Bootstrapping Spectral Functions

Reference: arXiv:2605.20509

In collaboration with Sarah Fields, Will Jay, Patrick Oare, and Matteo Saccardi

Ryan Abbott (Ernest Kempton Adams Fellow, Columbia University)

The RBC & UKQCD collaborations

BNL & RBRC
Peter Boyle
Taku Izubuchi
Chulwoo Jung
Christopher Kelly
Patrick Oare
Amarjit Soni (emeritus)
Xin-Yu Tuo
Tianle Wang
Shuhei Yamamoto

CERN
Matteo Di Carlo
Felix Erben
Andreas Jüttner (Southampton)

Columbia University
Ryan Abbott
Norman Christ
Jonah Eick
Rajiv Eranki
Sarah Fields
Ceran Hu
Yikai Huo
Erik Lundstrum
Robert Mawhinney

University of Connecticut
Thomas Blum
Jonas Hildebrand
Luchang Jin
Vaishakhi Moningi
Anton Shcherbakov
Douglas Stewart
Joshua Swain

DESY Zeuthen
Raoul Hodgson

University of Edinburgh
Matthew Black
Luigi Del Debbio
Vera Gülpers
Maxwell T. Hansen
Nils Hermansson Truedsson
Ryan Hill
Rajnandini Mukherjee
Antonin Portelli
Teseo San Jose

Johannes Gutenberg University of Mainz
Alessandro Barone

KEK
Shigemi Ohta (emeritus)

Liverpool Hope
Nicolas Garron (Liverpool)

University of Liverpool
Tobias Tsang
Aurelia Zuchanke

Massachusetts Institute of Technology
En-Hung Chao

University of Milano Bicocca
Francesca A. Bresciani
Mattia Bruno
Gabriele Morandi

Peking University
Xu Feng

University of Regensburg
Daniel Knüttel
Christoph Lehner
Julian Parrino
Andreas Völklein
Haoyu Wang

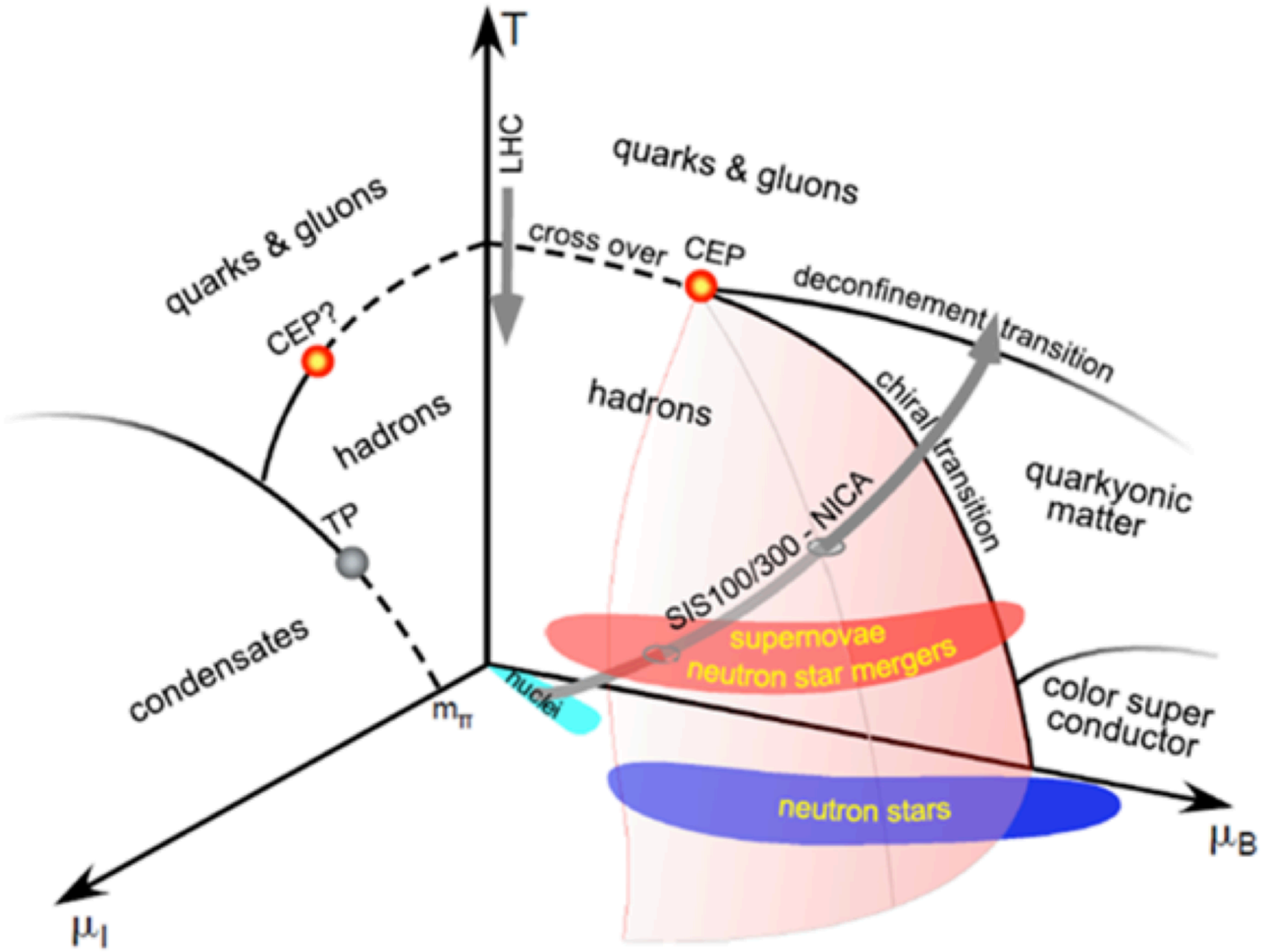
RIKEN R-CCS
Masaaki Tomii

University of Siegen
Oliver Witzel

University of Southampton
Bipasha Chakraborty
Ahmed Elgaziari
Jonathan Flynn
Ollie Heald
Joe McKeon
Chris Sachrajda

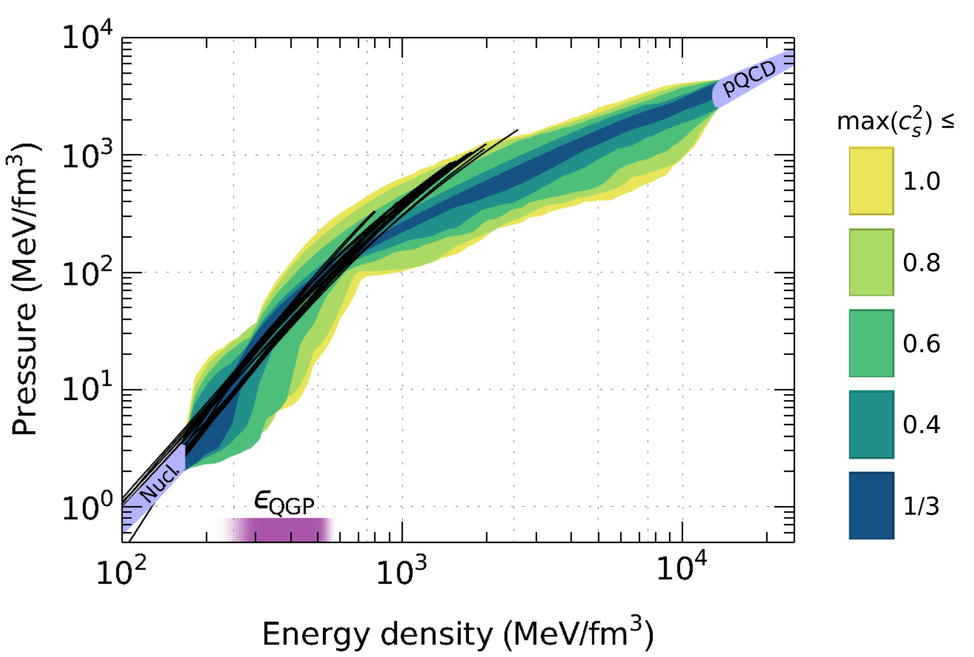
Stony Brook University
Sergey Syritsyn (RBRC)

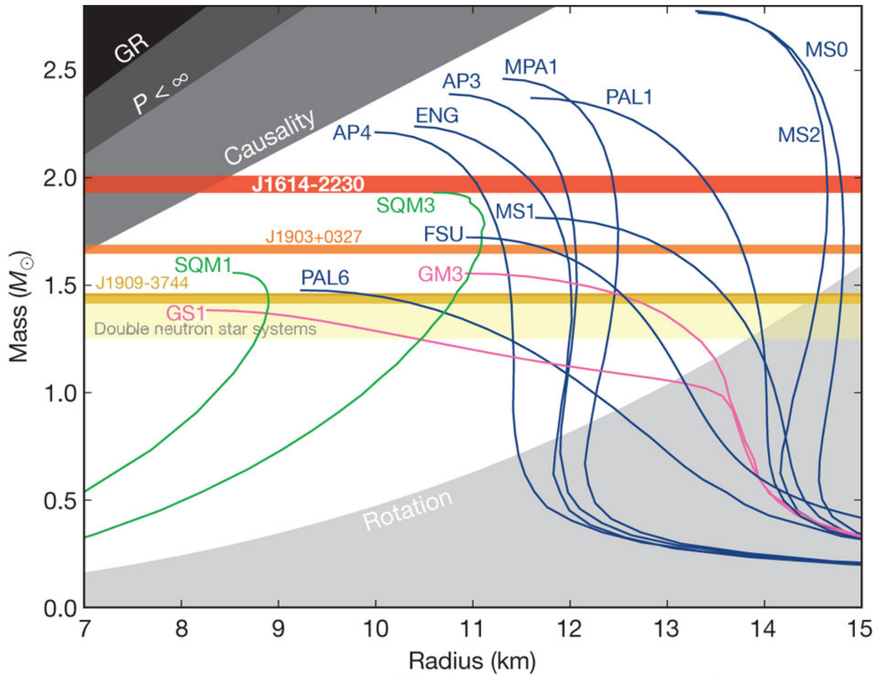
QCD Phase diagram



Equation of State

Equation of State → Mass-Radius Relations





[Annala et al., 1903.09121]

[Demorest et al, 1010.5788]

Ideally compute $P(\epsilon)$ directly, but $\mu_B \neq 0 \Longrightarrow$ sign problem

What we know rigorously

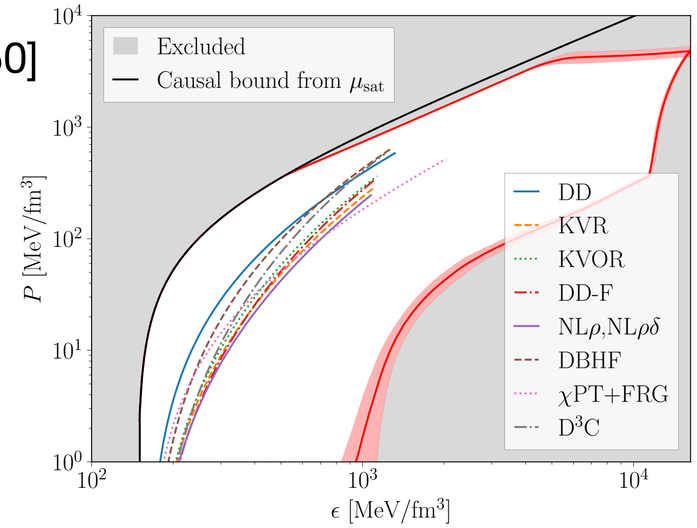
Causality $c_s^2 \leq 1$

pQCD (large μ_B)

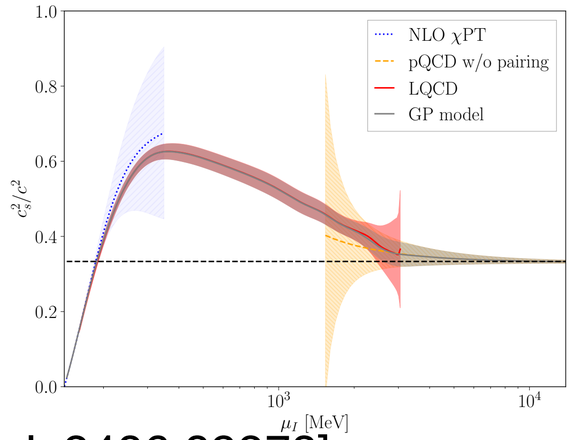
Chiral EFT (small μ_B)

[Komoltsev & Kurkela, 2111.05350]

[Abbott et al, 2406.09273]



Isospin dense QCD



[Abbott et al, 2406.09273]

$$p_B(\mu_B) \leq p_I(\mu_I = 2\mu_B/3)$$

[Fujimoto & Reddy, 2310.09427]
[Moore & Gorda, 2309.15149]
[Cohen, hep-ph/0304024]

(see also: Phase-quenched, Bastian Brandt's talk Monday)

What we know rigorously

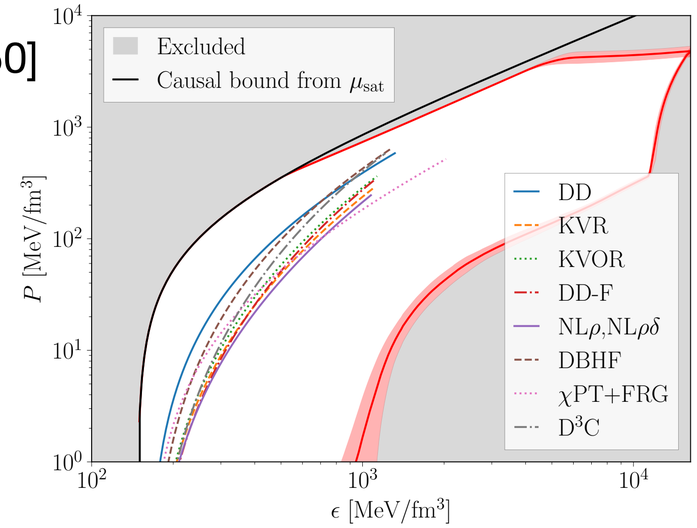
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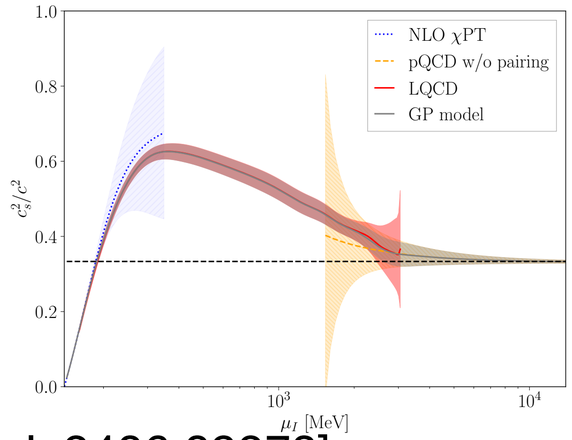
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Frustratingly finite — no systematic improvement

Dense vs Real-time

Dense QCD

$$\int \mathscr{D}U\, e^{-S(U)} \det D(U;\mu)$$

- Phase $\sim e^{i\mu_B N} \Rightarrow$ sign problem
- Perturbative at large μ_B
- Hard at intermediate μ_B

Real-time QCD

$$\int \mathscr{D}U\, e^{iS[U]}$$

- Phase $\sim e^{-iEt} \Rightarrow$ sign problem
- Perturbative at large E
- Hard at intermediate E

Dense vs Real-time

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Key upshot: Spectral functions are becoming tractable (but hard)

Still open: Generalizing to Dense QCD

Spectral functions

- Källén-Lehmann Spectral Representation: Spectral function/density

$$\langle 0 | T\{\mathcal{O}(x)\mathcal{O}(y)^\dagger\} | 0\rangle = \int_0^\infty d(E^2)\rho(E) \int \frac{d^4p}{(2\pi)^4} \frac{ie^{ip\cdot(x-y)}}{p^2 - E^2 + i0}$$

- Other forms:

$$\rho(E) = \int e^{iEt} dt \langle 0 | [\mathcal{O}(t), \mathcal{O}(0)^\dagger] | 0\rangle \sim \sum_X |\langle X | \mathcal{O}^\dagger | 0\rangle|^2 \delta(E_X - E)$$

- Real-time quantity, closely related to inclusive processes
- Many generalizations: $T \neq 0$, dense, matrix-valued

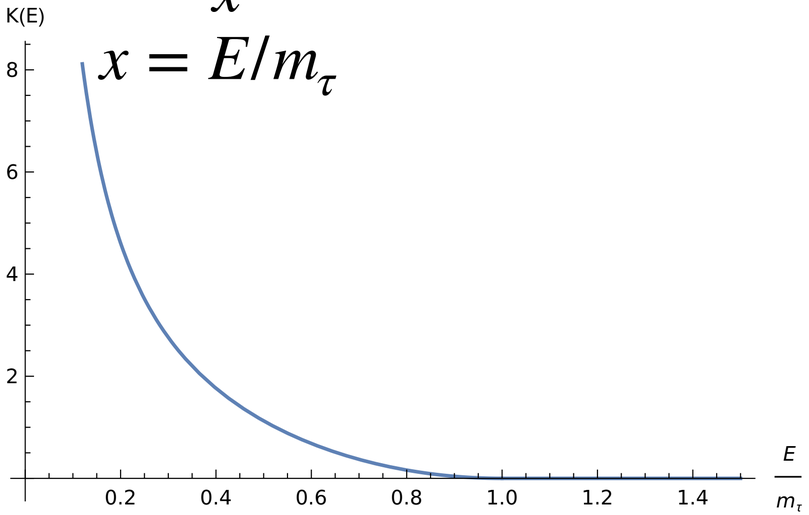
Example: Inclusive Decays & Spectral Functions

$$\sum_X \left| \tau \to J^\mu \; X \; \nu_\tau \right|^2 = \int K(E)\rho(E)dE$$

Kernel, Leptonic (known)

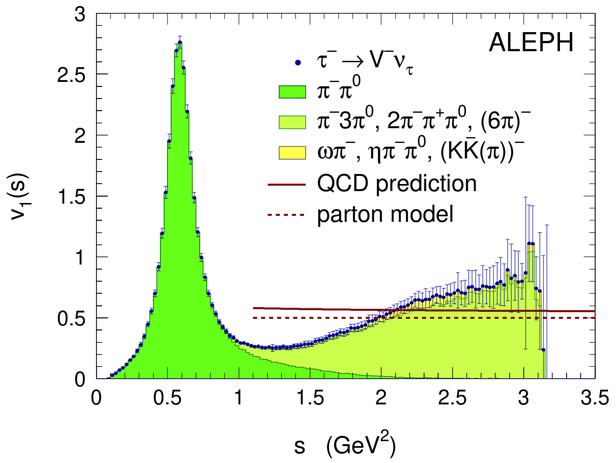
$$K(E) = \frac{1}{x}\left(1 - x^2\right)^2 \theta(1-x)$$

$$x = E/m_\tau$$



Spectral function, Hadronic

$$\rho(E) \sim \sum_X \left| \langle 0 | J | X \rangle \right|^2 \delta(E_X - E)$$



Spectral Functions from Lattice QCD

- Numerically determining ρ requires inverting *Laplace transform*
  - Often very ill-conditioned/ill-posed

$$C(t) = \langle \mathcal{O}(t)\mathcal{O}(0)^\dagger \rangle = \int \rho(E) e^{-Et} dE$$

- Problems: ρ is a distribution, finite vs infinite volume
  - Key idea: work with *smeared* spectral functions

$$\mathcal{K}[\rho] = \int \rho(E) K(E) dE$$

- Standard methods: Backus-Gilbert/Hansen Lupo Tantalo (HLT)

HLT: Exact Data

Some kernels are easy:

$$K_g(E) = \sum_t g_t e^{-Et} \qquad \Longrightarrow \qquad \int K_g(E)\rho(E)dE = \sum_t g_t C_t$$

Idea: tune $\{g_t\}$ so $K_g(E) \approx K(E)$, e.g.

$$\text{minimize } A[g] = \int dE \, |\, K_g(E) - K(E) \,|^2$$

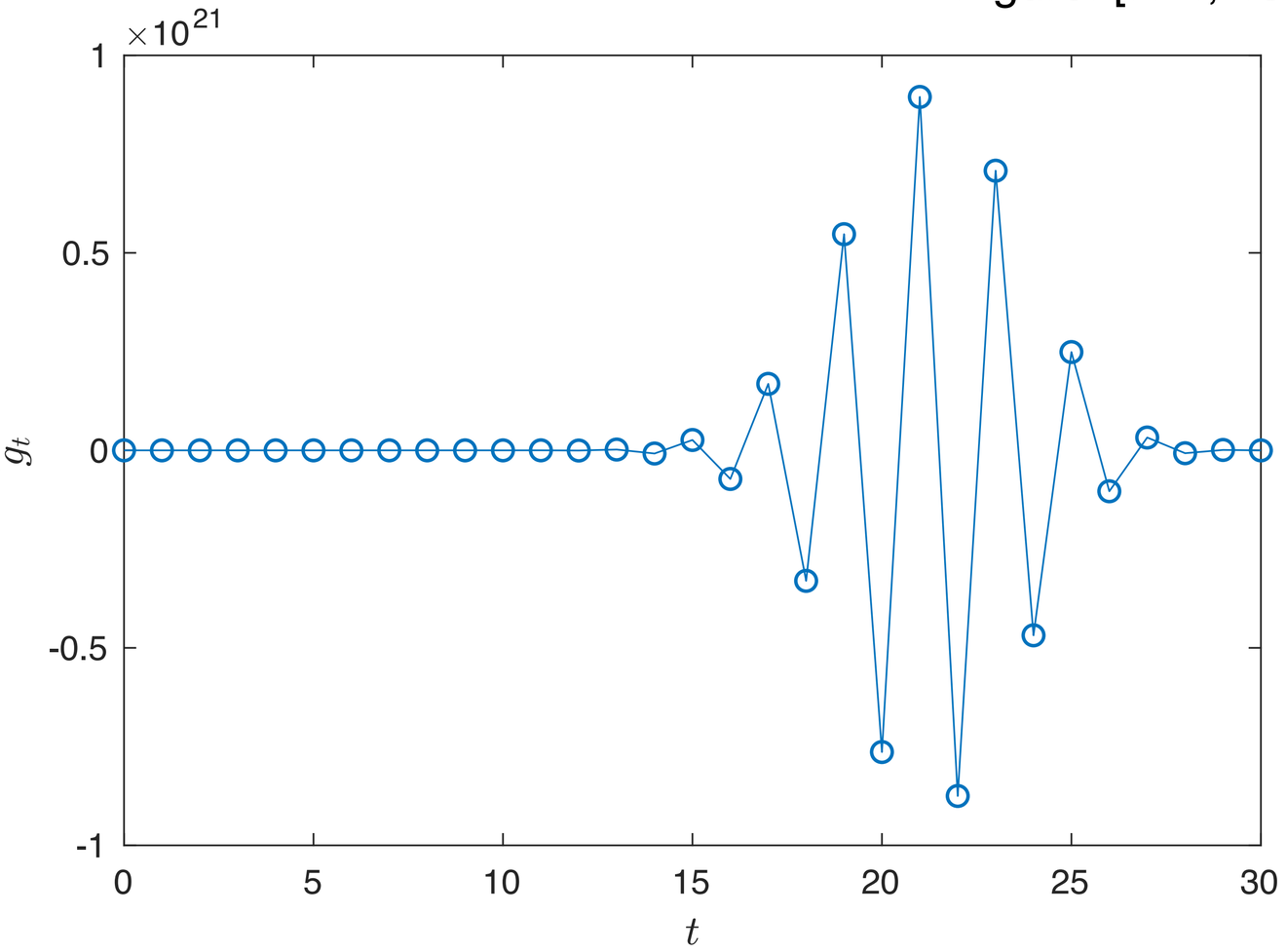
Works very well, with exact data…

HLT & Noise

Problem:

Figure: [HLT, 1903.06476]

$$\mathscr{K}[\rho] \approx \sum_t g_t C_t$$

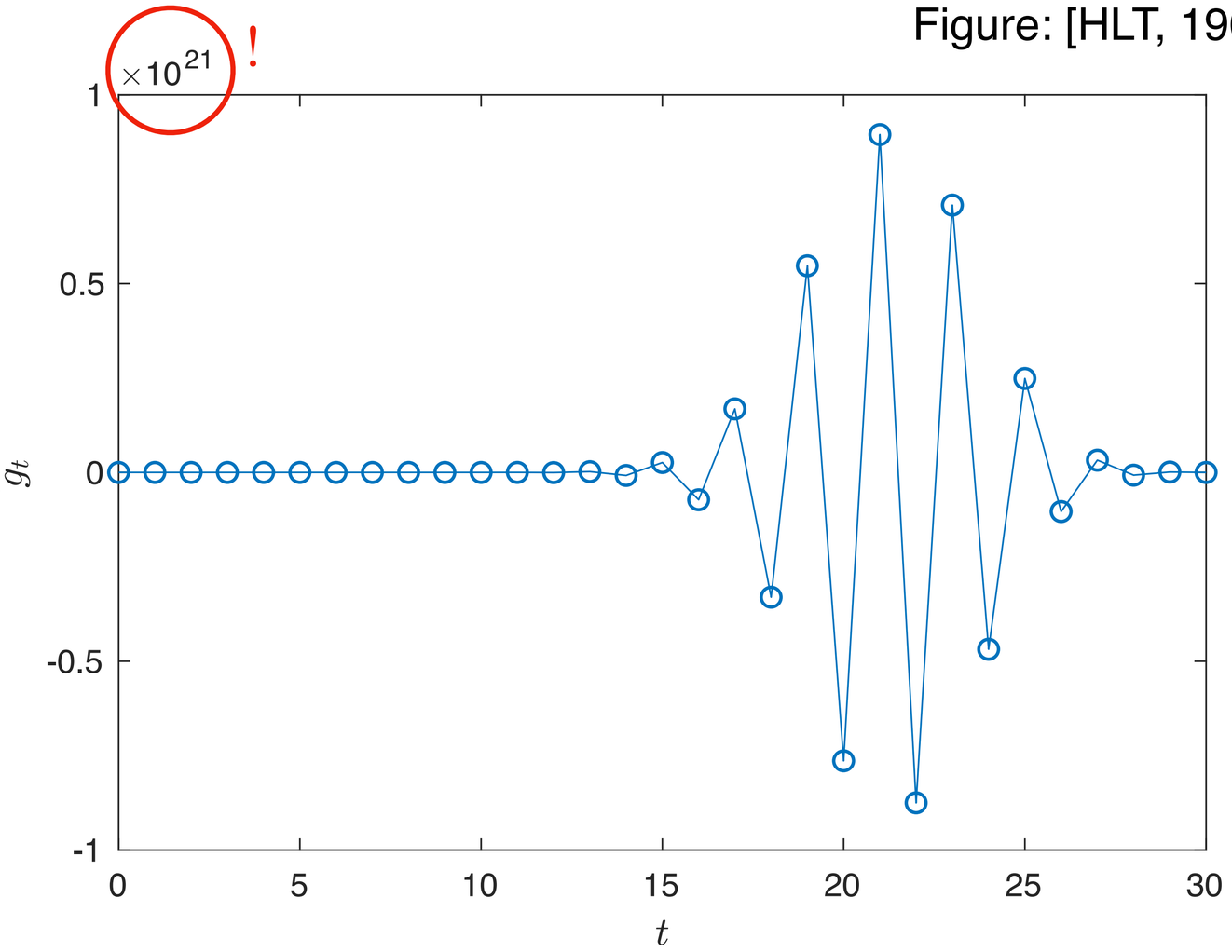


HLT & Noise

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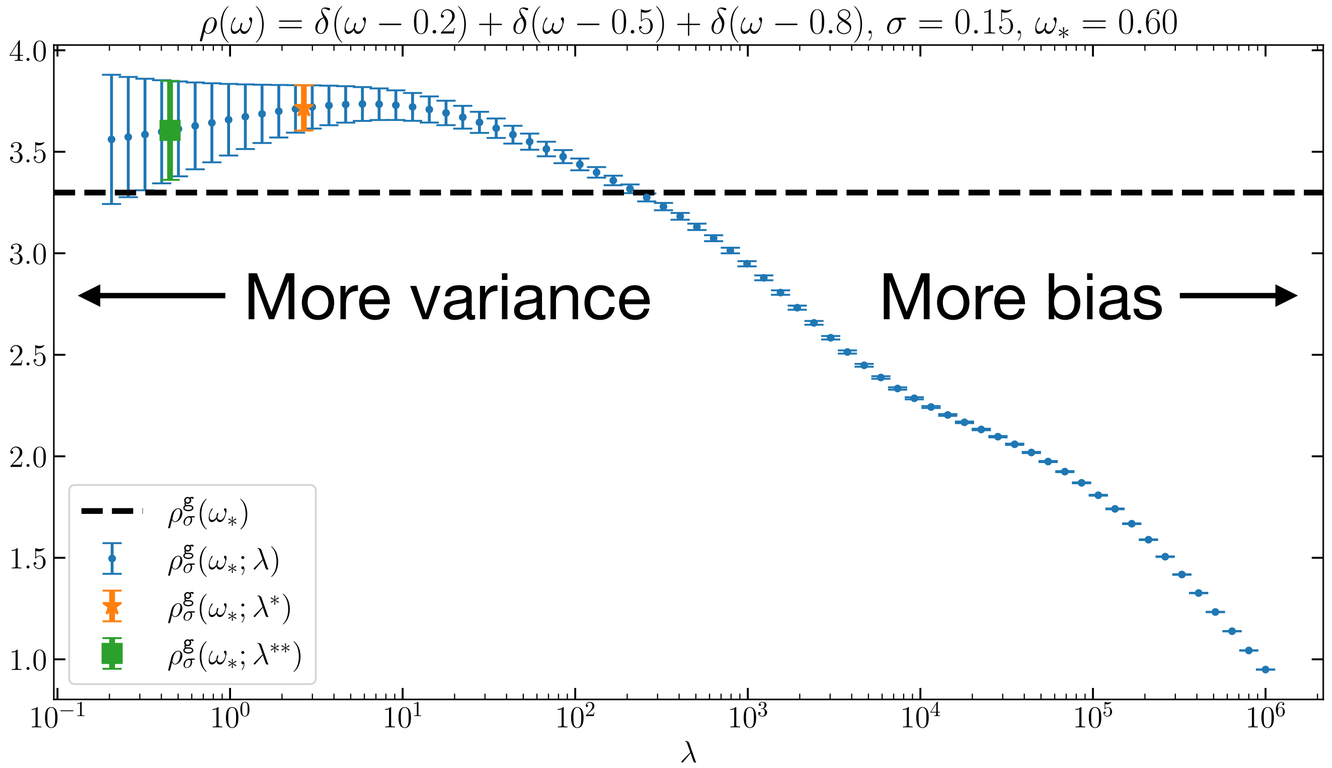
Figure: [HLT, 1903.06476]



HLT & Noise (cont'd)

Introduce covariance Σ, regulator λ, new functional

minimize $A[g] + \lambda g^\dagger \Sigma g$



Question: rigorous systematics?

Rigorous Bounds

- Question: how to get rigorous systematics?
- Several methods proposed
- Non-exhaustive, biased list
  - Nevanlinna-Pick interpolation [Bergamaschi, Jay, Oare 2305.16190]
  - Moment Problems [Abbott, Jay, Oare 2508.01377]
  - Linear/convex programming [Lawrence 2408.11766]

Key idea: the above are closely related/equivalent, and generalize HLT

- Also closely related: Conformal bootstrap, S-matrix bootstrap

$$\Longrightarrow \text{The Causal Bootstrap}$$

What's a Bootstrap, Anyways?

"Pull yourself up by your bootstraps"

$\Longrightarrow$ getting something out of nothing

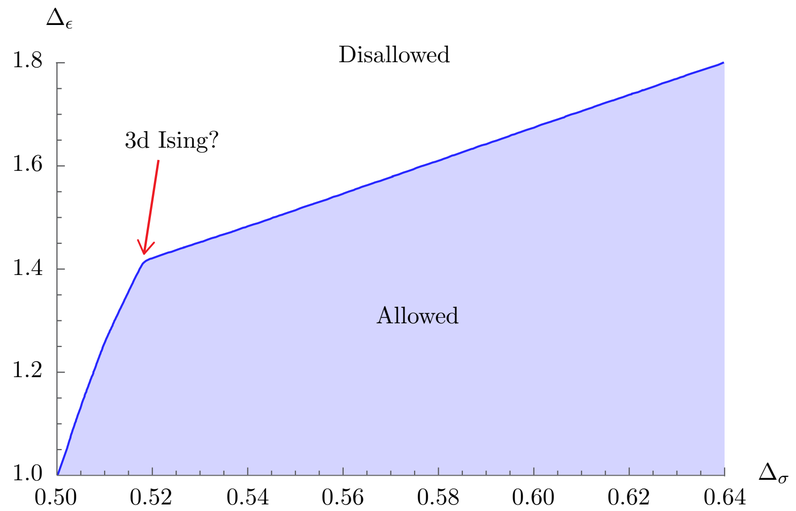


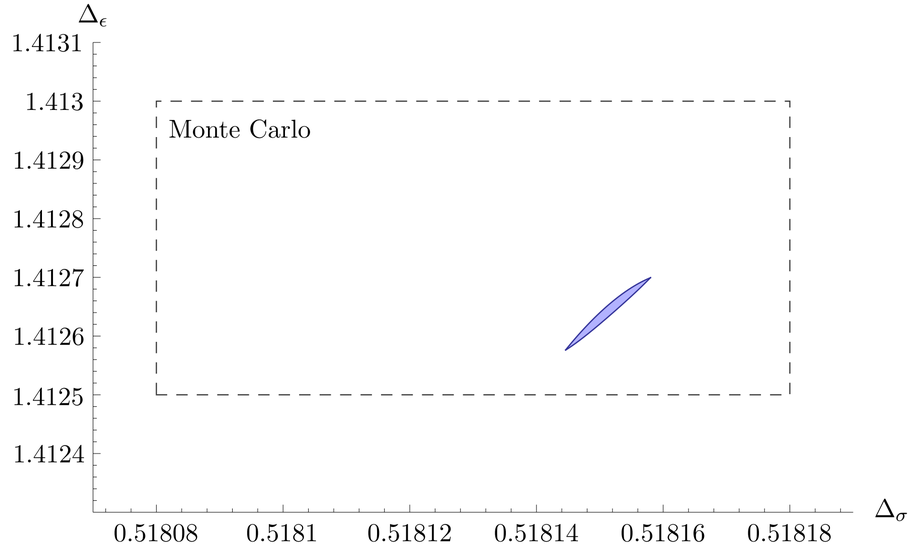
Statistical Bootstrap:
Resampling data to determine uncertainties

QFT Bootstrap:
Imposing causality, unitarity, symmetry to constrain a QFT
Ex: Conformal bootstrap, S-matrix bootstrap

Example: The Conformal Bootstrap

- Choose Conformal Symmetry + # relevant scalars, apply constraints
- Example: 3d Ising model





Figures: [Simmons-Duffin 1602.07982]

- Can we apply this to QCD?
  - Can't (usually) fully constrain theory (especially in 4d)
  - QCD is not conformal
- Key point: need Monte-Carlo data

Reflection Positivity

- Reflection positivity ~ Unitarity in Euclidean time

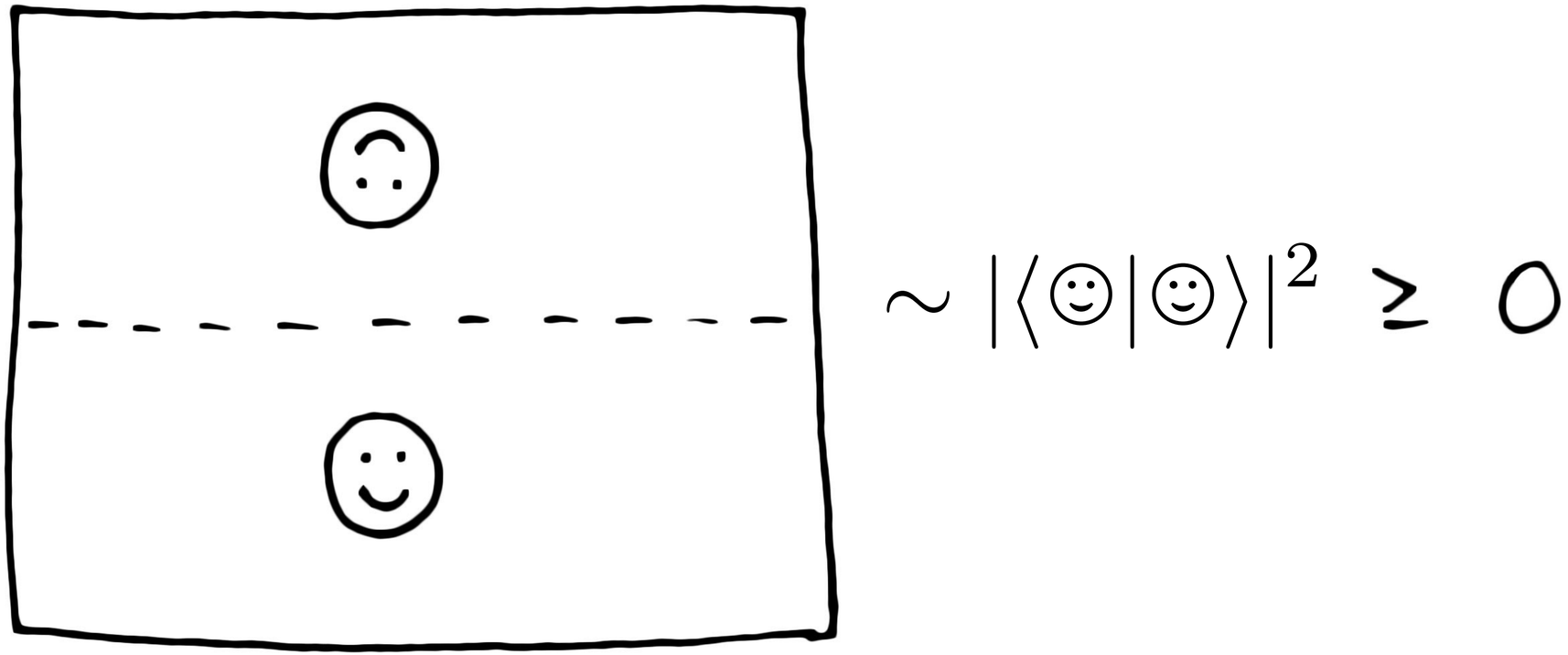


Figure: [Simmons-Duffin 1602.07982]

- Corollary: spectral positivity $\rho(E) \geq 0$

The Causal Bootstrap

- Idea: minimize/maximize target with $\rho(E) \geq 0$ (reflection positivity/causality)
- Formulate as a (convex) optimization problem

$$\text{minimize} \int K(E)\rho(E)dE$$

$$\text{subject to} \int \rho(E)e^{-Et} = \hat{C}_t$$

$$\text{and } \rho(E) \geq 0$$

Monte-Carlo data

- Referred to as a *semidefinite program* (SDP)
- Note: also need to maximize, equivalently $K(E) \mapsto -K(E)$

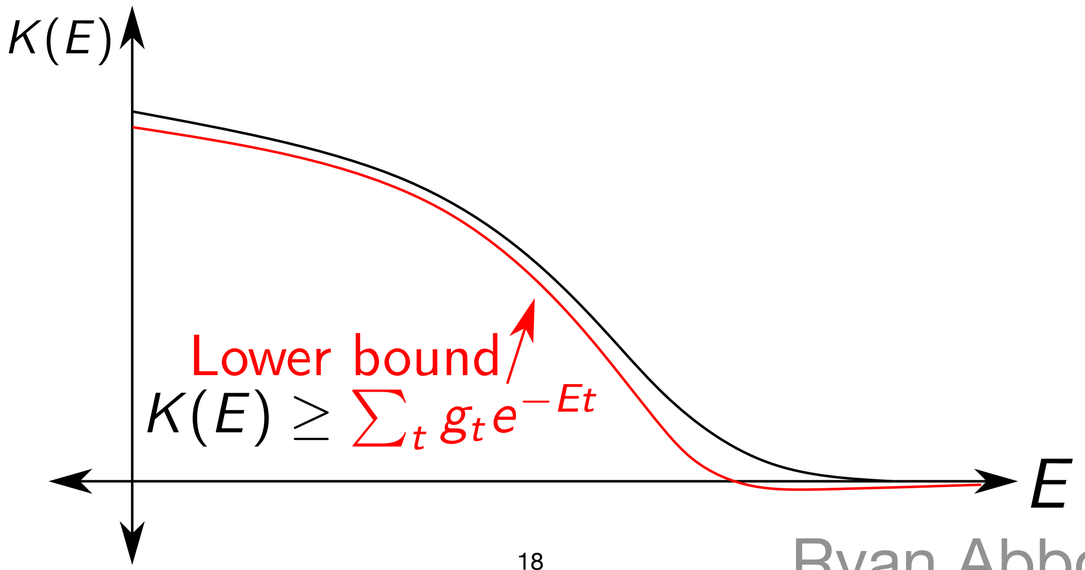
Lagrange duality

- General duality theory $\Longrightarrow$ equivalent dual problem [Lawrence 2408.11766]

$$\text{maximize} \sum_t g_t \hat{C}_t$$

$$\text{subject to } K(E) \geq \sum_t g_t e^{-Et}$$

- Infinite # of conditions but can make finite (conformal bootstrap)



Don’t Approximate, Bound

Linear methods (e.g. HLT):

$$K(E) \approx \sum_t g_t e^{-Et}$$

$$\Downarrow \int \rho$$

$$\int K(E)\rho(E) \approx \sum_t g_t C_t$$

- Choose g_t via regulator λ
- Bias-variance tradeoff

Causal Bootstrap

$$K(E) \geq \sum_t g_t e^{-Et}$$

$$\int \rho \Downarrow [\rho \geq 0]$$

$$\int K(E)\rho(E) \geq \sum_t g_t C_t$$

- Optimize over g_t (convex)
- Key fact: can saturate

More benefits: matrix correlators, noise/inexact data, …

Matrix Correlators

- Single operator $\mathcal{O} \mapsto$ variational set $\{\mathcal{O}_0, \ldots, \mathcal{O}_{r-1}\}$

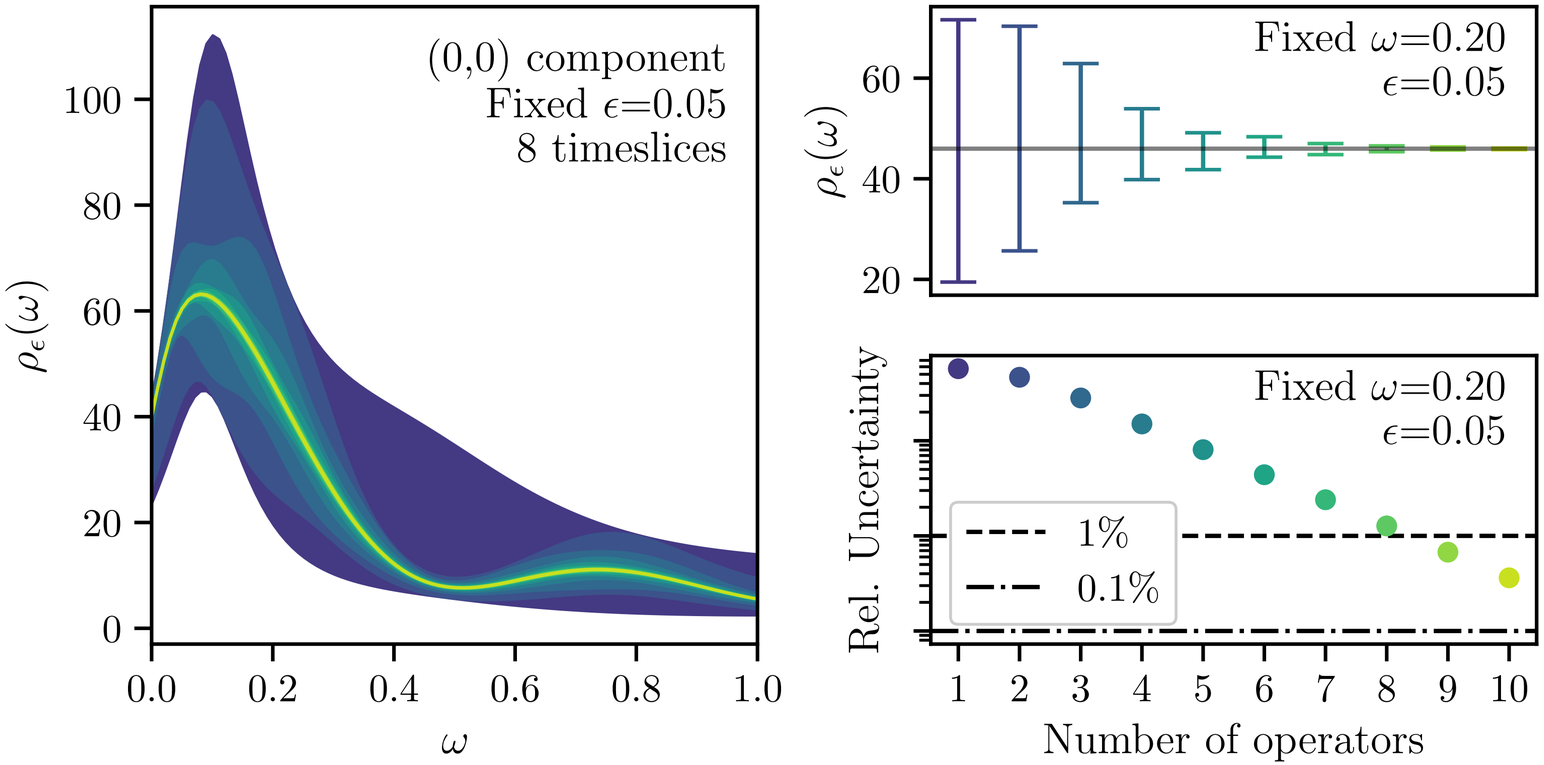
$r \times r$ matrix (Hermitian)

$$C_{ab}(t) = \langle O_a(t) O_b(0)^\dagger \rangle = \int dE\, \rho_{ab}(E) e^{-Et}$$

- More operators $\Longrightarrow$ more information (systematically improvable)
- Critical for excited state spectroscopy (usually via GEVP)
  - Example: $\mathcal{O}_0 = J^i_{\text{weak}}$, $\mathcal{O}_1 = \pi(\vec{p})\pi(-\vec{p})$, …,
- Difficult for linear methods (e.g. HLT) to integrate
- Spectral positivity $\rho(E) \geq 0 \mapsto \rho_{ab}(E) \succeq 0$ (positive semi-definite)
  - $\Longrightarrow$ Causal bootstrap can use matrix correlators directly

Matrix Correlators (cont.)

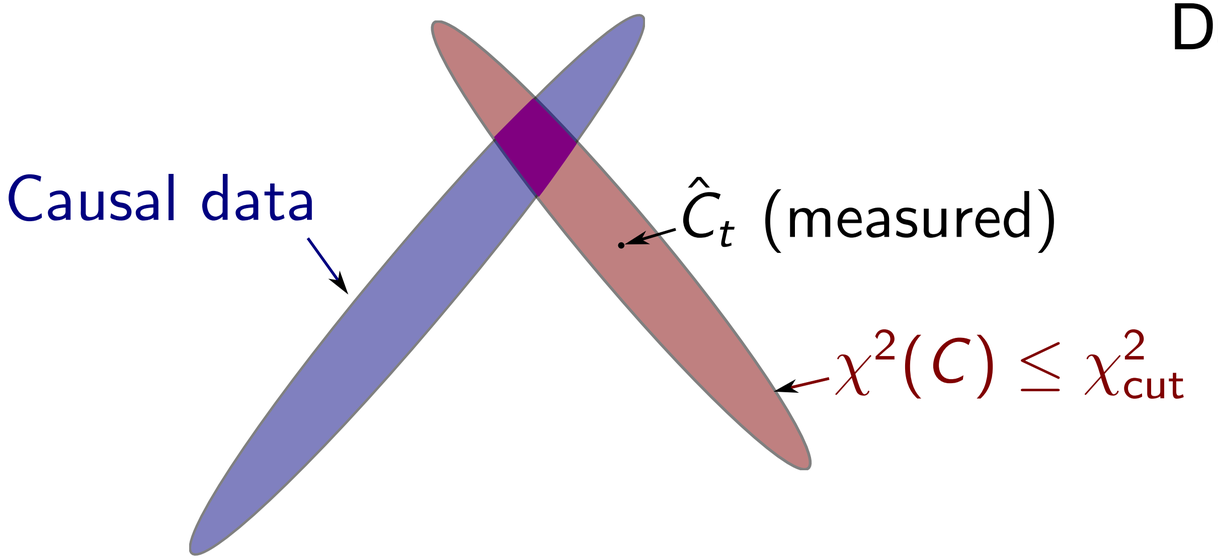
- Test on exact data [Abbott, Jay, Oare 2508.01377]:



- Exponential (!) improvement w/ more operators
- Key point: even if only want ρ_{00}, more operators still help
  - Compare with a_μ^{HVP} long-distance window

Inexact Data

- Problem: measured data $\hat{C}_t$ almost never causal
- Solution: Optimize subject to χ^2 cutoff (convex!)



Dual Formulation:

$$\text{maximize} \sum_t g_t \hat{C}_t - \chi_{\text{cut}} \sqrt{g^\dagger \Sigma g}$$

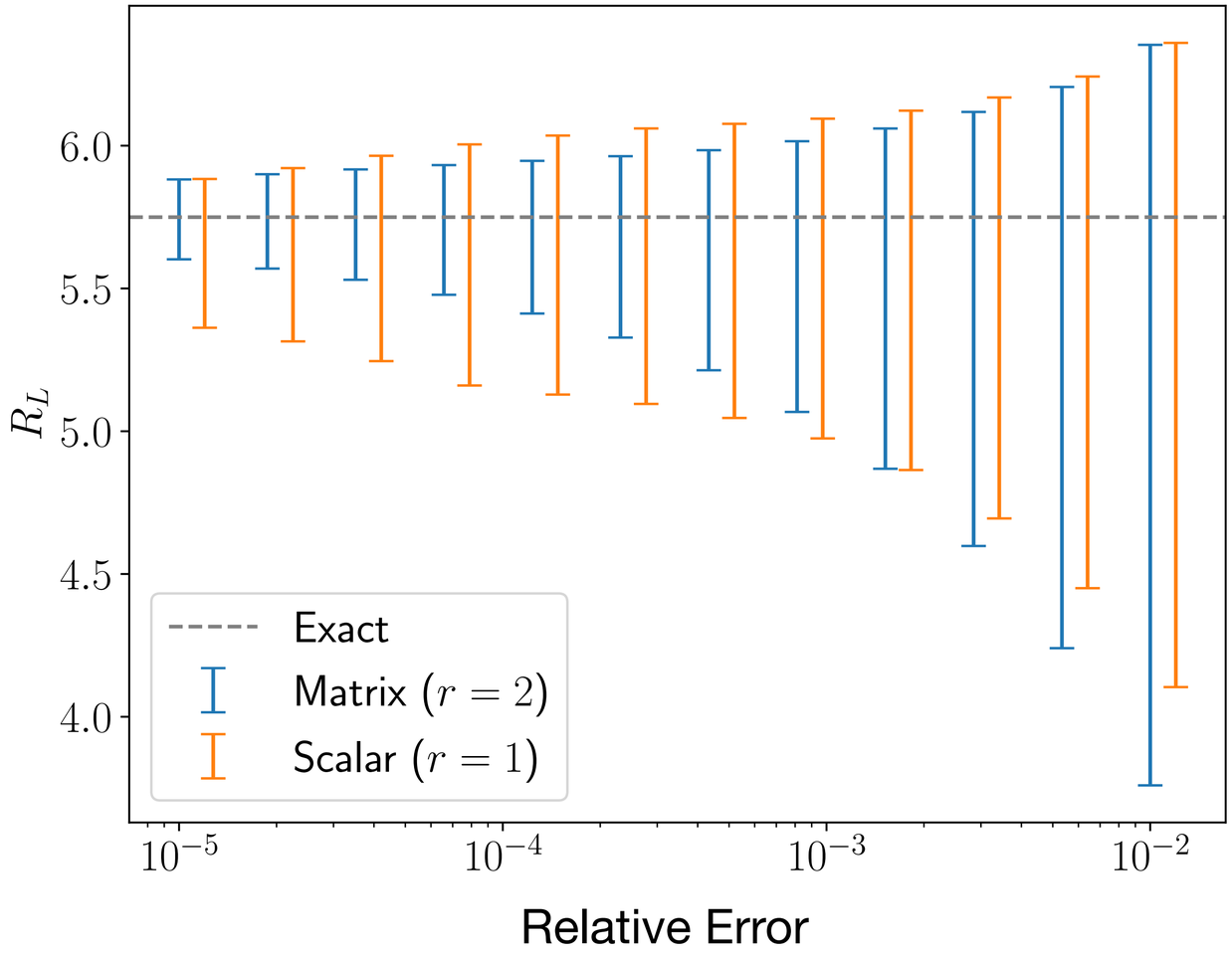
$$\text{subject to } K(E) \geq \sum_t g_t e^{-Et}$$

Results

Inclusive τ decay (toy data)

Toy model for τ decay [Abbott et al., 2605.20509]

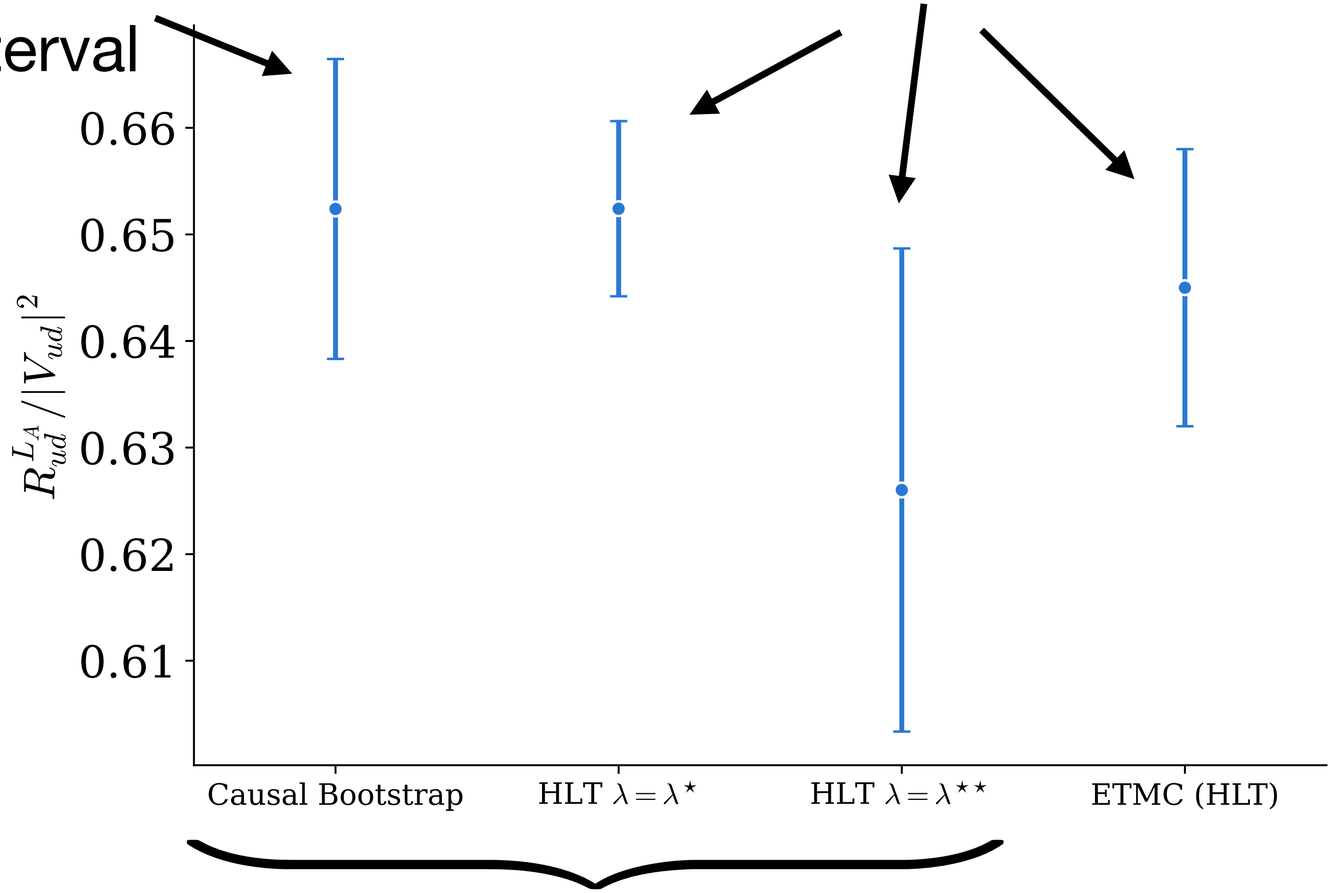
$R_L \sim \tau$ width



Inclusive τ Decay (real data)

95% conservative confidence interval

1-σ error

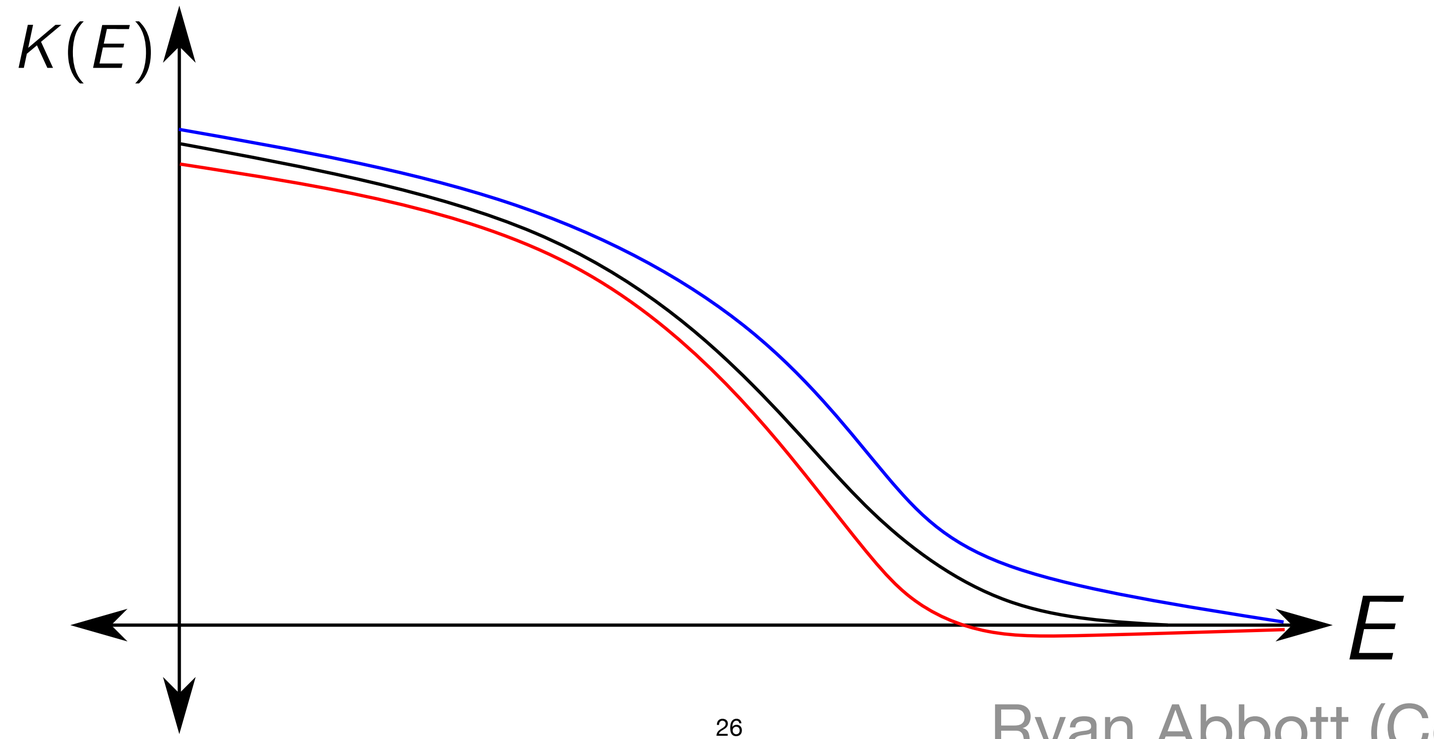


RBC/UKQCD Ensemble C ($a^{-1} = 1.73\,\mathrm{GeV}$, Physical m_π)

Caveat: single-ensemble

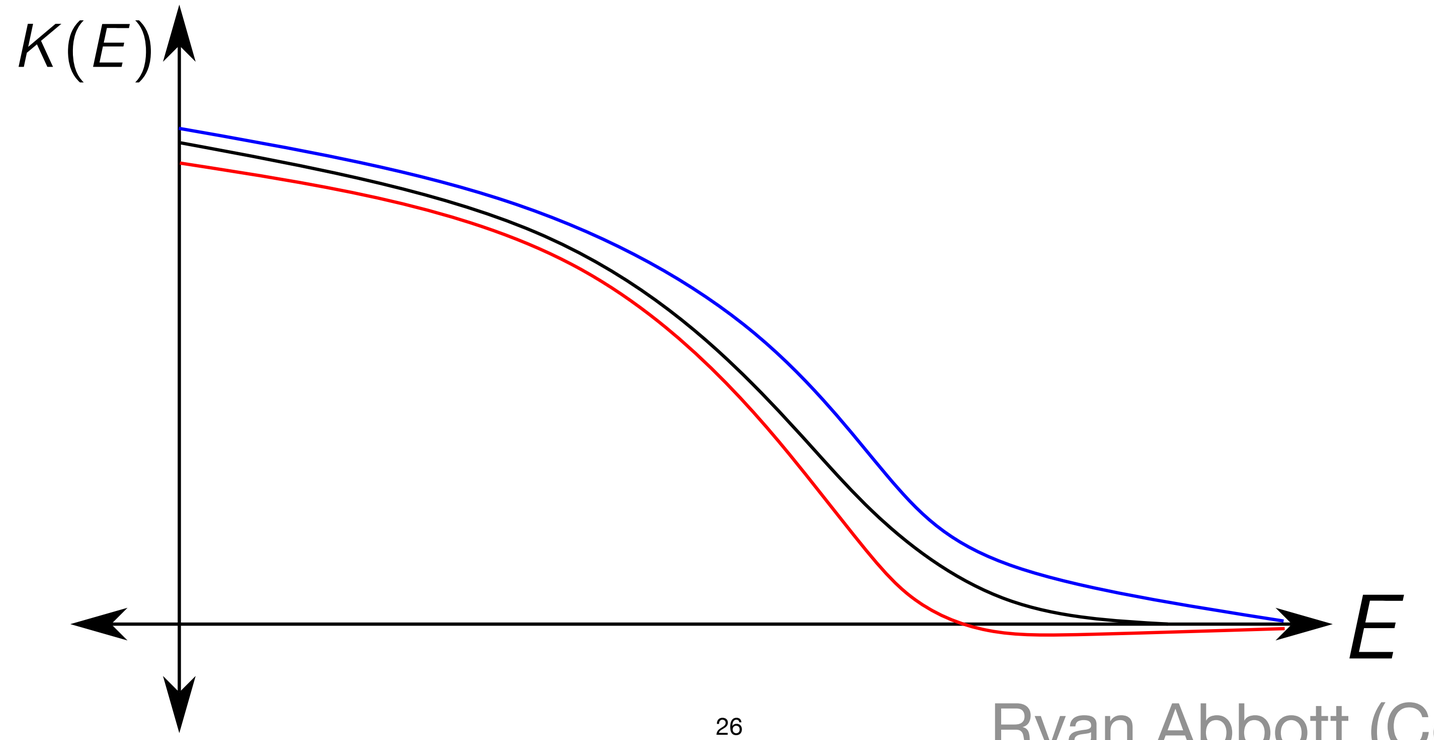
Outlook

- Bootstrap ideas link many spectral reconstruction methods
- Promising early results for inclusive τ decay
  - Future work: More channels, continuum limit, matrix correlators
- More speculative: dense QCD?



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  - Future work: More channels, continuum limit, matrix correlators
- More speculative: dense QCD?
- Thanks for listening! Questions?

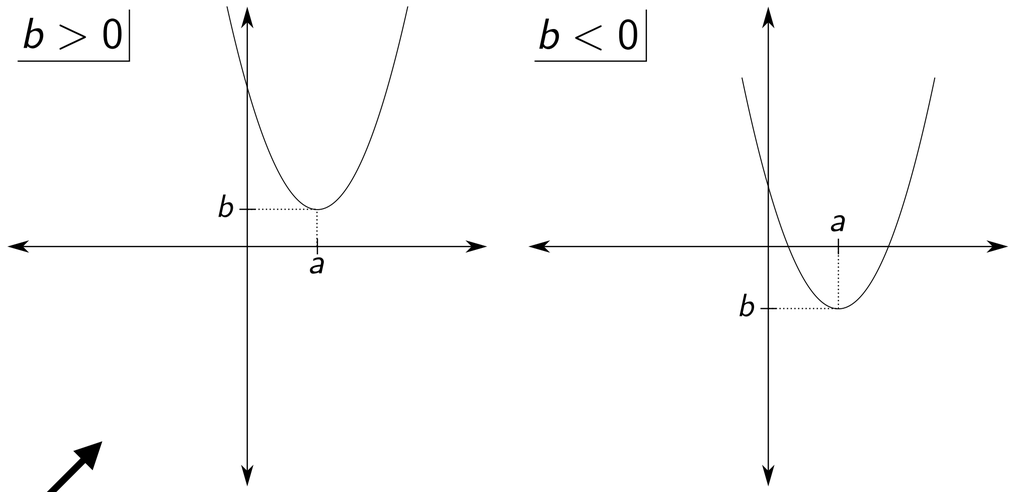


Backup

Polynomial Structure

- Substitute $\lambda = e^{-E} \Longrightarrow e^{-Et} = \lambda^t$
- Dual problem condition:

$$K(-\log \lambda) - \sum_t g_t \lambda^t \geq 0$$



- Polynomials make positivity easier
  - Example: $p(x) = (x-a)^2 + b$ positive $\iff b \geq 0$

- More generally: $p(x)$ positive $\iff p(x) =$ sum of squares
  - Provides finite-dimensional reduction
  - Public code: SDPB [Simmons-Duffin 1502.02033]
  - Slight generalizations $\Longrightarrow$ custom implementation

Toy model for τ decay

- Toy model:

$$\rho_{ab}^{\text{toy}}(E) = \sum_{k=0}^{N_{\text{state}}-1} Z_{ka}^* Z_{kb} \delta(E - E_k) \implies C_{ab}(t) = \sum_{k=0}^{N_{\text{state}}-1} Z_{ka}^* Z_{kb} e^{-E_k t}$$

- Energies $E_k = 0.1(k+1), N_{\text{state}} = 96$
- Overlap factors

$$Z_{k0} = \begin{cases} +\frac{1}{5} & n = 0 \\ +1 & n \text{ odd} \\ -\frac{1}{5} & n > 0 \text{ even} \end{cases} \qquad Z_{k1} = \begin{cases} +1 & n \text{ even} \\ \frac{1}{5} & n \text{ odd} \end{cases}$$

- Target Kernel: inclusive τ decay

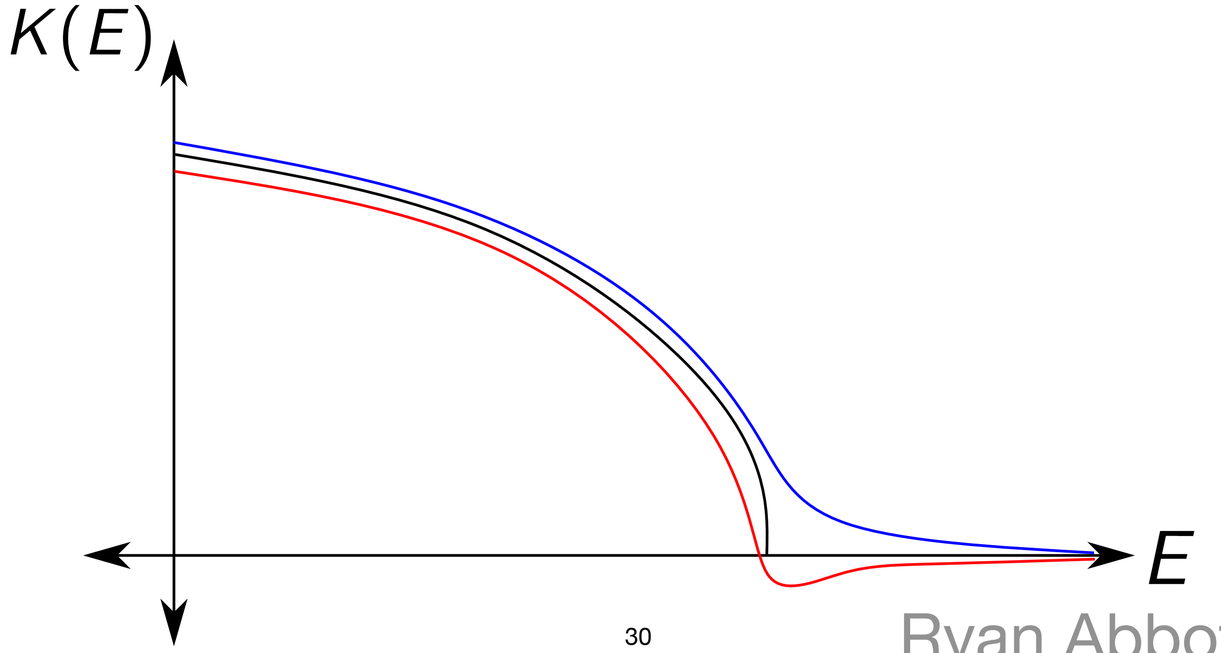
$$K(E) = \frac{1}{x}(1 - x^2)\theta(1 - x); \qquad x = E/m_\tau$$

Thresholds

- Problem: $K(E) \propto \theta(m_\tau - E)$, non-smooth
- Typical solution: regulate

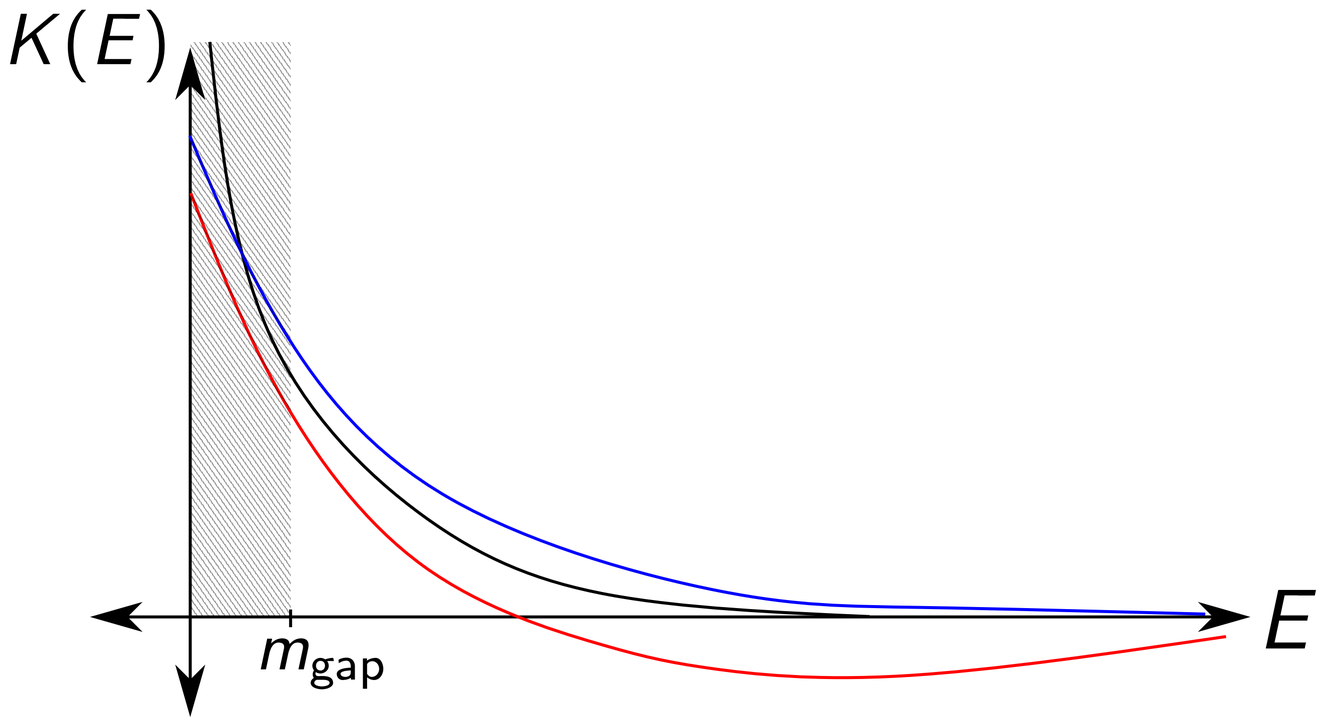
$$\theta_\sigma(x) = \frac{1}{1 + e^{-x/\sigma}} \xrightarrow{\sigma \to 0} \theta(x)$$

- No issue for constructing bounds



IR Divergences

- Problem: $K(E) \propto 1/E$ for small E
- Assume gap: $\rho(E) = 0$ for $E < m_{\text{gap}}$
- Note: only needed for upper bounds, not lower bounds



Wertevorrat vs SDP

- Bounds on correlation function:

$$G(z) = \int \frac{\rho(E)dE}{E - z}$$

- Matches exactly with Nevanlinna-Pick (NP) and moment problems

