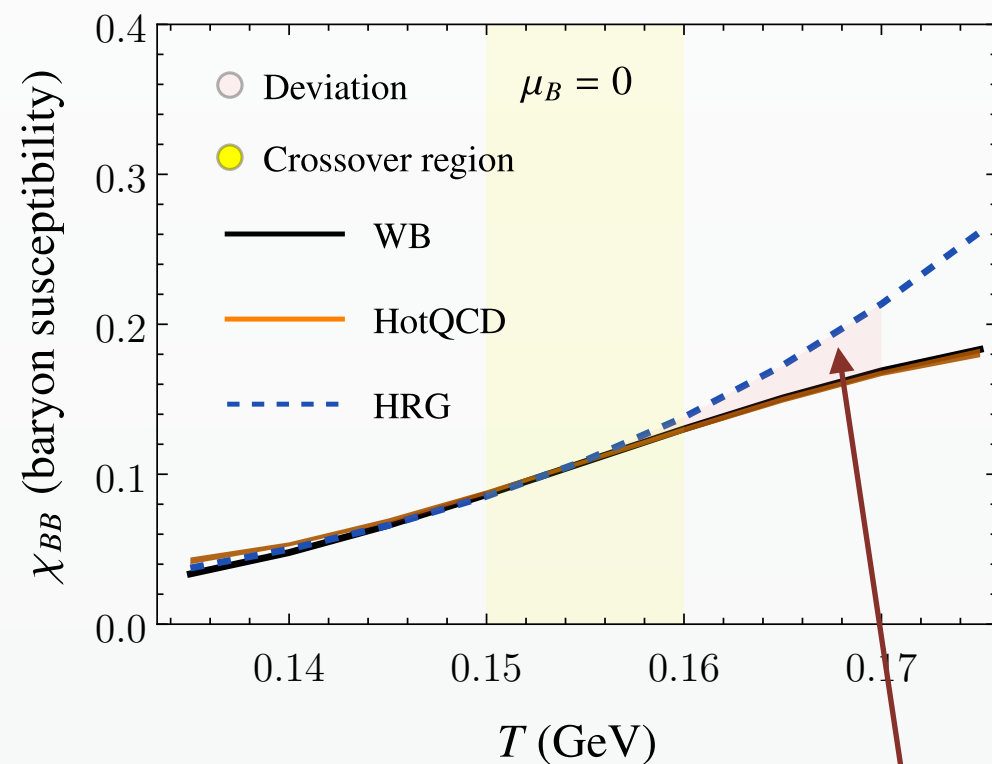


[Andronic et al, Nature 561 '18]

Motivations

At higher temperatures / nonzero density, QCD and HRG EoSs deviate

Signatures of the deconfinement transition



Focus on this deviation

Translate susceptibilities of conserved charges into proton number fluctuations

From baryon number to proton number

Local conservation laws:

$$\psi_a = \phi_A P_a^A$$

conserved charge densities
(baryon number, energy, charge, strangeness)

particle number densities
(e.g. proton number)

Hadronic variables:

$$A \equiv \{\mathbf{x}, \mathbf{p}, m, q, \dots\}$$

Hydro variables:

$$a \equiv \{E, \mathbf{x}\}, \{N_B, \mathbf{x}\}, \dots$$

P_a^A : contribution of a particle occupying the state labeled by A to the local value of the conserved quantities labeled by a .

$$P_{\{E, \mathbf{y}\}}^{\{\mathbf{x}, \mathbf{p}, m, q, \dots\}} \equiv \sqrt{\mathbf{p}^2 + m^2} \delta^3(\mathbf{x} - \mathbf{y}), \quad P_{\{Q, \mathbf{y}\}}^{\{\mathbf{x}, \mathbf{p}, m, q, \dots\}} \equiv q \delta^3(\mathbf{x} - \mathbf{y}), \dots$$

Ps are projectors: a denotes a coarser grained set of variables than A

More generally

H : connected correlators of ψ
``hydro correlators''

$$H_{a_1 \dots a_N} = G_{A_1 \dots A_N} P_{a_1}^{A_1} \dots P_{a_N}^{A_N}$$

G : connected correlators of ϕ
``hadronic correlators''

repeated indices $\equiv \sum_{\alpha=m, q, \dots} \int_{\mathbf{x}, \mathbf{p}}$

From baryon number to proton number

Susceptibilities / hydro correlators
can calculate from lattice / hydro

Projectors: *not invertible!*

$$H_{a_1 \dots a_N} = G_{A_1 \dots A_N} P_{a_1}^{A_1} \dots P_{a_N}^{A_N} \quad \text{cannot invert this equation!}$$

Hadronic correlators we want to know
(related to particle number fluctuations)

QCD susceptibilities differ from HRG

Hadronic correlators, $G_{A_1 \dots A_N}$ differ from the equilibrium HRG correlators, $\bar{G}_{A_1 \dots A_N}$

What is the most likely distribution where the correlators $G_{A_1 \dots A_N}$ satisfy this constraint?

Maximum Entropy Method

[Pradeep, Stephanov '24 + a bit of new stuff..]

Consider a set of fields ϕ_A with a distribution $P[\phi]$

entropy: $S = - \int \mathcal{D}\phi e^{s[\phi]} P \log P$

Equilibrium distribution:
(maximum of S for given $\langle \phi \rangle$)

$$\bar{P} = \frac{1}{\bar{Z}[J]} e^{J^A \phi_A} \quad \bar{Z}[J] \equiv - \int \mathcal{D}\phi e^{s[\phi]} e^{J^A \phi_A}$$

Equilibrium correlators:

$$\langle \phi_A \rangle = \frac{\delta}{\delta J^A} \log \bar{Z}[J], \quad \bar{G}_{A_1 A_2} = \frac{\delta}{\delta J^{A_1} \delta J^{A_2}} \log \bar{Z}[J], \quad \bar{G}_{A_1 A_2 A_3} = \frac{\delta}{\delta J^{A_1} \delta J^{A_2} \delta J^{A_3}} \log \bar{Z}[J], \dots$$

Maximum Entropy Method

What is the most likely distribution where the correlators $G_{A_1 \dots A_N}$ satisfy an external constraint?

maximum of S:
$$P = \frac{1}{Z[J, K]} \exp \left(J^A \phi_A + \sum_{n=2}^{\infty} \frac{1}{n!} K^{A_1 \dots A_N} \phi_{A_1} \dots \phi_{A_N} \right)$$

$$Z[J, K] = \int \mathcal{D}\phi e^{s[\phi]} \exp \left(J^A \phi_A + \sum_{n=2}^{\infty} \frac{1}{n!} K^{A_1 \dots A_N} \phi_{A_1} \dots \phi_{A_N} \right)$$

Intuitive explanation: in order to “tune” each $G_{A_1 \dots A_N}$ to a desired value (given by an external constraint) one must turn on a separate (non-local) source for $\phi_{A_1} \dots \phi_{A_N}$

$\log Z[J, K]$: generating functional

$$\langle \phi_A \rangle = \frac{\delta}{\delta J^A} \log Z \quad \langle \phi_{A_1} \dots \phi_{A_N} \rangle = \frac{\delta}{\delta K^{A_1 \dots A_N}} \log Z[J, K] \quad G_{A_1 \dots A_N} = \frac{\delta}{\delta J^{A_1} \dots \delta J^{A_N}} \log Z[J, K]$$

Maximum Entropy Method

Goal: express the difference $\Delta G_{A_1 \dots A_N} \equiv G_{A_1 \dots A_N} - \bar{G}_{A_1 \dots A_N} \longrightarrow$ *equilibrium hadronic corr.*
in terms of the hydro correlators $\Delta H_{a_1 \dots a_N} \equiv H_{a_1 \dots a_N} - \bar{H}_{a_1 \dots a_N}$



$$\bar{H}_{a_1 \dots a_N} \equiv \bar{G}_{A_1 \dots A_N} P_{a_1}^{A_1} \dots P_{a_N}^{A_N}$$

*HRG contribution to susceptibilities,
given by derivatives of p_{HRG} wrt β & μs*

What higher-point sources, $K^{A_1 \dots A_N}$, must be turned on to ensure

$$\text{conservation } H_{a_1 \dots a_N} = G_{A_1 \dots A_N} P_{a_1}^{A_1} \dots P_{a_N}^{A_N} ?$$

Maximum Entropy Method

- In the freeze-out region, the difference between QCD and HRG EoSs is small
- Linearize the generating functional, $W[J,K]=\log Z[J,K]$, in K
- Neglect loops which are suppressed in the hydro regime by central limit theorem

$$G_{AB} = \bar{G}_{AB} + \bar{G}_{AA'} \bar{G}_{BB'} K^{A'B'}$$

$$\vdots$$

$$A_1 \text{ --- } \begin{array}{c} \nearrow A_2 \\ \vdots \\ \searrow A_N \end{array} \equiv G_{A_1 \dots A_N}$$

$$A_1 \text{ --- } \begin{array}{c} \nearrow A_2 \\ \vdots \\ \searrow A_N \end{array} \equiv \bar{G}_{A_1 \dots A_N}$$

$$\begin{array}{c} A_1 \\ \text{---} \square \text{---} \\ A_1 \end{array} \equiv K^{A_1 \dots A_N}$$

Diagrammatically:

$$\text{---} = \text{---} + \text{---} \bigcirc \text{---}$$

$$\text{T} = \text{T} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} \text{Y}$$

$$\text{+} = \text{+} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} \text{Y} + \text{---} \bigcirc \text{---} \text{+} + \text{Y} \bigcirc \text{---} \text{Y}$$

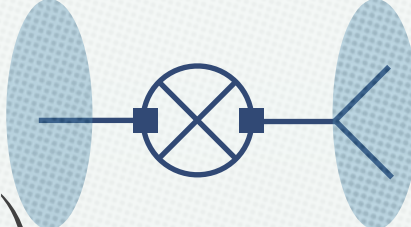
Maximum Entropy Method

$$\text{thick line} = \text{thin line} + \text{thin line with a circle containing an 'X' and a square at each end}$$

$$\text{thick T-junction} = \text{thin T-junction} + \text{thin T-junction with a circle containing an 'X' and a square at each end}$$

all possible groupings of 3 elements into 2 nonempty subsets

$\binom{3}{2} = 3$ diagrams



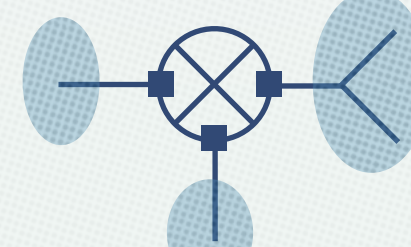
$$\text{thick cross} = \text{thin cross} + \text{thin cross with a circle containing an 'X' and a square at each end}$$

4 elements into 3 nonempty subsets

$\binom{4}{2} = 6$ diagrams

$6 = S(4,3)$

$S = 2\text{nd Stirling \#}$

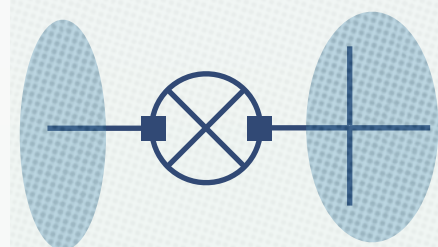
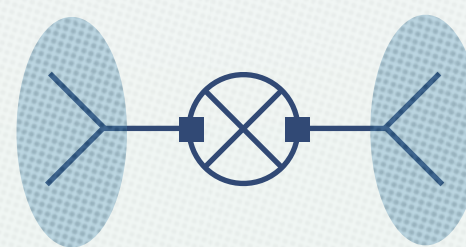


4 elements into 2 nonempty subsets

$\binom{4}{3} = 4$ diagrams

$\frac{1}{2!} \binom{4}{2} = 3$ diagrams

$7 = 4 + 3 = S(4,2)$

Maximum Entropy Method

$$K_{A_1 A_2} = \Delta G_{A_1 A_2} \equiv \hat{\Delta} G_{A_1 A_2}$$

inverse correlator $\leftarrow \bar{G}^{A_1 A'_1} \bar{G}_{A'_1 A_2} \equiv \delta_{A_2}^{A_1}$

$$K_{A_1 A_2 A_3} = \left[\Delta G_{A_1 A_2 A_3} - 3 \Delta G_{A_1 A'_1} \bar{G}_{A'_1 A_2 A_3} \right]_{123} \equiv \hat{\Delta} G_{A_1 A_2 A_3}$$

$$\bar{G}_{A_2 \dots A_N}^{A_1} \equiv \bar{G}^{A_1 A'_1} \bar{G}_{A'_1 A_2 \dots A_N}$$

Raising indices =
amputation

$\vdots$

$$K_{A_1 \dots A_N} = \hat{\Delta} G_{A_1 \dots A_N} \equiv \Delta G_{A_1 \dots A_N} - \sum_{n=2}^{N-1} \hat{\Delta} G_{A'_1 \dots A'_n} \sum_{p_i \in \Sigma} \left(\bar{G}_{p_1}^{A'_1} \times \dots \times \bar{G}_{p_n}^{A'_n} \right)$$

$\Sigma = \{ \text{all partitions of } A_1, \dots, A_N \text{ into } n \text{ subsets} \}$

“Irreducible Relative Cumulants”

- Turning on a source for an n -point correlator affects all the higher order ($N \geq n$) correlators
- The N -point IRCs is the correlator that cannot be turned on by lower order sources
(genuine N -point correlator that can only be turned on by an N -point source)

Maximum Entropy Method

In order to satisfy the constraint $H_{a_1 \dots a_N} = G_{A_1 \dots A_N} P_{a_1}^{A_1} \dots P_{a_N}^{A_N}$,
the sources have to be “coarse grained” as well: $K^{A_1 \dots A_N} = \Lambda^{a_1 \dots a_N} P_{a_1}^{A_1} \dots P_{a_N}^{A_N}$

$$\Lambda^{a_1 \dots a_N} = P_{A_1}^{a_1} \dots P_{A_N}^{a_N} K^{A_1 \dots A_N} \quad \text{inverse hydro corr.} \leftarrow \bar{H}^{aa'} \bar{H}_{a'b} = \delta_b^a$$

$$P_A^a \equiv \bar{H}^{aa'} P_{a'}^{A'} \bar{G}_{A'A}, \quad P_A^a P_b^A = \delta_b^a$$

At tree level: $\Lambda_{a_1 \dots a_N} = \hat{\Delta} H_{a_1 \dots a_N} \longrightarrow$ Related to $\Delta H_{a_1 \dots a_N}$ s via the same algebraic relations as $\hat{\Delta} G_{a_1 \dots a_N}$ s

From which it follows that

$$\hat{\Delta} G_{A_1 \dots A_N} = \hat{\Delta} H_{a_1 \dots a_N} P_{A_1}^{a_1} \dots P_{A_N}^{a_N}$$

Hadronic IRCs

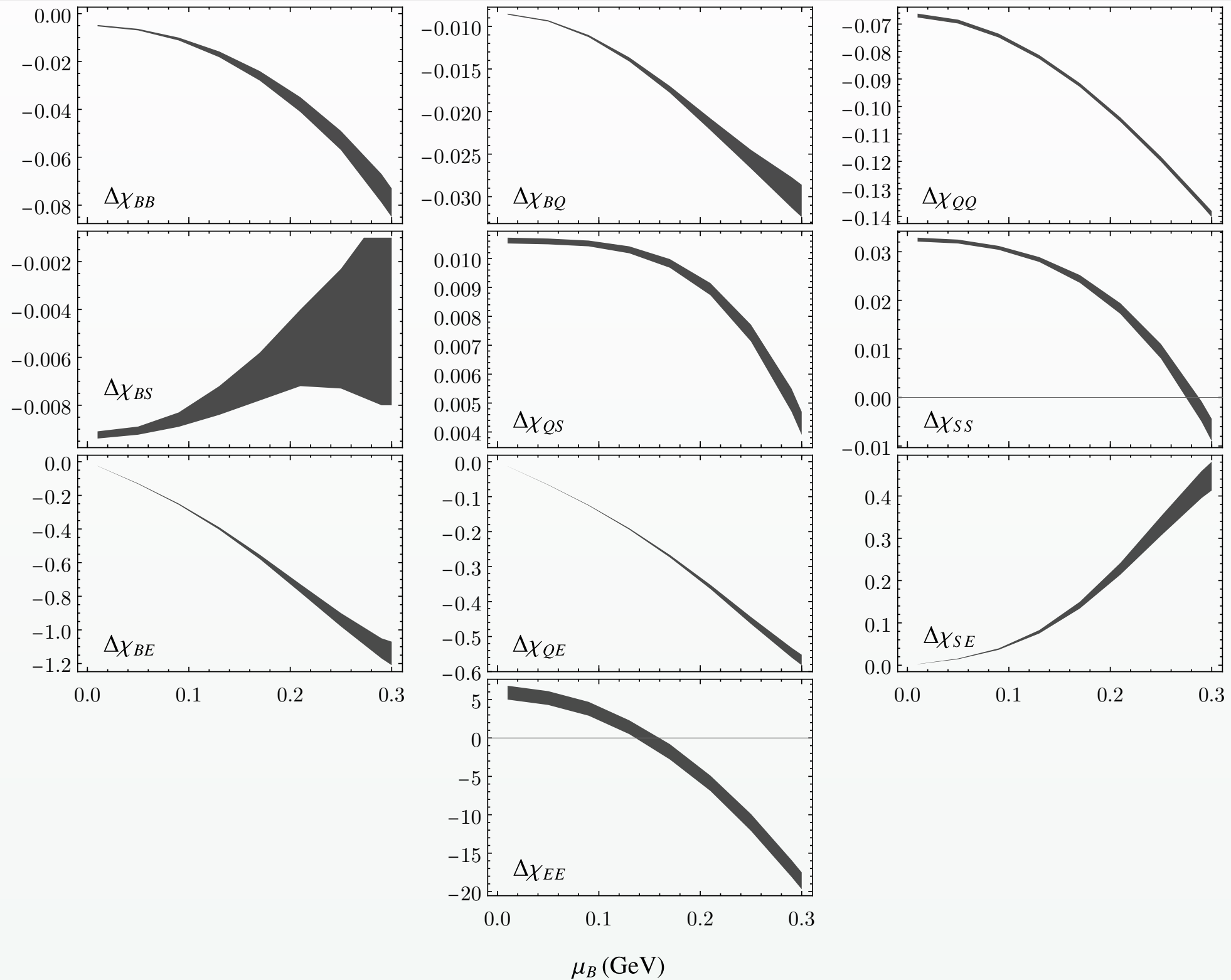
(For classical ideal HRG = factorial cumulants)

Hydro IRCs

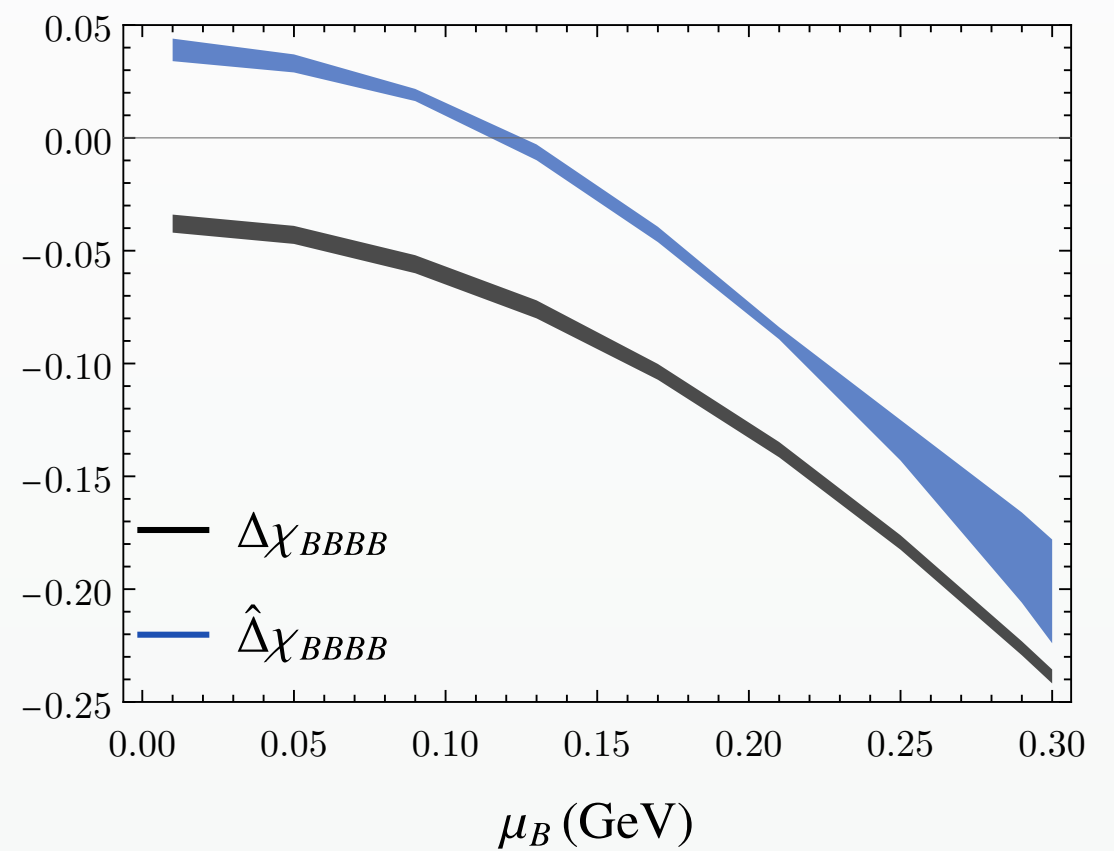
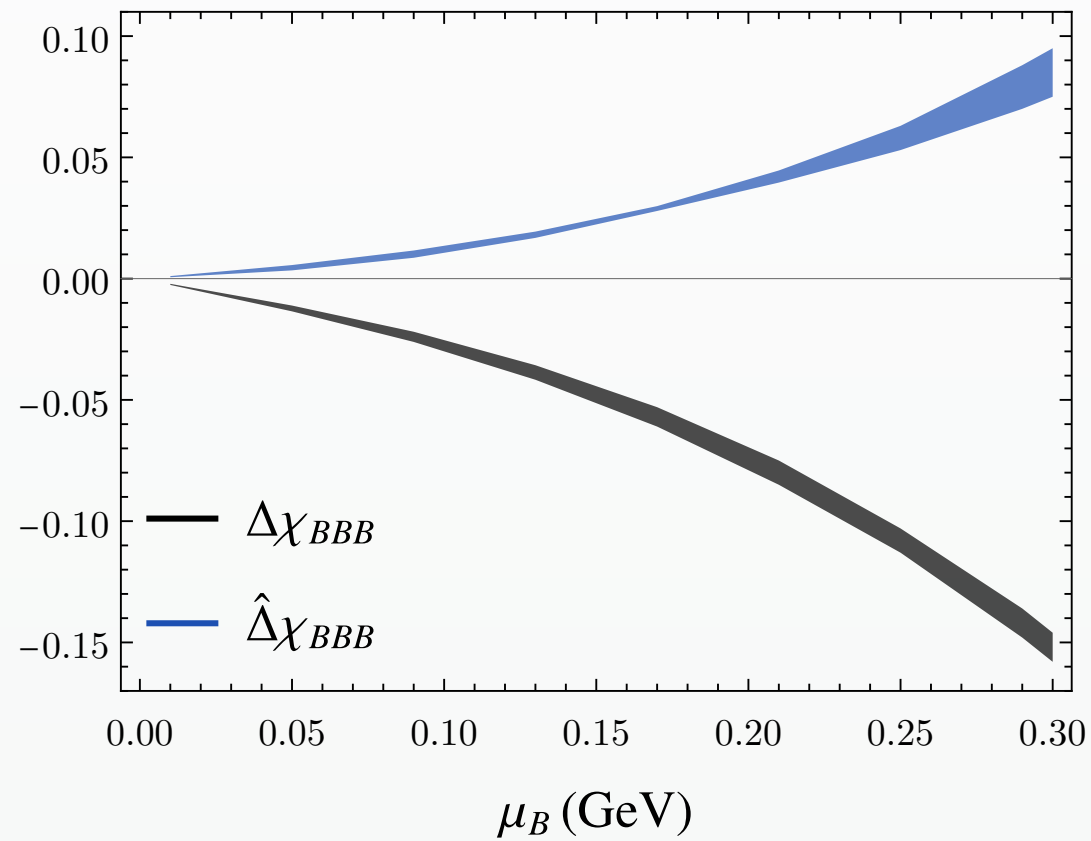
Used lattice QCD EoS (Wuppertal-Budapest) 4th order Taylor series in μ_B, μ_Q, μ_S

[A. Abuali, et al PRD 112 (2025) 5, 054502 ,
Bellwied et al. PRD 92 (2015) 11, 114505]

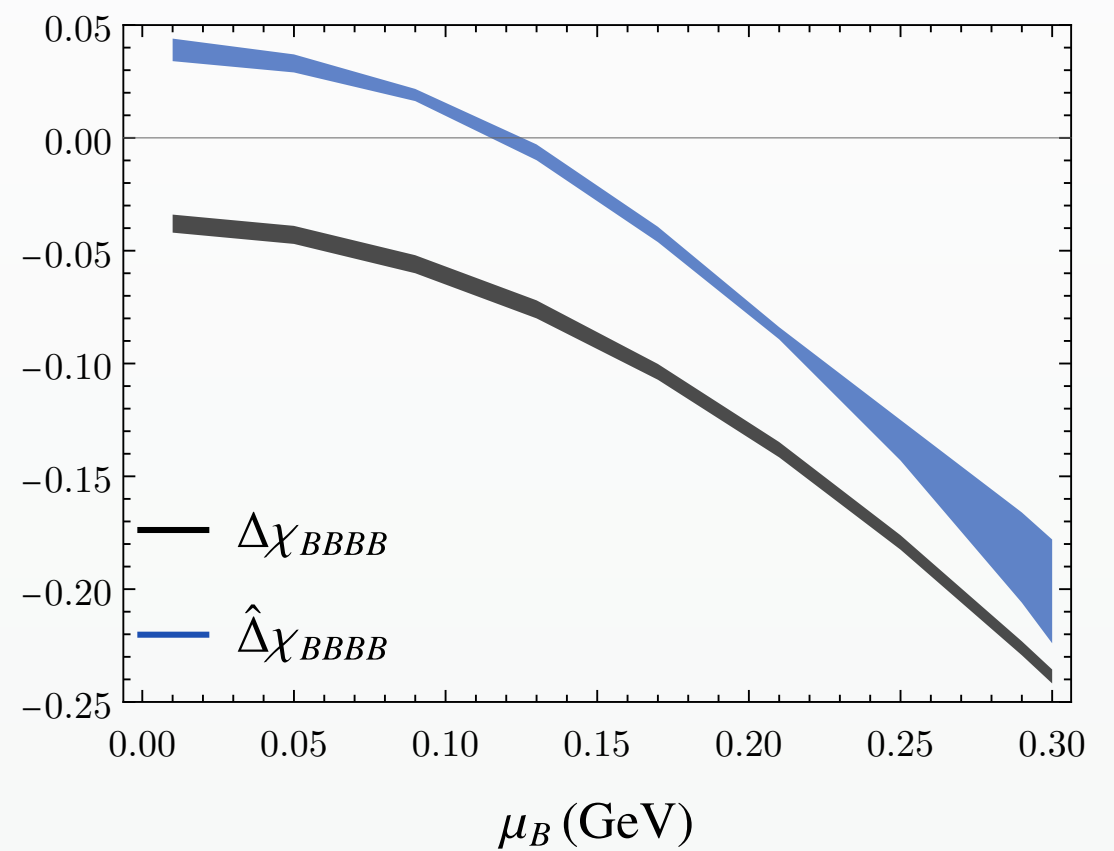
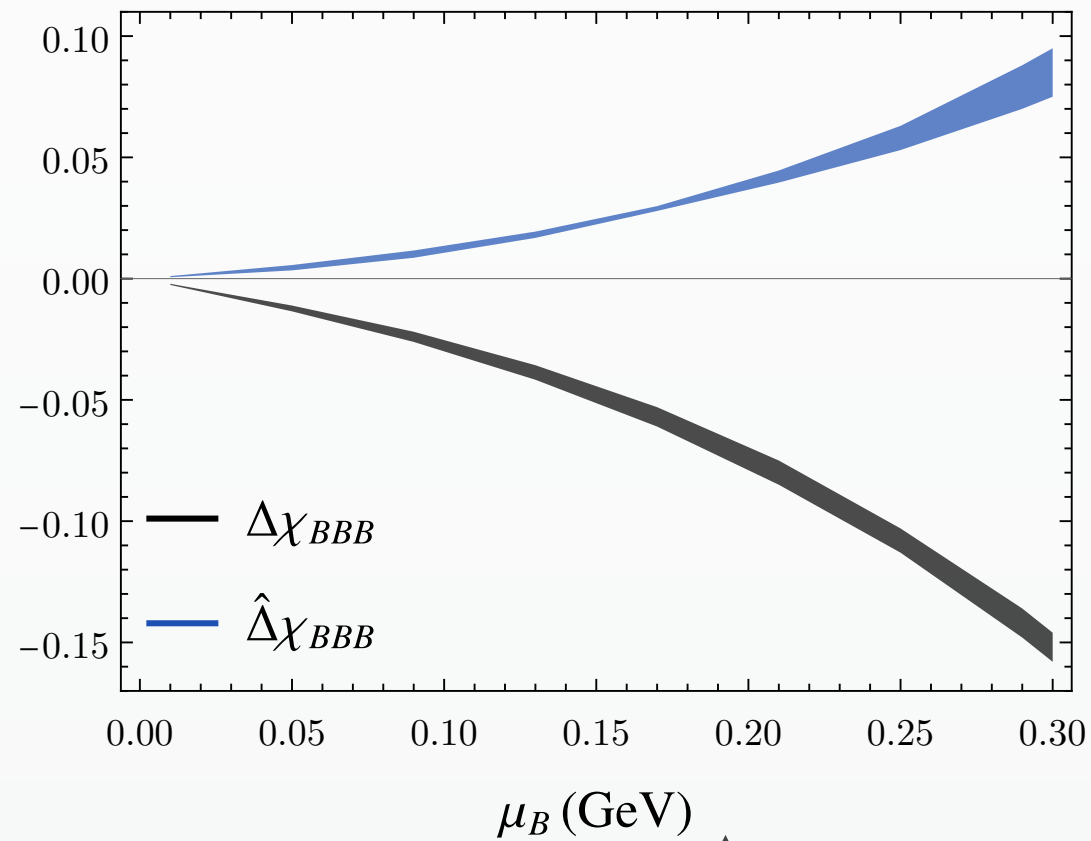
Deviation between QCD and HRG along freeze-out



Deviation between QCD and HRG along freeze-out



Deviation between QCD and HRG along freeze-out

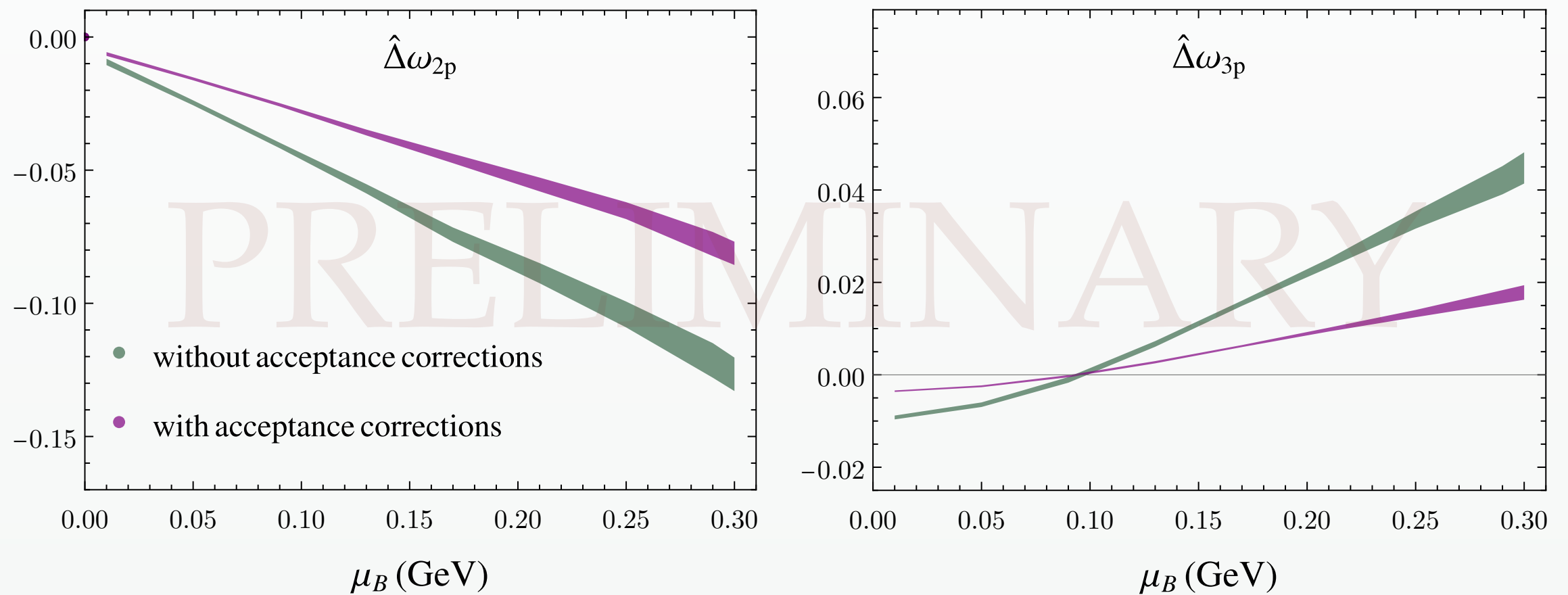


$\Delta\chi_{BBB}$ and $\hat{\Delta}\chi_{BBB}$ differ qualitatively
 $\Delta\chi_{BBB}$ dominated by $\Delta\chi_{ab}s$

Deviation between QCD and HRG along freeze-out

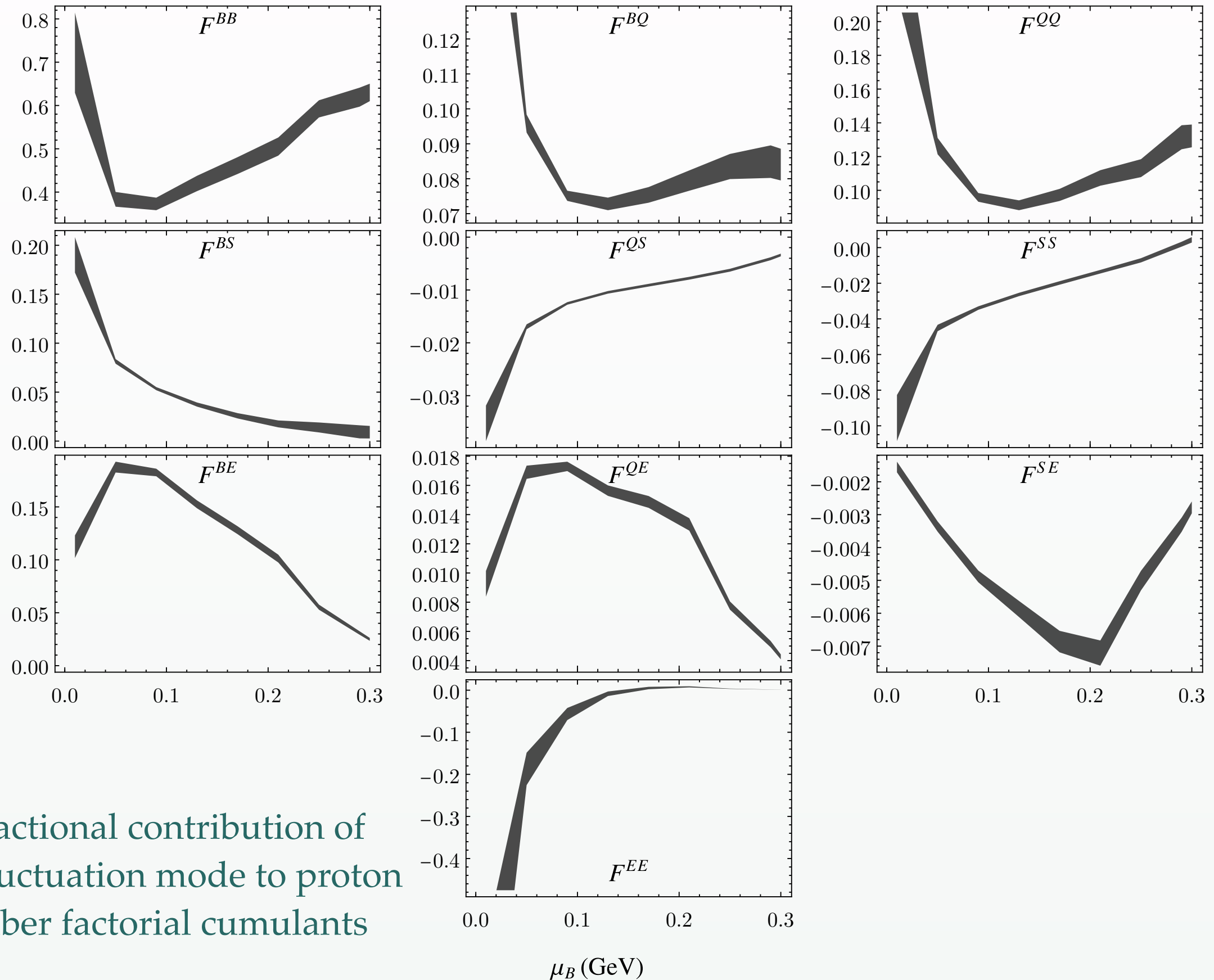
Proton number factorial cumulants

$$\hat{\Delta}\omega_{kp} = \frac{1}{\langle N_p \rangle} \int_{\text{phase sp.}} \hat{\Delta}G_{kp} = \kappa_k^p + (\text{quantum corrections})$$



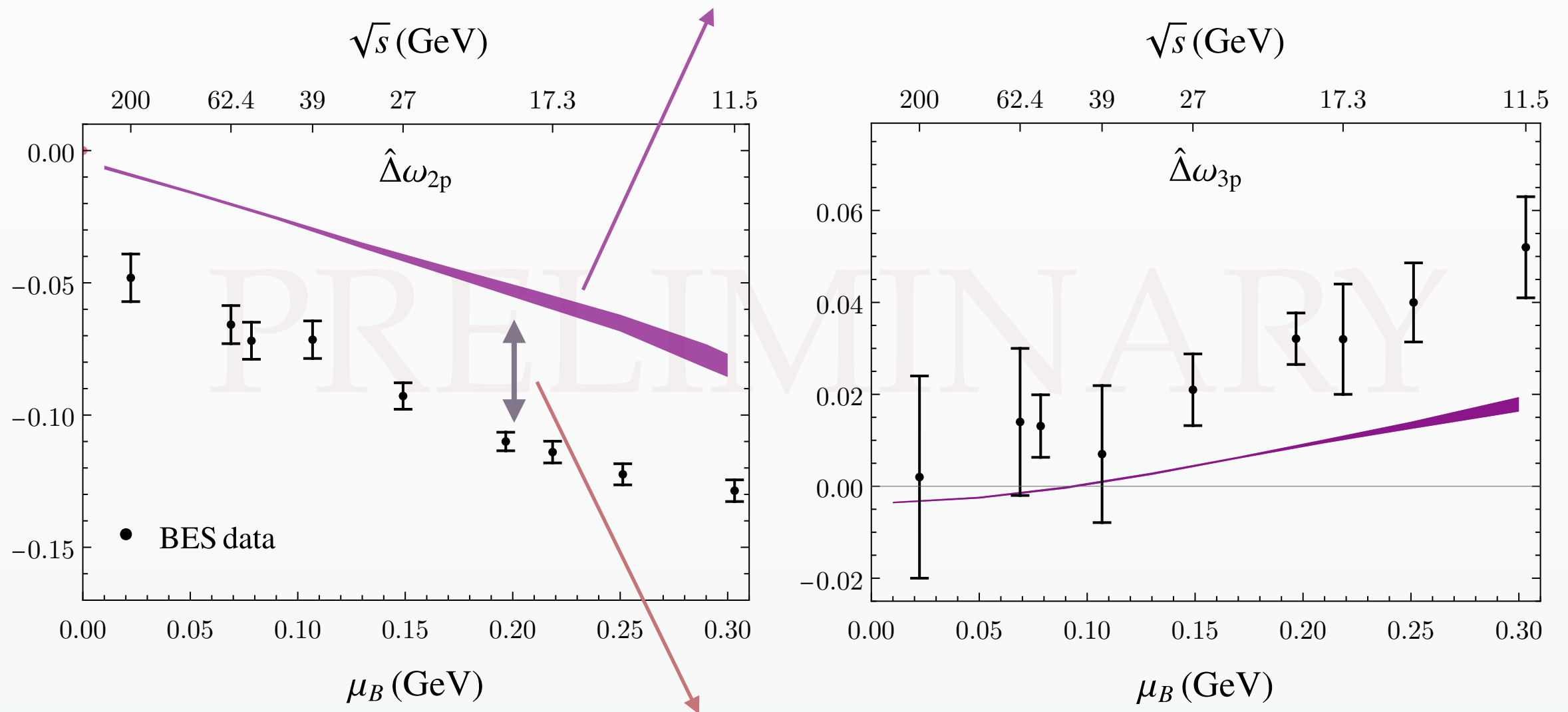
* $\hat{\Delta}\omega_{kp}$ s includes decays

All fluctuation modes contribute to proton number fluctuations



Comparison to data

Equilibrium QCD prediction



Non-equilibrium / dynamical effects ?

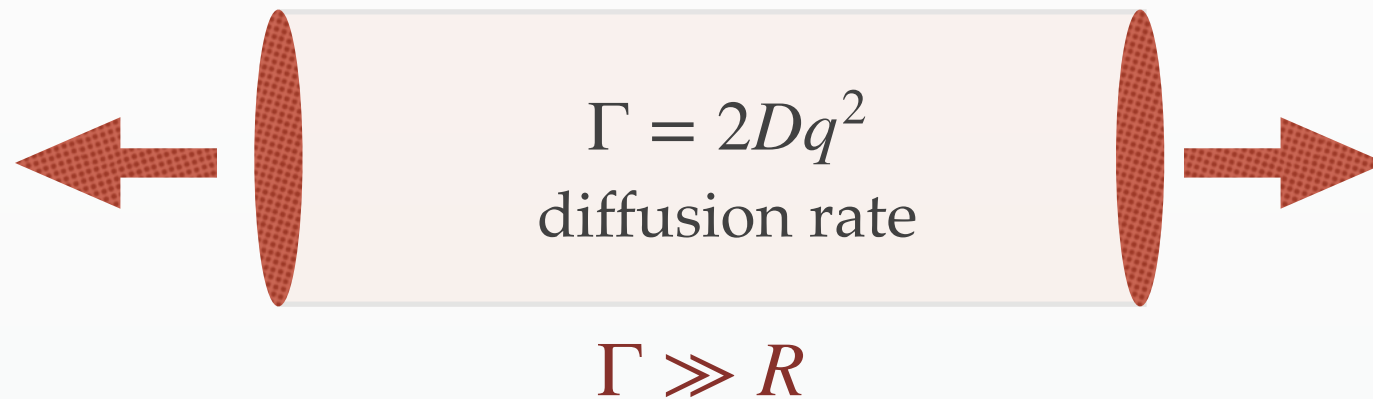
Memory effect

A simple model

$$\partial_\tau W(\tau) = -\Gamma(W(\tau) - W_{\text{eq}}(\tau))$$

W = Wigner transform of
fluctuation mode
with wavenumber q

$R = \partial_\tau \log W_{\text{eq}}$
expansion rate



Time lag: $W(\tau) = W_{\text{eq}}(\tau + \Delta\tau)$

determined by diffusion

$\Delta\tau \approx -\frac{1}{\Gamma}$

lag (with an arrow pointing to the negative sign)

Bjorken expansion:

$$\frac{\partial \log T}{\partial \log s} = c_s^2, \partial_\tau(\tau s) = 0 \quad \Rightarrow \quad \Delta T \approx -\Delta\tau T c_s^2 = \frac{T c_s^2}{\Gamma \tau}$$

fluctuations freeze-out at a slightly higher temperature

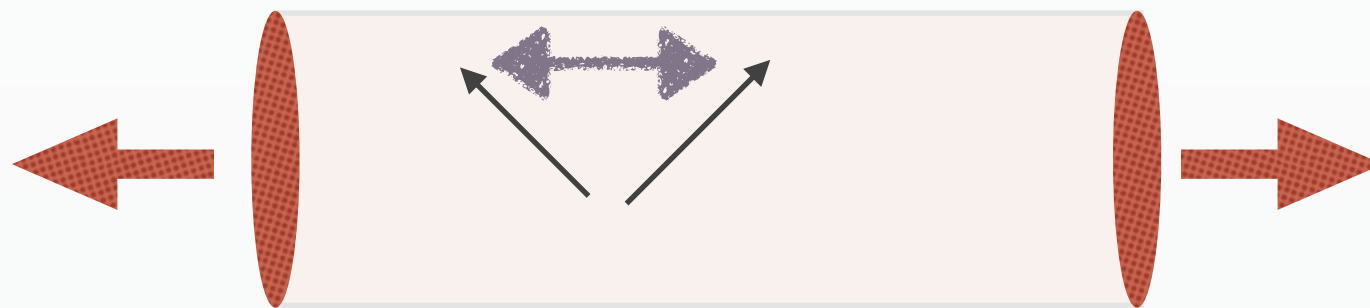
[see also: Berdnikov Rajagopal '99; Mukherjee Venugopalan, Yin '15; An, GB, Stephanov, Yee '20, ...]

[Koch, Jeon, Asakawa '00; Heinz, Muller '00;
Stephanov, Shuryak '01, ...]

Memory effect

A quick estimation of the order of magnitude of the memory effect

thermal smearing: $\Delta z \sim \tau \bar{v} \sim (10 \text{ fm}) \sqrt{\frac{8T}{\pi m_p}} \approx 6 \text{ fm}$



$$q \approx \frac{2\pi}{\Delta z} \approx 1 \text{ fm}^{-1}$$

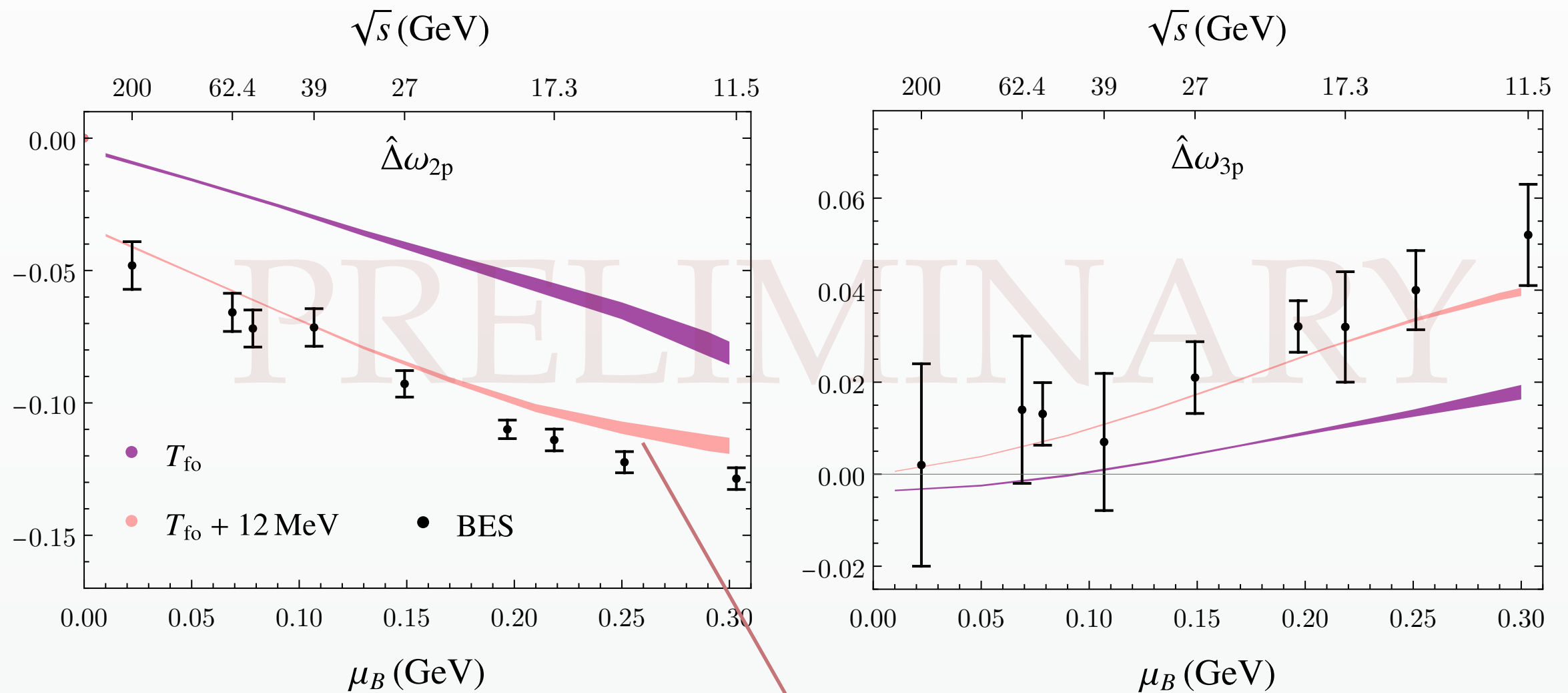
$$\Delta T \approx \frac{T c_s^2}{\Gamma \tau} \approx 12 \text{ MeV}$$

*fluctuations freeze-out at
roughly 8% above T_{fo}*

$$\Gamma \approx 2 \frac{1}{2\pi T} q^2 \approx 0.4 \text{ fm}^{-1}$$

This is only an estimate, need a full dynamical simulation
[see talk by Schaefer]

Comparison to data

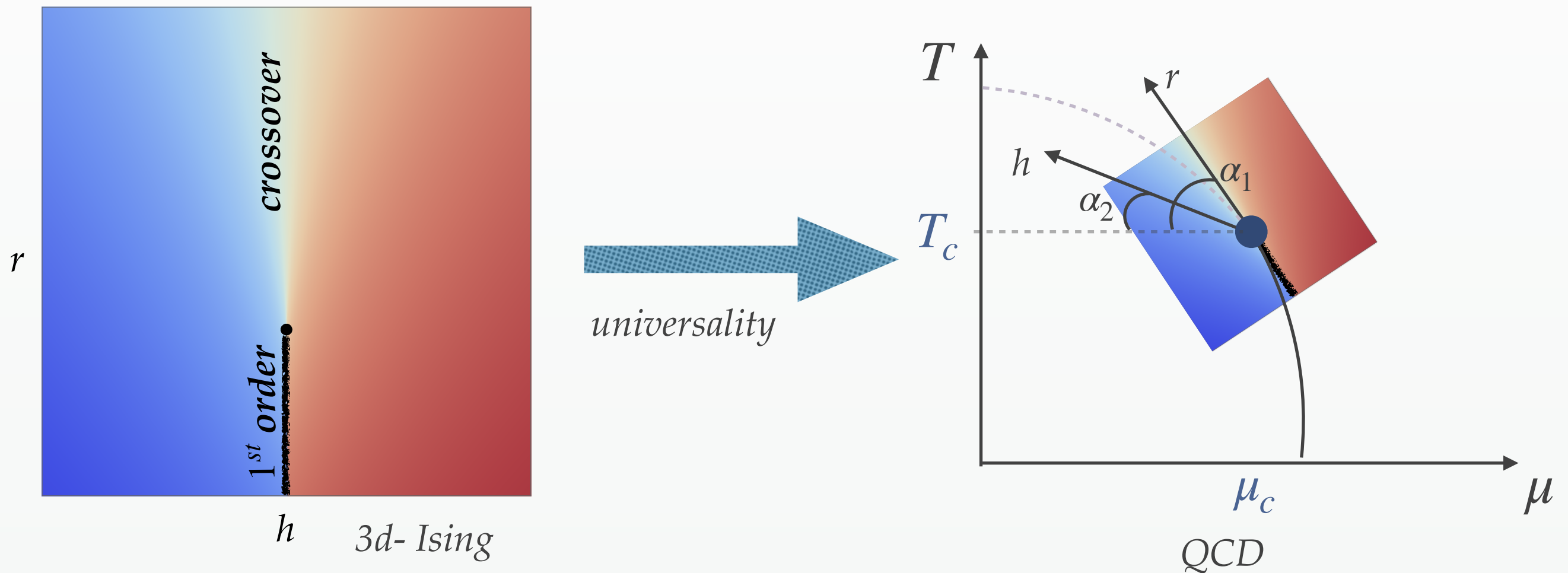


Memory effect?

Other effects? (e.g. charge cons.) [see talk by Vovchenko]

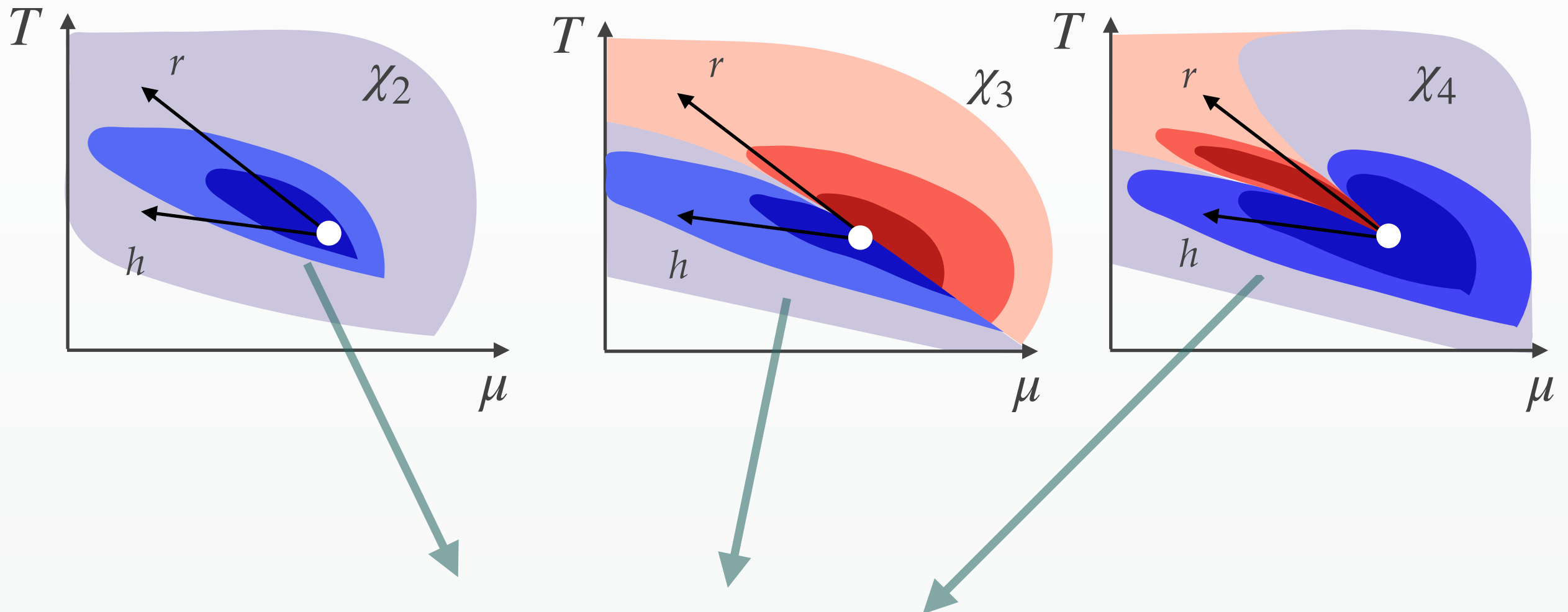
Critical fluctuations

$$\begin{pmatrix} r \\ h \end{pmatrix} = \frac{1}{T_c w \rho \sin(\alpha_1 - \alpha_2)} \begin{pmatrix} \cos \alpha_2 & \sin \alpha_2 \\ -\rho \cos \alpha_1 & -\rho \sin \alpha_1 \end{pmatrix} \begin{pmatrix} T - T_c \\ \mu - \mu_c \end{pmatrix}$$



Non-universal parameters

Cartoon images of susceptibilities (red <0 , blue >0)



shape & size: depend on non-universal parameters. $\rho, w, \alpha_1, \alpha_2$

How does the experimental signal depend on the mapping parameters?

Extract the non-universal mapping parameters from QCD

Scaling of the factorial cumulants

How does the experimental signal depend on the mapping parameters?

In the vicinity of the critical point

$$\hat{\Delta}\omega_{kp}(\bar{\rho}, w, \dots) = \left(\frac{w'}{w}\right)^{1+\frac{1}{\delta}} \hat{\Delta}\omega_{kp}(\bar{\rho}, w', \dots)$$

$\bar{\rho} = \rho w^{1-\frac{1}{\beta\delta}}$ determines the shape of the cumulant

w determines its overall scale

Yang-Lee Trajectory

QCD EoS has singularities in complex μ_B plane [Yang-Lee edge singularity: Yang, Lee '56]

In the vicinity of the critical point, it takes the form [Stephanov, '06]

$$\mu_{LY}(T) \approx \mu_c - c_1(T - T_c) \pm ix_{LY}c_2(T - T_c)^{\beta\delta}$$

$\text{Re}\mu_{LY}$
Crossover slope
 $c_1 = \tan \alpha_1$

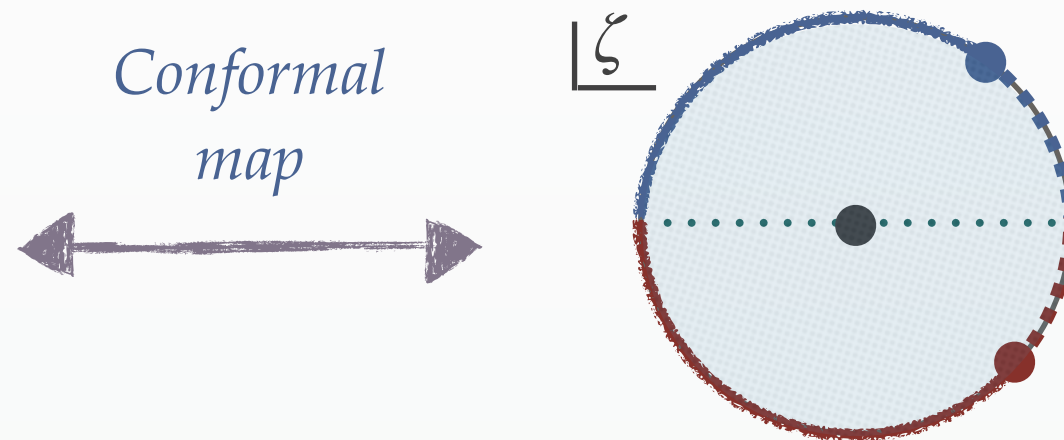
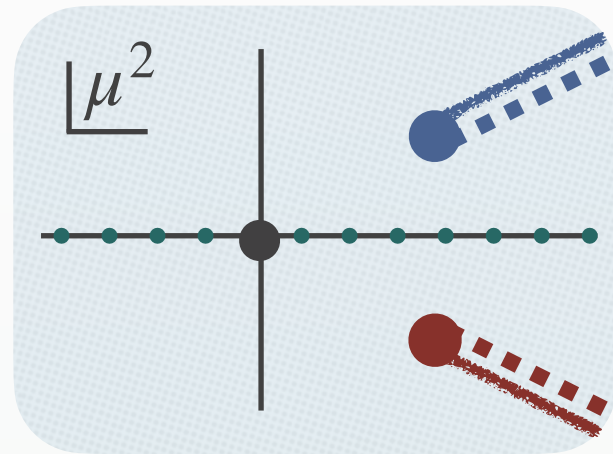
$$\bar{\rho} = \left(\frac{1}{c_2 T_c^{\beta\delta-1}} \frac{|\sin(\alpha_1 - \alpha_2)|}{|\sin \alpha_1|^{\beta\delta+1}} \right)^{1/\beta\delta}$$

For fixed α_1 and α_2 the imaginary part of the YL trajectory fixes the shape of the critical contribution to factorial cumulants!

Estimation of the YL trajectory

[GB, *PRL* 127 17, 171603; *PRC* 110 1, 015203; GB, Dunne, Z. Yin *PRD* 105 10, 105002]

Estimate the singularity from Taylor coefficients



Do Padé
here

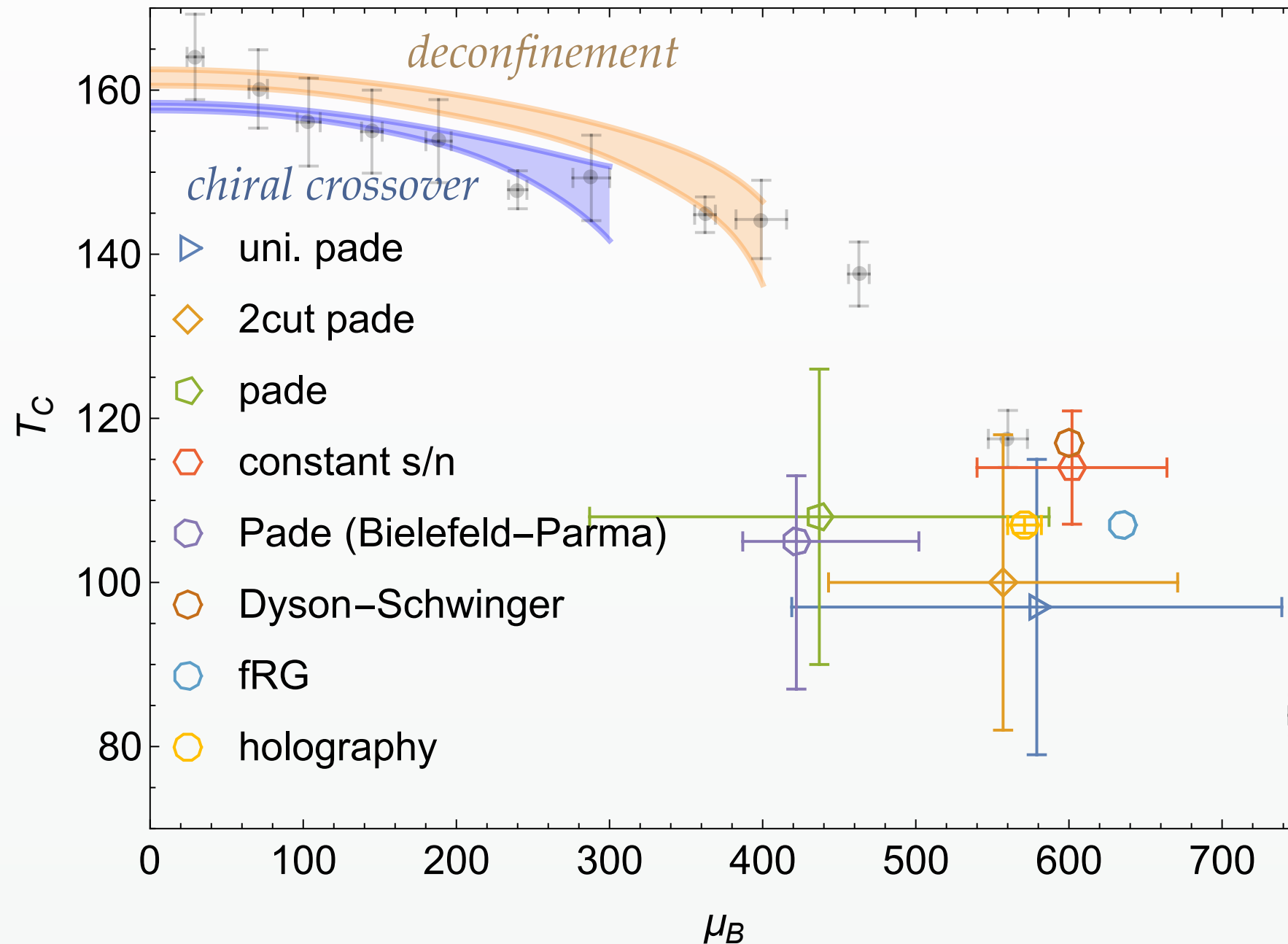
“conformal Padé”

$$P\chi(T, \phi(\zeta)) = \frac{\tilde{p}_0(T) + \tilde{p}_1(T)\zeta + \dots + \tilde{p}_{N/2}(T)\zeta^N}{\tilde{q}_0(T) + \tilde{q}_1(T)\zeta + \dots + \tilde{q}_{N/2}(T)\zeta^N} \Big|_{\zeta=\phi^{-1}(\mu^2)}$$

$$T_C = 97 \pm 20 \text{ MeV} \quad \mu_C = 579 \pm 200 \text{ MeV}$$

$$\alpha_1 = 9.40^\circ_{-3.81}^{+3.89} \quad c_2 = 2.22_{-0.86}^{+0.52}$$

Comparison with other estimates



constant s/n
[Shah et al '24 2410.16206]

Bielefeld-Parma
[Di Renzo et al. Lattice 23]

Dyson-Schwinger
[Gunkel et al '21 PRD 104 054022]

Functional RG
[Fu et al '20 PRD 101 054032]

Holographic model
[Hippel et al '24 PRD 110 094006]

Chiral crossover / deconfinement
[WP data: 2410.06216v1]

Pale black points:
freeze-out (various)

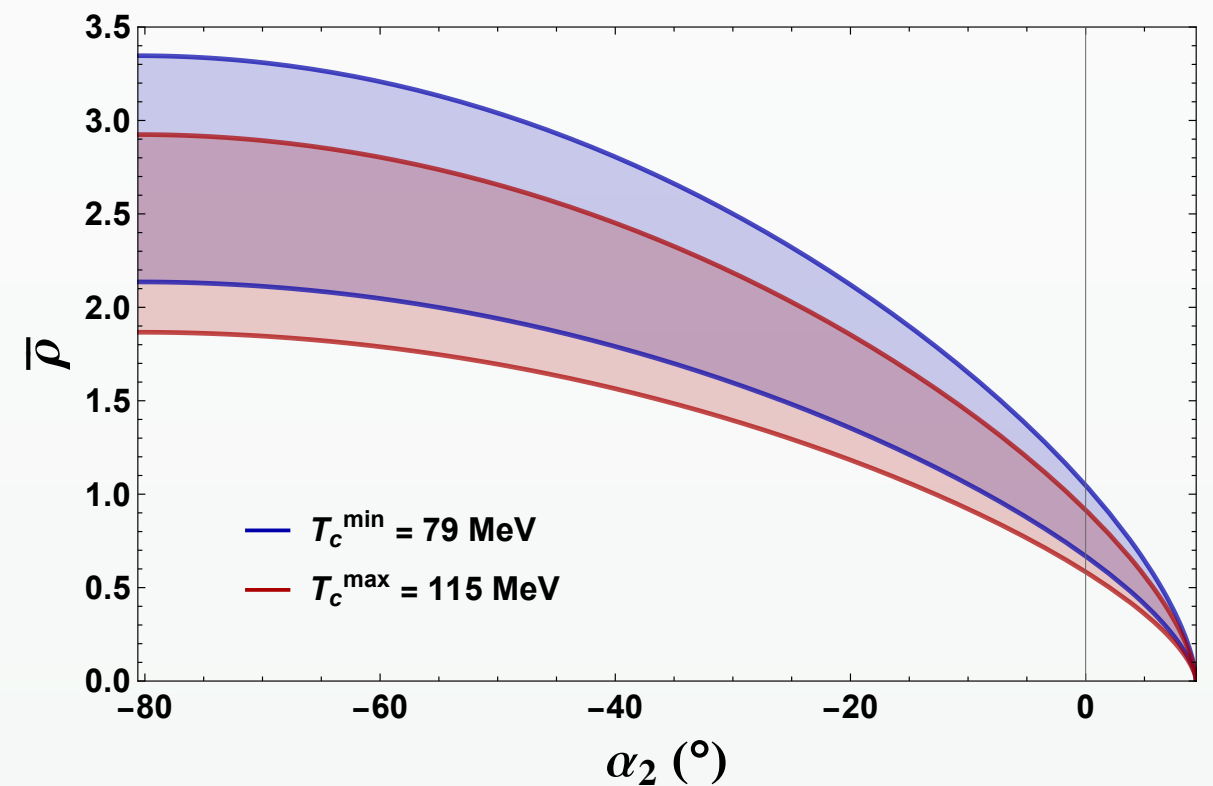
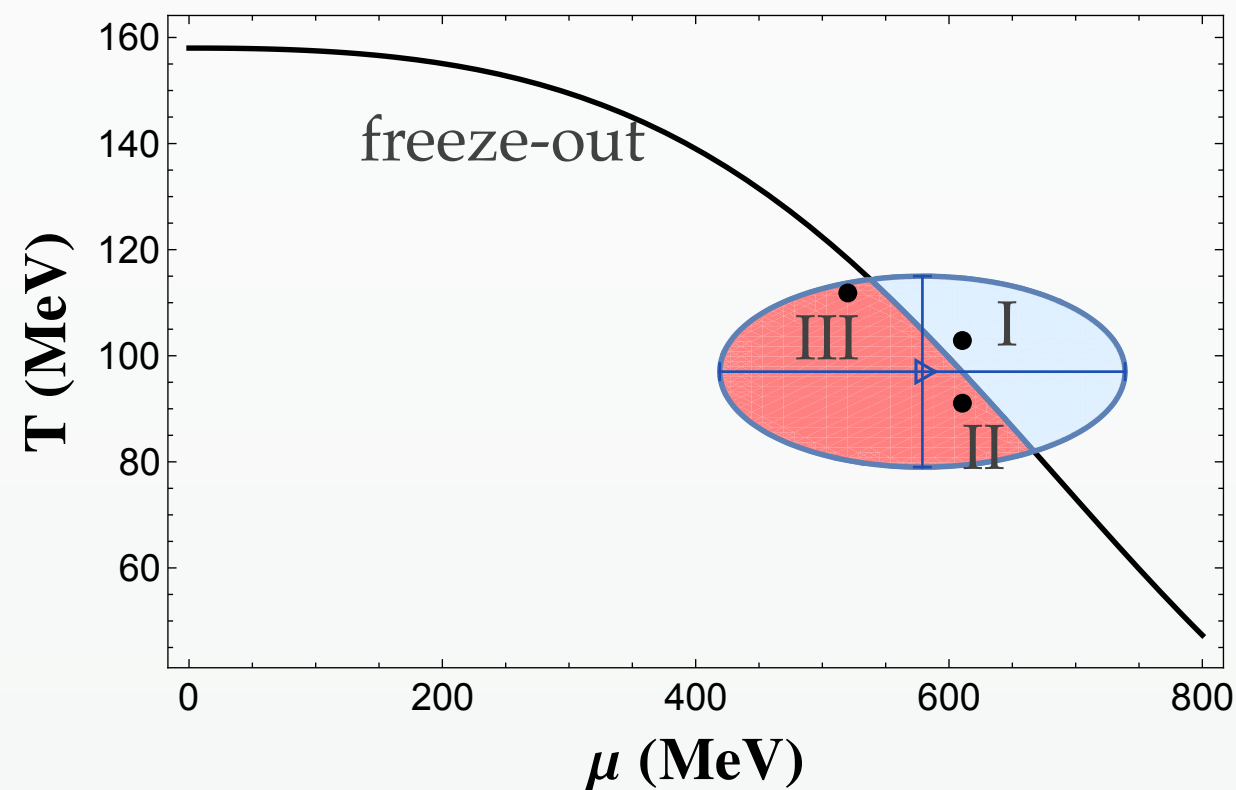
Some examples

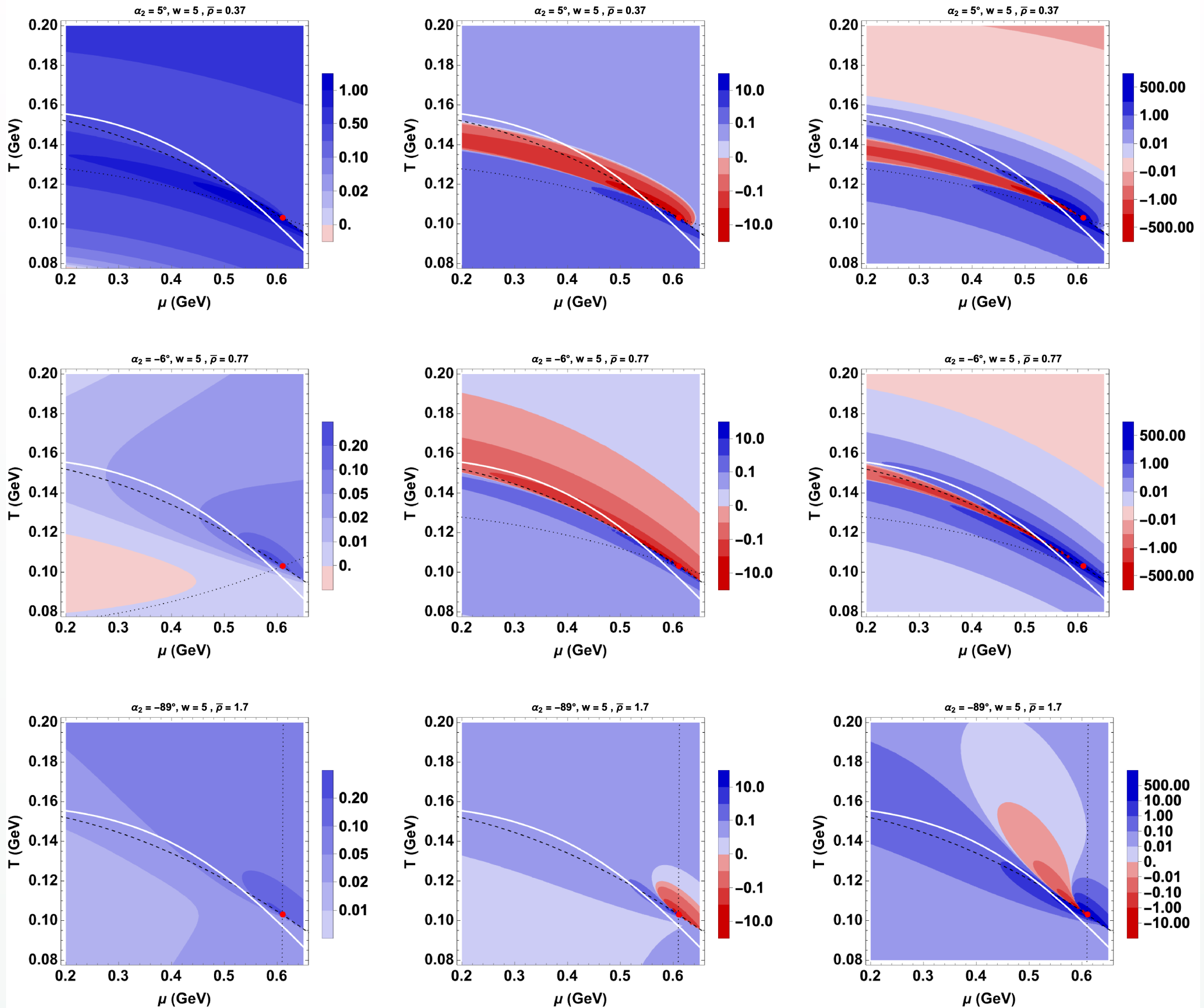
We consider three values for the critical point for illustration

Use the values for $\bar{\rho}$ and α_1 from conformal Padè

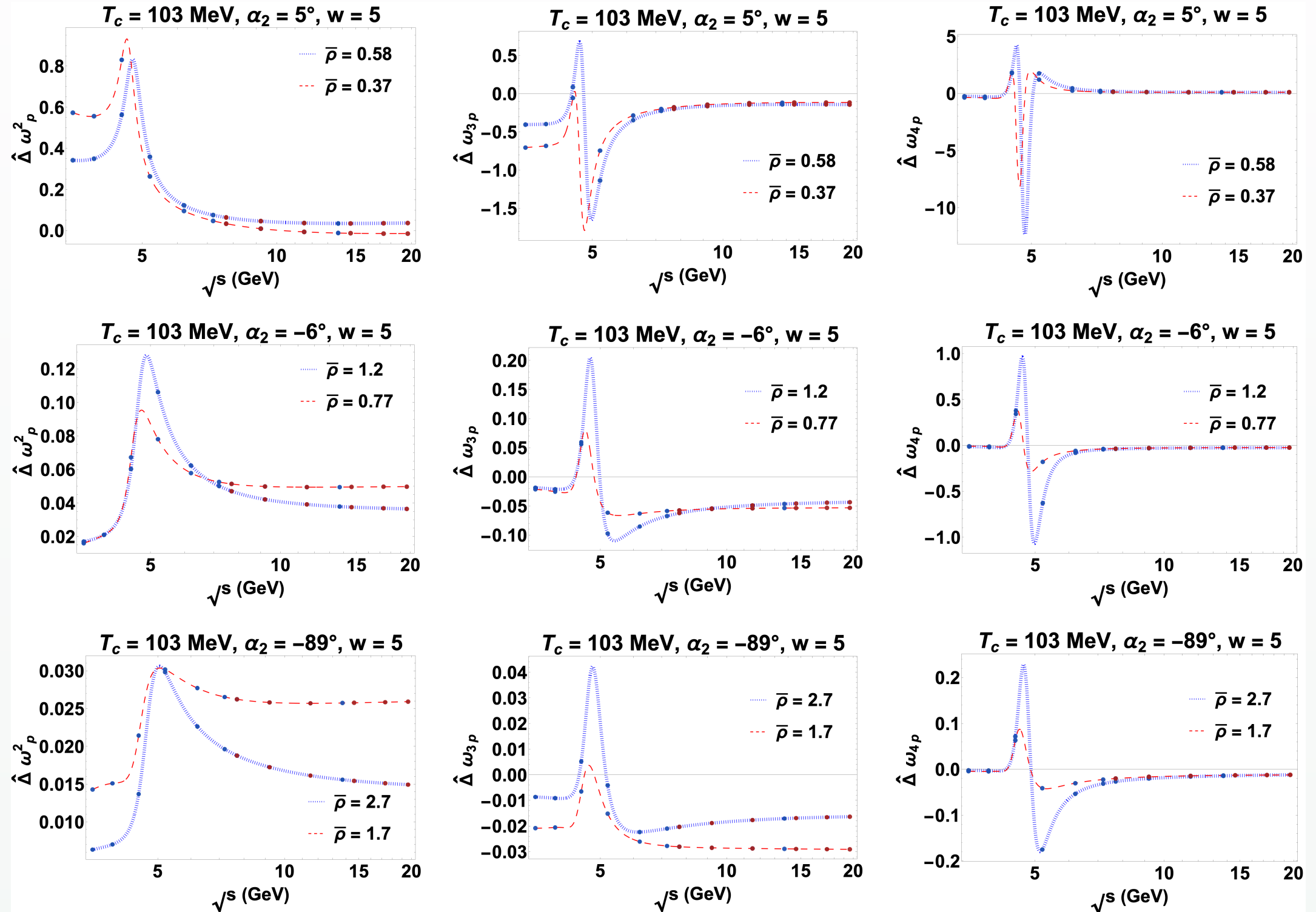
Take a range of α_2 s

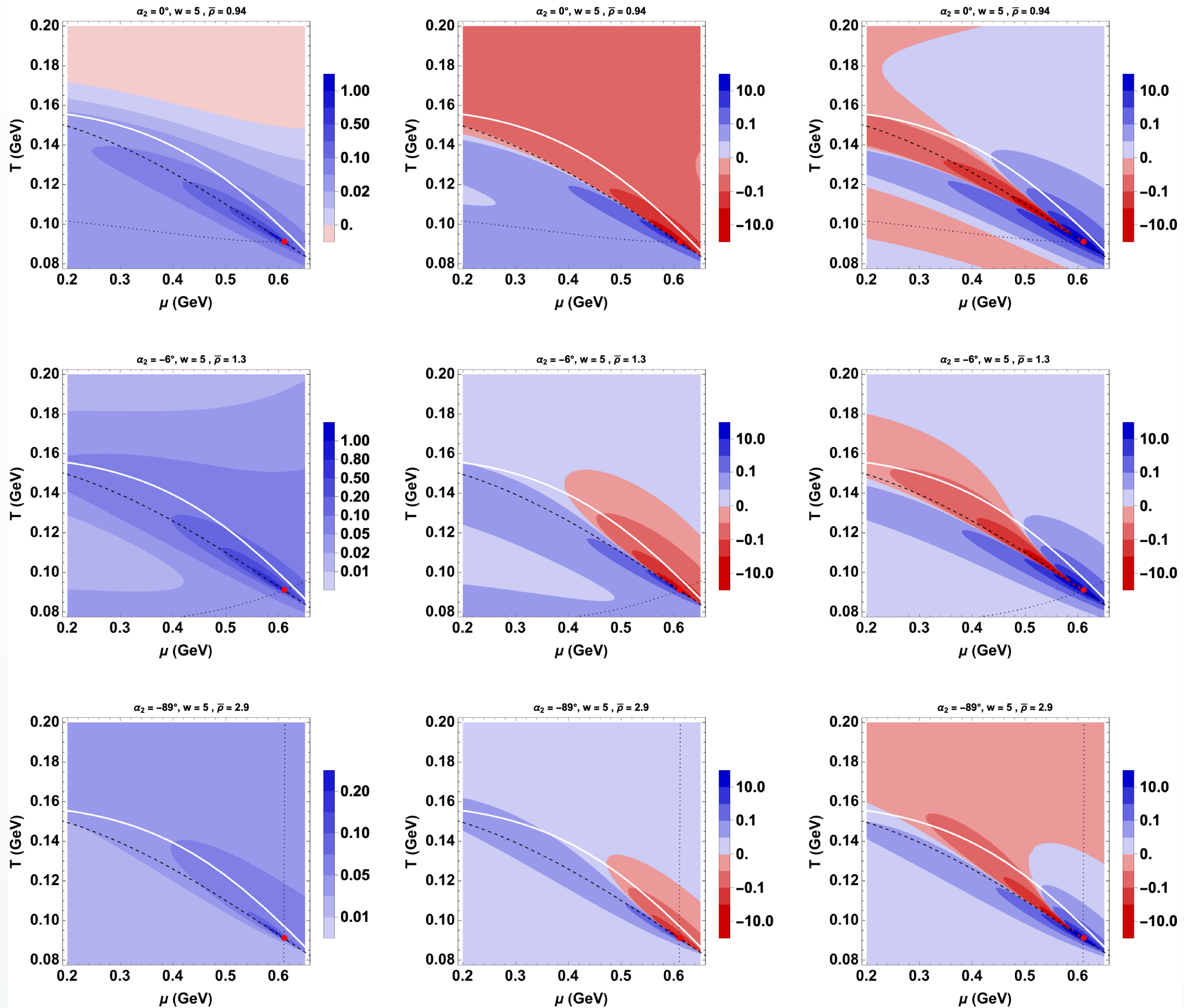
[GB, PRC 110 1, 015203]



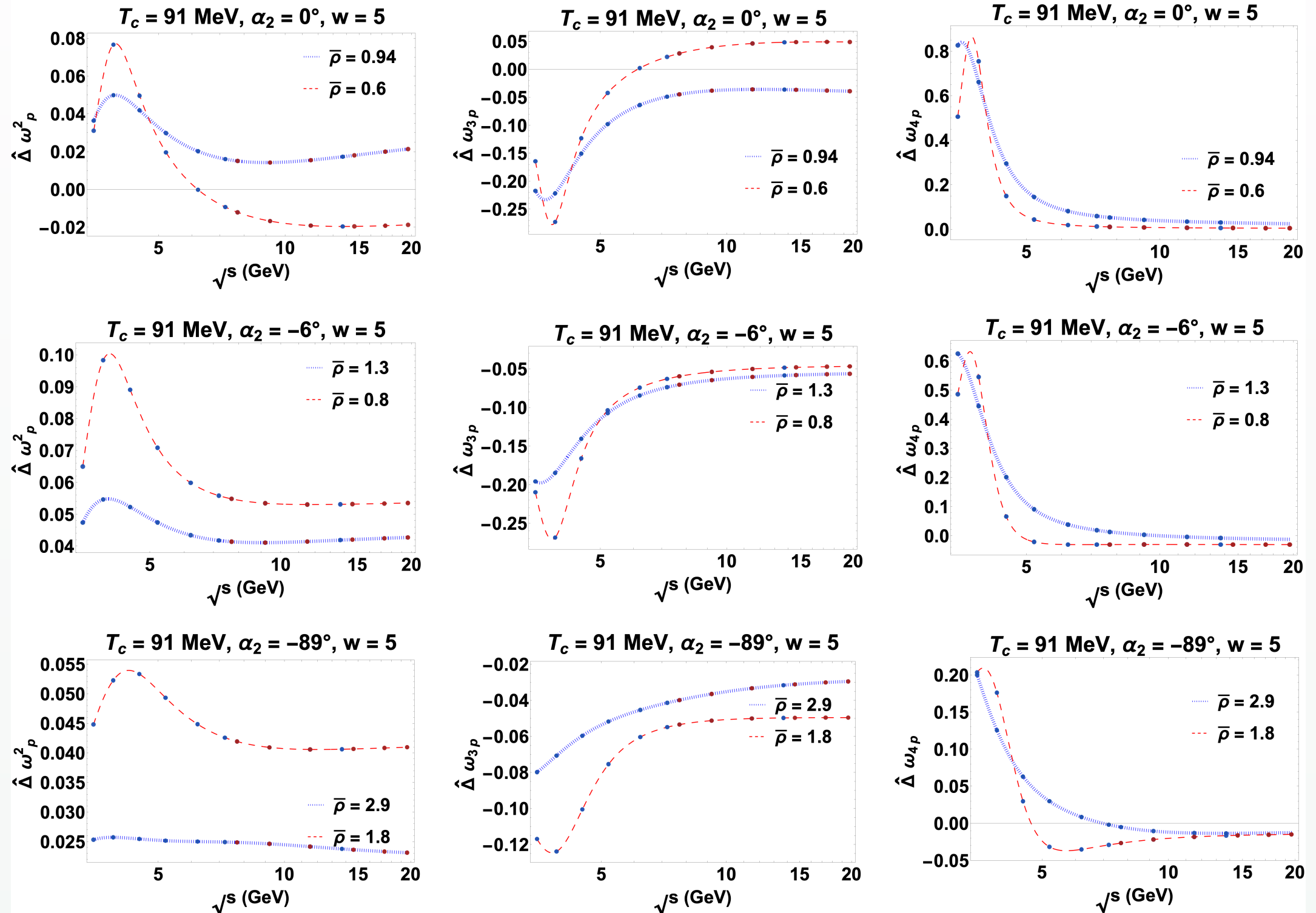


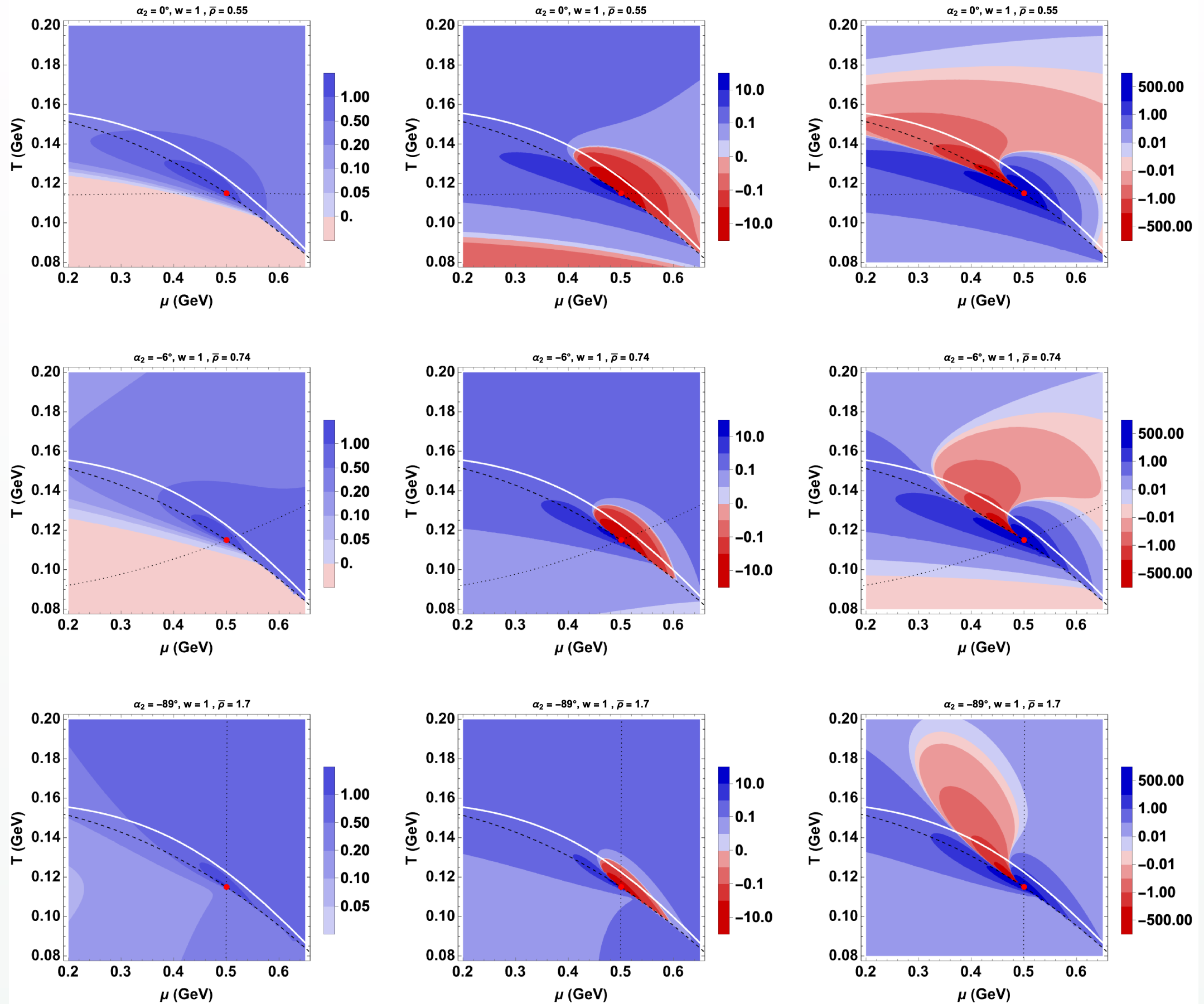
Factorial Cumulants



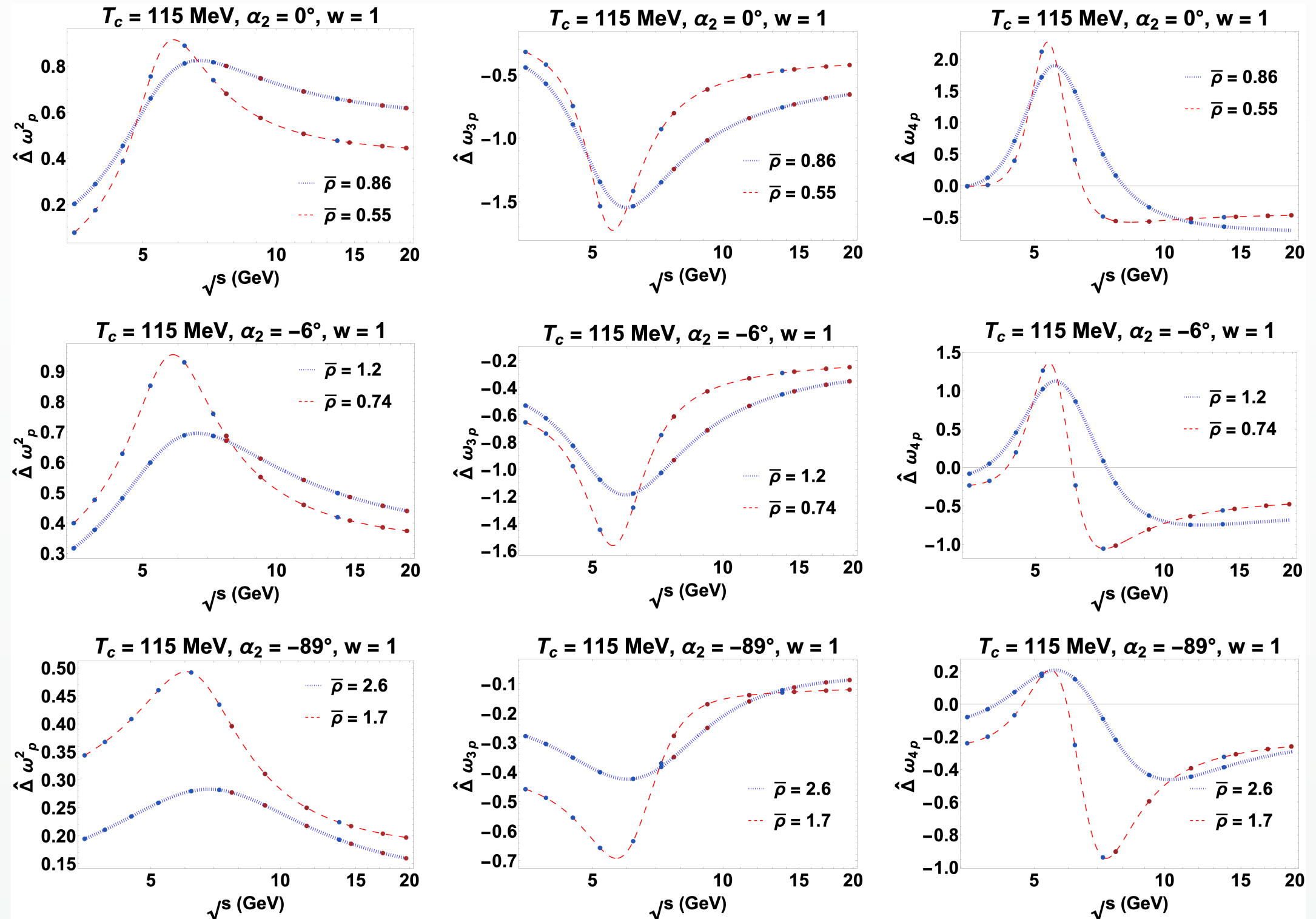


Factorial Cumulants





Factorial Cumulants



Conclusions

- Calculation of the proton number factorial cumulants from the lattice
- The signal is driven by the difference between QCD and HRG EoSs: *signs of the deconfinement transition*
- *All fluctuation modes (BB, BQ, QQ, etc...) contribute to proton number fluctuations*
- Equilibrium QCD EoS leads to a smaller signal for proton number cumulants compared to the BES data: *dynamical effects*
- In the critical region, the proton number cumulants depend sensitively on the non-universal mapping parameters between 3D Ising and QCD
- The location of the critical point in reference to the freeze-out curve as well as the values of the mapping parameters dramatically change the critical signal both qualitatively (dips and peaks) and quantitatively