

What does the lattice QCD Dirac spectrum tell us about hot QCD?

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Motivation: why would you care about the Dirac spectrum?

- $m_u \approx m_d \approx 0$
- Symmetries: $SU(2)_L \times SU(2)_R \times U(1)_V \times U(1)_A$
 - $U(1)_A$ anomalous
 - $SU(2)_L \times SU(2)_R \rightarrow SU(2)_V$ spontaneously broken below T_c
- Order parameter of the symmetry breaking (Banks-Casher formula):

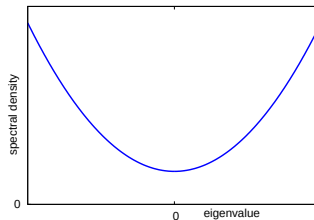
$$\langle \bar{\psi} \psi \rangle \propto \int d\lambda \frac{m}{\lambda^2 + m^2} \rho(\lambda, m) \xrightarrow{m \rightarrow 0} \rho(0, 0)$$

λ : spectrum of the Dirac operator, $\rho(\lambda)$: spectral density

The finite temperature transition

Below T_c

- Chiral symmetry broken
- Order parameter: $\rho(0) \neq 0$



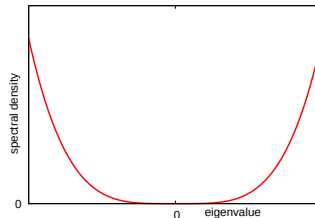
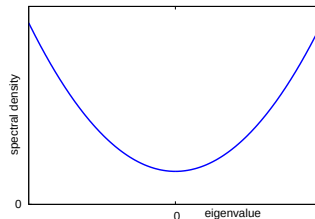
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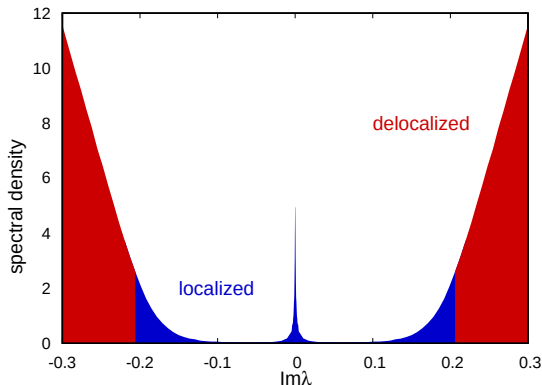
Above T_c

- Polyakov loop (temporal b.c.) gets ordered
- Matsubara gap appears
- Order parameter $\rho(0) = 0$
- Chiral symmetry restored



gap at $\lambda = 0 \iff$ chiral symmetry

Spectral density of the lattice Dirac operator for $T > T_c$



mobility edge at $\pm\lambda_c$

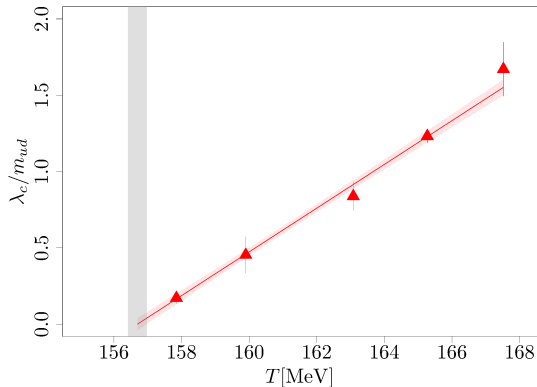
Anderson-type localization transition in the spectrum with same critical exponent

Giordano, TGK, Pittler, PRL 112 (2014) 102002

Mobility edge vs. temperature

$N_f = 2 + 1$ lattice, staggered quarks, physical masses

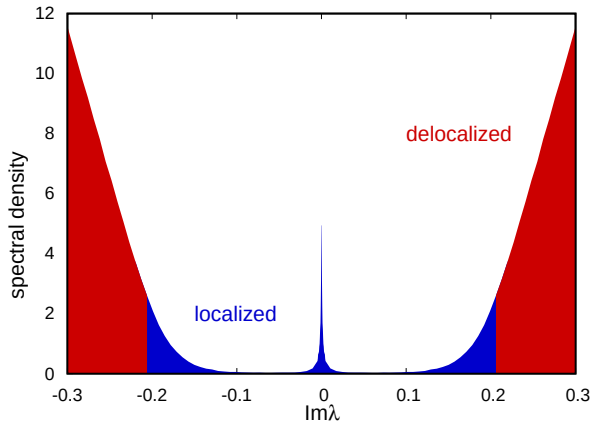
The mobility edge vanishes at the crossover $\rightarrow$ all states delocalized below T_c



Giordano, TGK, Pittler, PRD 113 (2026) 11, 114514.

Spectral density of the lattice Dirac operator for $T > T_c$

Overlap operator with exact chiral symmetry



Peak at $\lambda = 0$ in the spectral density

Edwards et al. PRD 61 (2000); Alexandru & Horvath, PRD 92 (2015); JLQCD, PRD 103 (2021) 074506

Kaczmarek, Mazur, Sharma, PRD 104 (2021) 2021; Fodor, Kotov, TGK, Szabó, PoS QCHSC24 (2025) 057

- Why is there a peak at the center of the gap?
- What happens to it in the chiral limit?
- How does the peak affect chiral symmetry?

Why should you care about the spectral peak

- Observables related to chiral symmetry dominated by $\lambda \lesssim m$.

- Chiral condensate:
 $SU(2)_L \times SU(2)_R \rightarrow SU(2)_V$
$$\langle \bar{\psi} \psi \rangle \propto \int_{-\Lambda}^{\Lambda} d\lambda \frac{m}{\lambda^2 + m^2} \rho(\lambda, m)$$

- $\pi - \delta$ susceptibility:
 $U(1)_A$ anomalous breaking
$$\chi_{\pi} - \chi_{\delta} \propto \int_{-\Lambda}^{\Lambda} d\lambda \frac{m^2}{(\lambda^2 + m^2)^2} \rho(\lambda, m)$$

“Instantons” → spectral peak

- (Anti)instanton
→ zero eigenvalue of $D(A)$ with $(-)+$ chirality eigenmode
- High T :
large instantons “squeezed out” in the temporal direction
→ dilute gas of instantons and anti-instantons
- Zero modes exponentially localized:

$$\psi(r) \propto e^{-\pi Tr}$$

- Condensed matter analogy:
modes in a gap, bound to an impurity are exponentially localized

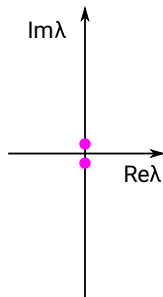
Instanton-antiinstanton pair

The Dirac operator in the subspace of zero modes

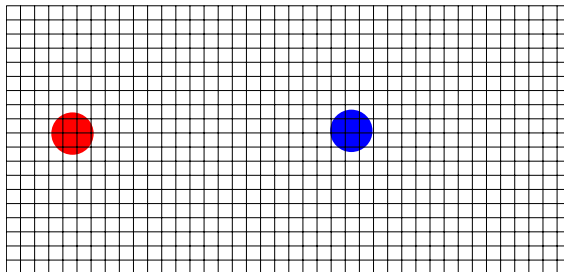
$$D(A) = \begin{pmatrix} 0 & iw \\ iw & 0 \end{pmatrix}$$

$$w \propto e^{-\pi Tr}$$

Spectrum of $D(A)$



Instanton and antiinstanton



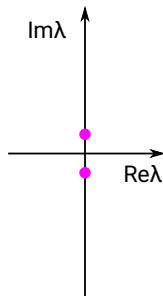
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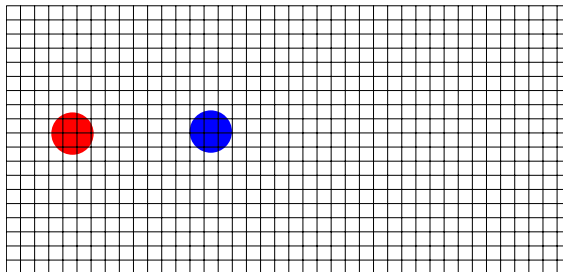
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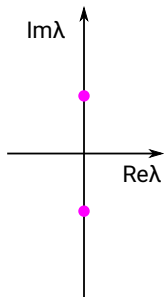
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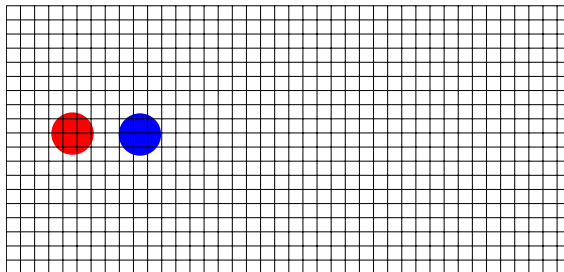
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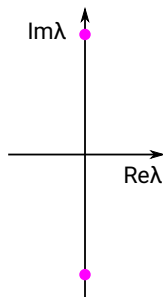
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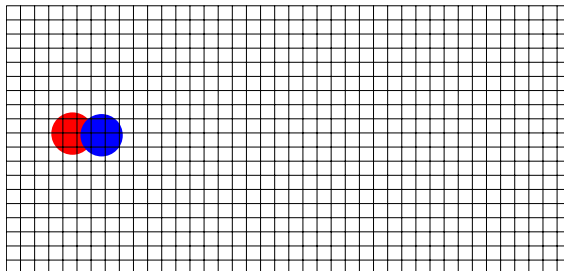
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Spectrum of $D(A)$



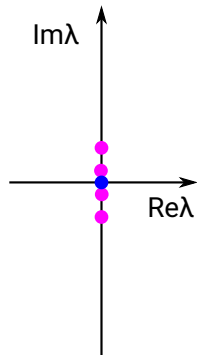
Instanton and antiinstanton



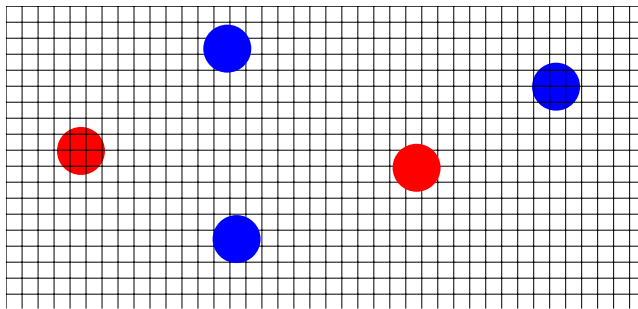
Spectrum of $D(A)$ in dilute gas of instantons

The Dirac operator in the subspace of zero modes

Spectrum of $D(A)$



Instantons and antiinstantons



n_i instantons n_a antiinstantons

→ $|n_i - n_a|$ exact zero modes + mixing near zero modes

Subspace of zero modes = spectral peak

Instanton-based random matrix model of the peak

Work by E.V. Shuryak, J.J.M. Verbaarschot, T. Schäfer... (1990-2000)

- n_i instantons, n_a antiinstantons in 3d box of size L^3
- Construct $(n_i + n_a) \times (n_i + n_a)$ matrix:

$$D = \begin{pmatrix} \overbrace{0}^{n_i} & \overbrace{iW}^{n_a} \\ iW^\dagger & 0 \end{pmatrix}$$

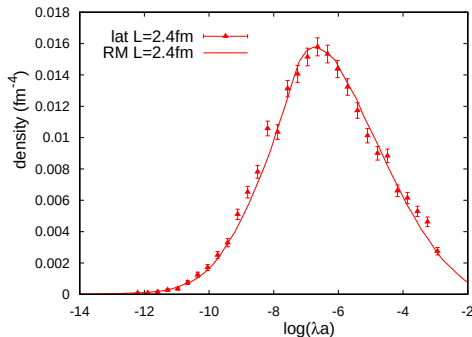
- $w_{ij} = A \cdot \exp(-\pi T \cdot r_{ij})$ r_{ij} is the distance of instanton i and antiinstanton j

Random matrix model of the spectral peak

One-parameter fit to the lattice Dirac spectrum

- Two parameters:
 - χ – topological susceptibility: from exact zero modes $\rightarrow \chi = \langle Q^2 \rangle / V$
 - A – prefactor of the exponential mixing between zero modes
- Fit A to distribution of Dirac eigenvalues (lowest eigenvalue)

$L = 2.4\text{fm}$ fit



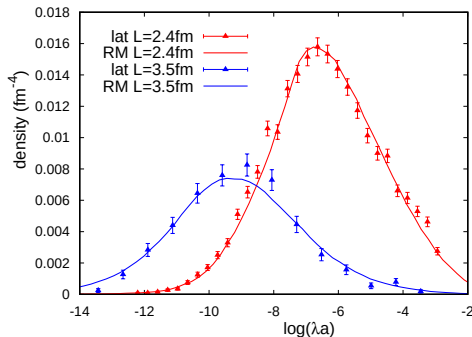
Random matrix model of the spectral peak

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$L = 2.4\text{fm}$ fit

$L = 3.5\text{fm}$ prediction



Random matrix model of the spectral peak

- Perfectly reproduces spectral peak of chiral lattice Dirac operator
- $\approx 10^6 \times$ fewer degrees of freedom than lattice
- Easily simulated in large volumes with light dynamical chiral quarks
- Simulations with $N_f = 2$

TGK PRL 132 (2024) 131902

Simulation results

Random matrix model with $N_f = 2$ dynamical quarks

- Ideal instanton gas + I-A molecules (dynamical quarks $\rightarrow$ I-A attraction)
- Free instanton gas $\rightarrow$ singular spectral density $\rho(0) = ?$ Banks-Casher?
- $SU(2)_V \rightarrow SU(2)_L \times SU(2)_R$ restored
- $U(1)_A$ breaking remains in $\chi_\pi - \chi_\delta$

“Banks-Casher” for singular spectral density

$$\langle \bar{\psi} \psi \rangle \propto \left\langle \sum_i \frac{m}{m^2 + \lambda_i^2} \right\rangle \approx \underbrace{\left(\text{avg. number of instantons in free gas} \right)}_{\propto \chi_{top} \propto m^{N_f}} \cdot \frac{1}{m} \propto m^{N_f-1}$$

$|\lambda_i| \ll m \longleftarrow |\lambda_i|$ exponentially small in typical I-A distance

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Is $U(1)_A$ breaking present in $\chi_\pi - \chi_\delta$?

Yes!

$$\chi_\pi - \chi_\delta \propto \left\langle \sum_i \frac{m^2}{(m^2 + \lambda_i^2)^2} \right\rangle \approx \underbrace{\left(\text{avg. number of in-stantons in free gas} \right)}_{\propto \chi_{top} \propto m^{N_f}} \cdot \frac{1}{m^2} \propto m^{N_f-2}$$

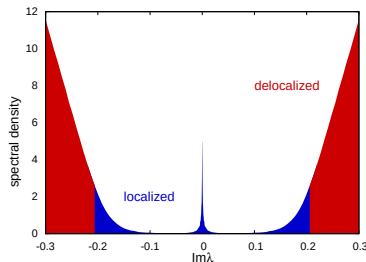
$\rightarrow \lim_{m \rightarrow 0} (\chi_\pi - \chi_\delta) \neq 0$ for $N_f = 2$

Further consistency check

- If $\langle \bar{\psi}\psi \rangle = 0$ and $\chi_\pi - \chi_\delta \neq 0 \implies$ eigenmodes around $\lambda = 0$ delocalized

Exact proof: [Giordano, arXiv:2510.24381](#) and [arXiv:2510.24392](#)

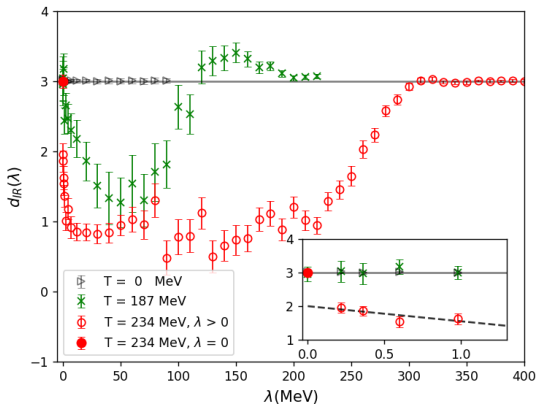
- But aren't modes in spectral peak all localized?



Zoom into the peak: lattice QCD

Dimension of eigenmodes across the spectrum

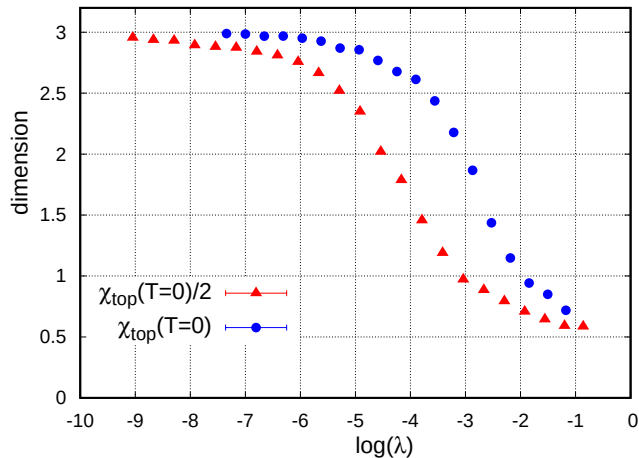
- eigenmode “volume”
 $\propto (\text{box linear size})^d$
 - $d = 0$ localized
 - $d = 3$ delocalized
- Modes close to zero delocalize
- Is there another mobility edge?
- Talk by I. Horváth, next Wed.



Meng, Sun, Alexandru, Horváth, Liu, Wang, Yang, JHEP 12 (2024) 101.

Dimension of the eigenmodes across the spectrum

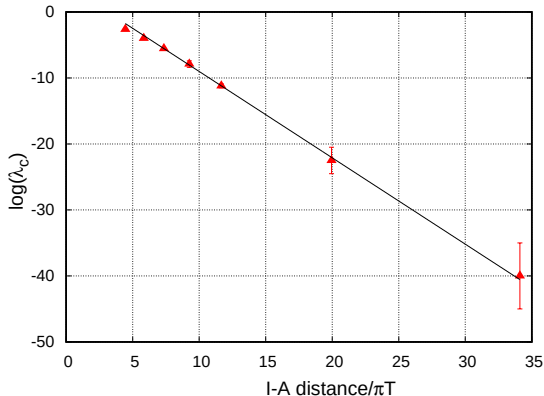
Instanton-based random matrix model



Mobility edge exponentially small for dilute system

Finite size scaling $\longrightarrow$ mobility edge λ_c

- Two length scales
- Closest I-A distance $\propto \chi_{\text{top}}^{-1/3}$
- $1/\pi T$



Condensed matter analogy

- Bipartite lattice $\rightarrow$ chiral symmetry $\{H, \gamma_5\} = 0$
- On-site energy 0, uniform hopping
- Random vacancies $\rightarrow$ zero modes
- Zero modes in the gap $\rightarrow$ mixing exponential in distance
- Examples: graphene with vacancies, diluted Kitaev model etc.

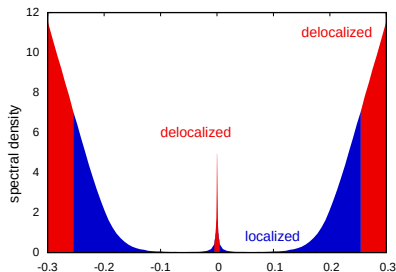
Conclusions

- Above T_c ideal gas of unit charge topological lumps $\rightarrow$ chiral properties
- Dirac spectral density has singular peak and delocalized modes at zero
- $N_f = 2$ “Effectively” $\rho(\lambda) \propto m^2 \delta(\lambda)$ since peak width $\ll m$

Chiral symmetry:

$$\langle \bar{\psi} \psi \rangle \propto \frac{\chi_{\text{top}}}{m} \propto m^{N_f-1}$$

$$\chi_\pi - \chi_\delta \propto \frac{\chi_{\text{top}}}{m^2} \propto m^{N_f-2}$$



BACKUP SLIDES

Random matrix model of $D(A)$ in the zero mode zone

- How to choose instanton numbers (n_i, n_a) and locations?

- Quenched lattice $T > 1.05 T_c \rightarrow$ free instanton gas

Bonati et al. PRL 110 (2013); Vig R. & TGK, PRD 103 (2021)

- n_i and n_a independent identical Poisson-distributed

$$p(n_i, n_a) = e^{-\chi V} \cdot \frac{(\chi V/2)^{n_i}}{n_i!} \cdot \frac{(\chi V/2)^{n_a}}{n_a!}$$

χ is the topological susceptibility

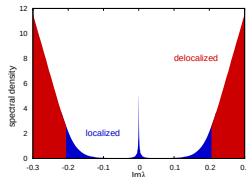
- Locations random (uniform)

- $\rightarrow D(A)$ in quenched QCD $\Leftrightarrow$ ensemble of random matrices

Random matrix model of full QCD zero mode zone

- Include $\det(D + m)^{N_f}$ in Boltzmann weight
- $\det(D + m) = \prod_{\text{zms}} (\lambda_i + m) \times \prod_{\text{bulk}} (\lambda_i + m)$
- Bulk weakly correlated with zero mode zone
- Approximate det with $\prod_{\text{zms}} (\lambda_i + m)$
- Consistently included in RM model:

$$P(n_i, n_a) = \underbrace{e^{-\chi_0 V} \frac{1}{n_i!} \frac{1}{n_a!} \left(\frac{\chi_0 V}{2} \right)^{n_i + n_a}}_{\text{free instanton gas with random locations}} \times \det(D + m)^{N_f}$$



Free instanton gas component remains

- Quark determinant for n_i instantons and n_a anti-instantons:

$$\det(D + m)^{N_f} = \prod_{n_i, n_a} (\lambda_i + m)^{N_f} \approx m^{N_f(n_i + n_a)}$$

if $|\lambda_i| \ll m$

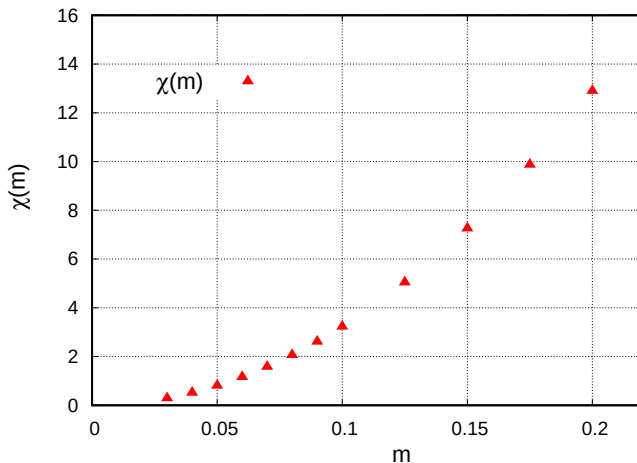
- Reweighting depends on number of topological objects, not on their type or location

$$P(n_i, n_a) \propto \left(\frac{\chi_0 V}{2}\right)^{n_i + n_a} \times \det(D + m)^{N_f} \approx \left(\frac{m^{N_f} \chi_0 V}{2}\right)^{n_i + n_a}$$

- Free gas, but susceptibility suppressed as $\chi_0 \rightarrow m^{N_f} \chi_0$
- As $m \rightarrow 0$ instanton gas more dilute $\Rightarrow |\lambda_i|$ smaller
- Even in the chiral limit $|\lambda_i| \ll m \Rightarrow$ free instanton gas

Random matrix simulation: results for $N_f = 2$

Topological susceptibility:



Random matrix simulation: results for $N_f = 2$

Topological susceptibility:

$$\chi(m) = m^2 \chi_0 \quad \text{not a fit!}$$

↑ quenched susceptibility

