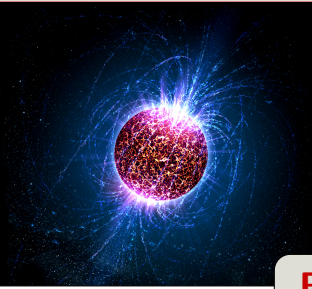
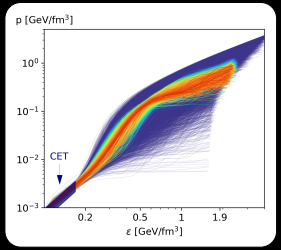
Bounds on the high density QCD equation of state from phase quenched simulations





Bastian B. Brandt

in collaboration with

Gergely Endrődi, Guy Moore,
Daniele Cattaneo, Arpith Kumar
& Johannes Weber

CRC-TR 211
Strong-interaction matter under extreme conditions

UNIVERSITÄT BIELEFELD
Faculty of Physics

Why phase quenched QCD?

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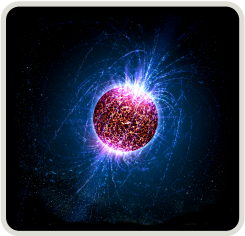
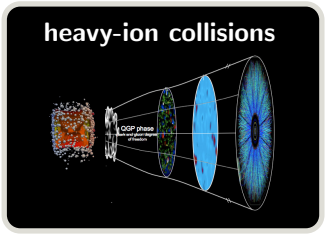
Physical systems and the equation of state





many physical systems dominated by strong interaction:

feature (large) baryon densities n_B

- **neutron stars**
- heavy-ion collisions

for phenomenological description:

need QCD properties at $\mu_B > 0$

of particular relevance:

equation of state (EoS) – pressure $p(T, \mu_B, \mu_I, \mu_S, \ldots)$

standard way to compute non-perturbatively:

compute via lattice QCD

Why phase quenched QCD?

Lattice QCD and the sign problem

$$\langle O \rangle_{\mu_f} = \frac{1}{Z} \int D[U]\, O[U] \underbrace{e^{-S_g[U]} \prod_f \det\{D_f[U](\mu_f)\}}_{=p[U]}$$

computations in lattice QCD

importance-sampling Monte-Carlo integration with weight factor $p[U]$

necessary: $\boldsymbol{p[U]}$ **is real and positive** $\iff$ $\boldsymbol{\prod_f \det\{D_f[U]\}}$ **is real and positive**

- $\mu_f = 0$: $\quad \gamma_5 D_f \gamma_5 = D_f^\dagger \quad \Longrightarrow \quad \det\{D_f\} = \det\{D_f^\dagger\} = \det\{D_f\}^*$ is real

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- $\mu_f \neq 0$: $\quad \gamma_5 D_f(\mu_f) \gamma_5 = D_f^\dagger(-\mu_f) \quad \Longrightarrow \quad \det\{D_f(\mu)\} = \det\{D_f(-\mu_f)\}^* \quad$ is complex

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$\Rightarrow$ not suitable for importance sampling! – **sign problem**

Why phase quenched QCD?

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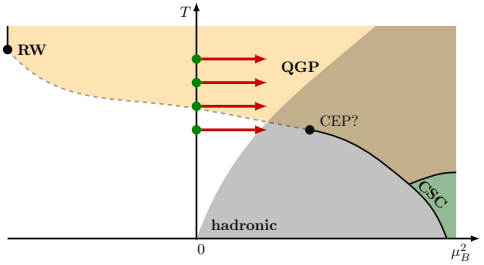
Lattice QCD and the equation of state of cold and dense QCD matter

Sign problem: prohibits direct simulations at $\mu_f \neq 0$
(exception: pure isospin chemical potential $\mu_u = -\mu_d$)

for small $\mu_B/T \lesssim 4$: EoS can be accessed indirectly

- Taylor expansion around $\mu_f = 0$
- Analytic continuation from imaginary μ_f



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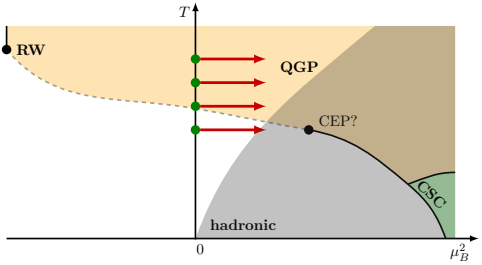

→ **many interesting systems remain inaccessible**

in particular: **EoS for neutron stars**

Proposal: **change paradigm to compute bounds for the EoS**

→ Bounds can be obtained from **phase quenched QCD!**

... and what is that?

Bounding the pressure with phase quenched QCD

Phase quenched QCD:

- provides exact upper bound for QCD pressure

$$p_{\mathrm{QCD}}(T,\mu_f) \leq p_{\mathrm{PQ}}(T,\mu_f)$$

$$\begin{aligned} Z_{\mathrm{QCD}} &= \int dU\, e^{-S_g} \det D(\mu) \\ &= \int dU\, e^{-S_g} e^{i\varphi} |\det D(\mu)| \\ &\leq \int dU\, e^{-S_g} |\det D(\mu)| = Z_{\mathrm{PQ}} \end{aligned}$$

$$p = \frac{T}{V} \log Z$$

... and what is that?

Bounding the pressure with phase quenched QCD

Phase quenched QCD:

- provides exact upper bound for QCD pressure
  $p_{\rm QCD}(T,\mu_f) \le p_{\rm PQ}(T,\mu_f)$
- bound can be computed without sign problem
- **bound is close to $p_{\rm QCD}$ at large μ_B**
  perturbation theory: $p_{\rm PQ}/p_{\rm QCD} = 1 + 0.11\alpha_s^3$
  Moore, Gorda, JHEP 12 (2023) Navarrete *et al* '24

$$
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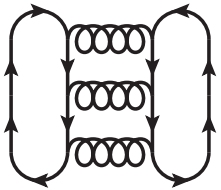
leading pQCD:



Why should bounds be useful?

Model agnostic sampling of the neutron star EoS

idea: get EoS through inference from constraints of data/theory

Guillot *et al*, Astrophys.J. 772 (2013); ... Annala *et al*, PRL 120 (2018); ...

sample causal EoSs with constraints from:

- nuclear effective theory (CET) – at small n_B

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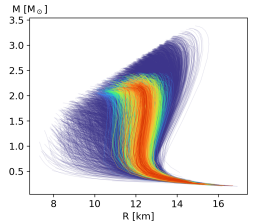

Gorda, Komoltsev, Kurkela, Astrophys.J. 950 (2023)

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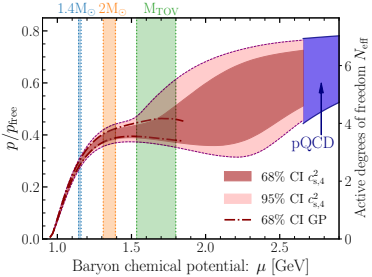

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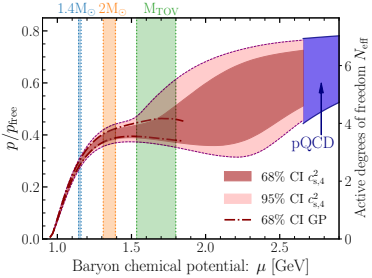
- renormalization and matching point dependence
- potential presence of non-perturbative gap effects



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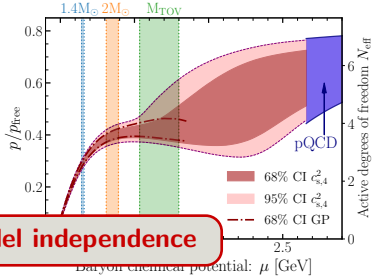

Alternative constraints at large μ_B are needed for model independence

Annala *et al*, Nature Commun. 14 (2023)

but: **matching involves obstacles/uncertainties**

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Has it been tried already?

A first attempt for two-flavour QCD from the isospin EoS

First attempt:

- use EoS from large pure isospin density
  - Abbott *et al*, PRD 108 (2023); Abbott *et al*, PRL 134 (2025)

  (available only for 2 lattice spacings)
- and thermodynamic relations
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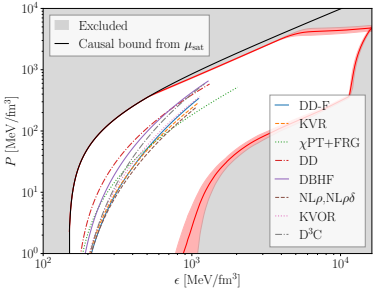

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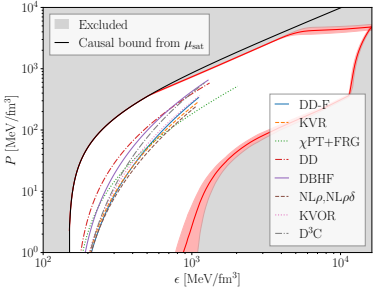


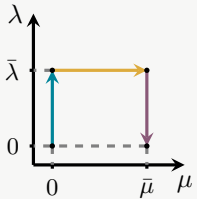
Abbott *et al*, PRL 134 (2025)

stricter upper bound for p with physical quark content would be very useful!

Outline

CRC-TR 211

1



Phase quenched QCD in practice

- connection to isospin
- simulation settings
- integration contours

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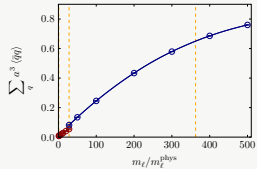

First results for $T = 0$ and large μ_B

- integrations in practice
- towards the continuum
- systematic effects

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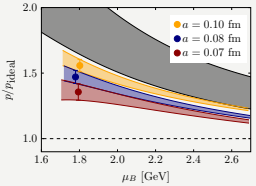

Prospects

- correcting gap effects
- pQCD contact at $T \neq 0$

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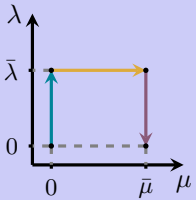

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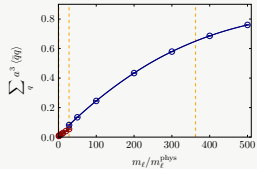

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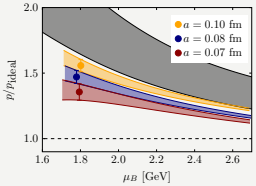

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Phase quenching and isospin chemical potential

How to simulate phase quenched QCD

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absolute value: inconvenient for simulations

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apart from square root: resembles expectation value with N_f isospin doublets

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reuse extensive knowledge from $\mu_I \neq 0$ studies with physical quark masses

Kogut, Sinclair, PRD 66 (2002); ... Endrődi, PRD 90 (2014); ... BB, Endrődi, Schmalzbauer, PRD 97 (2018)

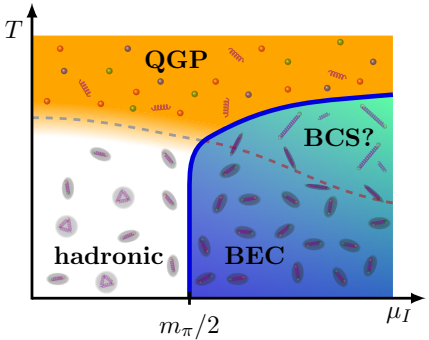
Phase quenching and isospin chemical potential

Meson condensation and pseudoscalar source

non-zero $\mu_I = \mu_u = -\mu_d$: **BEC of charged pions for $\mu_I > m_\pi/2$**

Son, Stephanov, PRL 86 (2001); ... Kogut, Sinclair, PRD 66 (2002); ... BB, Endrődi, Schmalzbauer, PRD 97 (2018)

recent review: BB, Endrődi, arXiv:2604.26715



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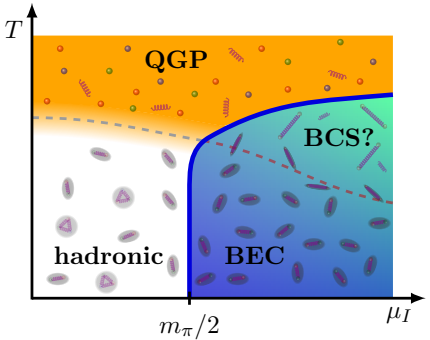
chiral symmetry: $\mathrm{SU}_V(2) \xrightarrow{\mu_I \neq 0} \mathrm{U}_Q(1) \xrightarrow{\mu_I = \frac{m_\pi}{2}} \varnothing$

$\Rightarrow$ Goldstone mode in the spectrum



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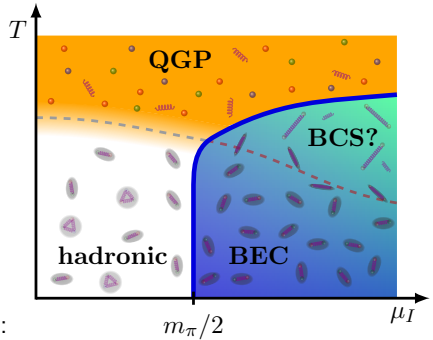
simulations: need introduction of an **infrared regulator**

pion source with prefactor λ

Kogut, Sinclair, PRD 66 (2002); PRD 66 (2002)



$\Rightarrow$ necessary replacement for every phase quenched flavour f:

$$\left[\det\{D_f(-\mu_f)D_f(\mu_f)\}\right]^{\frac{1}{2}} \;\to\; \left[\det\{D_f(-\mu_f)D_f(\mu_f) + \lambda_f^2\}\right]^{\frac{1}{2}}$$

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PQ simulations and inequality in the presence of λ

Summary for phase quenched:

condensation of pseudoscalar meson of artificial quark doublet f at $\mu_f = \frac{m_{P,f}}{2}$

$\Rightarrow$ introduction of infrared regulator λ_f mandatory

- simulations done at $\lambda \neq 0$
- physical bounds: send $\lambda \to 0$

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Further inequality: $\det\{B\} \leq \det\{B + \lambda^2 \mathbf{1}\}$ for positive-hermitian B and real λ

$$\Rightarrow \quad Z_{\mathrm{QCD}} \leq Z_{\mathrm{PQ}} \leq Z_{\mathrm{PQ}}(\lambda_f) = \int dU\, e^{-S_g} \prod_f \left[\det\{D_f(-\mu_f) D_f(\mu_f) + \lambda_f^2\} \right]^{\frac{1}{2}}$$

consequently: $\boldsymbol{p_{\mathrm{QCD}}(T, \mu_f) \leq p_{\mathrm{PQ}}(T, \mu_f) \leq p_{\mathrm{PQ}}(T, \mu_f, \lambda_f)}$

Phase quenched bounds for the neutron star EoS

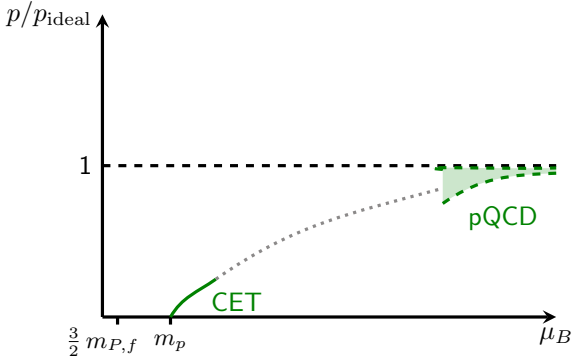
What we expect for the pressures – where bounds might have impact

Neutron stars: $T = 0$

- silver blaze: $p = 0$ for $\mu_B = 0$

QCD pressure:

- $p \neq 0$ at $\mu_B = m_p$
  pressure known through CET
- $\dfrac{p}{p_{\text{ideal}}} \xrightarrow{\mu_B \to \infty} 1$
  pressure known through pQCD



Phase quenched bounds for the neutron star EoS

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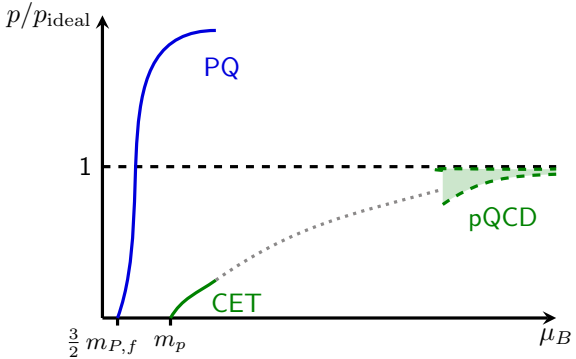

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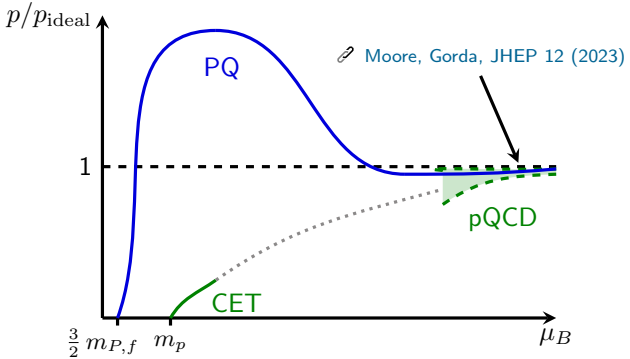

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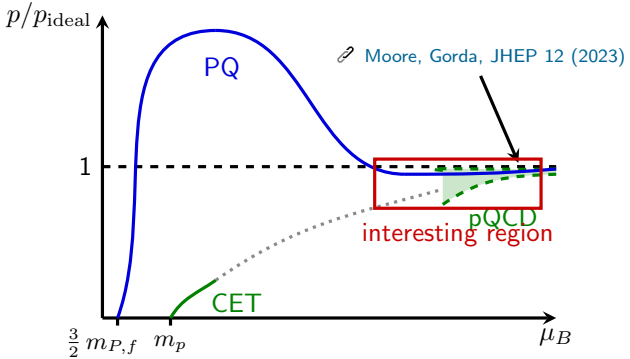


Phase quenched bounds for the neutron star EoS

The generalized integral method for pressure computations

pressure: $p = \frac{T}{V} \ln Z$

in lattice QCD simulations: ***Z* not directly available**

instead: use integral method Engels *et al*, PLB 252 (1990); ...

generalized form: compute pressure from know starting point $\vec{\nu}_0$

$$p(\vec{\nu}^*) = p(\vec{\nu}_0) + \int_{\vec{\nu}_0}^{\vec{\nu}^*} ds \left\langle \vec{\nabla}_\nu p \right\rangle \frac{\partial \vec{\nu}}{\partial s}$$

$\vec{\nu}$: generalized parameter vector (e.g.: $\vec{\nu} = (T, \mu_u, \mu_d, \mu_s)^T$)

s: parameterizes the integration contour in the space of parameters

suitable starting point: $p(\vec{0}) = 0$

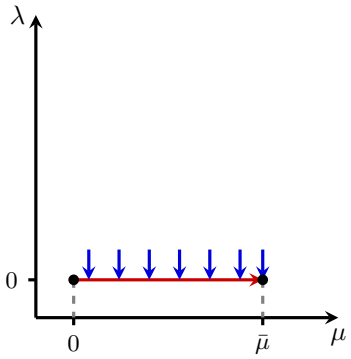
Phase quenched bounds for the neutron star EoS

Efficient integration paths

The case at hand: reach high $\bar{\mu}_B$ at $T = 0$

- standard path: integrate in μ_B
  used e.g. for EoS at $\mu_I \neq 0$ BB, Cuteri, Endrődi, JHEP 07 (2023)

integrand: $\dfrac{\partial p}{\partial \mu_B} = \langle n_B \rangle$



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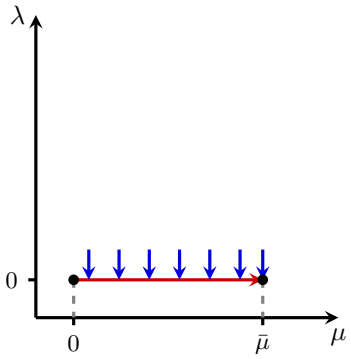

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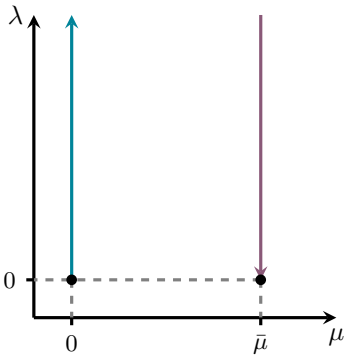
- **maybe more efficient: integrate in $\lambda \to \infty$**

- path1: $\lambda = 0 \to \infty$ at $\mu_B = 0$
- path2: $\lambda = \infty \to 0$ at $\mu_B = \bar{\mu}_B$

integrand: $\dfrac{\partial p}{\partial \lambda_f} = \langle \pi_f \rangle$



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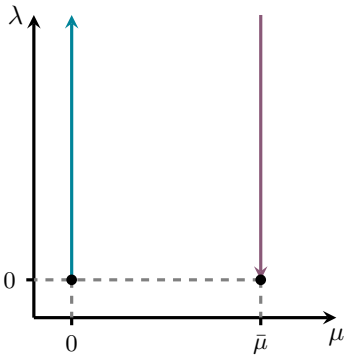

- path1: $\lambda = 0 \to \infty$ at $\mu_B = 0$
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note at $\mu_f = 0$: λ and m_f equivalent – $m_{\text{eff}} = \sqrt{\lambda^2 + m^2}$

Phase quenched bounds for the neutron star EoS

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  however: demands small λ and $\lambda \to 0$ for each point
  - simulation cost decreases with growing λ
- **maybe more efficient: integrate in $\lambda \to \infty$**
- **other options: stop at $\bar{\lambda}$ & include path in μ_B direction**

- path1: $\lambda = 0 \to \bar{\lambda}$ at $\mu_B = 0$
- path2: $\lambda = \bar{\lambda} \to 0$ at $\mu_B = \bar{\mu}_B$
- path3: $\mu_B = 0 \to \bar{\mu_B}$ at $\lambda = \bar{\lambda}$

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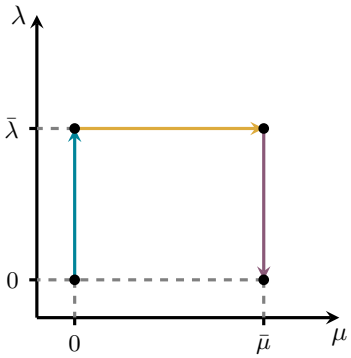

Outline

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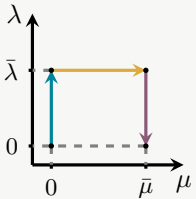
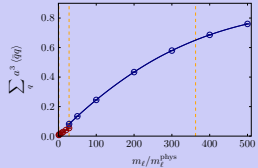
1



Phase quenched QCD in practice

- connection to isospin
- simulation settings
- integration contours

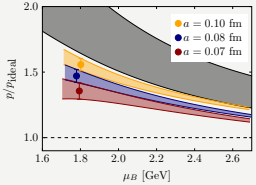
2



First results for $T = 0$ and large μ_B

- integrations in practice
- towards the continuum
- systematic effects

3



Prospects

- correcting gap effects
- pQCD contact at $T \neq 0$

Numerical details

Simulation setup

- $N_f = 2 + 1$ two-stout staggered fermions with rooting
  pure baryon chemical potential: $\mu_u = \mu_d = \mu_s = \mu_B/3$ & $\lambda_f = \lambda \quad \forall\, f$
  - optimization: combine degenerate phase-quenched flavours to regular isospin doublet
- Symanzik-improved gauge action
- physical quark masses (except for $m_u = m_d$)

Integration setup

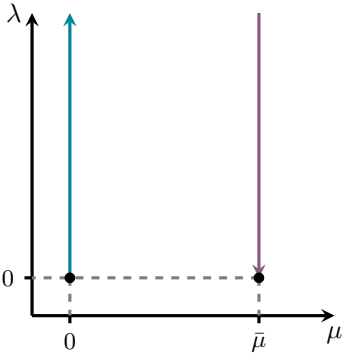
- For interpolations: cubic spline interpolations
  (different analyses – including nodepoint Monte-Carlo)
  BB, Endrődi, PoS LAT2016; JHEP 07 (2023)
- integral method for pressure computation: analytic integration via spline

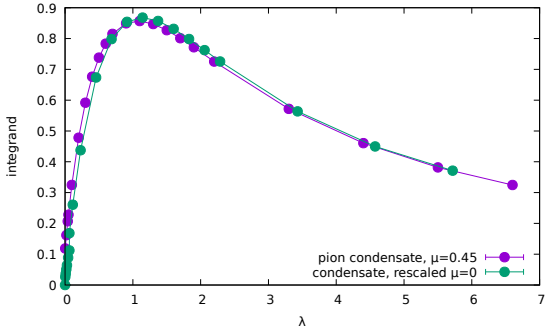
Optimizing integration contours

Testing integration to infinite mass and back

first test with lattice setup: $a = 0.15$ fm – $24^3 \times 32$

proposed integration contour: up in λ at $\mu_f = 0$ – down in λ at $\mu_f > 0$





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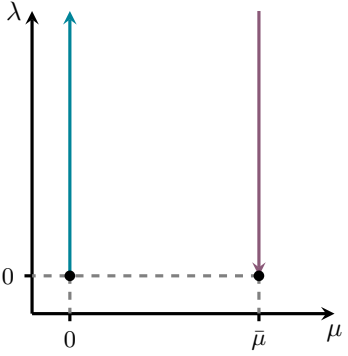

integrand: $\pi_{\rm tot} = \dfrac{\lambda}{2}\,\mathrm{Tr}\,K^{-1}_{\mu,m_\ell,\lambda} + \dfrac{\lambda}{4}\,\mathrm{Tr}\,K^{-1}_{\mu,m_s,\lambda}$

expand in "hopping" parameter: λ^{-1}

$$\pi_f = \frac{N_c}{2\lambda} + \frac{1}{2}\sum_{i=1}^{\infty}(-1)^i \frac{1}{\lambda^{2i+1}}\,\mathrm{Tr}|\not{D}_\mu + m_f|^{2i}$$

leading order:

$$\pi_{\rm tot} = \frac{3N_c}{4\lambda}\left[1 - \frac{9+3\,\cosh(2\mu)}{6\lambda^2} - \frac{1}{3\lambda^2}\left\{2m_\ell^2 + m_s^2\right\}\right]$$

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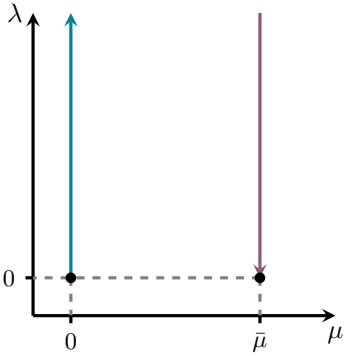
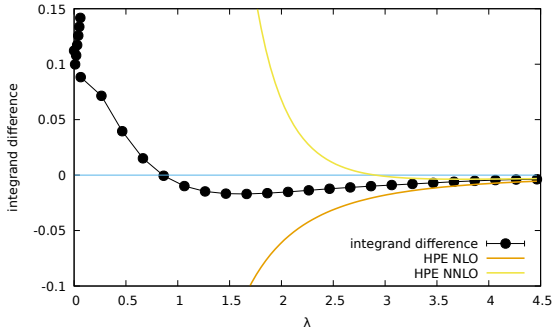
extrapolation: matching to "hopping-parameter" expansion – but: **slow convergence**





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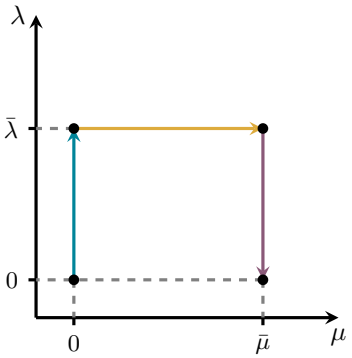
benfitial: include path in μ_f-direction at large $\lambda = \bar{\lambda}$

shown here: results for lattice setup $a = 0.10$ fm – $40^3 \times 48$



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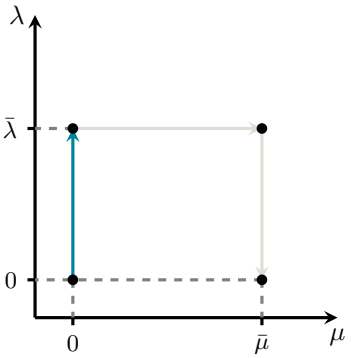

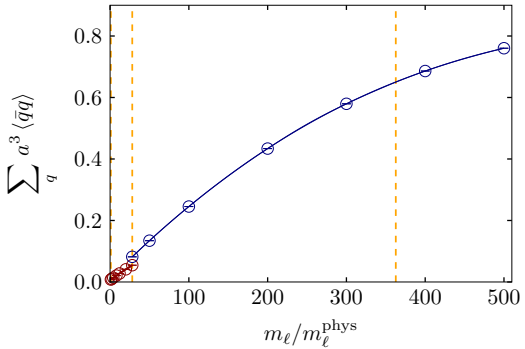

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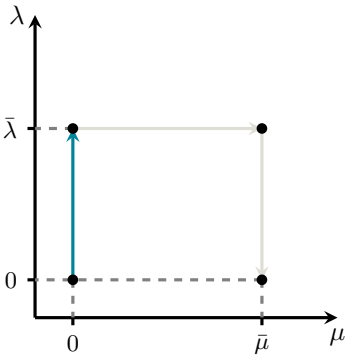

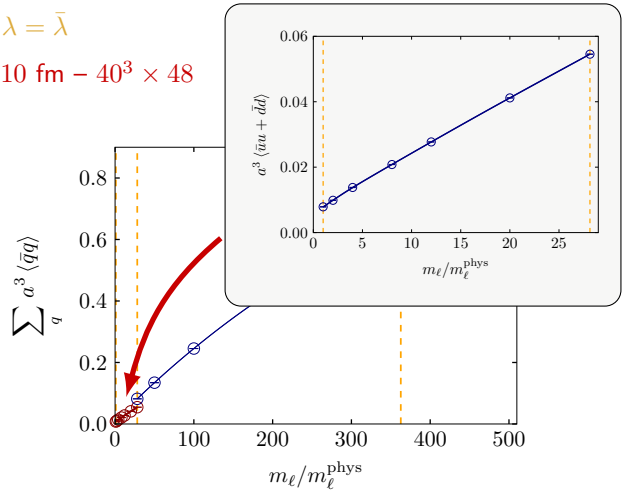

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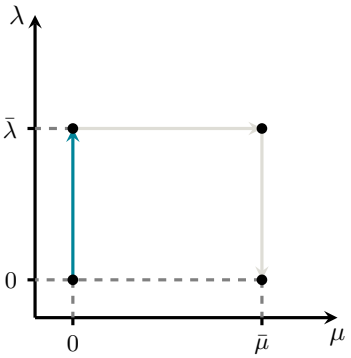

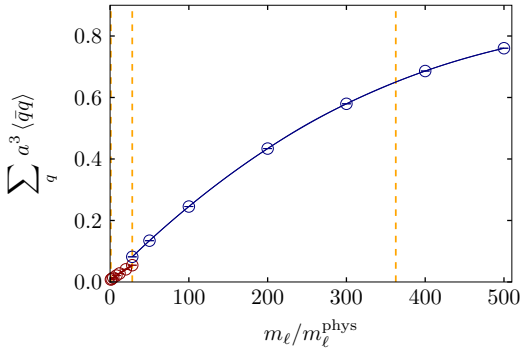

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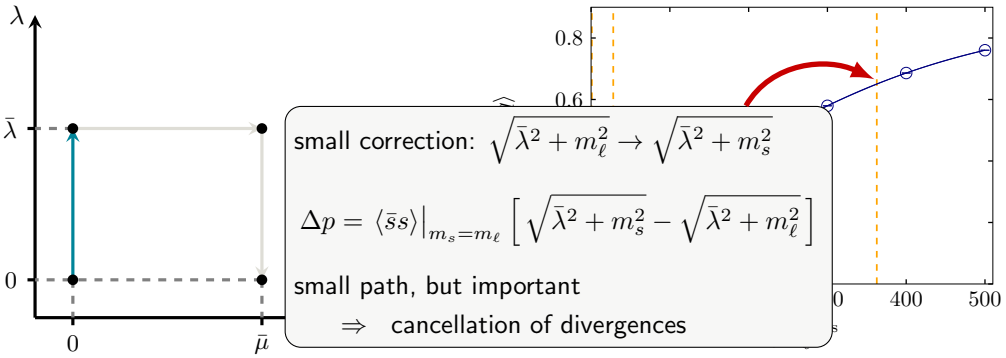

small correction: $\sqrt{\bar{\lambda}^2 + m_\ell^2} \to \sqrt{\bar{\lambda}^2 + m_s^2}$

$$\Delta p = \langle \bar{s}s \rangle\big|_{m_s = m_\ell} \left[\sqrt{\bar{\lambda}^2 + m_s^2} - \sqrt{\bar{\lambda}^2 + m_\ell^2} \right]$$

small path, but important

$\Rightarrow$ cancellation of divergences

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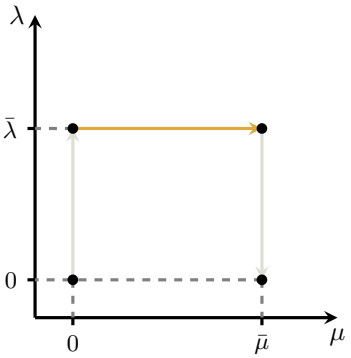

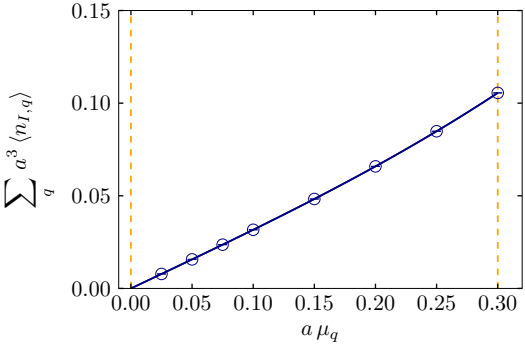

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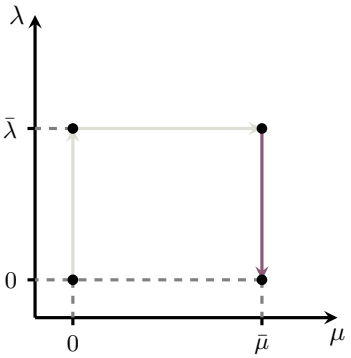

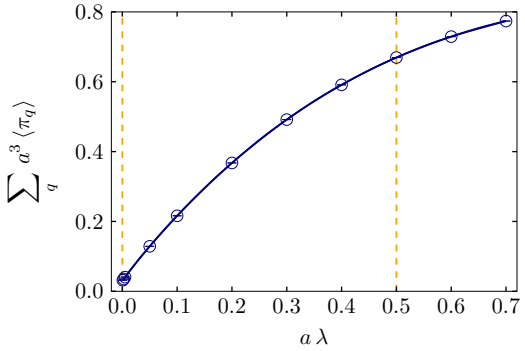

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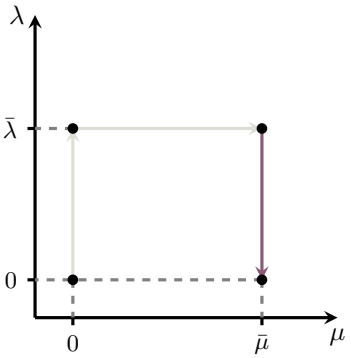

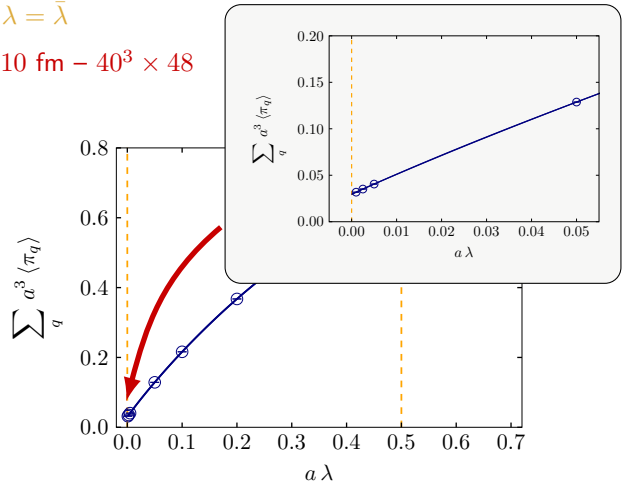

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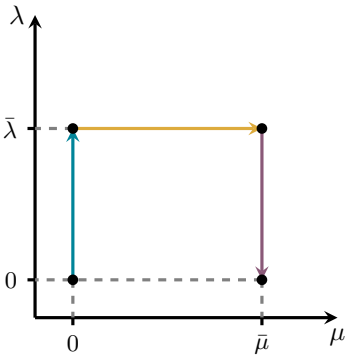
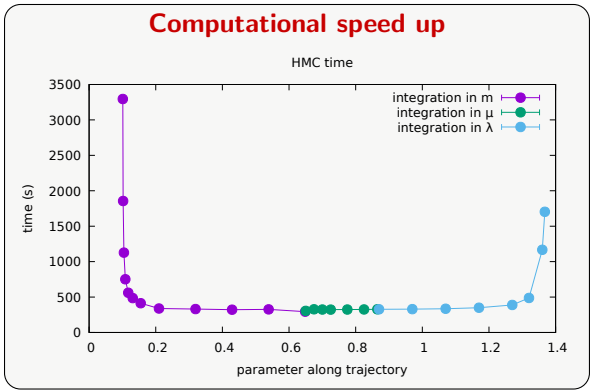
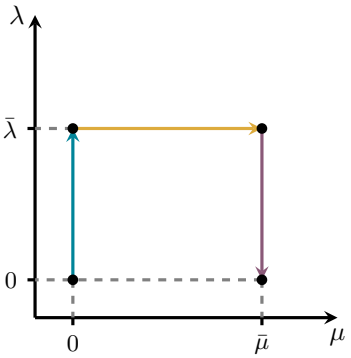


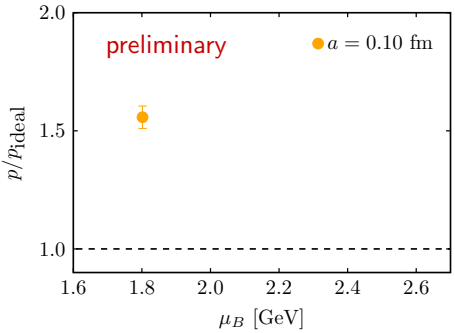


Final results for the pressure at large μ_B

First point for $a = 0.10$ fm

- first full result: $a = 0.10$ fm – $40^3 \times 48$





Final results for the pressure at large μ_B

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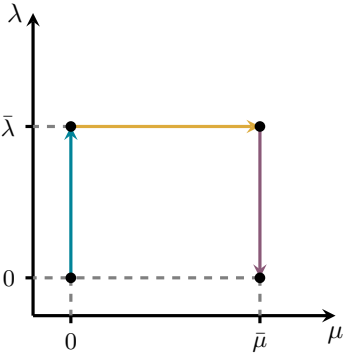
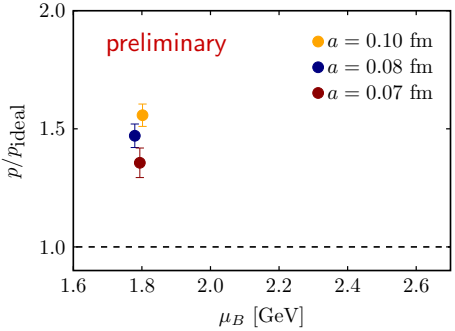
Adding other lattice spacings

- first full result: $a = 0.10$ fm – $40^3 \times 48$
- additional lattices: • $a = 0.08$ fm – $40^3 \times 60$ • $a = 0.07$ fm – $48^3 \times 72$





Final results for the pressure at large μ_B

Using these as seeds for regular integration in μ_B direction

- first full result: $a = 0.10$ fm – $40^3 \times 48$
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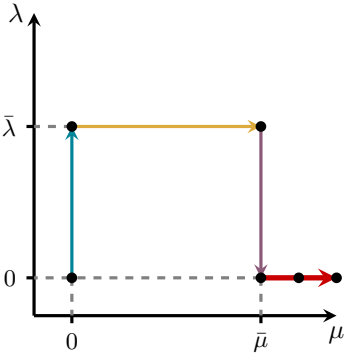
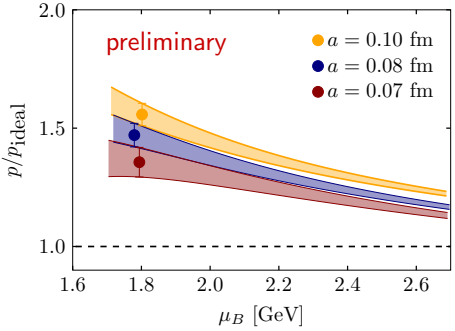
- use as seed for further points at large μ_B





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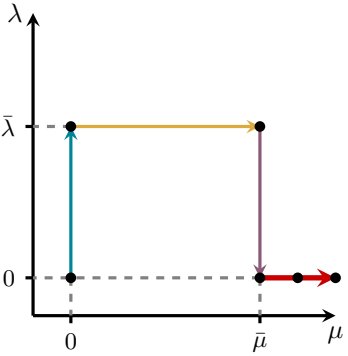
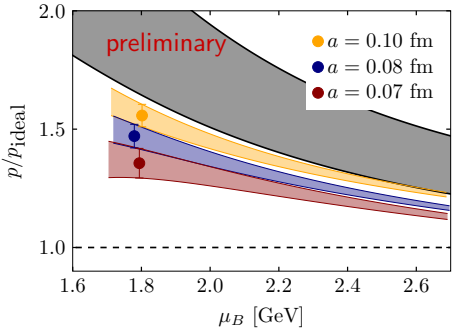
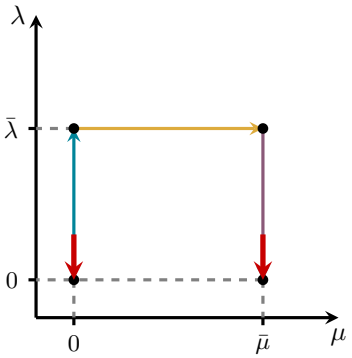




Finite size and temperature effects

correlation length: decrease with growing λ $\Rightarrow$ effects most relevant in the small-λ region



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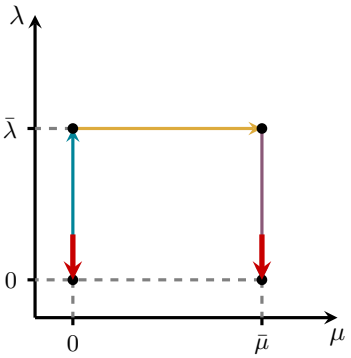

effective description: theory of 3 pions

correction to free energy per 4-volume with length L_μ:

$$\Delta F = \sum_{\mu=xyzt} \frac{3m_\pi^2}{2\pi^2 L_\mu^2} K_2(m_\pi L_\mu)$$

$\Rightarrow$ tiny correction of $O(10^{-4})$ for the pressure!

Finite size and temperature effects

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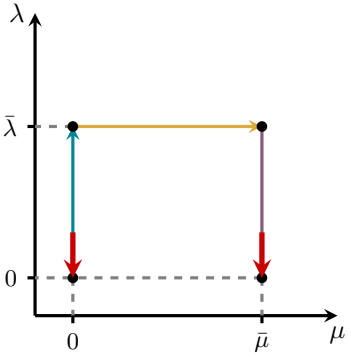
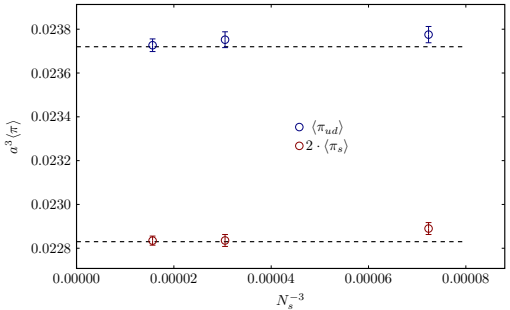
- at $\mu_B = 0$: mass gap and 4d rotation invariance – analytical treatment
- at $\mu_B \neq 0$: massless excitations and distinct t and $\vec{x}$ directions



Finite volume effects



Systematic effects

Finite size and temperature effects

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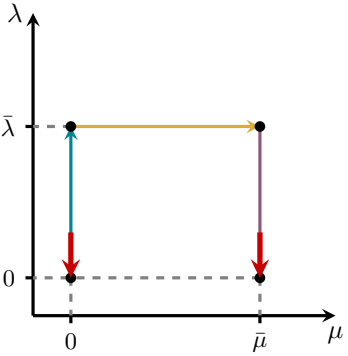
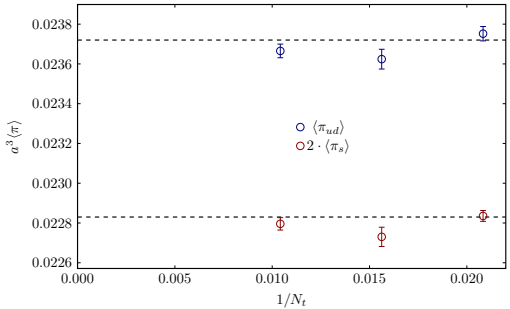


Finite temperature effects



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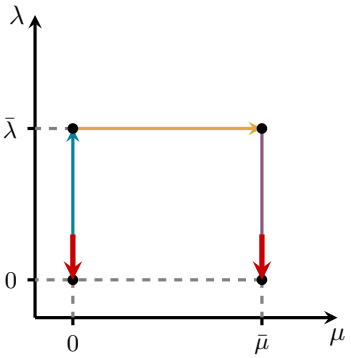


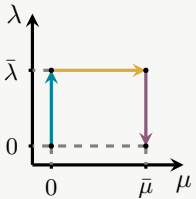
Final task:

Take continuum limit!

Outline

CRC-TR 211

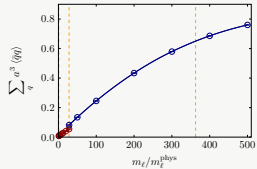
1



Phase quenched QCD in practice

- connection to isospin
- simulation settings
- integration contours

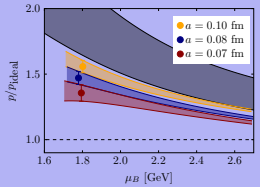
2



First results for $T = 0$ and large μ_B

- integrations in practice
- towards the continuum
- systematic effects

3



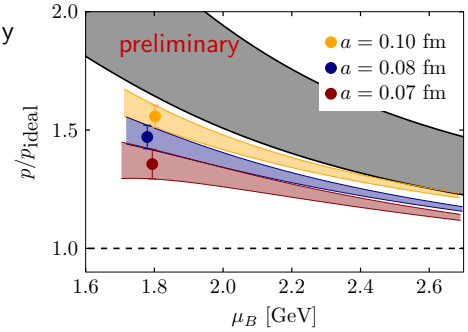
Prospects

- correcting gap effects
- pQCD contact at $T \neq 0$

Summary and discussion of results

Final results for the pressure at large μ_B

- first results in full phase quenched setting
- control over the main systematic effects
- "physical" case with $N_f = 2 + 1$ but currently at μ_B only
- fully include all non-perturbative effects
  - in particular: all gap effects on thermodynamics
- compared to perturbative bounds:
  there is NO assumption/approximation in these bounds
  $\Rightarrow$ continuum extrapolated result can be taken as is!
- can be systematically extended to chemical equilibrium
  β-equilibrium and charge neutrality: $\mu_d = \mu_s = \mu_u + \mu_e$
  (large effect???)



Superconducting gaps

CRC-TR 211

BCS gaps and their effect on the pressure

Asymptotically: perturbation theory will provide reliable results

however: in many situations the **ground state includes condensate of Cooper-pairs**

attractive interaction leads to gap Δ at Fermi surface

Barrois, NPB 129 (1977); Bailin Love, PRep. 107 (1984); ... Alford *et al* Rev. Mod. Phys. 80 (2008)

pairing effect on pressure: **shift to larger values – non-perturbative effect**

perturbative inclusion: $$p(\Delta) = p(0) + p_{\text{ideal}}\big(\gamma_1(g)\big|\frac{\Delta}{\mu}\big|^2 + \dots\big)$$

for non-perturbatively determined Δ

Superconducting gaps

BCS gaps in different regions of QCD parameter space

large μ_B: color superconductivity

asymptotically: CFL phase favoured

review: Alford *et al* Rev. Mod. Phys. 80 (2008)

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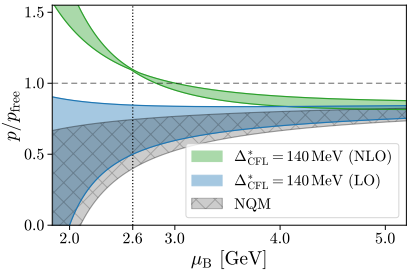
at $\mu_B \sim 2.5$ GeV: shift pressure to $\frac{p}{p_{\text{ideal}}} \sim 1$

e.g.: Geißel, Gorda, Braun, PRL 135 (2025); ...



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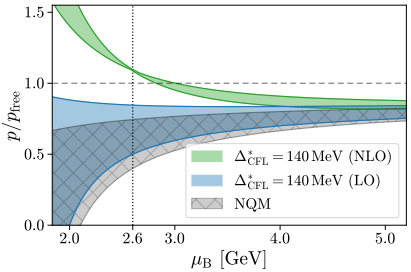

large μ_I: $\pi^{\pm}$-type Cooper pairs

Son, Stephanov, PRL 86 (2001)

gap is expected to be parametrically larger

Son, Stephanov, PRL 86 (2001); Fujimoto, PRD 109 (2024)

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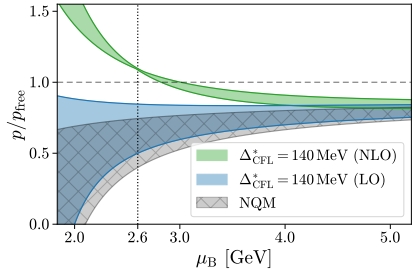

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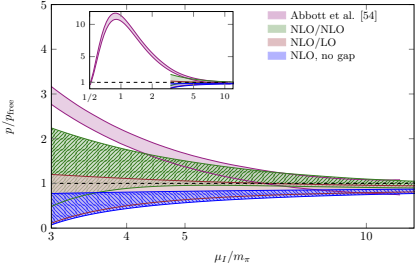
Son, Stephanov, PRL 86 (2001); Fujimoto, PRD 109 (2024)

$\Rightarrow$ effect larger at similar μ_f $(\mu_I/m_\pi \sim 6)$

e.g.: Rehberg-Juditzky, Geißel, Braun, arXiv:2610.25882; ...



Prospects: subtracting the gap effects

More accurate bounds through gap subtraction

Isospin gap $\Delta_{\mathrm{iso}} \geq \Delta_{\mathrm{QCD}}$ is present in PQ simulations

$\Rightarrow$ likely leads to strong excess of $\frac{p}{p_{\mathrm{ideal}}} > 1$

idea: **obtain tighter bounds by subtracting gap effects**

→ need to compute the gap in simulations

Computation of the gap $\Delta_{\mathrm{iso,\,f}}$

one possibility: Banks-Casher type relation

Kanazawa, Wettig, Yamamoto, EPJA 49 (2013)

$$\Delta^2_{\mathrm{iso,\,f}} = \frac{2\pi^3}{9} \left\langle \rho_{\mathrm{sing}}(0) \right\rangle \qquad (V \to \infty,\, m_f \to 0,\, \mu_f \text{ large})$$

→ difficult observable – first tests not conclusive – **currently: looking into alternatives**

Phase quenched bounds at non-zero temperature

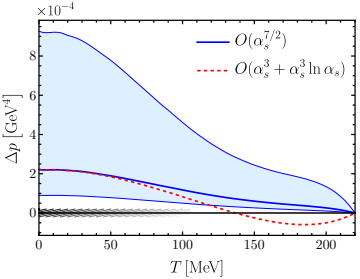
$T \neq 0$: no gap effects – direct comparison to perturbation theory possible

BCS condensate: "melts" at non-zero temperature (actually: $T_{\rm melt} \sim \Delta$)

→ gap effects are absent from the thermodynamics
& difference between QCD and PQ is only perturbatively

perturbative differences computed to $O(\alpha^{\frac{7}{2}})$ in Gorda *et al*, PRD 114 (2026)

proposal: compute $p_{\rm PQ}$ & perturbative Δp to obtain $p_{\rm QCD}$ Gorda *et al*, PRD 114 (2026)



Δp for arc $(4.338\pi T)^2 + \mu_B^2 = 3$ GeV^2
from Gorda *et al*, PRD 114 (2026)

See talk by Tyler Gorda
Wednesday at 10:00 am

Conclusions

- One application for pressure bounds from phase quenched QCD: constraints in model-agnostic sampling of neutron star EoS

  Computed via integral method with optimized integration paths

- Shown results for three lattice spacings in the relevant region with full control over systematic effects.

  To do: continuum limit

- To increase bound efficiency, gap effects might be subtracted!

  + other applications in the future

