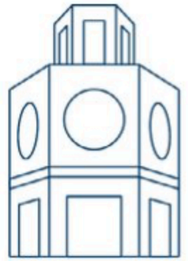
Heavy Quark Dynamics at Nonzero Temperature



UC SANTA BARBARA
Kavli Institute for Theoretical Physics

New Frontiers in QCD 2026
Yukawa Institute for Theoretical Physics
October 26, 2026

Bruno Scheihing-Hitschfeld (KITP), based on
2501.06289, 2504.21139, 2606.02693 with K. Rajagopal, U. A. Wiedemann
2604.21895, 2610.03863 with J. F. Du Plessis

$T > 0$

$|\mathbf{p}\rangle$

$M \gg T$



$T > 0$

$|\mathbf{p}\rangle$

$|\mathbf{p} - \mathbf{k}\rangle$

Propagate for a time t

$M \gg T$



$T > 0$

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Propagate for a time t

$M \gg T$



What is the probability distribution $P(\mathbf{k})$?

$T > 0$



What is the probability distribution $P(\mathbf{k})$?

$$P(\mathbf{k}) = P(\mathbf{k}; \mathbf{p}, T, ..., t)$$

Characterizing $P(\mathbf{k};\mathbf{v})$

the outline of this talk

1. Introduction: The momentum change probability in terms of a Wilson loop

2. Structural properties of the momentum change probability

   - *Universal Equilibration Condition for Heavy Quarks*
     (2504.21139 with K. Rajagopal and U. A. Wiedemann)

3. Weakly coupled calculation of the Wilson loop

   - *Heavy Quark Transport is Non-Gaussian Beyond Leading Log*
     (2604.21895, 2610.03863 with Jean F. Du Plessis)

4. A strongly coupled calculation: heavy quark transport in holography

   - *Dynamics of Heavy Quarks in Strongly Coupled $\mathcal{N} = 4$ SYM*
     (2501.06289 with K. Rajagopal and U. A. Wiedemann; see also 2606.02693)

5. Comparing weakly and strongly coupled environments

Dynamics of heavy particles

in gauge theories — a very short introduction to HQET

- We consider fermions coupled to a gauge field. In the large mass limit, the fermion kinetic term

$$\bar{\psi}(x)\left[i\gamma^{\mu}\left(\partial_{\mu}-igA_{\mu}\right)-M\right]\psi(x)\,,$$

$$\psi = e^{-iv_{\mu}x^{\mu}M}\left(\frac{1+v_{\mu}\gamma^{\mu}}{2}Q_{v}+\frac{1-v_{\mu}\gamma^{\mu}}{2}B_{v}\right)$$

becomes, after a field redefinition and expanding around a velocity v^{μ}

$$\bar{Q}(x)iv^{\mu}\left(\partial_{\mu}-igA_{\mu}\right)Q(x)+\mathcal{O}(M^{-1})\,.$$

- Corrections to this effective Lagrangian (HQET) are governed by the ratios of i) momentum transfer to the mass and ii) scales in A_{μ} to the mass.

$T > 0$

Heavy quark e.o.m.:

$\mu = 0$

$M \gg T$

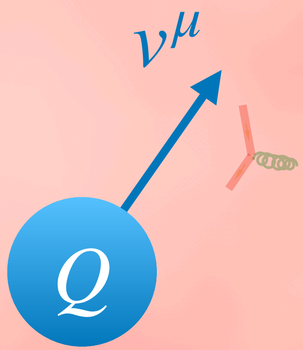
$$v^\mu \left(\partial_\mu - igA_\mu \right) Q(x) + \mathcal{O}\left(M^{-1}\right) = 0$$

$v^\mu = (1, \mathbf{v})$

$|\mathbf{p}\rangle = |M\gamma\mathbf{v}\rangle$

$\mathbf{x} \leftrightarrow \mathbf{k}$

($\mathbf{x}$ is conjugate to $\mathbf{k}$)



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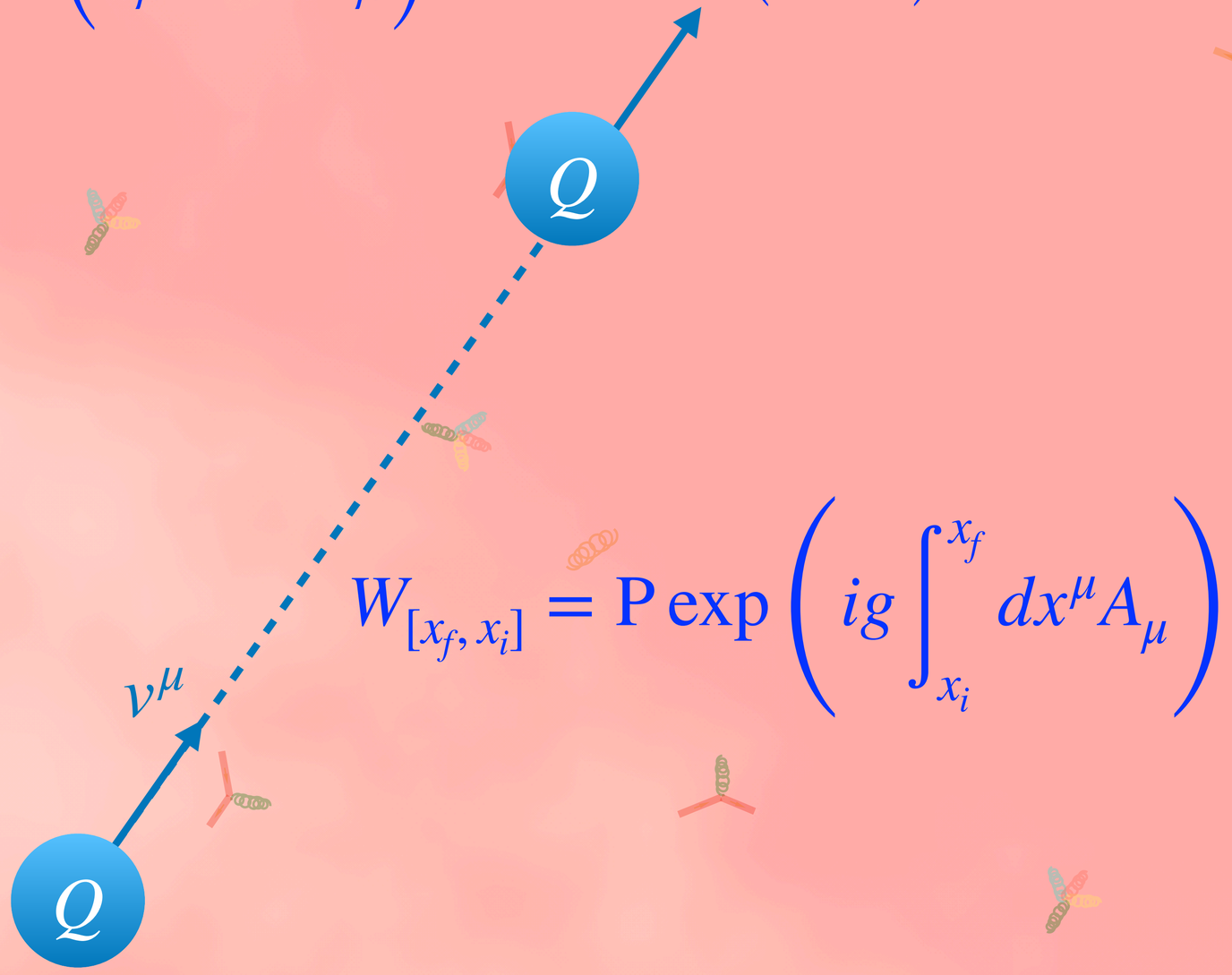
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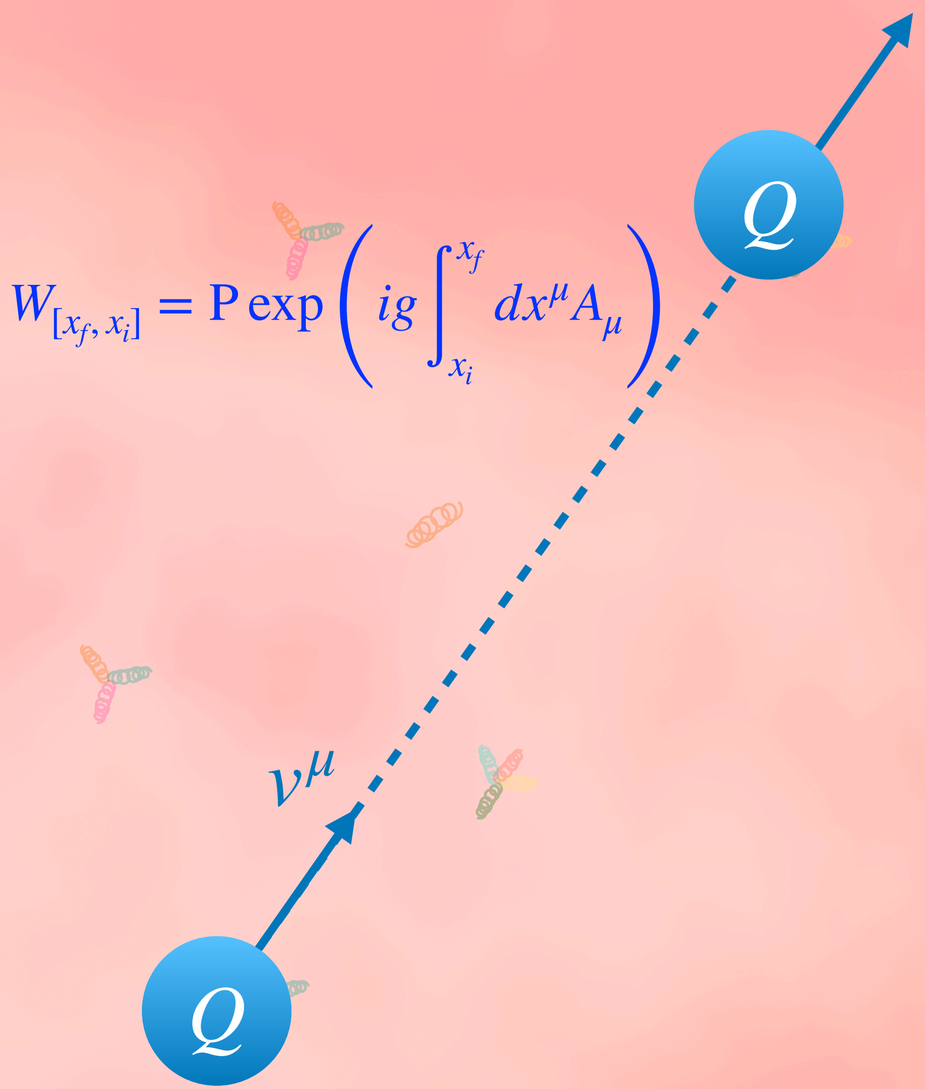
($\mathbf{x}$ is conjugate to $\mathbf{k}$)



$$W_{[x_f, x_i]} = \mathrm{P} \exp\left(ig \int_{x_i}^{x_f} dx^\mu A_\mu \right)$$

$T > 0$

$\mu = 0$



Going from position to momentum space,

$$\langle \mathbf{p} - \mathbf{k} \,|\, U(t) \,|\, \mathbf{p} \rangle = \int d^3x \, e^{i\mathbf{x}\cdot\mathbf{k}} \, W_{[x_f,\,x_i]}$$

Averaging over a (light d.o.f) thermal ensemble,

$$P(\mathbf{k};\mathbf{v}(\mathbf{p})) \propto \frac{\mathrm{Tr}\{\, |\langle \mathbf{p} - \mathbf{k} \,|\, U(t) \,|\, \mathbf{p} \rangle|^2 e^{-\beta H}\}}{\mathrm{Tr}\{e^{-\beta H}\}}$$

$$\propto \int d^3L \, e^{-i\mathbf{k}\cdot\mathbf{L}} \, \langle W_{\mathbf{v}} \rangle (\mathbf{L})$$

for more on this, see [Rajagopal, **BSH**, Wiedemann: 2504.21139, 2606.02693] and [Lin, **BSH**:2603.06784]

Initial and final state

what $|\mathbf{p}\rangle$ means in the previous slide

- What I should really be writing is

$$\langle W_{\mathbf{v}}\rangle(\mathbf{L}) = \frac{1}{Z}\mathrm{Tr}_{\mathscr{H}}\left[W^{lk}_{[(0,\mathbf{L}),(t,\mathbf{v}t+\mathbf{L})]}\,\tilde{\rho}_{ki}\,W^{ij}_{[(t,\mathbf{v}t),(0,\mathbf{0})]}\,e^{-\beta H}\rho_{jl}\right],$$

where ρ and $\tilde{\rho}$ describe the initial state preparation and the final state projection.

- For example, one can have $\tilde{\rho}_{ij} = \psi_i(t, \mathbf{v}t + \mathbf{L})\,\psi_j^\dagger(t, \mathbf{v}t)$ with ψ a light fermion.
- For purposes of this discussion, we will focus on the limit $t \gg T^{-1}$ (also $t \gg \Lambda_{\mathrm{QCD}}^{-1}, \ldots$ if applicable) in which case the leading behavior of the loop is $\langle W_{\mathbf{v}}\rangle(\mathbf{L}) = \exp\left(-t\,T\,S(\mathbf{L};\mathbf{v}) + O(t^0)\right)$, and the initial state preparation only contributes to the $O(1)$ prefactor.

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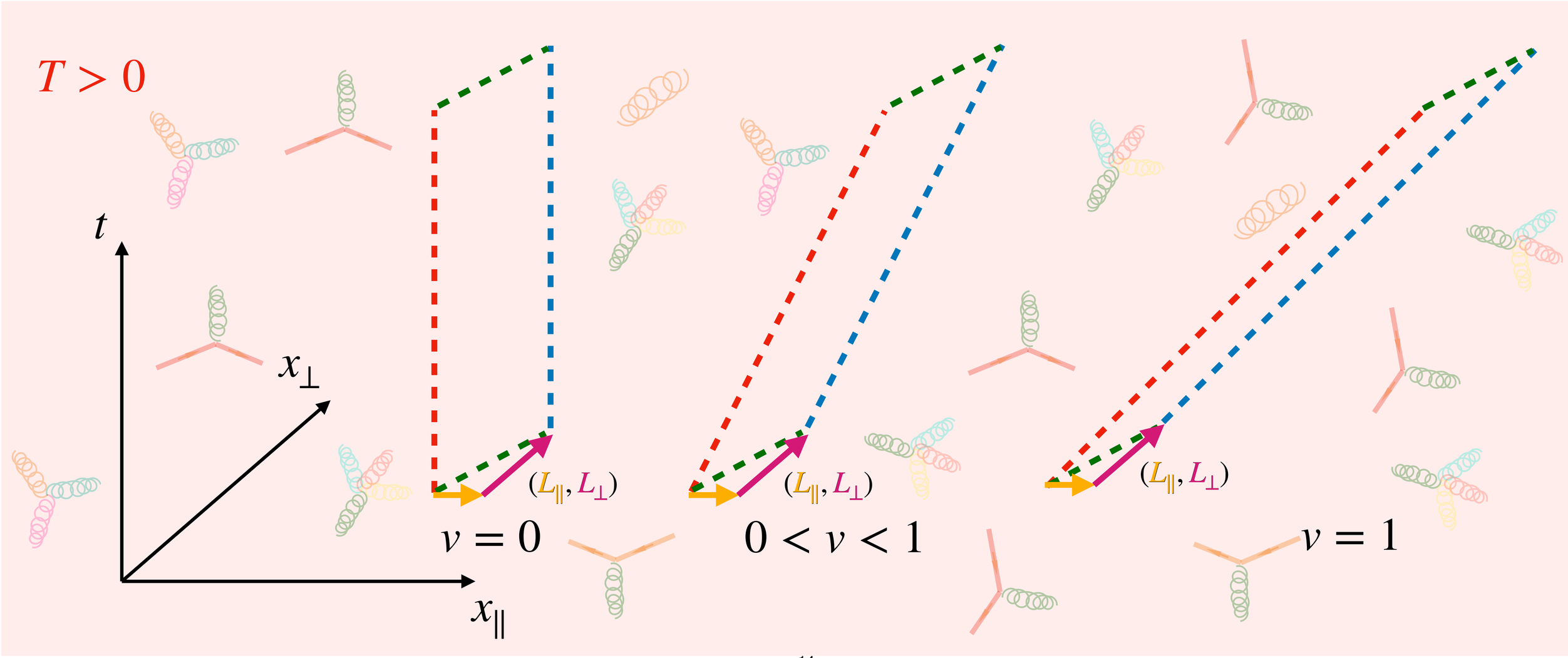
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Wilson loops

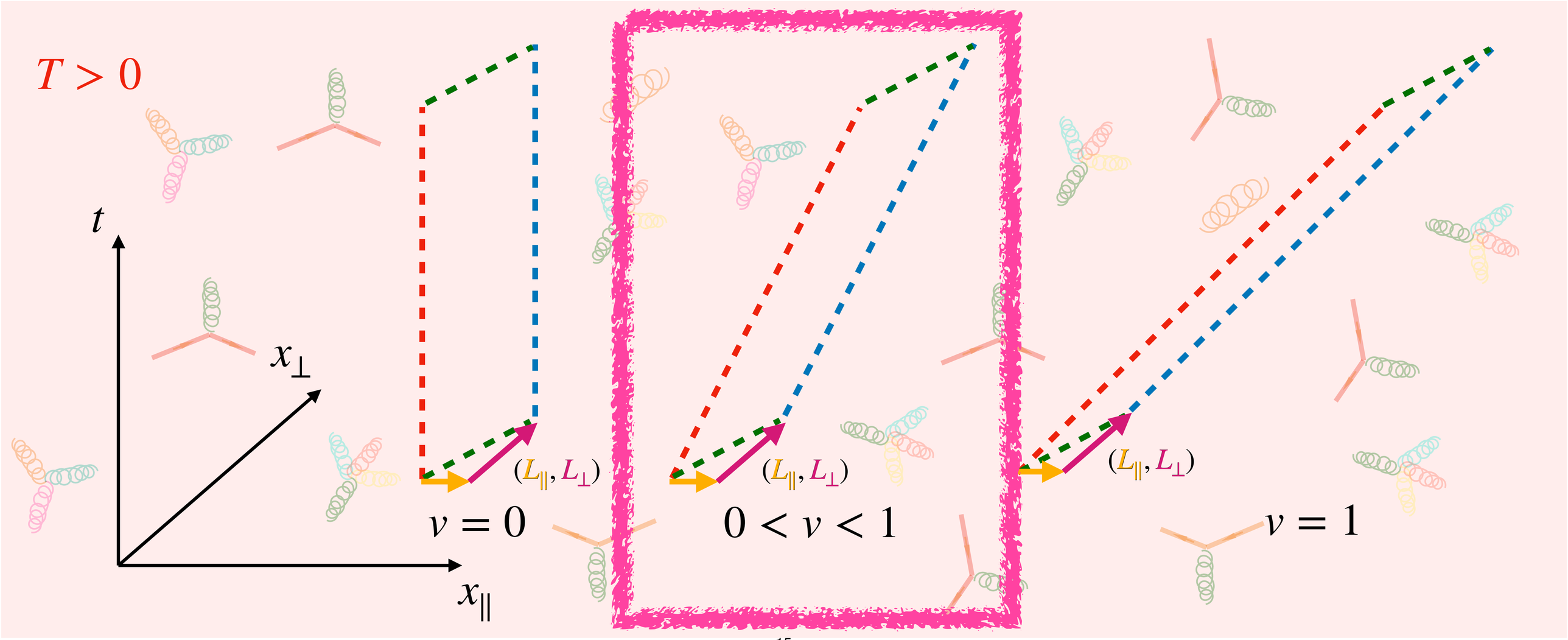
the configurations of interest



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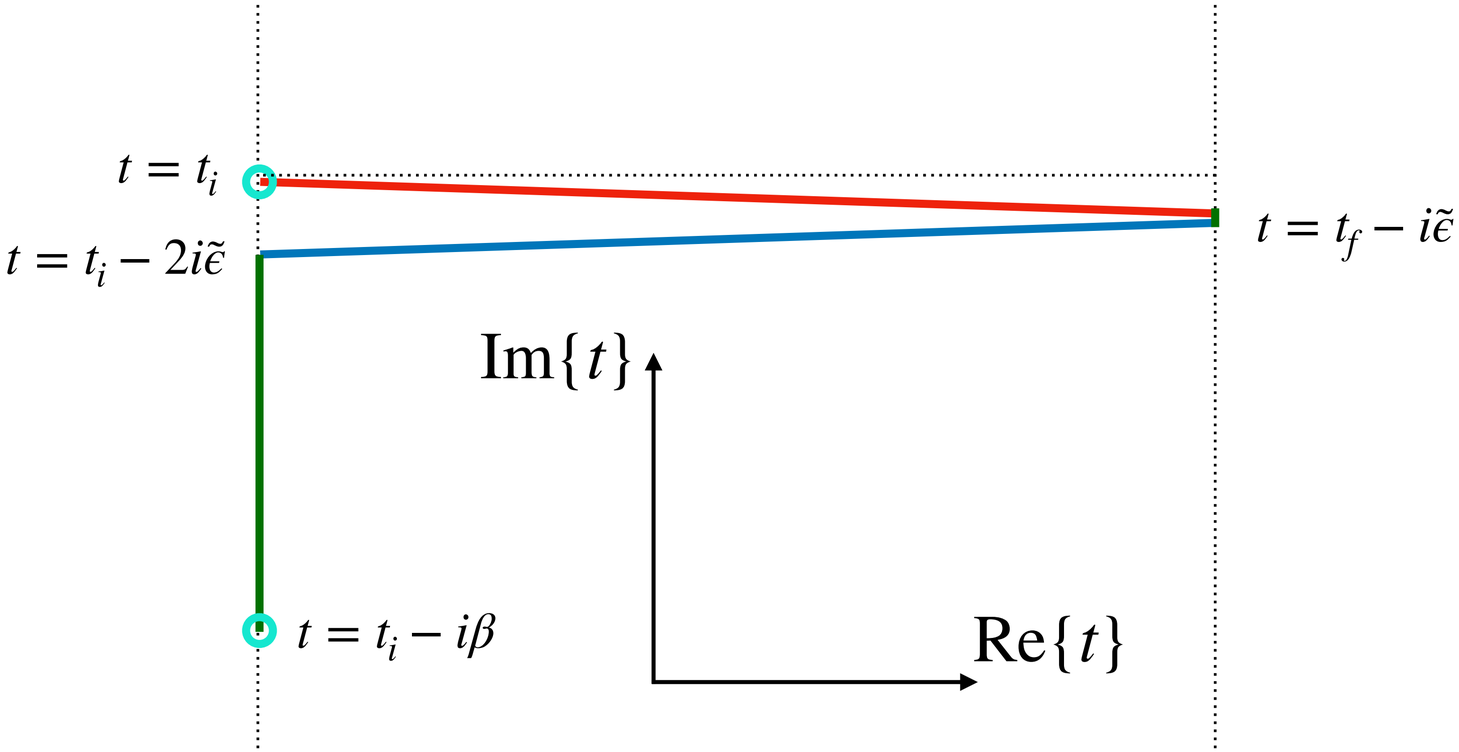
our main focus today

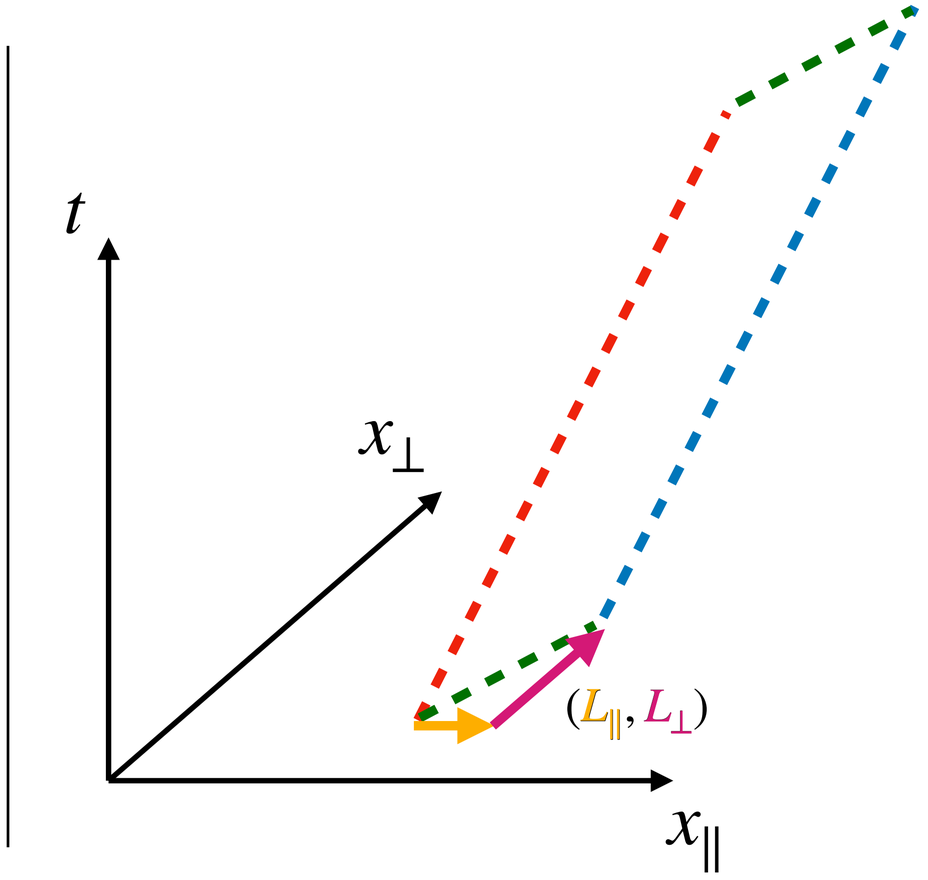


Operator ordering

and the Schwinger-Keldysh (SK) contour

- The expectation value we want to calculate is *not* a time-ordered correlation function. It is an extended operator of the form $\langle (\,\overline{\mathcal{T}}\,)\,(\,\mathcal{T}\,)\rangle_T$. We need to use the SK path integral.

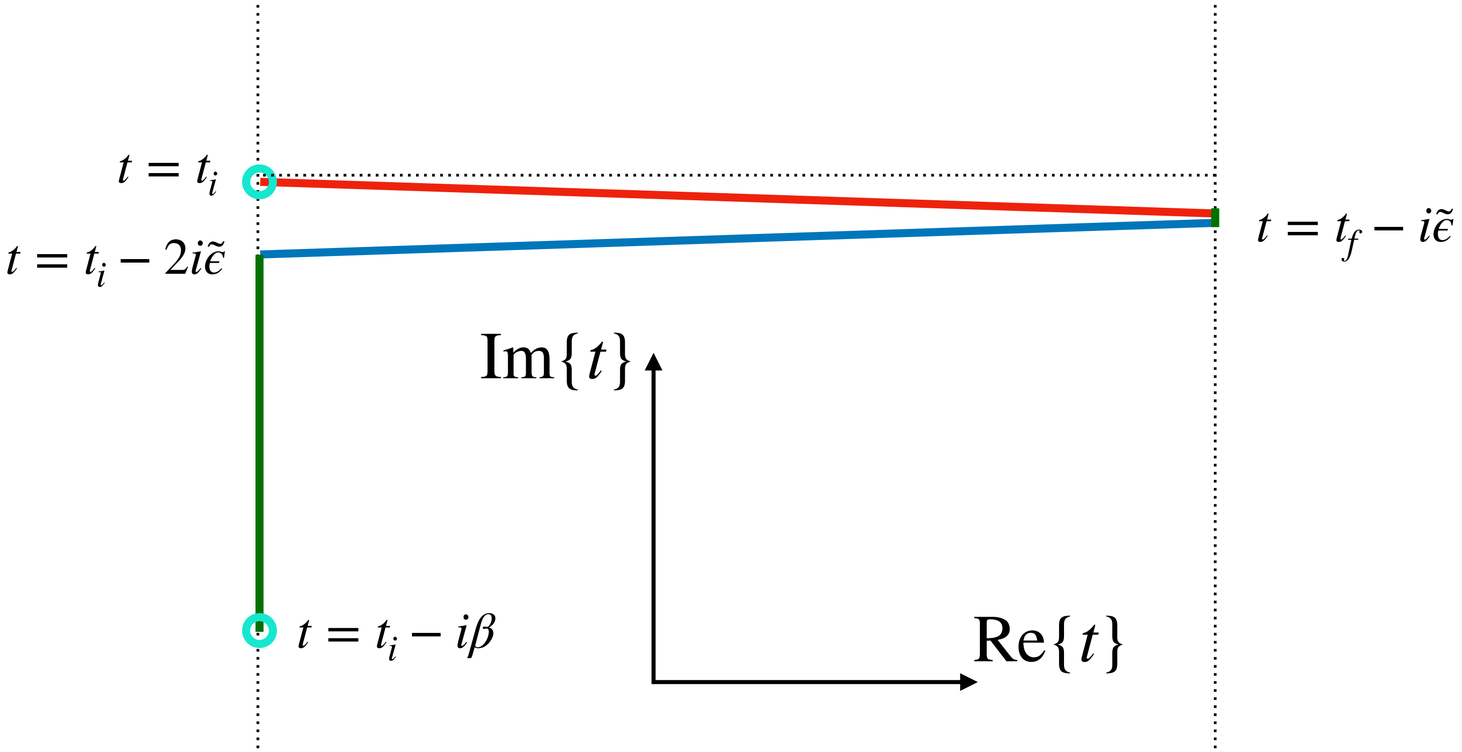


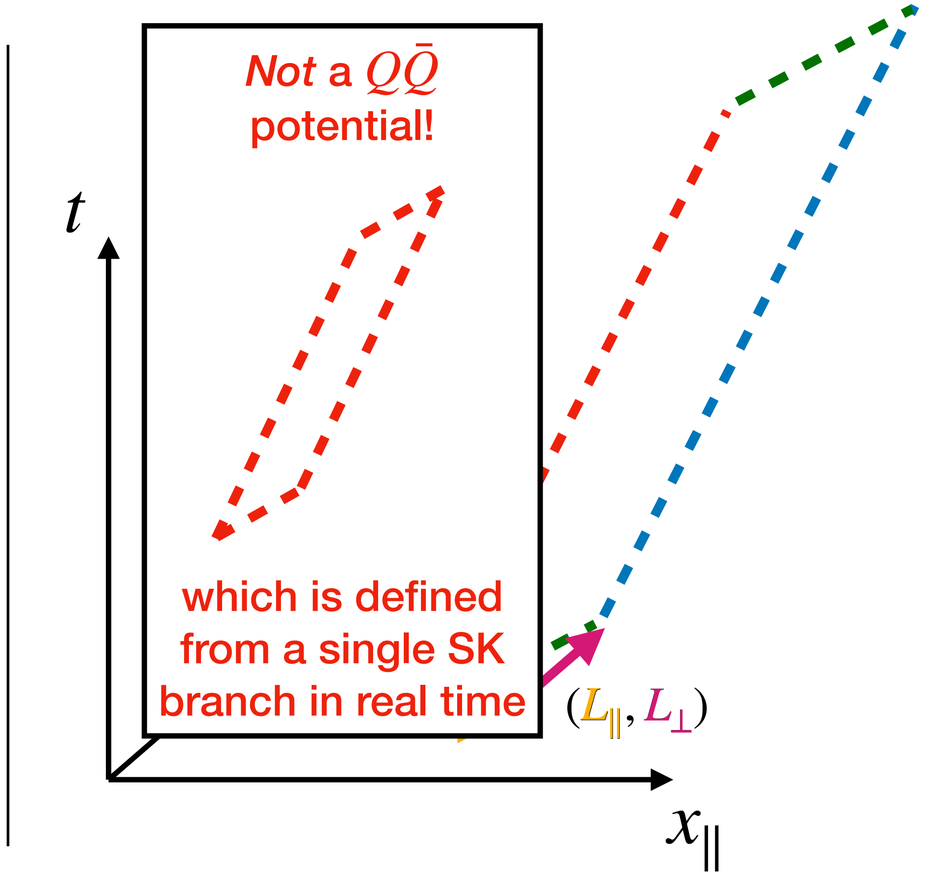


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Two remarks

regarding the definition of $P(\mathbf{k}; \mathbf{v})$

1. Provided the momentum transfer is small compared to M, the expression

$$P(\mathbf{k}; \mathbf{v}(\mathbf{p})) = \int \frac{d^3\mathbf{L}}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{L}} \langle W_{\mathbf{v}} \rangle(\mathbf{L})$$

   holds regardless of what the medium is — it only requires that the particle is coupled to the medium via a gauge field.

   - In particular, we can study it both at weak and strong coupling.

2. A very similar structure appears for light-like particles, although the EFT construction is different (SCET).

2. A structural, theory-independent, non-perturbative property of $P(\mathbf{k}; \mathbf{v})$

- *Universal Equilibration Condition for Heavy Quarks*
  (2504.21139 with K. Rajagopal and U. A. Wiedemann)

What we want to calculate

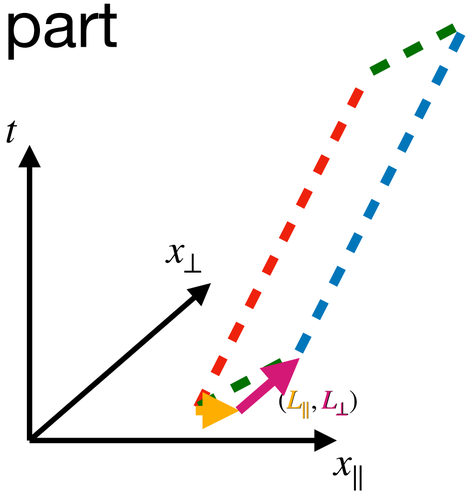
$$W_{[x_f,x_i]} = P\exp\left(ig\int_{x_i}^{x_f} dx^\mu A_\mu\right)$$

- We are interested in the late-time limit $tT \gg 1$.
  - We want to see how kicks from the medium change the HQ momentum *cumulatively* at long times (so that eventually the HQ velocity may change).
- In the large t limit, the emergence of a Markovian regime is equivalent to

$$\langle W_{\mathbf{v}}\rangle(\mathbf{L}) = \exp\left(-t\,T\,S(\mathbf{L};\mathbf{v}) + \mathcal{O}(t^0)\right).$$

- This exponential dependence is encoded in the time-dependent part

$$\langle W_{\mathbf{v}}\rangle(\mathbf{L}) \approx \frac{1}{Z}\mathrm{Tr}\left[W_{[(0,\mathbf{L}),(t,\mathbf{v}t+\mathbf{L})]}W_{[(t,\mathbf{v}t),(0,\mathbf{0})]}e^{-\beta H}\right].$$



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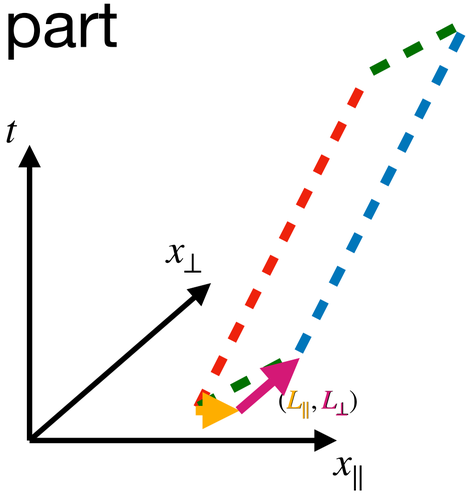
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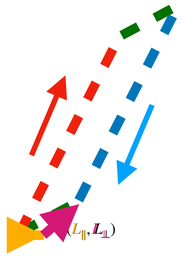


- Let us look at this more closely.

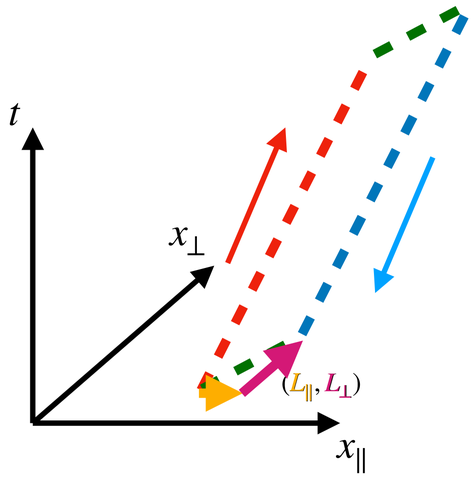
A property of line operators in equilibrium

$$W_{[x_f,x_i]} = P\exp\left(ig\int_{x_i}^{x_f} dx^\mu A_\mu\right)$$

(Rajagopal, BSH, Wiedemann 2504.21139)



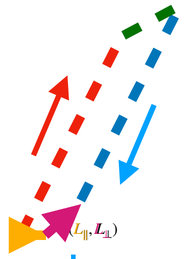
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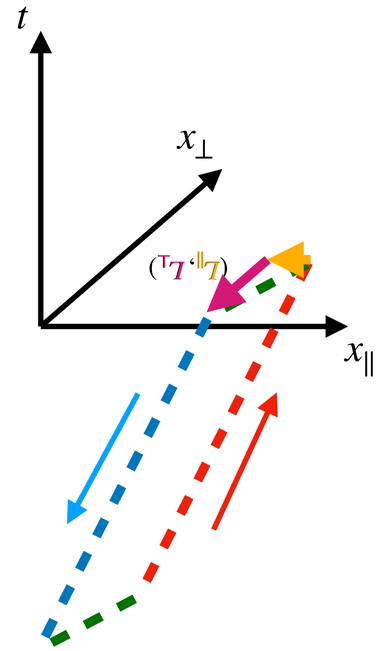
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Parity and time reversal: $\mathscr{P}\mathscr{T} \implies$

$$= \frac{1}{Z}\mathrm{Tr}\left[W_{[(0,\mathbf{0}),(-t,-\mathbf{v}t)]}W_{[(-t,-\mathbf{v}t-\mathbf{L}),(0,-\mathbf{L})]}e^{-\beta H}\right]$$



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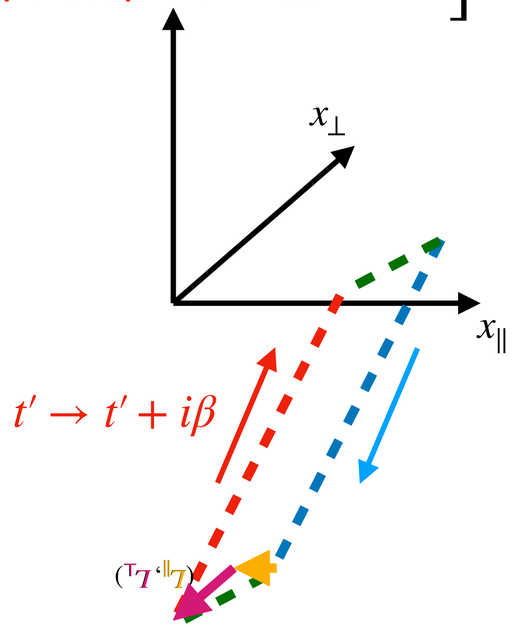
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Time evolution:

$$e^{-\beta H}\mathcal{O}(t) = \mathcal{O}(t+i\beta)e^{-\beta H} \implies$$

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A property of line operators in equilibrium

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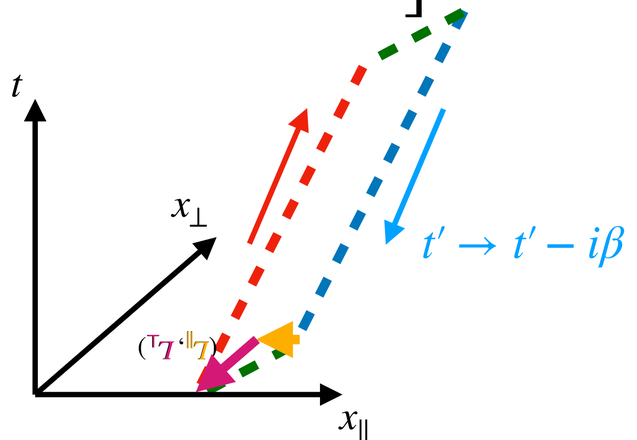
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Translation symmetry: $[P_i, H] = 0 \implies$

$$= \frac{1}{Z}\mathrm{Tr}\left[W_{[(-i\beta,-\mathbf{L}),(t-i\beta,\mathbf{v}t-\mathbf{L})]}W_{[(t,\mathbf{v}t),(0,\mathbf{0})]}e^{-\beta H}\right]$$



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Reparametrization invariance for $t \to \infty$
+ $e^{-iP_\mu x^\mu}$ analytic function of x^μ $\implies$

$$\approx \frac{1}{Z}\mathrm{Tr}\left[W_{[(0,-\mathbf{L}+i\beta\mathbf{v}),(t,\mathbf{v}t-\mathbf{L}+i\beta\mathbf{v})]}W_{[(t,\mathbf{v}t),(0,\mathbf{0})]}e^{-\beta H}\right]$$

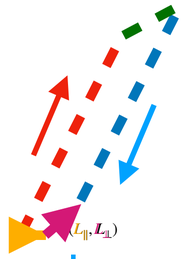
$$= \langle W_{\mathbf{v}}\rangle(-\mathbf{L}+i\mathbf{v}/T)\,.$$



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$$= \langle W_{\mathbf{v}}\rangle(-\mathbf{L}+i\mathbf{v}/T)\,.$$



$$P(\mathbf{k};\mathbf{v}) = \int \frac{d^3L}{(2\pi)^3}\, e^{-i\mathbf{k}\cdot\mathbf{L}}\, \langle W[\mathscr{C}]\rangle_T(\mathbf{L})$$

- This condition

$$\langle W_{\mathbf{v}}\rangle(\mathbf{L}) = \langle W_{\mathbf{v}}\rangle(-\mathbf{L} + i\mathbf{v}/T)\,.$$

is in fact equivalent to

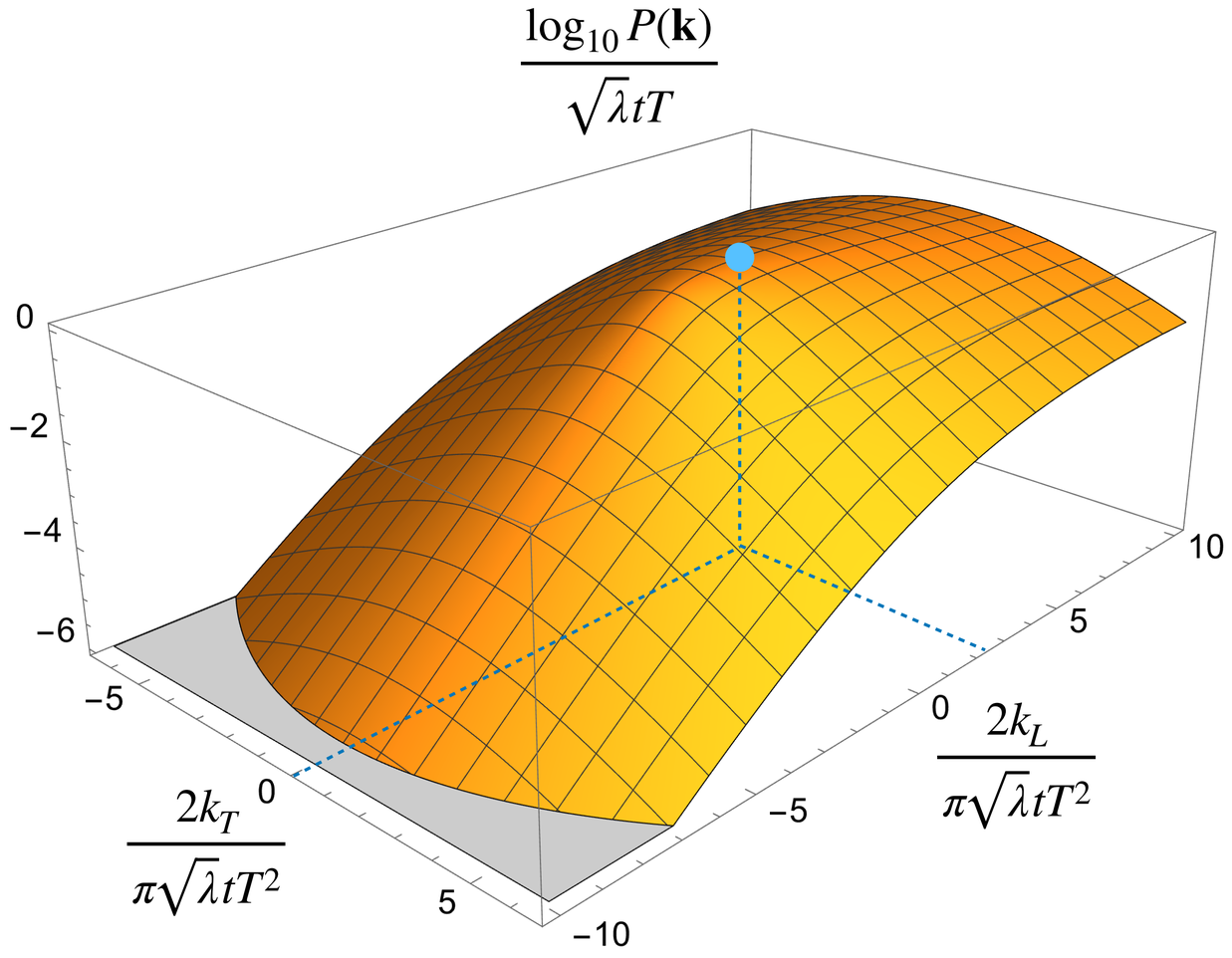
$$P(\mathbf{k};\mathbf{v})\, \exp\left[-\frac{\mathbf{v}\cdot\mathbf{k}}{2T}\right]$$

$$= P(-\mathbf{k};\mathbf{v})\, \exp\left[\frac{\mathbf{v}\cdot\mathbf{k}}{2T}\right]\,,$$

a universal symmetry property of $P(\mathbf{k};\mathbf{v})$.

($\Longleftrightarrow$ detailed balance)

- Note: $P(\mathbf{k};\mathbf{v})$ can be highly non-Gaussian!



Example $P(\mathbf{k};\mathbf{v})$

The Universal Equilibration Condition

a QFT derivation of detailed balance for heavy particles

- The symmetry property

$$P(\mathbf{k};\mathbf{v})\,\exp\left[-\frac{\mathbf{v}\cdot\mathbf{k}}{2T}\right] = P(-\mathbf{k};\mathbf{v})\,\exp\left[\frac{\mathbf{v}\cdot\mathbf{k}}{2T}\right]$$

is exactly the detailed balance condition

$$\exp\left(-\frac{E(\mathbf{p})}{T}\right)P(\mathbf{k};\mathbf{v}(\mathbf{p})) = \exp\left(-\frac{E(\mathbf{p}-\mathbf{k})}{T}\right)P(-\mathbf{k};\mathbf{v}(\mathbf{p}-\mathbf{k}))$$

that *ensures* that the Boltzmann distribution $\exp\left(-E(\mathbf{p})/T\right)$ is a stationary solution to the stochastic process defined by $P(\mathbf{k};\mathbf{v}(\mathbf{p}))$. (up to $\mathcal{O}(\mathbf{k}/M)$ corrections)

$$P(\mathbf{k};\mathbf{v}) = \int \frac{d^3L}{(2\pi)^3}\, e^{-i\mathbf{k}\cdot\mathbf{L}} \langle W[\mathscr{C}]\rangle_T(\mathbf{L})$$

Summary

Universal Equilibration Condition for Heavy Quarks 2504.21139

- QFT calculations of the momentum change probability of a charged heavy particle in a thermal medium are guaranteed to verify

$$\langle W_{\mathbf{v}}\rangle(\mathbf{L}) = \langle W_{\mathbf{v}}\rangle(-\mathbf{L} + i\mathbf{v}/T)\,,$$

or equivalently,

$$P(\mathbf{k};\mathbf{v})\, \exp\left[-\frac{\mathbf{v}\cdot\mathbf{k}}{2T}\right] = P(-\mathbf{k};\mathbf{v})\, \exp\left[\frac{\mathbf{v}\cdot\mathbf{k}}{2T}\right]\,.$$

This condition guarantees that kinetic equilibrium can be attained.

3. Calculating the Wilson loop at weak coupling

- *Heavy Quark Transport is Non-Gaussian Beyond Leading Log*
  (2604.21895, 2610.03863 with Jean F. Du Plessis)

Weakly coupled calculation of the Wilson loop

a brief overview [Jean F. Du Plessis & BSH: 2604.21895, 2610.03863]

- We calculate $S(\mathbf{L};\mathbf{v})$ defined as

$$\langle W_{\mathbf{v}}\rangle(\mathbf{L}) = \exp\left(-t\,T\,S(\mathbf{L};\mathbf{v}) + \ldots\right)$$

  in weakly coupled $\mathrm{SU}(N_c)$ gauge theories with N_f Dirac fermions and N_X real adjoint scalars.

- In perturbation theory, the leading contribution is proportional to g^4. We show the diagrams that give rise to $\mathbf{L}$ dependence on the right.

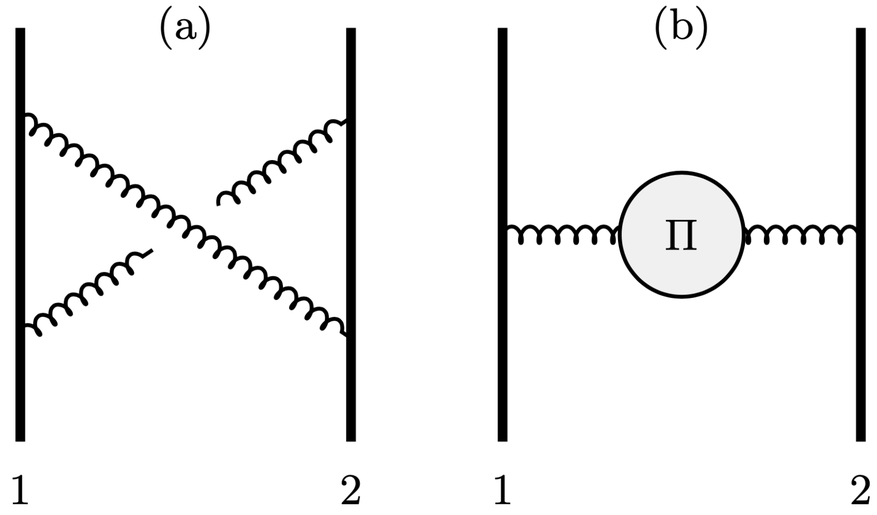


FIG. 2. Contributing order-g^4 perturbative topologies in Feynman gauge. Panel (a) denotes the crossed one-gluon exchanges between opposite contour branches; panel (b) denotes the self-energy correction to a single exchanged gluon, including gauge, fermion, and scalar contributions.

Weakly coupled calculation of the Wilson loop

the details [Jean F. Du Plessis & BSH: 2604.21895, 2610.03863]

- Technically, the most challenging aspect of the calculation is that one needs to use HTL resummation to regulate the IR physics without introducing higher-order g^n contamination. This means

$$S(\mathbf{L};\mathbf{v}) = -\frac{g^2 C_F}{T}\int_{\mathbf{p}} \frac{e^{i\mathbf{p}\cdot\mathbf{L}}-1}{1-e^{-\mathbf{p}\cdot\mathbf{v}/T}}\Big\{ g^2 I_{\rm Hard}(\mathbf{p},\mathbf{v};T,N_c,N_X,N_f)$$

$$+I_{\rm Soft}(\mathbf{p},\mathbf{v};m_D^2) - I_{\rm Match}(\mathbf{p},\mathbf{v};m_D^2)\Big\},$$

where “hard” is a reflection of the fact that the characteristic scale in that term is $p\sim T$. “Soft” stands for $p\sim gT$, and requires HTL resummation. “Match” avoids double-counting.

Weakly coupled calculation of the Wilson loop

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Individually **IR** divergent, finite in total

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Individually **UV** divergent, finite in total

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Weakly coupled calculation of the Wilson loop

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Individually **UV** divergent

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The structure of the result

log-enhanced and fixed-order terms

- The result of the calculation can be divided into two terms:

$$S(\mathbf{L};\mathbf{v}) = g^4 \ln(1/g)\, \bar{S}_{\rm LL}(\mathbf{L};\mathbf{v}) \; + g^4\, \bar{S}_{\rm FO}(\mathbf{L};\mathbf{v})\,,$$

  which differ in their coupling dependence by $\ln(1/g)$.

- $\bar{S}_{\rm LL}$ is a quadratic function of $\mathbf{L}$, meaning that if all we kept is this term, we would have a Gaussian $P(\mathbf{k};\mathbf{v})$. (However, in practice, the magnitude of g is never *that* small…)

- $\bar{S}_{\rm FO}$ is *not* a quadratic function of $\mathbf{L} \Longrightarrow P(\mathbf{k};\mathbf{v})$ is non-Gaussian.

- The equilibration condition $S(\mathbf{L};\mathbf{v}) = S(-\mathbf{L} + i\mathbf{v}/T;\mathbf{v})$ is satisfied separately for each contribution.

Aside: how a kinetic theory emerges

in terms of the HQ phase space distribution

- The heavy quark (HQ) phase space distribution $\mathscr{P}(\mathbf{p}, t)$, i.e., the "number" of HQs with momentum $\mathbf{p}$ at time t evolves following

$$\mathscr{P}(\mathbf{p}, t + \Delta t) = \int d^3\mathbf{k}\, P(\mathbf{k}; \mathbf{v}(\mathbf{p}+\mathbf{k}), \Delta t)\, \mathscr{P}(\mathbf{p}+\mathbf{k}, t)\ .$$

- The separation of scales $T \ll M$ allows for a Markovian regime, where

$$\partial_t \mathscr{P} = -\,T\,S(-i\partial_{\mathbf{p}}; \mathbf{p})\mathscr{P} = \int d^3\mathbf{k}\,\Big[\mathscr{C}(\mathbf{k}; \mathbf{p}+\mathbf{k})\mathscr{P}(\mathbf{p}+\mathbf{k}) - \mathscr{C}(\mathbf{k}; \mathbf{p})\mathscr{P}(\mathbf{p})\Big].$$

where $\mathscr{C}$ is (up to an additive shift) the Fourier transform of S. In this sense, it encodes the rate of interactions (intuitive in a weak coupling expansion).

Aside: how a kinetic theory emerges

in terms of the HQ phase space distribution

- The heavy quark (HQ) phase spac
  HQs with momentum $\mathbf{p}$ at time t e

$$\mathscr{P}(\mathbf{p}, t + \Delta t) = \int d^3\mathbf{k}\, K$$

- The separation of scales $T \ll M$

$$\partial_t \mathscr{P} = -\mathscr{C}\, S(-i\partial_{\mathbf{p}}; \mathbf{p})\mathscr{P} = \int d^3\mathbf{k}$$

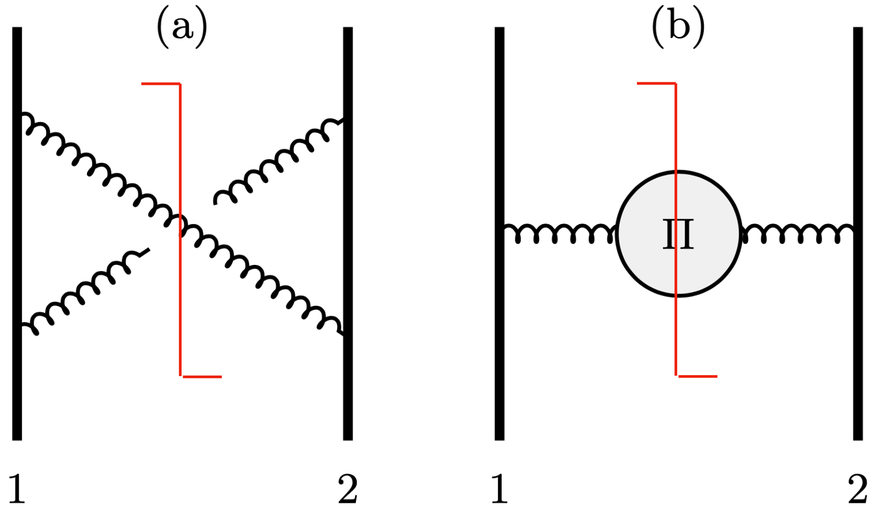


FIG. 2. Contributing order-g^4 perturbative topologies in Feynman gauge. Panel (a) denotes the crossed one-gluon exchanges between opposite contour branches; panel (b) denotes the self-energy correction to a single exchanged gluon, including gauge, fermion, and scalar contributions.

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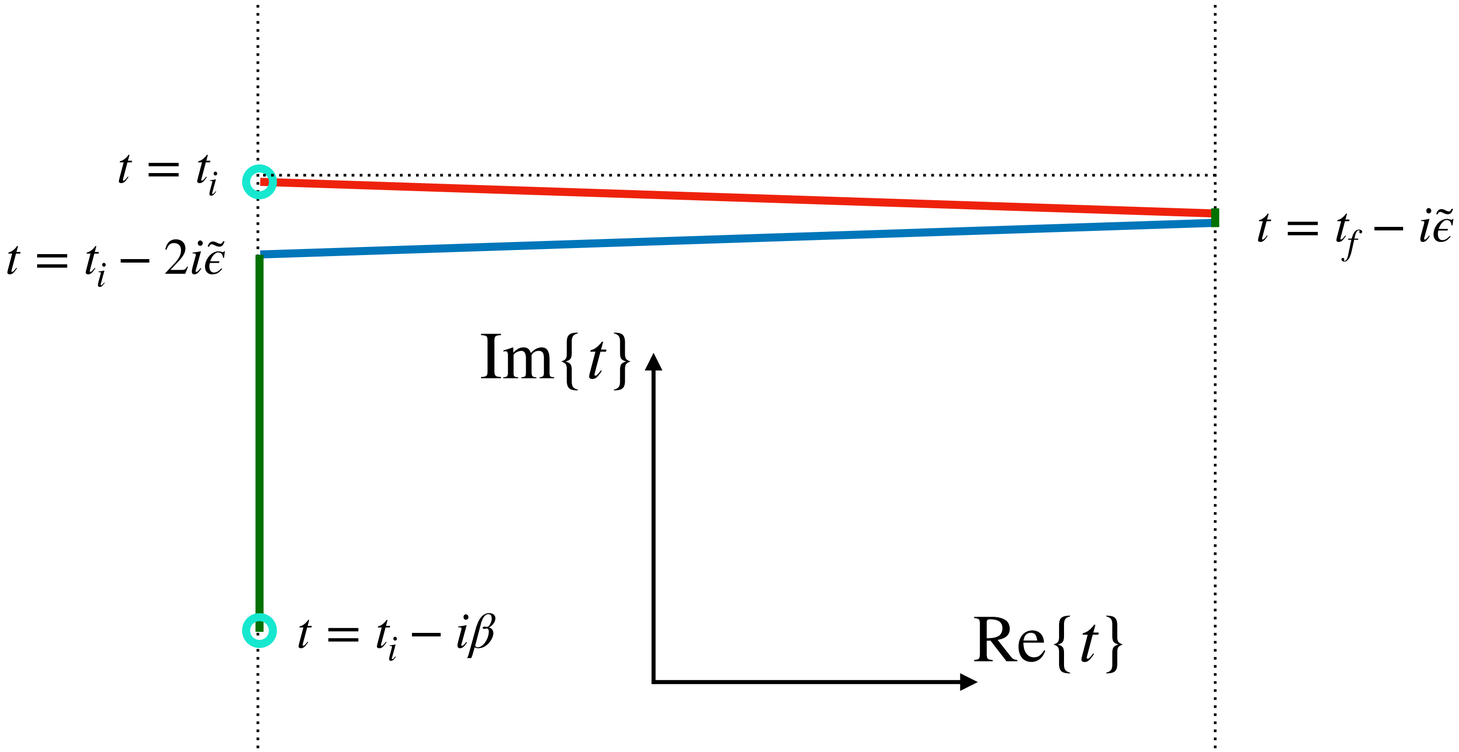
4. Calculating the Wilson loop at strong coupling via AdS/CFT

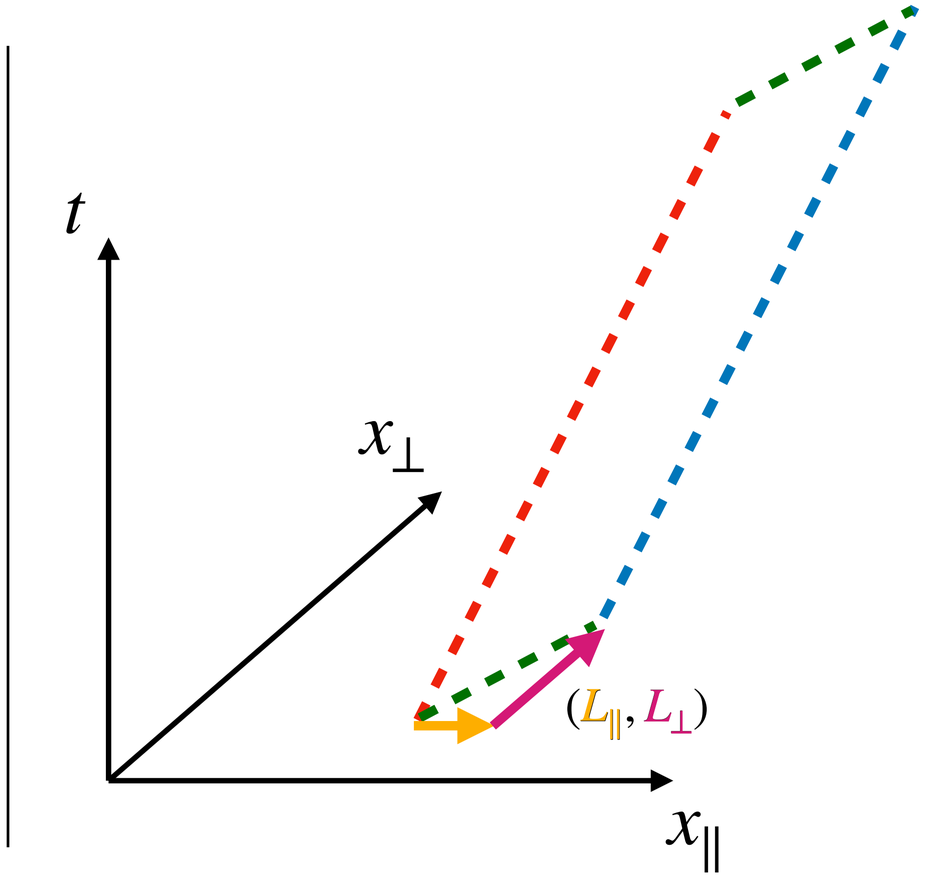
- *Dynamics of Heavy Quarks in Strongly Coupled $\mathcal{N} = 4$ SYM*
  (2501.06289 with K. Rajagopal and U. A. Wiedemann)

Operator ordering

and the Schwinger-Keldysh (SK) contour

- The expectation value we want to calculate is *not* a time-ordered correlation function. It is an extended operator of the form $\langle (\,\overline{\mathcal{T}}\,)\,(\,\mathcal{T}\,)\rangle_T$. We need to use the SK path integral.

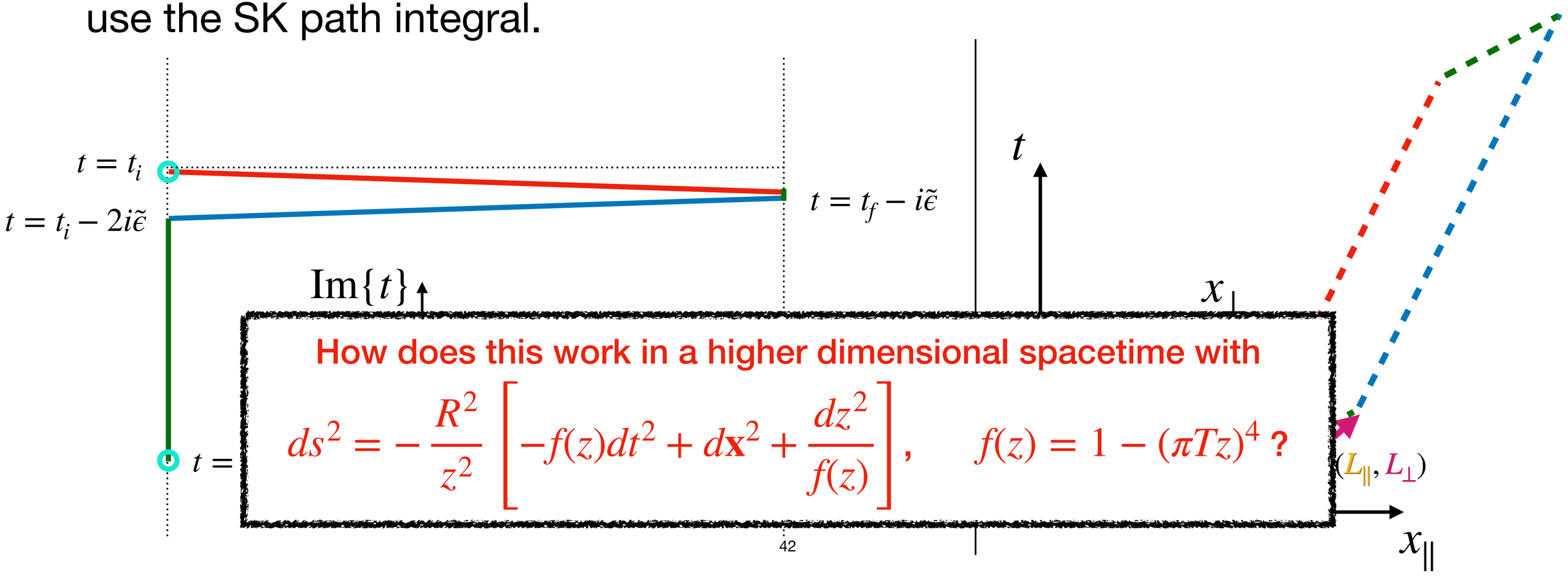




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The Schwinger-Keldysh contour

in holography [Skenderis and B. C. van Rees, 0805.0150, 0812.2909]

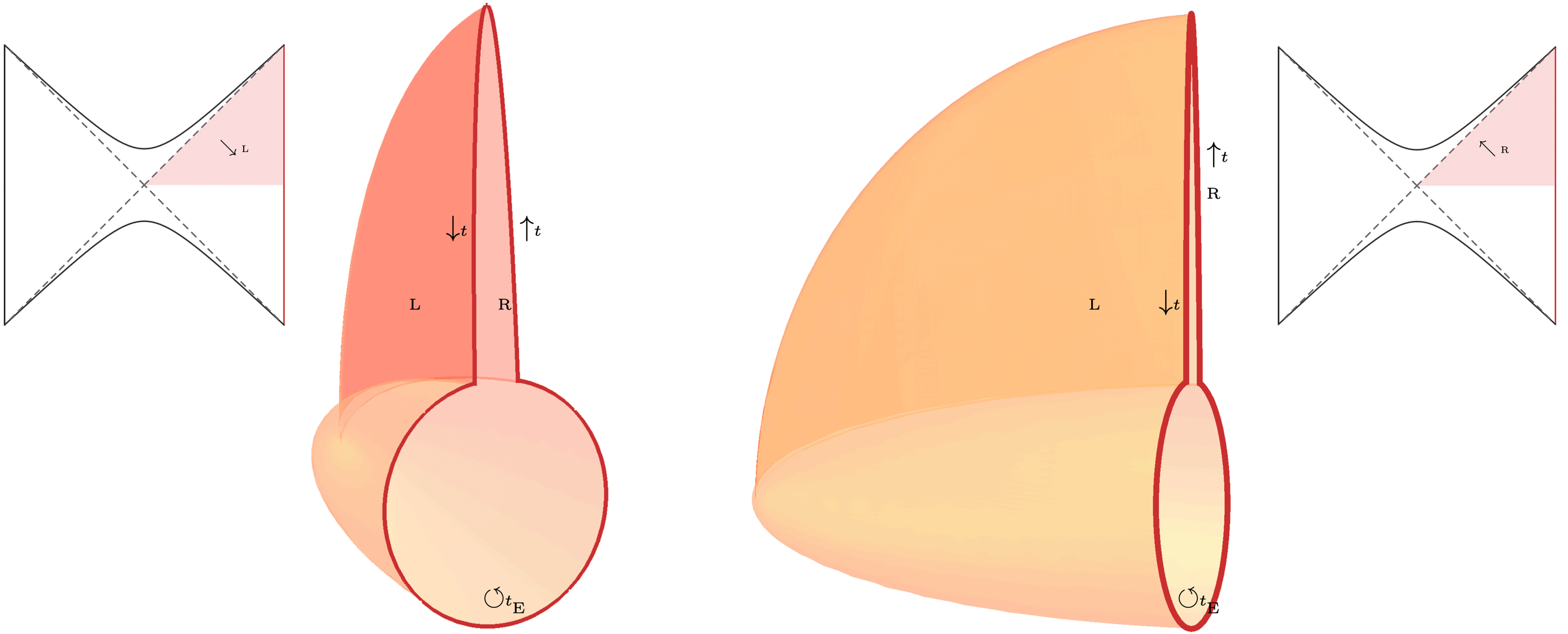
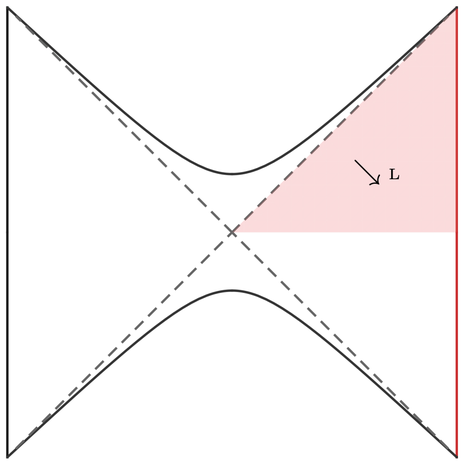
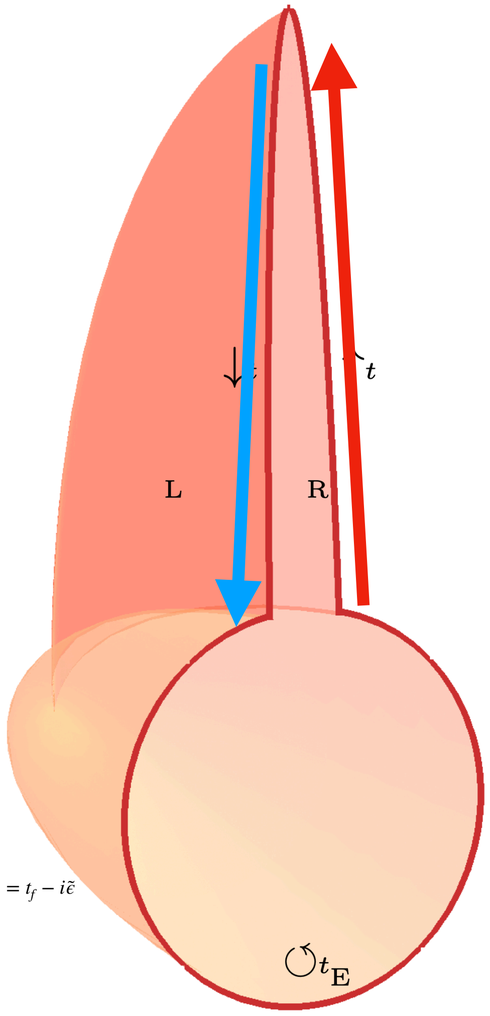


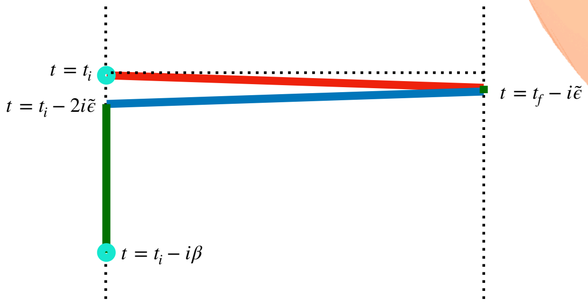
figure credit: Jana, Loganayagam, Rangamani 2004.02888

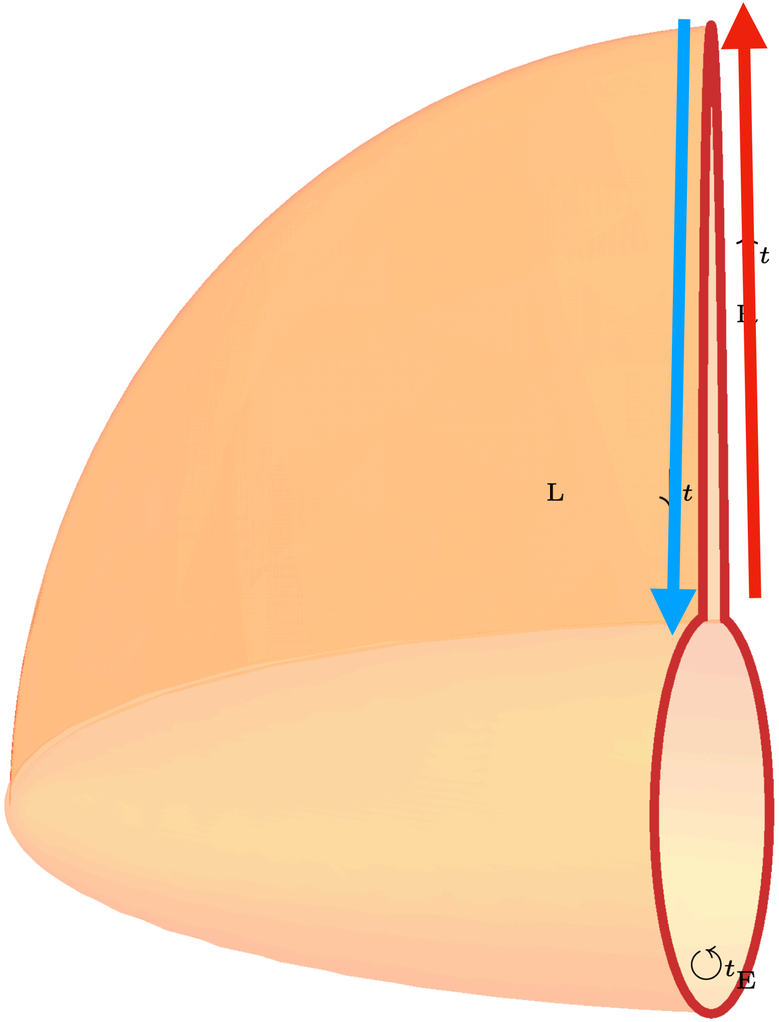
The Schwinger-Keldysh contour

in holography [Skenderis and B. C. van Rees, 0805.0150, 0812.2909]









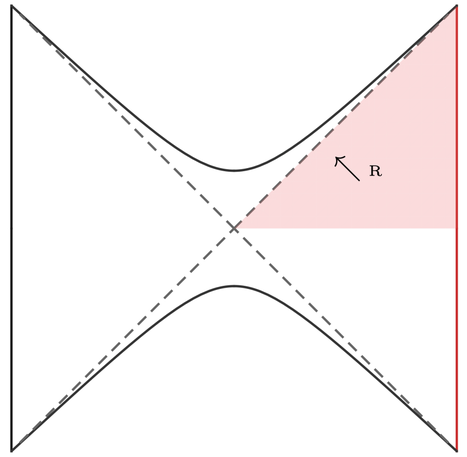
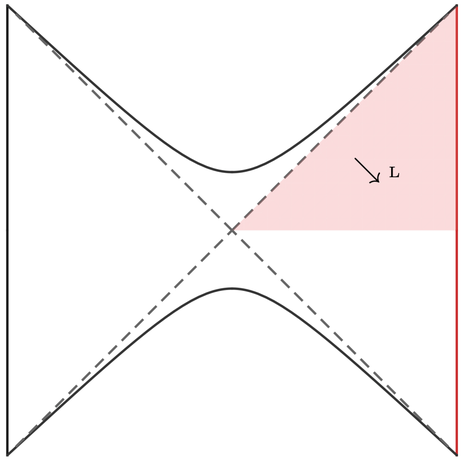


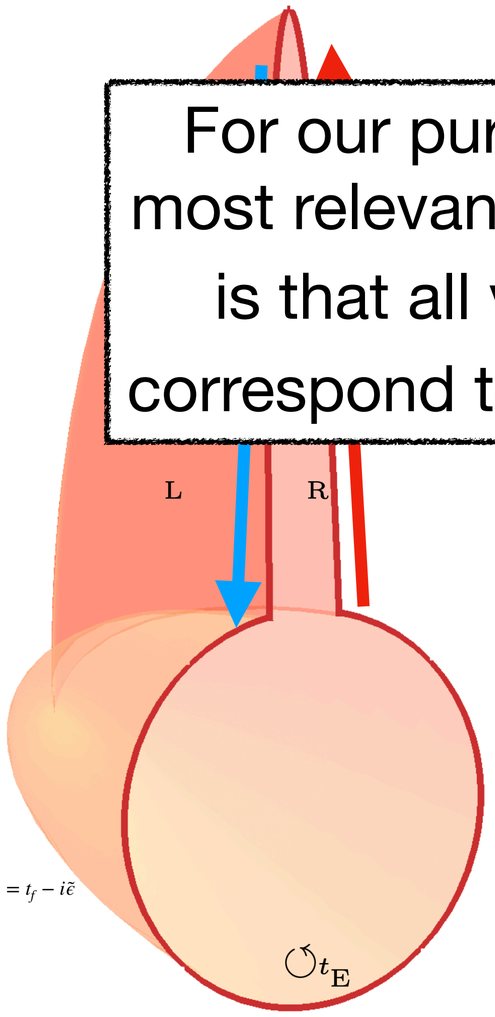
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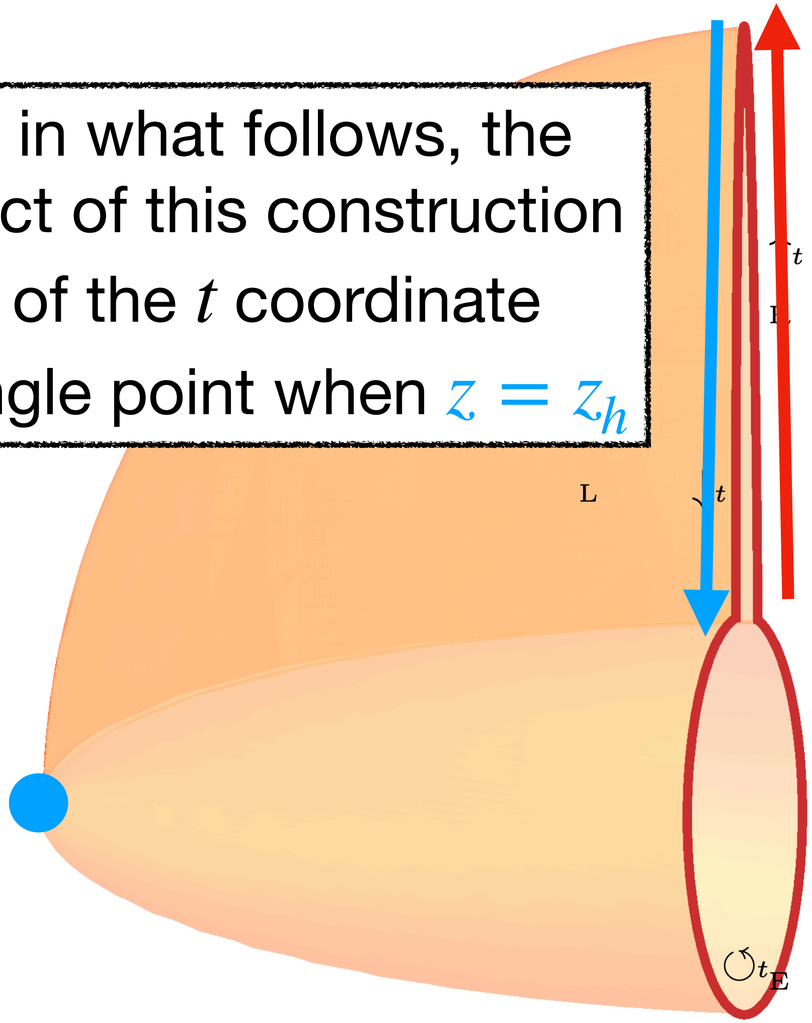
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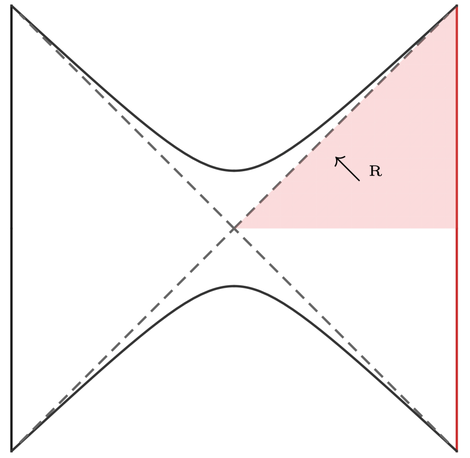
in holography [Skenderis and B. C. van Rees, 0805.0150, 0812.2909]



For our purposes in what follows, the most relevant aspect of this construction is that all values of the t coordinate correspond to a single point when $z = z_h$







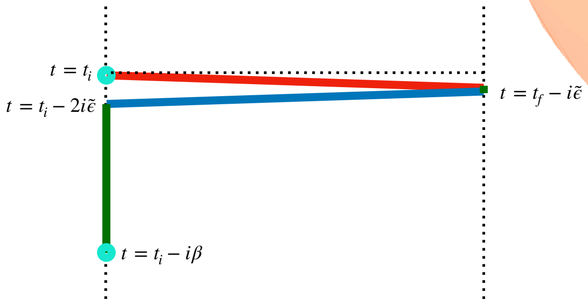


figure credit: Jana, Loganayagam, Rangamani 2004.02888

[1]Rey & Yee hep-th/9803001
[2]Maldacena hep-th/9803002
[3]Drukker, Gross & Ooguri hep-th/9904191

How to calculate Wilson loops

in strongly coupled $\mathcal{N} = 4$ SYM

- The expectation value of Wilson loops can be calculated as[1,2,3]

$$\langle W[\mathscr{C}] \rangle = \exp\left\{ -(\mathcal{S}_{\mathrm{NG}}[\Sigma(\mathscr{C})] - \mathcal{S}_0) \right\} ,$$

where $\mathcal{S}$ is the action of a string (with boundary conditions set by $\mathscr{C}$)

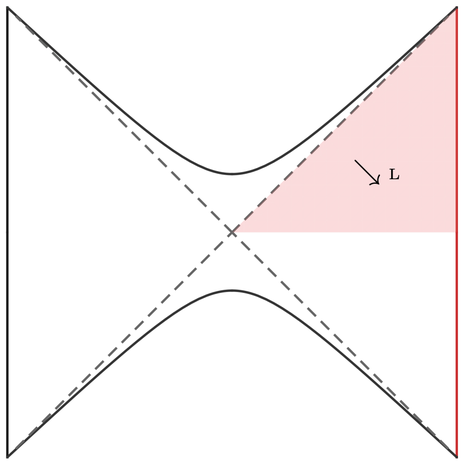
$$\mathcal{S}_{\mathrm{NG}}[\Sigma] = \frac{\sqrt{\lambda}}{2\pi} \int d\sigma\, d\tau \sqrt{\det\left(g_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \right)} ,$$

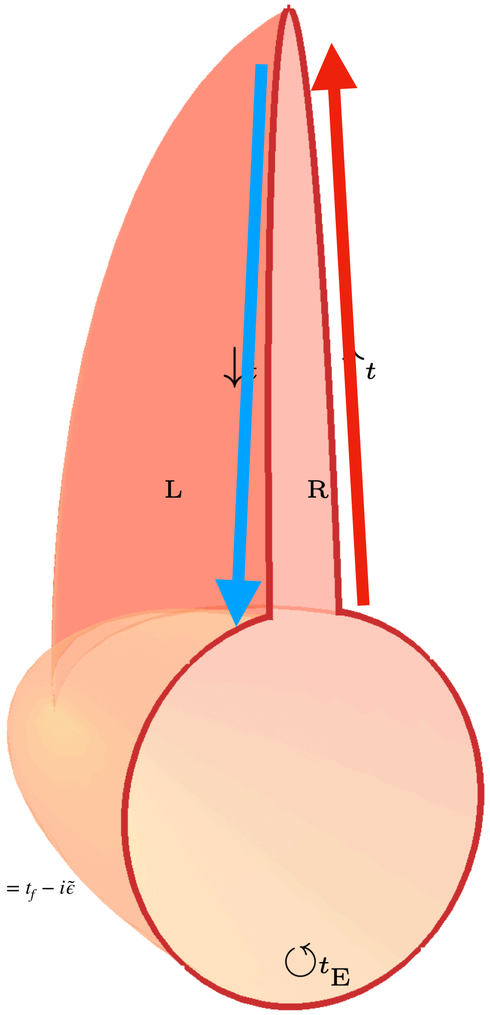
in a higher dimensional spacetime with a black brane with temperature T

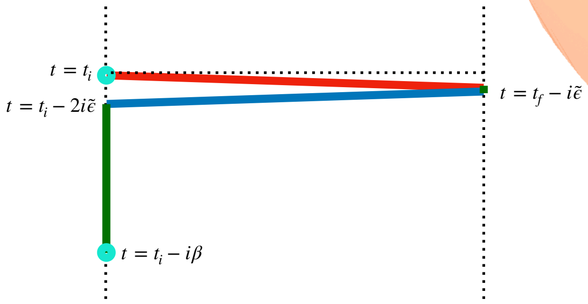
$$ds^2 = \frac{1}{z^2} \left[f(z)\, d\tau^2 + d\mathbf{x}^2 + \frac{dz^2}{f(z)} + z^2 d\Omega_5^2 \right] , \qquad f(z) = 1 - (\pi T z)^4 .$$

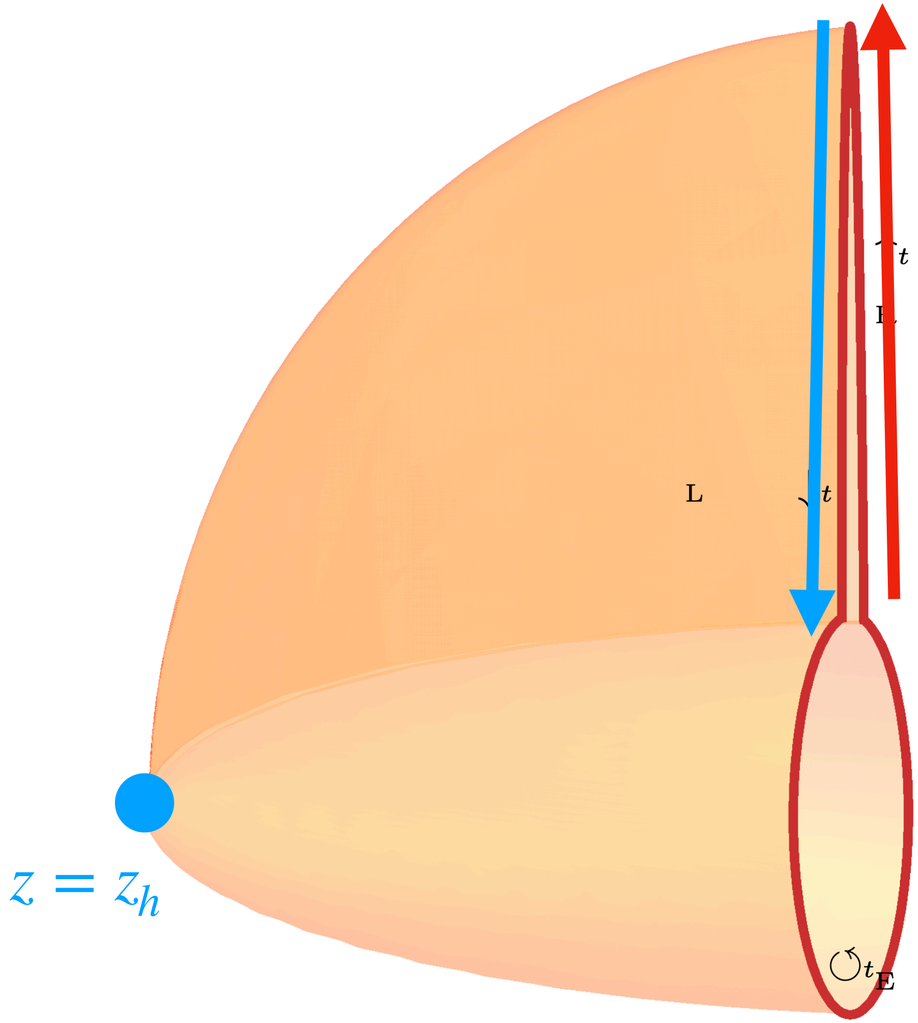
The Schwinger-Keldysh contour

in holography [Skenderis and B. C. van Rees, 0805.0150, 0812.2909]









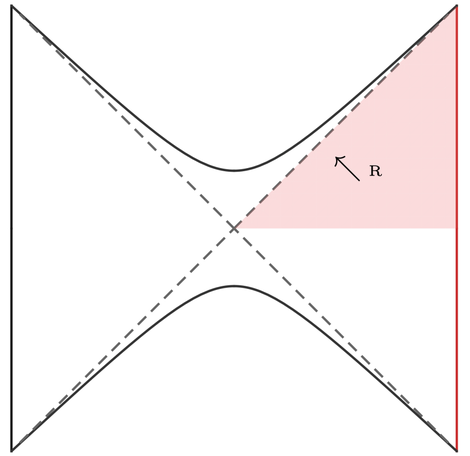
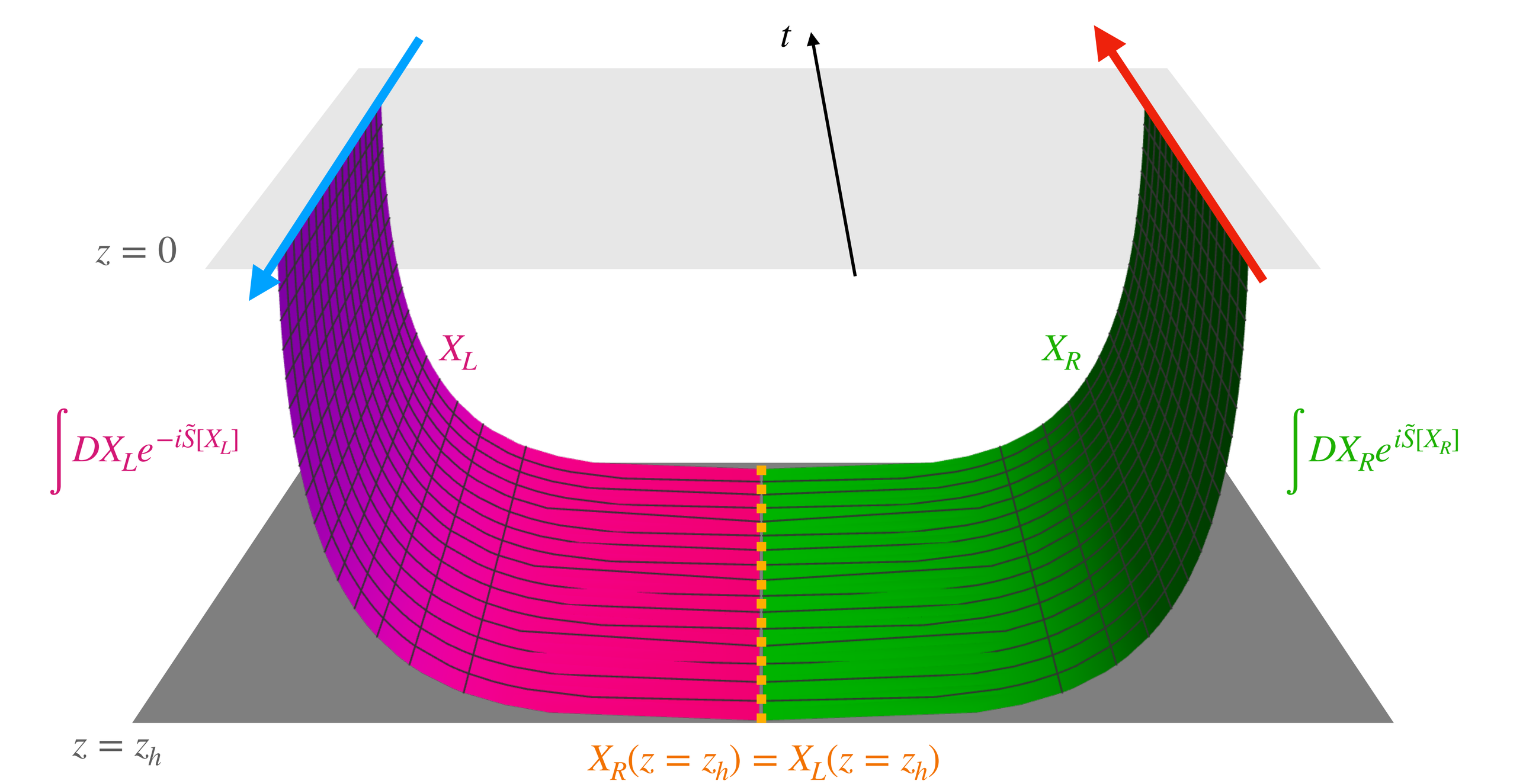
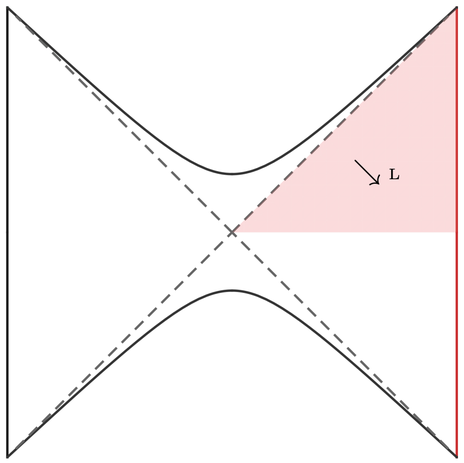


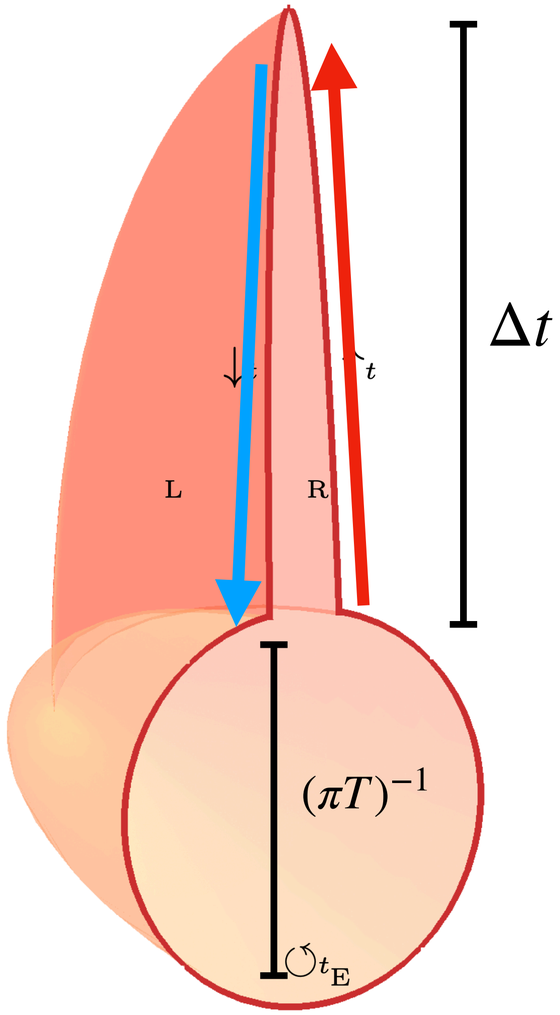
figure credit: Jana, Loganayagam, Rangamani 2004.02888


t
$z = 0$
X_L
X_R
$\int DX_L e^{-i\tilde{S}[X_L]}$
$\int DX_R e^{i\tilde{S}[X_R]}$
$z = z_h$
$X_R(z = z_h) = X_L(z = z_h)$

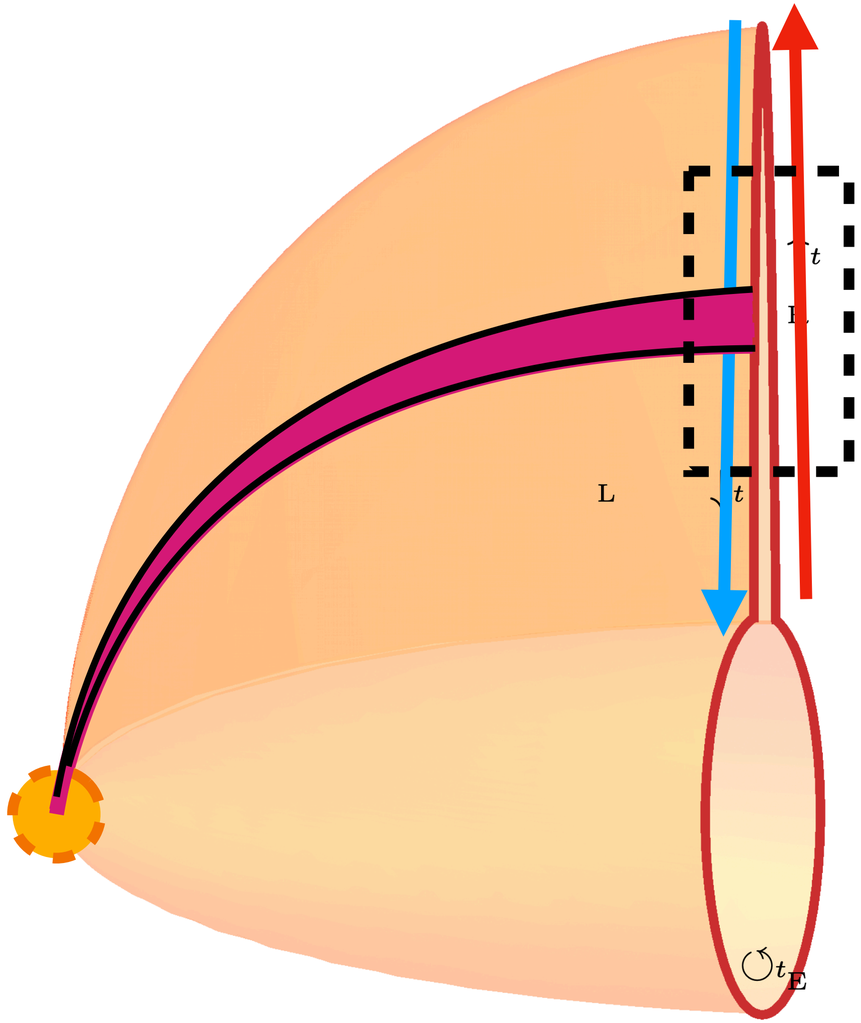
The Schwinger-Keldysh contour

in holography [Skenderis and B. C. van Rees, 0805.0150, 0812.2909]





$$T\Delta t \gg 1$$



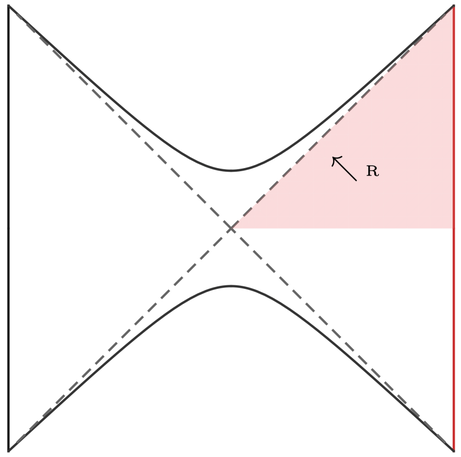


figure credit: Jana, Loganayagam, Rangamani 2004.02888

4. A. A 20-year-old puzzle

- *Momentum Fluctuations of heavy quarks in the gauge-string duality* (S. Gubser, hep-th/0612143)

Average energy loss

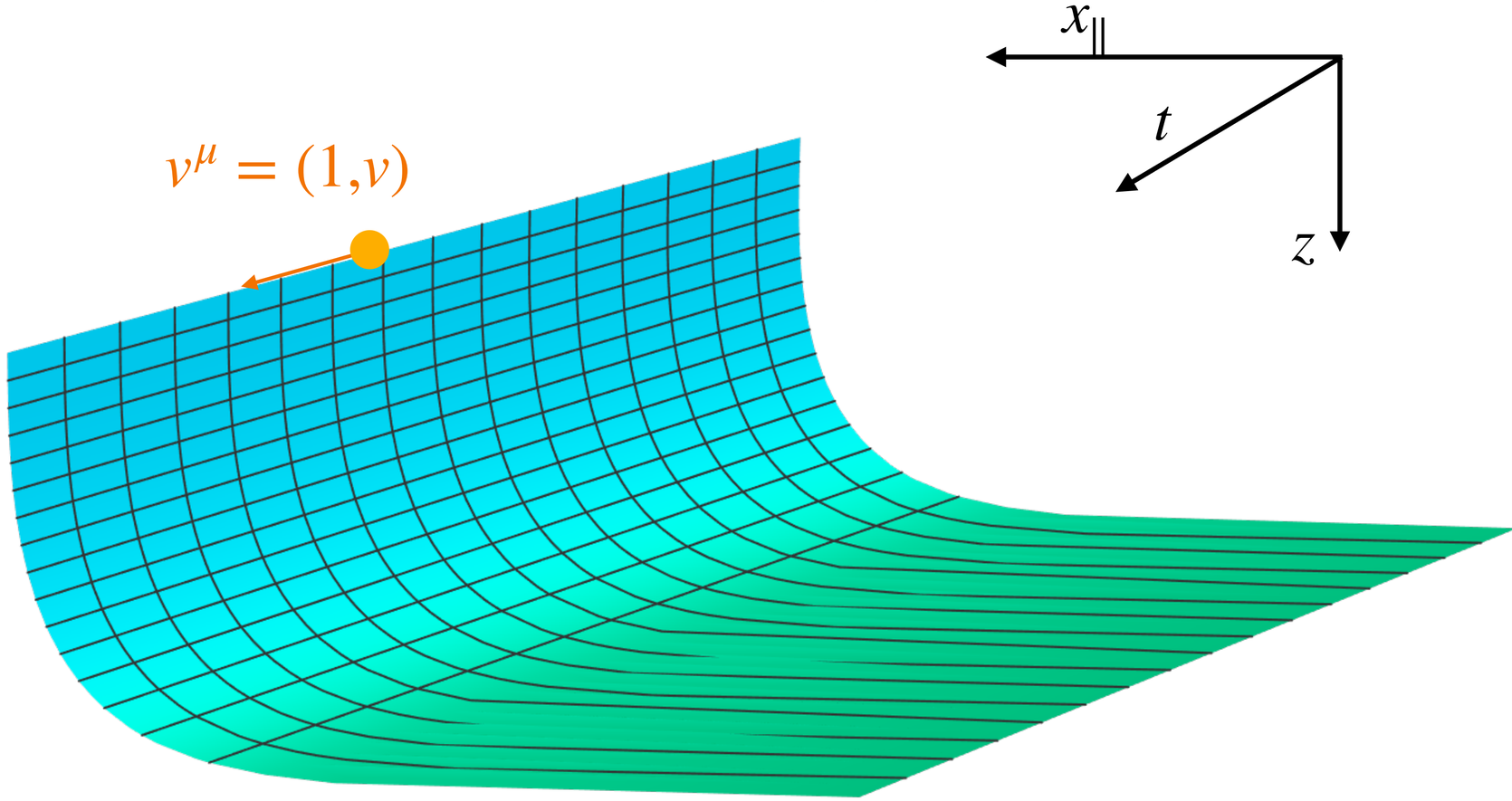
The trailing string [Herzog et al. hep-th/0605158; Gubser hep-th/0605182] — the configuration that maximizes $P(\mathbf{k})$ [2501.06289]

The energy-momentum flow down the string gives the drag force F.

One obtains

$$F = \frac{\pi}{2}\sqrt{\lambda}T^2 v\gamma \equiv \eta_D p,$$

where $p = M\gamma v$.



Momentum broadening

Fluctuations on the trailing string [Casalderrey-Solana & Teaney hep-ph/0605199, hep-th/0701123; Gubser hep-th/0612143]

The force-force correlator along the trajectory of the heavy quark determines the average momentum fluctuations.

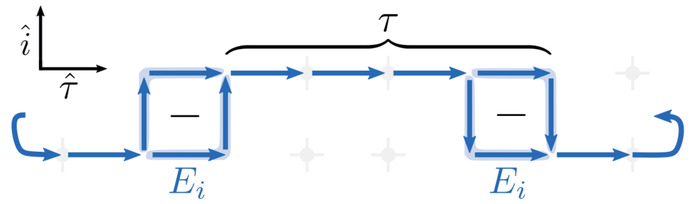
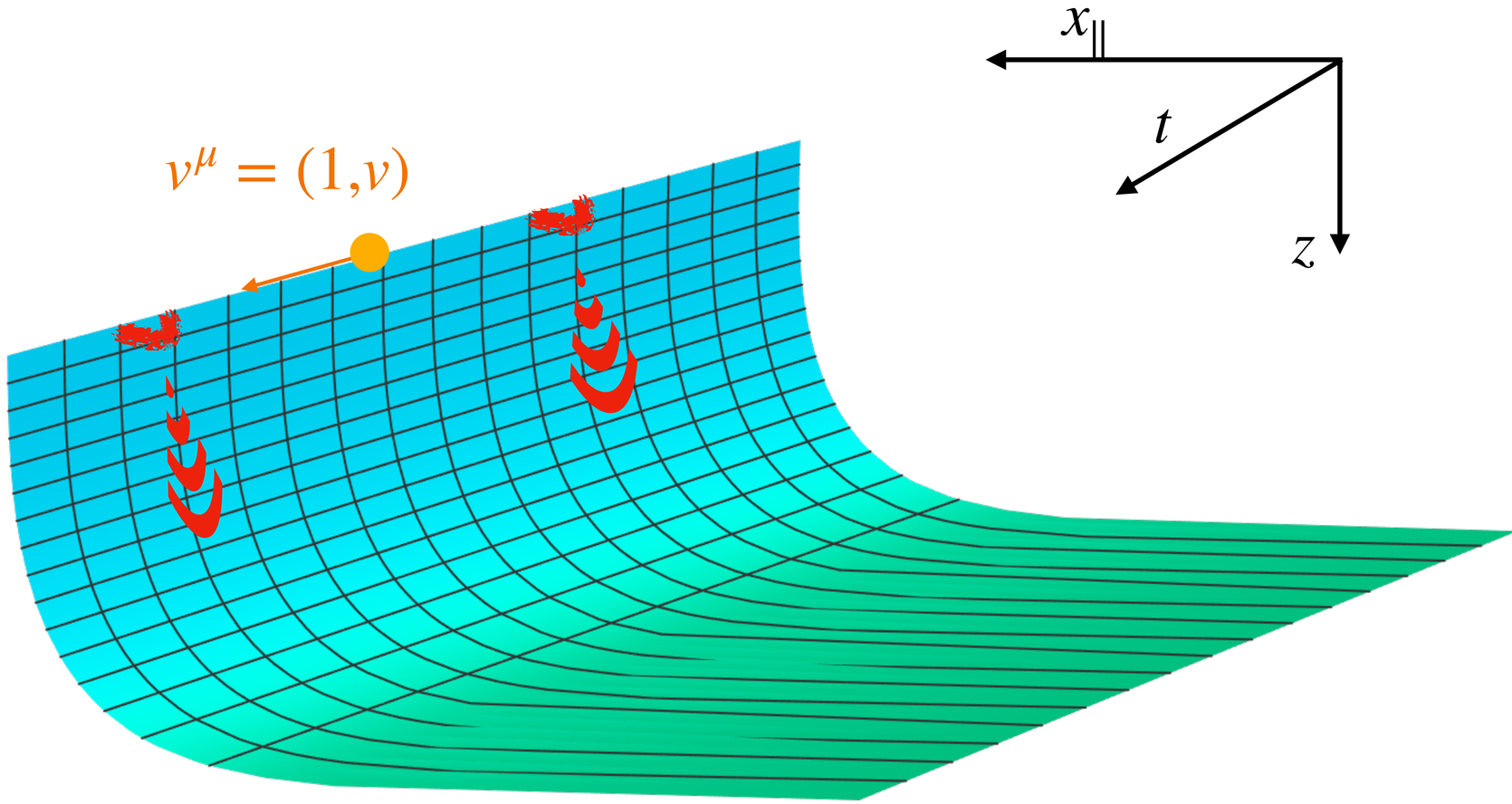


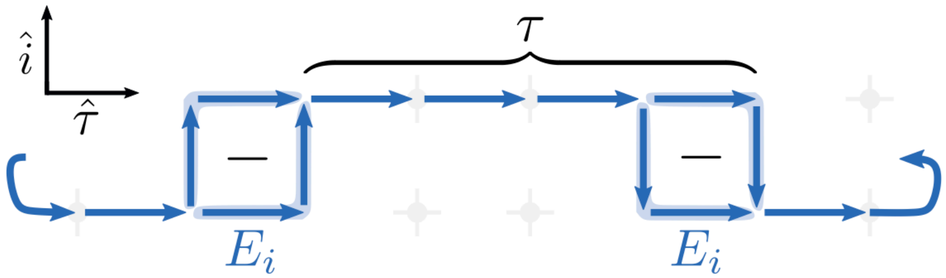
figure credit: L. Altenkort



HQ diffusion from lattice QCD

QCD at $T \sim \Lambda_{\rm QCD}$ vs $\mathcal{N} = 4$ SYM with $N_c \to \infty, \lambda = g^2 N_c \to \infty$

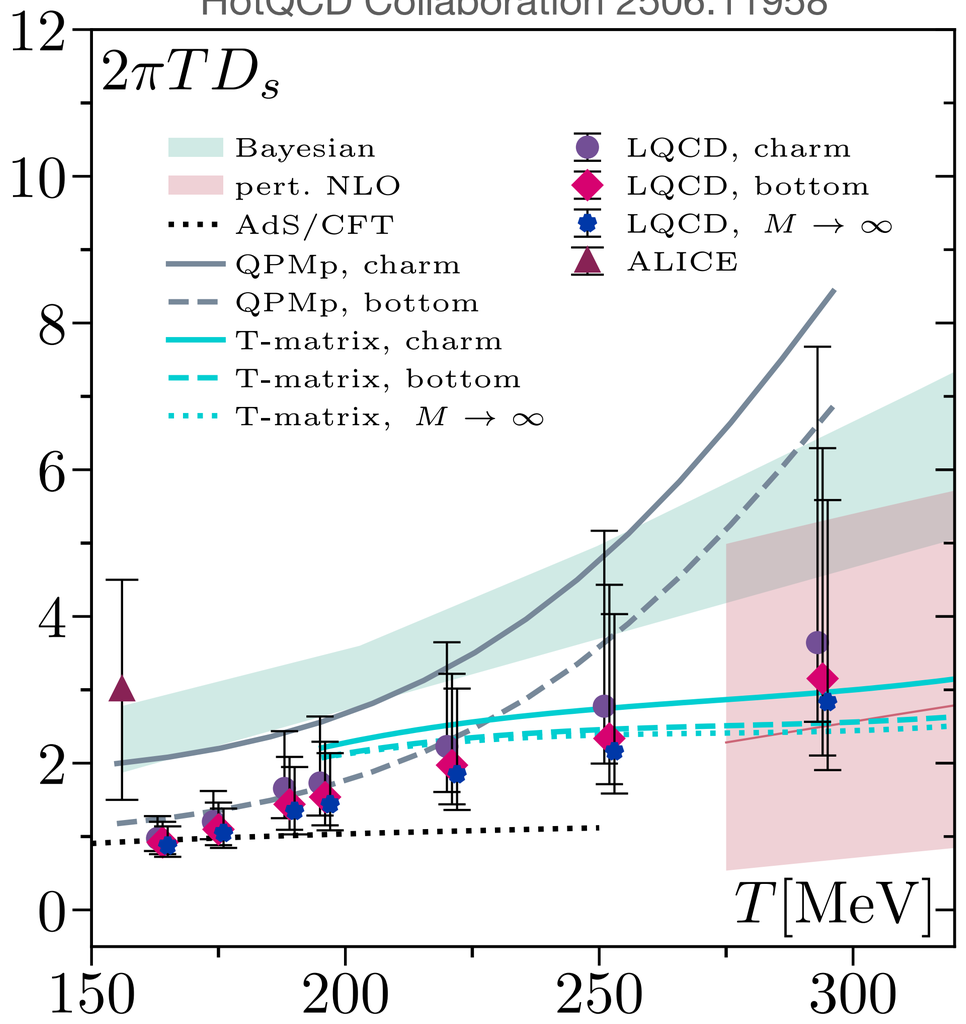
- Heavy quark diffusion coefficient at $v = 0$: low frequency limit of



$\mathcal{N} = 4$ prediction: $\kappa = \pi\sqrt{\lambda}T^3$ [Casalderrey-Solana & Teaney hep-ph/0605199]

Lattice QCD calculation: figure on the right, in terms of $D_s = 2T^2/\kappa$.

HotQCD Collaboration 2506.11958



Momentum broadening

Fluctuations on the trailing string [Casalderrey-Solana & Teaney hep-ph/0605199, hep-th/0701123; Gubser hep-th/0612143]

The force-force correlator along the trajectory of the heavy quark determines the average momentum fluctuations.

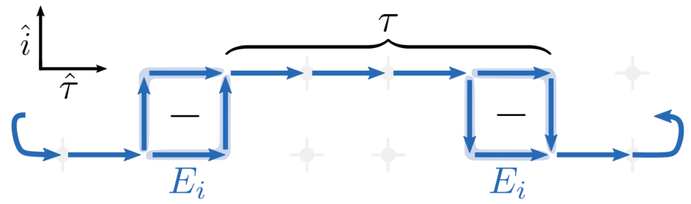
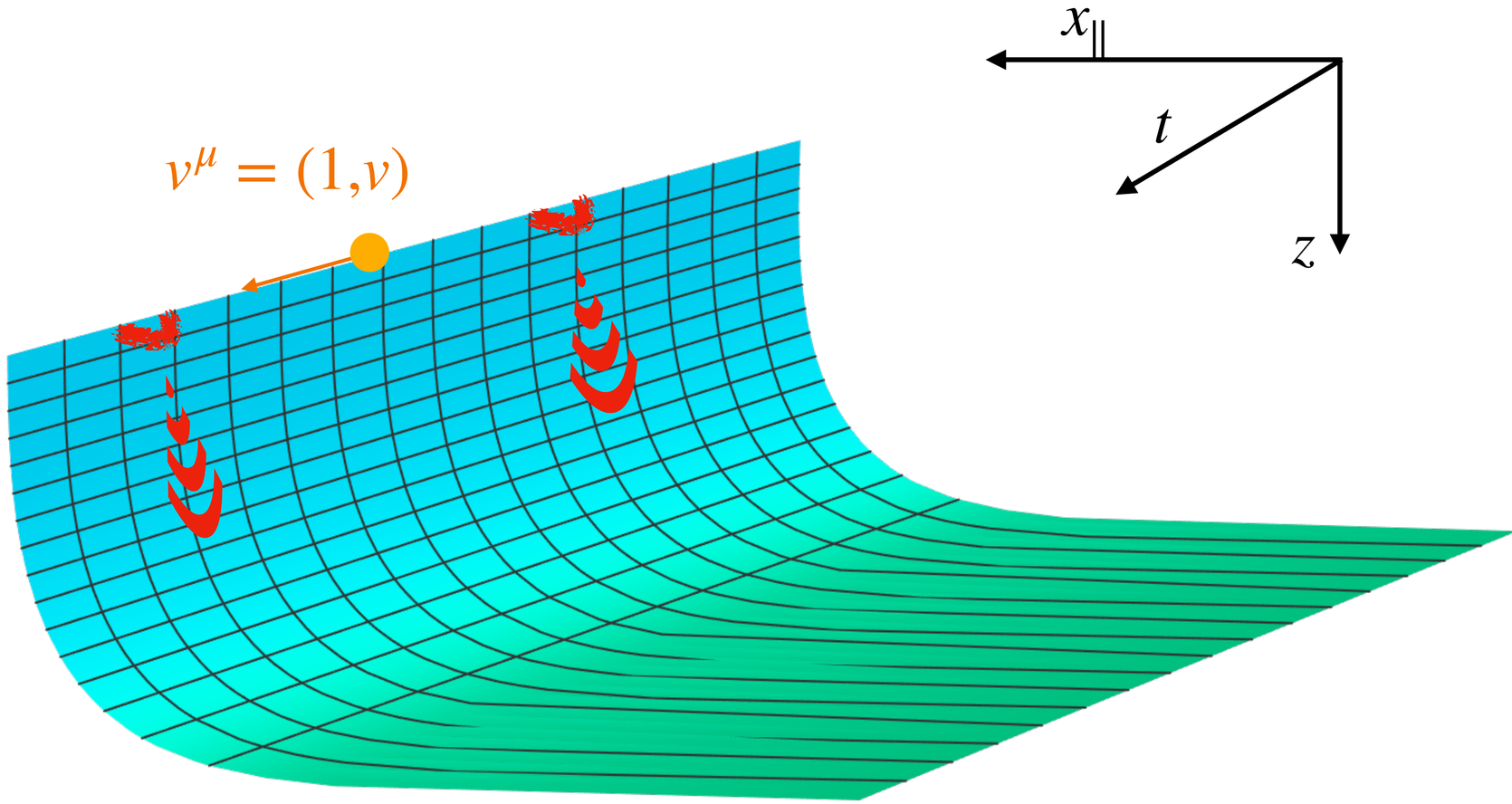


figure credit: L. Altenkort



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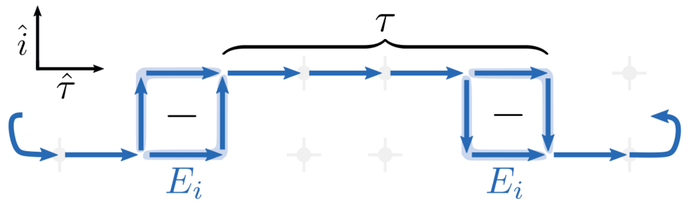


figure credit: L. Altenkort

The result is

$$\frac{1}{2}\frac{d\langle \mathbf{p}_\perp^2 \rangle}{dt} \equiv \kappa_T = \pi\sqrt{\lambda}\gamma^{1/2}T^3,$$

$$\frac{d\langle (\Delta p_L - \langle \Delta p_L \rangle)^2 \rangle}{dt} \equiv \kappa_L = \pi\sqrt{\lambda}\gamma^{5/2}T^3.$$

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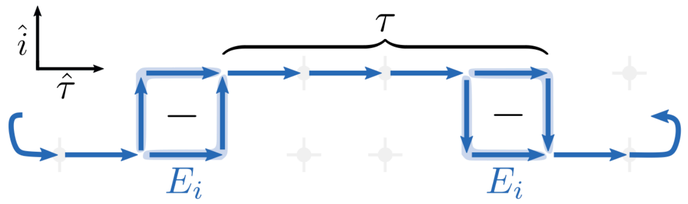


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Puzzle [Steven Gubser hep-th/0612143]:
it was not clear how to derive the process of heavy quark equilibration at strong coupling, because the truncation to Gaussian characteristics is inconsistent with equilibration!

4. B. The resolution

- *Dynamics of Heavy Quarks in Strongly Coupled $\mathcal{N} = 4$ SYM*
  (2501.06289 with K. Rajagopal and U. A. Wiedemann)

Non-Gaussian contributions

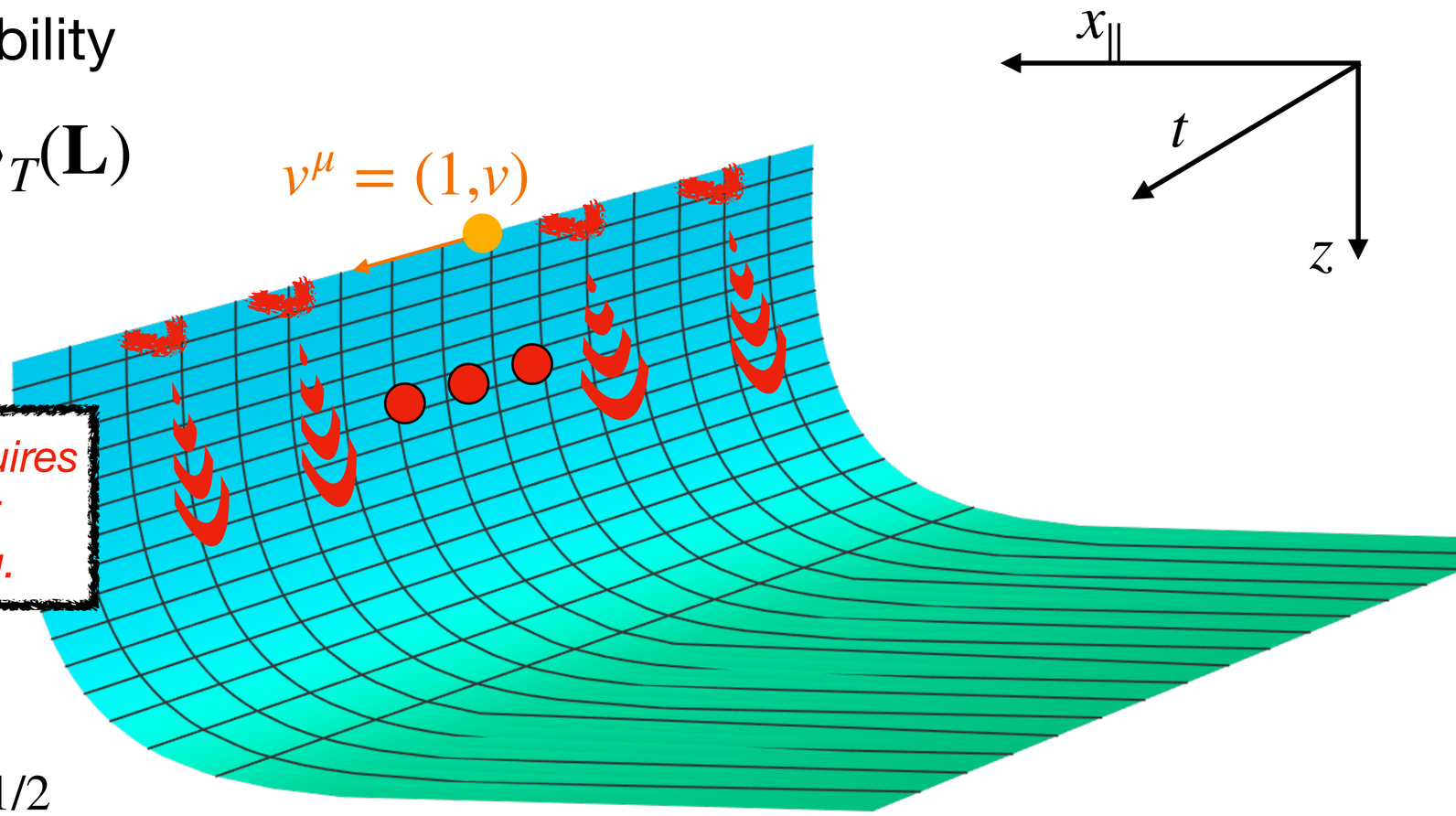
As higher-order fluctuations on the trailing string [K. Rajagopal, BSH, U. A. Wiedemann 2501.06289]

The momentum change probability

$$P(\mathbf{k};\mathbf{v}(\mathbf{p})) \propto \int_{\mathbf{L}} e^{-i\mathbf{k}\cdot\mathbf{L}} \langle W[\mathscr{C}]\rangle_T(\mathbf{L})$$

is *not only* determined by its Gaussian characteristics.

A characterization in terms of moments requires all of them. These describe higher order correlation functions on the trailing string.



We found

$$\frac{d\langle k_L^n\rangle_c}{dt} \propto \sqrt{\lambda} T^{n+1} \gamma^{3n/2-1/2}$$

Non-Gaussian contributions

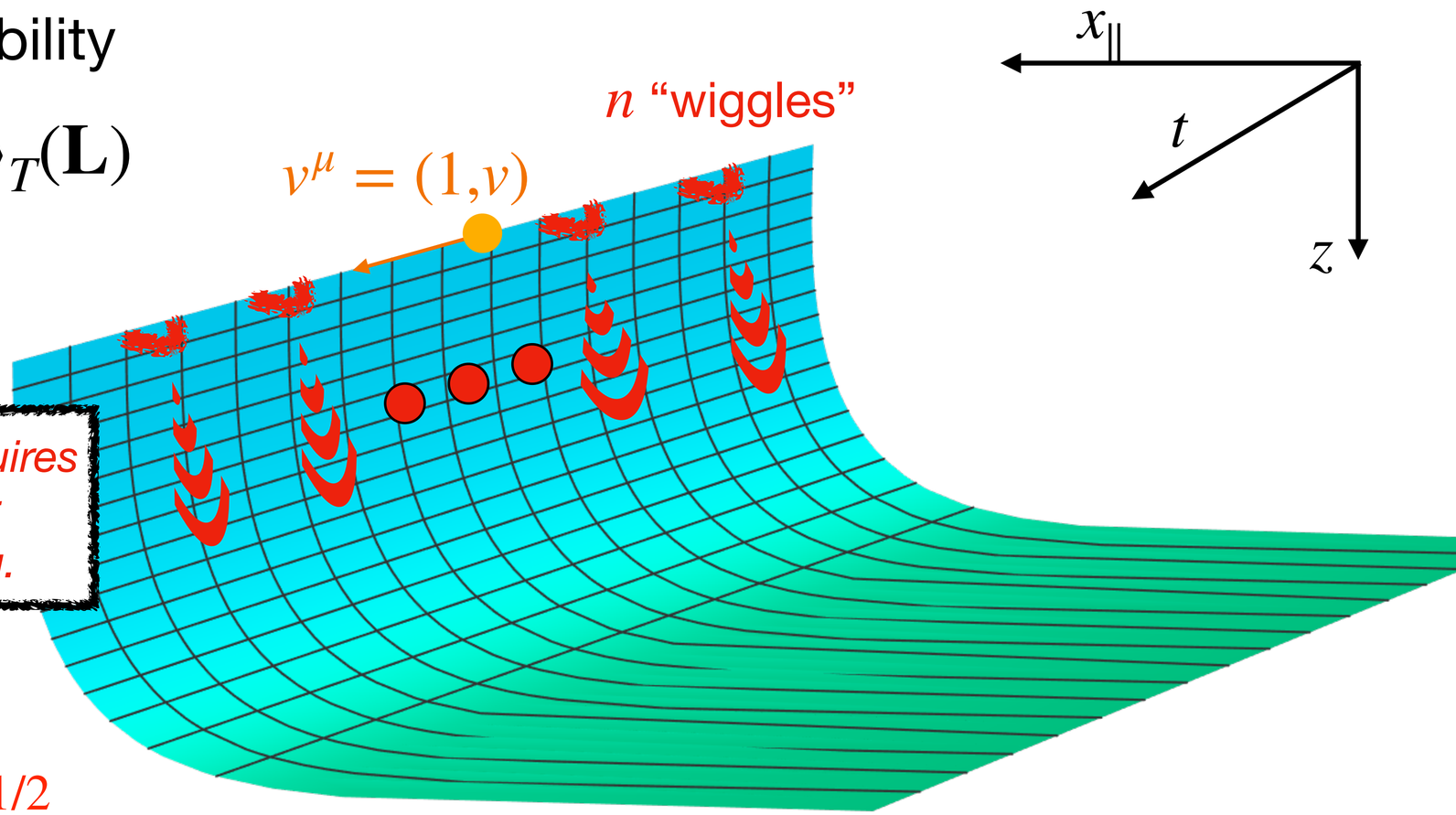
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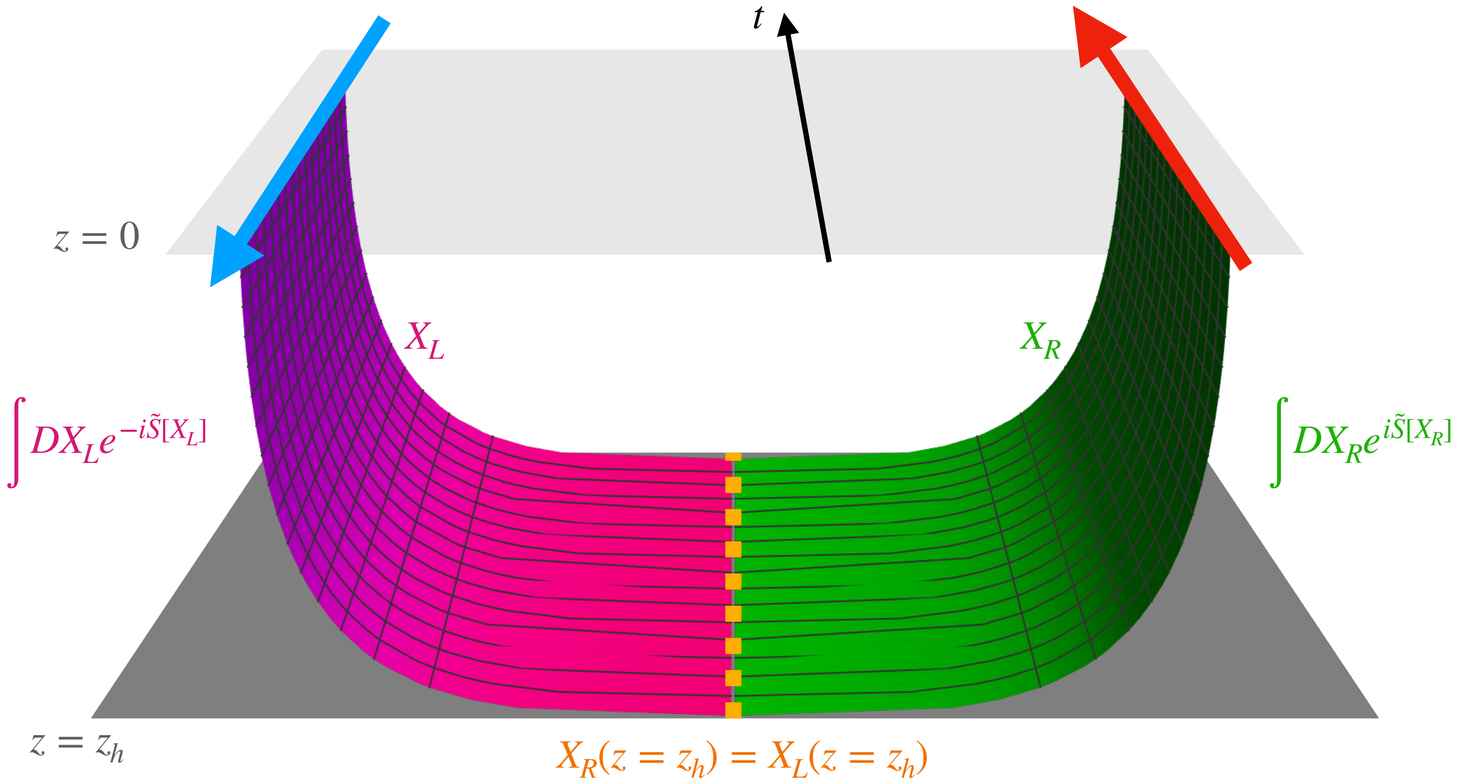
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A characterization in terms of moments requires all of them. These describe higher order correlation functions on the trailing string.



We found

$$\frac{d\langle k_L^n\rangle_c}{dt} \propto \sqrt{\lambda} T^{n+1} \gamma^{3n/2-1/2}$$



$$P(\mathbf{k}) \propto \exp\left[-\sqrt{\lambda}Tt\,\tilde{S}_{\rm tot}\left(\frac{2\mathbf{k}}{\sqrt{\lambda}Tt}\right)\right]$$

☑ Satisfies the universal equilibration condition ⟺ detailed balance

$$\tilde{S}_{\rm tot}(\mathbf{C}) = \sqrt{1+\mathbf{C}^2}\,\frac{\pi}{32}\,\frac{(z_+^4-z_-^4)^2}{z_-^5(1-z_-^4)}\,F_1\left(\frac{3}{2},\frac{5}{4},1;3;-\frac{z_+^4-z_-^4}{z_-^4},\frac{z_+^4-z_-^4}{1-z_-^4}\right)+\frac{\pi}{2}\,|\,vC_3\,|\,\theta(-vC_3)$$

Rajagopal, **BSH**, Wiedemann [arXiv:2501.06289]

$$z_\pm^4 = \frac{1+C_\perp^2+(1-v^2)(1+C_3^2)}{2(1+C_\perp^2+C_3^2)} \pm \frac{\sqrt{C_\perp^4+2C_\perp^2(v^2+C_3^2(1-v^2))+(v^2-(1-v^2)C_3^2)^2}}{2(1+C_\perp^2+C_3^2)}\,.$$

$$\nu = 0$$

$$e^{-\tilde{S}_{\rm tot}} = \left[P(\mathbf{k})\right]^{\frac{1}{\sqrt{\lambda}tT}} = \exp\left[\frac{\ln P(\mathbf{k})}{\sqrt{\lambda}tT}\right]$$

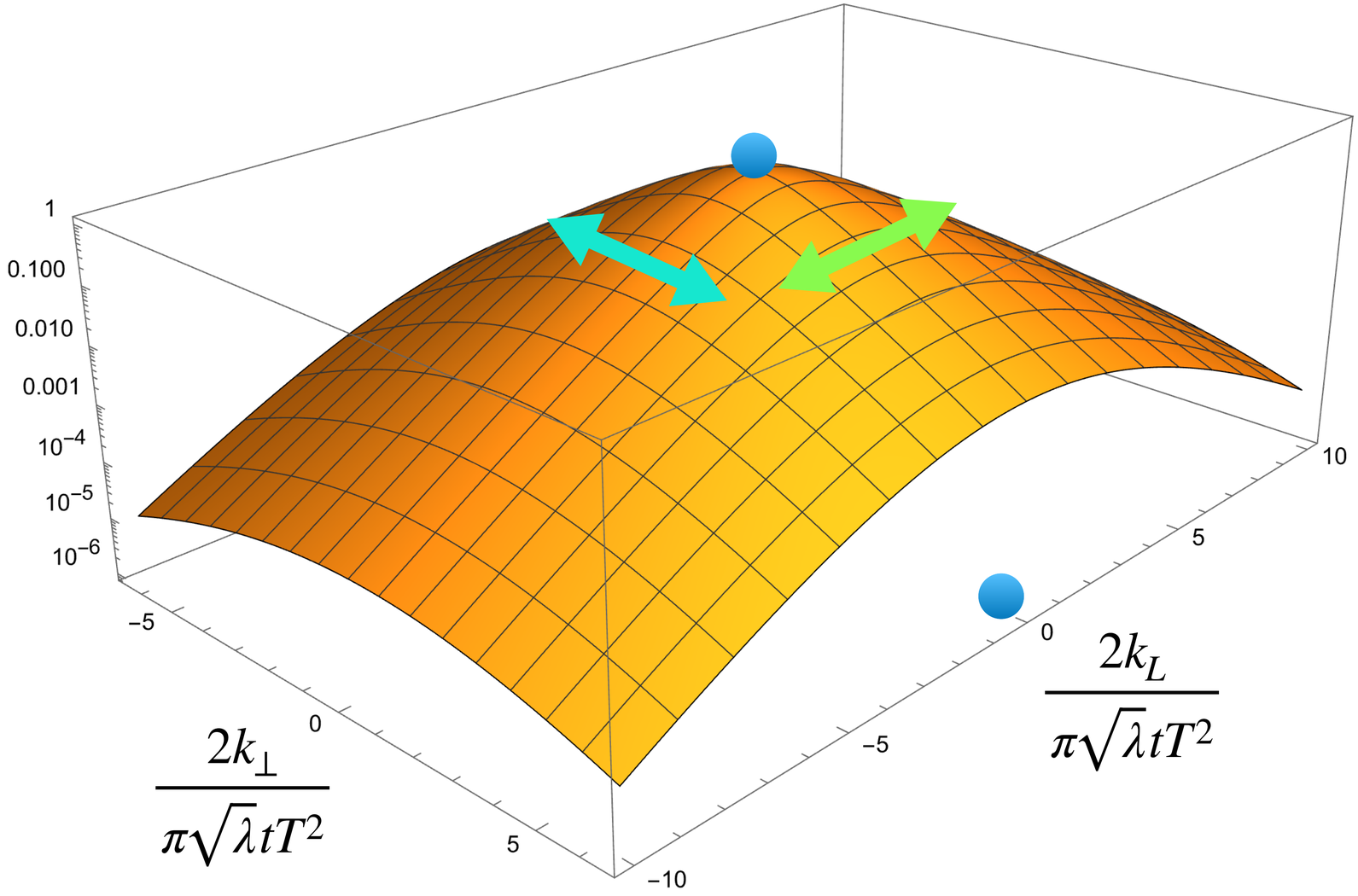
● : most likely momentum change

$\Longrightarrow$ most likely value of energy loss

$$\langle k_1 \rangle = \langle k_2 \rangle = \langle k_3 \rangle = 0$$

⟷ : $\kappa_T = \pi\sqrt{\lambda}T^3$

⟷ : $\kappa_L = \pi\sqrt{\lambda}T^3$



$$v = 0.9$$

$$e^{-\tilde{S}_{\rm tot}} = \left[P(\mathbf{k})\right]^{\frac{1}{\sqrt{\lambda}tT}} = \exp\left[\frac{\ln P(\mathbf{k})}{\sqrt{\lambda}tT}\right]$$

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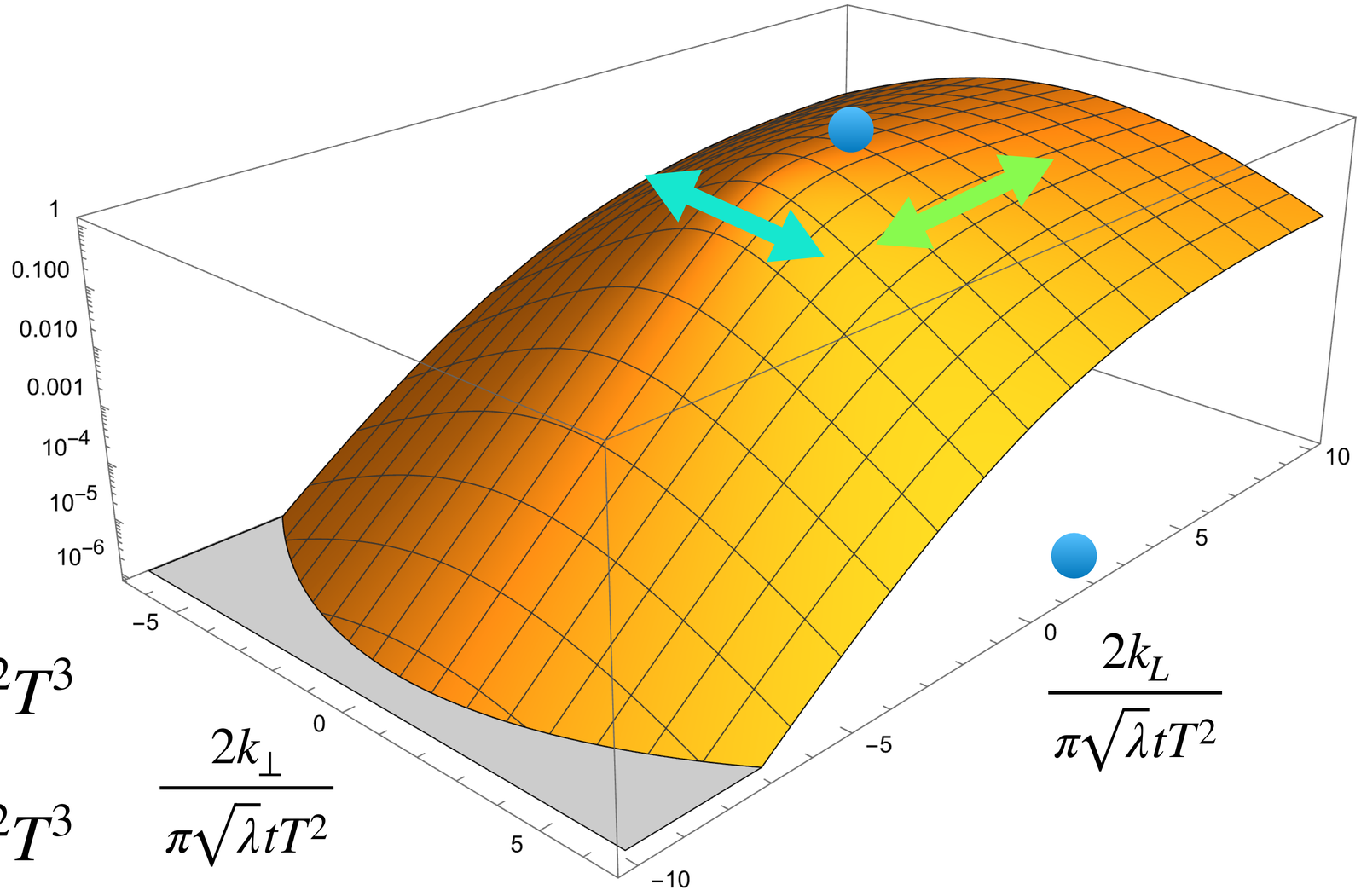
$\Longrightarrow$ most likely value of energy loss

$$\langle k_1 \rangle = \langle k_2 \rangle = 0$$

$$\langle k_3 \rangle = \frac{\pi\sqrt{\lambda}T^2 t}{2}\gamma v$$

⟷ : $\kappa_T = \pi\sqrt{\lambda}\gamma^{1/2}T^3$

⟷ : $\kappa_L = \pi\sqrt{\lambda}\gamma^{5/2}T^3$



$$\nu = 0.99$$

$$e^{-\tilde{S}_{\rm tot}} = \left[P(\mathbf{k})\right]^{\frac{1}{\sqrt{\lambda}tT}} = \exp\left[\frac{\ln P(\mathbf{k})}{\sqrt{\lambda}tT}\right]$$

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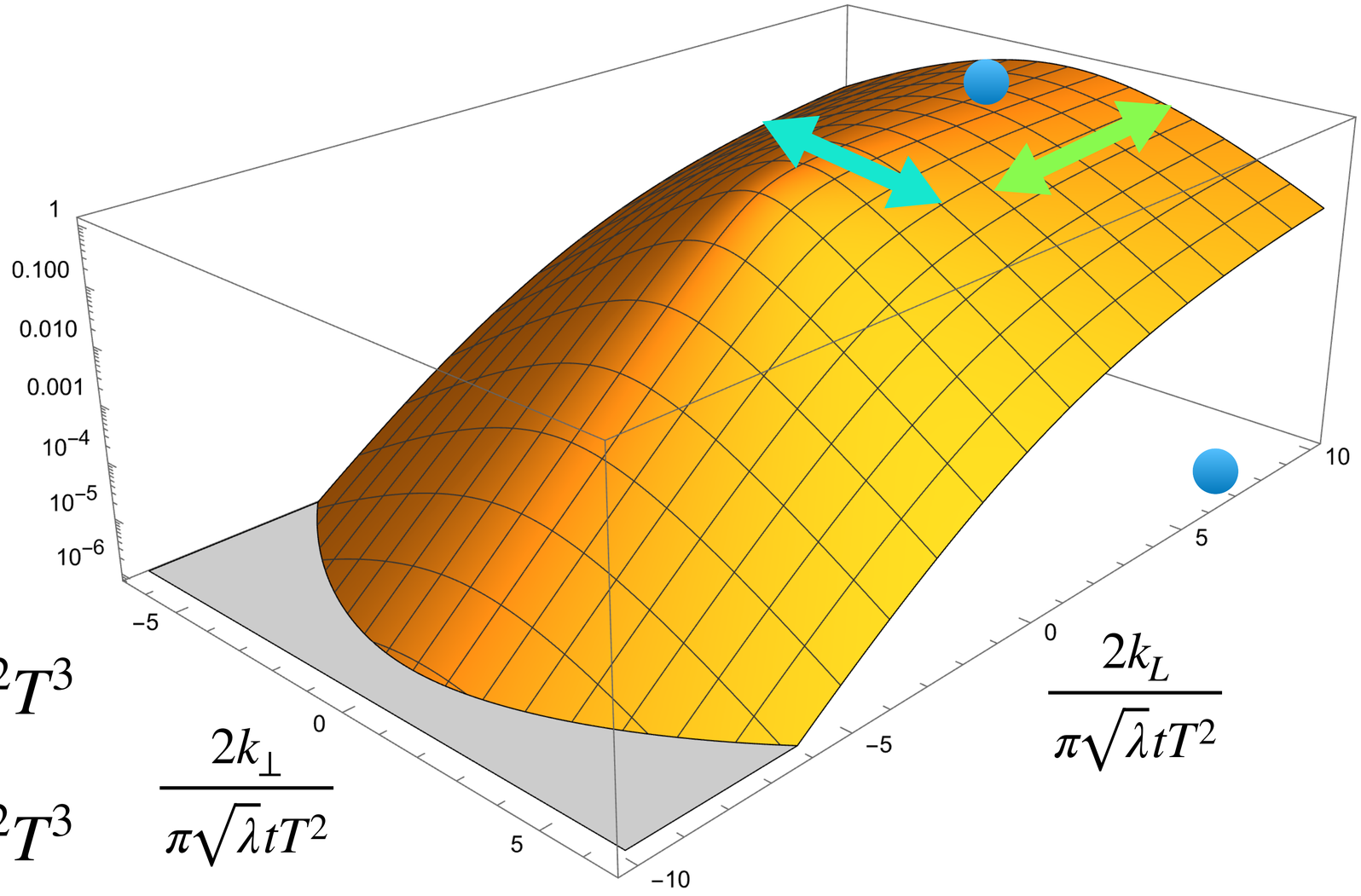
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5. Comparing weakly and strongly coupled environments

- *Heavy Quark Transport is Non-Gaussian Beyond Leading Log*
  (2604.21895 and 2610.03863 with Jean F. Du Plessis)

Non-Gaussianity is generic!

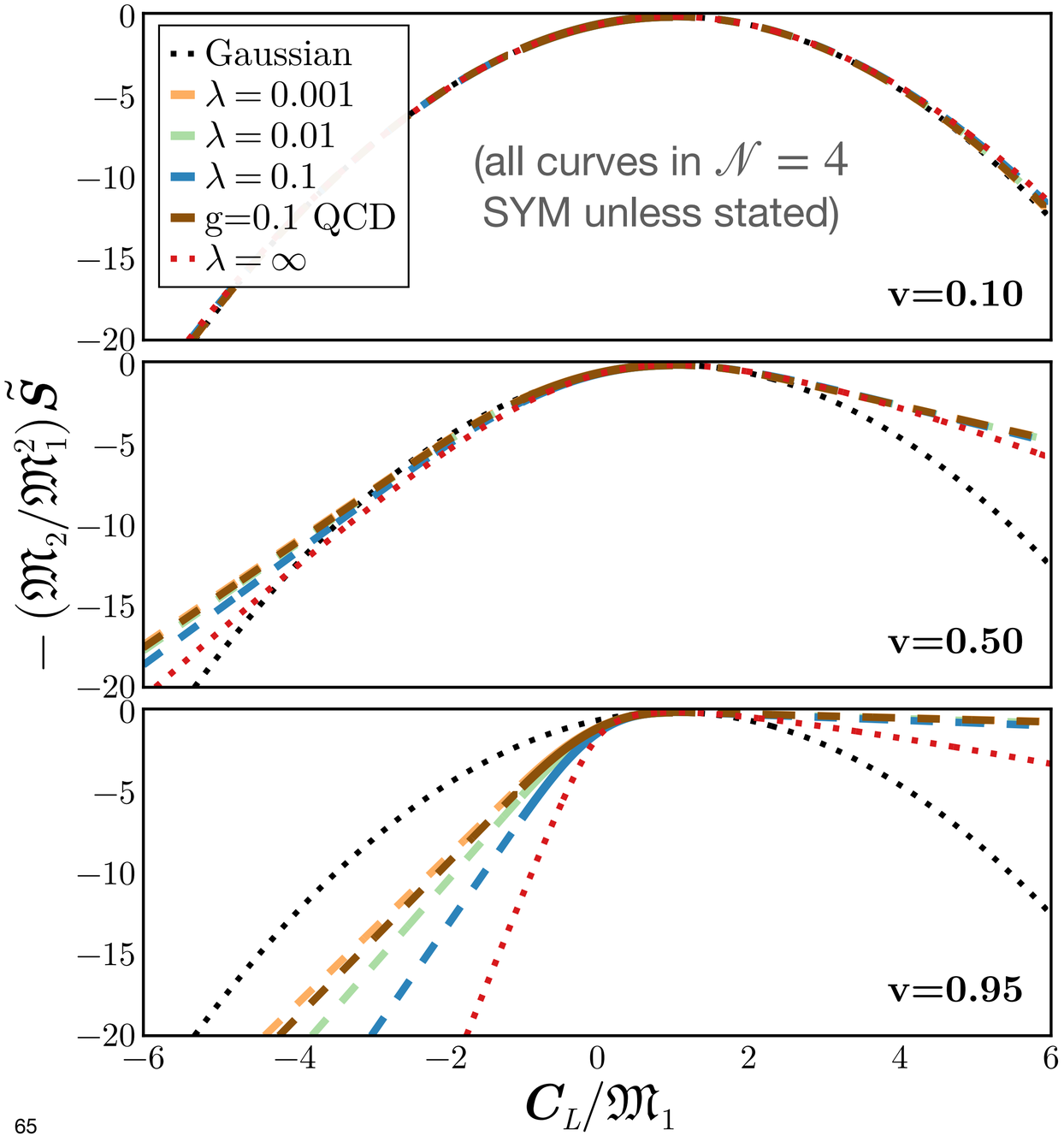
and exponential! [2604.21895]

- We plot $\tilde{S}(C_L; v)$, defined as

$$P(k_L; v) = \exp\left(-t\,T\,\tilde{S}\left(\frac{k_L}{tT}; v\right) + \dots \right),$$

rescaled such that it would look like a parabola $-\dfrac{(x-1)^2}{2}$ if it were quadratic.

- The tails are linear on this scale $\Longrightarrow P(k_L; v)$ has exponential tails!



How to verify these tails?

A: look at the moments $\langle k_L^n \rangle$

- We have checked by explicit calculation that, as a function of n,

$$\langle k_L^n \rangle = \int dk_L P(k_L; v)\, k_L^n \sim R(v)^{-n}\, n! \qquad \times (\text{low} - \text{order polynomial in } n)$$

in all theories where we have determined $P(k_L; v)$. Including

1. Generic weakly coupled $\mathrm{SU}(N_c)$ gauge theories
2. Strongly coupled $\mathcal{N} = 4$ SYM via holography

- This is indicative of exponential tails

$$\left(\int_0^\infty dx e^{-x} x^n = n! \right)$$

Conclusion of [2604.21895, 2610.03863]

the tails of $P(k_L)$ are exponential at weak and strong coupling

- We found that in both weakly and strongly coupled theories, the following asymptotic behavior holds

$$P(k_L) \propto e^{-R(v)k_L} \qquad \text{as } k_L \to \infty\,,$$
$$P(k_L) \propto e^{+[R(v)+v/T]k_L} \qquad \text{as } k_L \to -\infty\,,$$

  with theory-dependent *tail exponent* $R(v)$.

- Velocity and theory dependence:

$$R_{\text{w.c.}}(v=0) = \frac{1}{2T}\,, \quad R_{\text{w.c.}}(v\to 1) = \frac{1}{4T\gamma^2}\,, \quad R_{\text{s.c.}}(v\to 1) = \frac{1}{(1.7972...)T\gamma^{3/2}}$$

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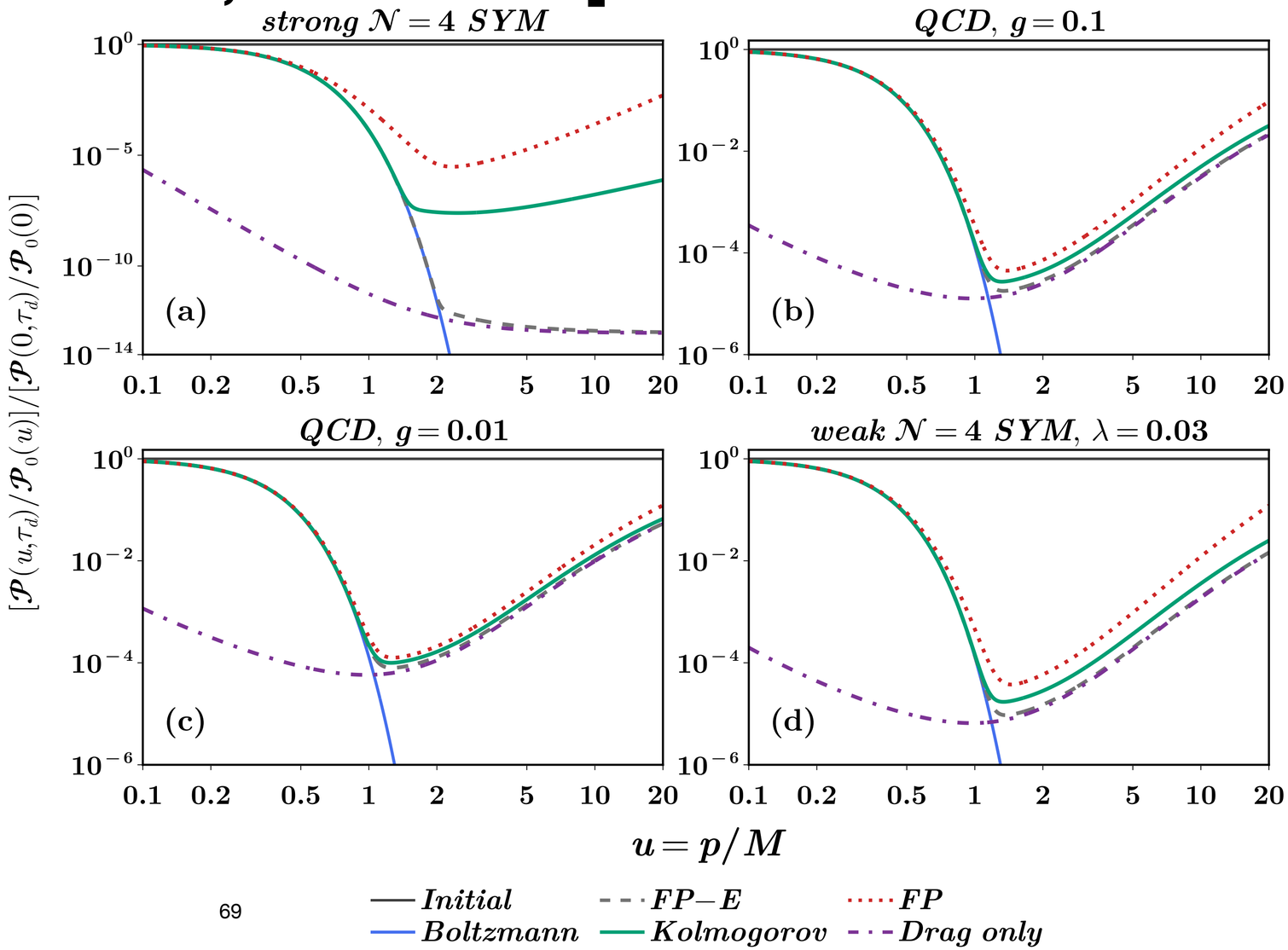
$\lambda = g^2 N_c \to 0$ $\qquad$ $\sqrt{\lambda} \to \infty$

A comparison across couplings and theories

survivor bias effects [2606.02693, 2610.03863]

To the right, a snapshot of (a double ratio of) solutions for the heavy quark phase space distributions in an infinite medium with constant T, across different theories and couplings, at $t = 5\eta_D^{-1}(0)$ for the same initial condition.

- Key takeaway: as the coupling increases, the tail of the distribution becomes more sensitive to fluctuations.

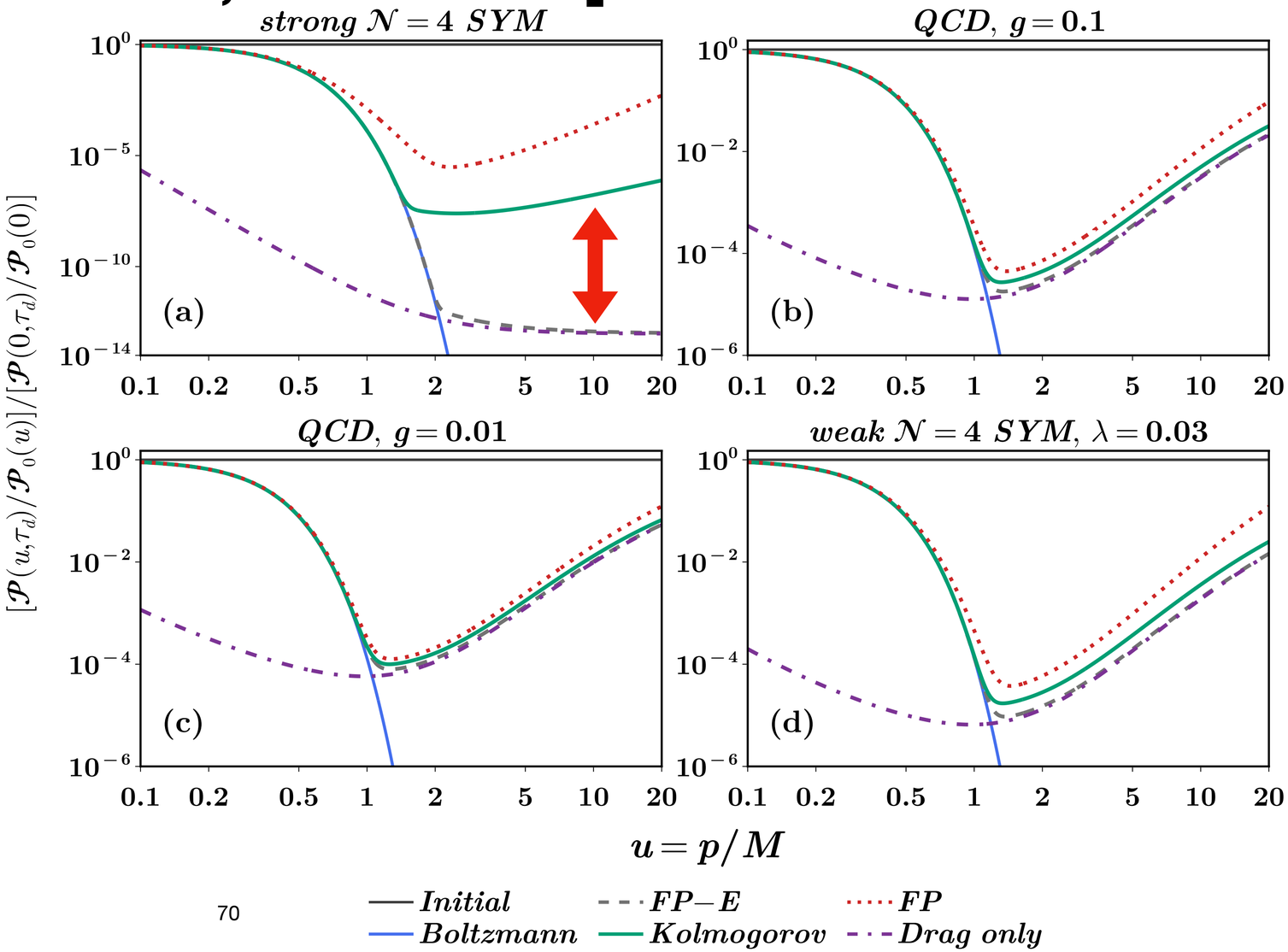


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Outlook

some challenges and opportunities for the future

- Phenomenology:
  - Gaussian *assumptions* for $P(\mathbf{k};\mathbf{v})$ are widely used in heavy ion collisions modeling.
  - How will the interpretation of the underlying physical mechanisms change once non-Gaussian effects are included? Specifically, $R(v)$.
    - Preliminary answer (2606.02693, 2610.03863): survivor bias effects!
  - How does a non-Gaussian $P(\mathbf{k};\mathbf{v})$ interplay with a hydrodynamic background?
- Theory:
  - Can we formulate calculable proxies of $P(\mathbf{k};\mathbf{v})$ in Euclidean time?
  - Can we explicitly bridge weak and strong coupling calculations in $\mathcal{N}=4$?
  - Is there additional analytic structure in $\langle W_{\mathbf{v}}\rangle(\mathbf{L})$? (for more see 2610.03863)

Extra Slides

Why is this a problem?

A: We want to explain data from Heavy-Ion Collisions

For example:

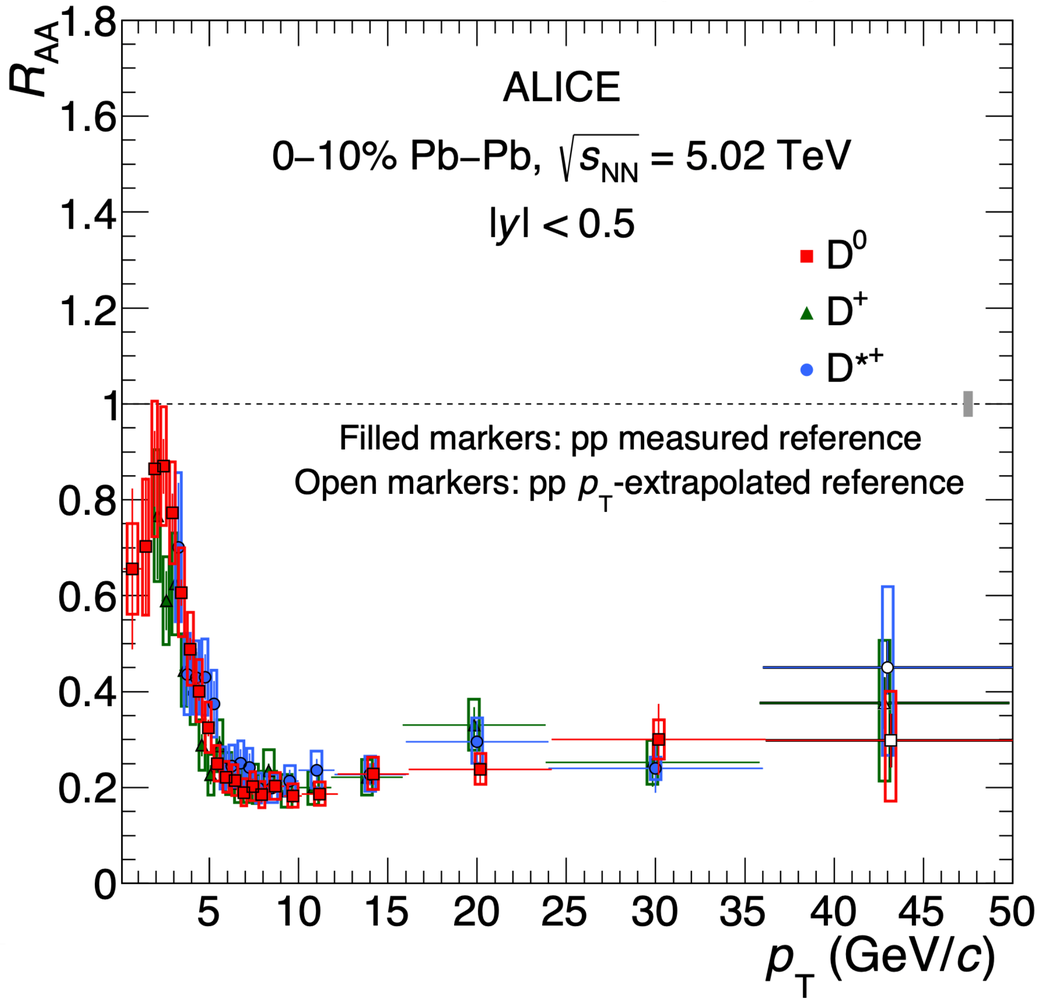
- On the right, we have a plot of

$$R_{AA} = \frac{[\#\,\text{of}\,D\text{'s}](p_T)\,\text{in}\,\text{Pb}-\text{Pb}\,\text{coll}.}{(\text{norm.}) \times [\#\,\text{of}\,D\text{'s}](p_T)\,\text{in}\,\text{p}-\text{p}\,\text{coll}.}$$

[a D is a hadron with a charm quark in it]

- The data we seek to explain is in the p_T range $1 - 50$ GeV.
- The mass of the charm quark is ~ 1.27 GeV.

ALICE *JHEP* 01 (2022) 174



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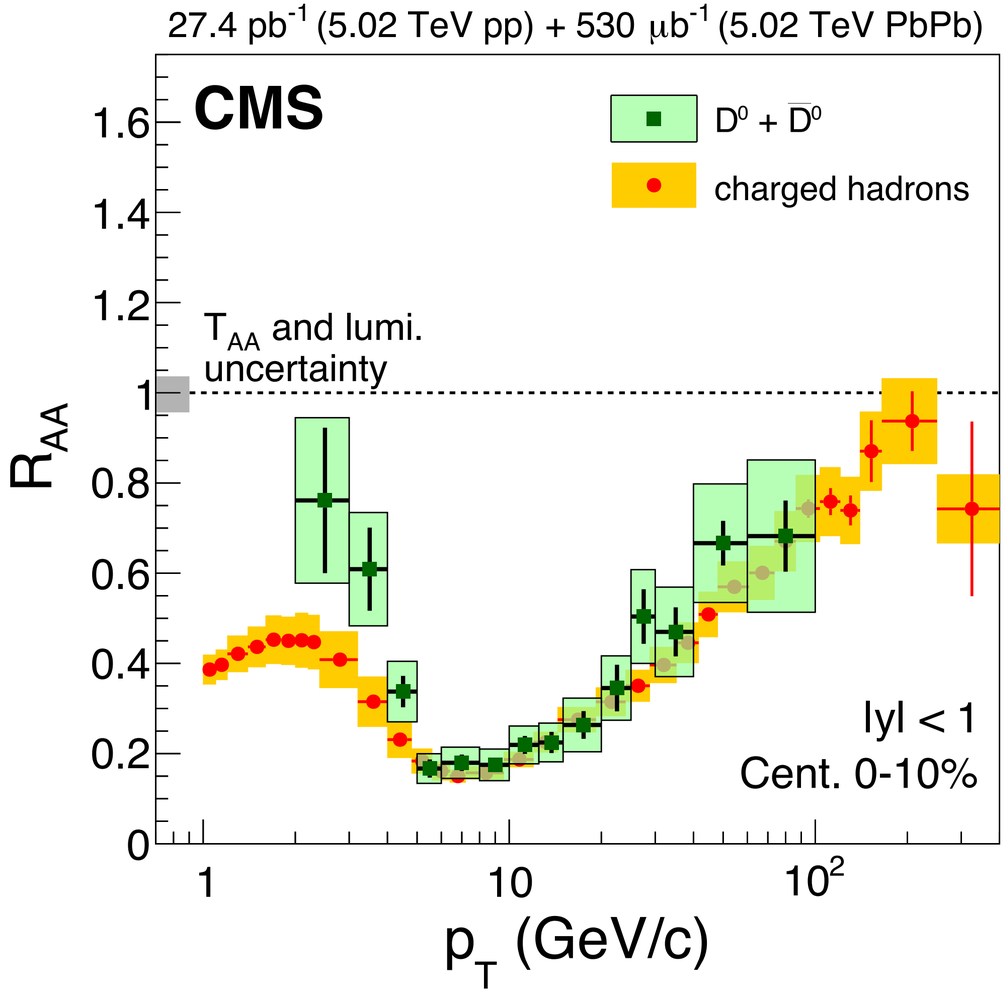
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[D^0 is a hadron with a charm quark in it]

- The data we seek to explain is in the p_T range $1 - 100$ GeV.
- The mass of the charm quark is ~ 1.27 GeV.

CMS *Phys.Lett.B* 782 (2018) 474-496



Why is this a problem?

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For example:

- On the right, we have a plot of

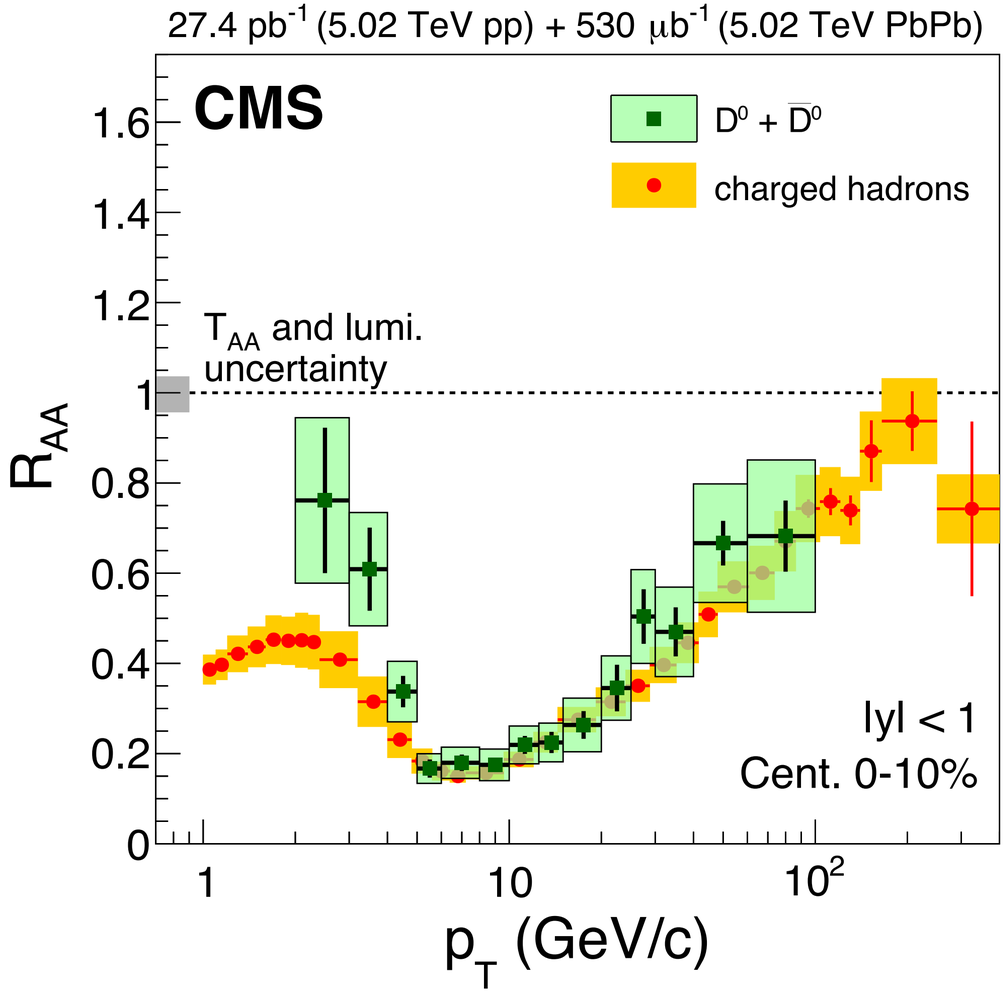
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[D^0 is a hadron with a charm quark in it]

- The data we seek to explain is in the p_T range $1 - 100$ GeV.
- The mass of the charm quark is ~ 1.27 GeV.

Far from the non-relativistic limit!

CMS *Phys.Lett.B* 782 (2018) 474-496



$$P(\mathbf{k};\mathbf{v}) = \int \frac{d^3L}{(2\pi)^3}\, e^{-i\mathbf{k}\cdot\mathbf{L}}\, \langle W[\mathscr{C}]\rangle_T(\mathbf{L})$$

Dynamics

in terms of the HQ momentum space distribution

- The heavy quark (HQ) momentum space distribution $\mathscr{P}(\mathbf{p}, t)$, i.e., the "number" of HQs with momentum $\mathbf{p}$ at time t evolves following

$$\mathscr{P}(\mathbf{p}, t + \Delta t) = \int d^3\mathbf{k}\, P(\mathbf{k}; \mathbf{v}(\mathbf{p}+\mathbf{k}), \Delta t)\, \mathscr{P}(\mathbf{p}+\mathbf{k}, t)\ .$$

- This can be rewritten* as $\partial_t \mathscr{P} = -\, T\, K(\partial_{\mathbf{p}}; \mathbf{p})\mathscr{P}$, where

$$K(\mathbf{x};\mathbf{p}) = -\lim_{t\to\infty} \frac{1}{tT} \log\left[\langle W[\mathscr{C}]\rangle_T\Big(\mathbf{L} = -i\mathbf{x}; \mathbf{v} = \mathbf{v}(\mathbf{p})\Big)\right].$$

*with some assumptions

How to evolve $\mathscr{P}_{\rm HQ}$

To appear soon 2606.XXXXX

Rajagopal, **BSH**, Wiedemann

$$K(\mathbf{x};\mathbf{p}) = -\lim_{t\to\infty}\frac{1}{tT}\log\Big[\langle W[\mathscr{C}]\rangle_T\big(\mathbf{L}=-i\mathbf{x};\mathbf{v}=\mathbf{v}(\mathbf{p})\big)\Big]$$

$$P(\mathbf{k};\mathbf{v}) = \int\frac{d^3L}{(2\pi)^3}\,e^{-i\mathbf{k}\cdot\mathbf{L}}\,\langle W[\mathscr{C}]\rangle_T(\mathbf{L})$$

The general evolution equation

$$\partial_t\mathscr{P} = -\,T\,K(\partial_{\mathbf{p}};\mathbf{p})\mathscr{P}$$

is guaranteed to drive heavy quarks to kinetic equilibrium if K is calculated from a QFT.

In practice, we introduce $\mathscr{P}(\mathbf{p},t) = \exp\left(\frac{M}{T}f(\mathbf{p},t)\right)$ and solve the simpler equation (leading order in T/M)

$$\partial_\tau f = -\,K\left(T^{-1}\,\partial_{\mathbf{u}}f,\,\mathbf{u}\right).$$

$$\tau = t\frac{\sqrt{\lambda}T^2}{M}$$

Calculation of $P(\mathbf{k})$

$$P(\mathbf{k}) \propto \int d^3L\, e^{i\mathbf{k}\cdot\mathbf{L}} \langle W[\mathscr{C}]\rangle_T(\mathbf{L})$$

Rajagopal, BSH, Wiedemann 2501.06289

*[Skenderis and B. C. van Rees, 0805.0150, 0812.2909]

The action to minimize

a string on the holographic SK manifold*

- The Wilson loop wraps around the SK contour, and therefore so does the string, meaning that

$$\langle W[\mathscr{C}]\rangle_T = \int DX_R DX_L DX_E \exp\left(-S[X_R] - S[X_L] - S[X_E]\right) .$$

- We are only interested in the part that is extensive on t. As such, we have

$$\langle W[\mathscr{C}]\rangle_{T,\, t-\text{extensive}} = \int DX_R DX_L \exp\left(-S[X_R] - S[X_L]\right) ,$$

supplemented by a matching condition $X_R(z = z_h) = X_L(z = z_h)$.

back

[1]Rey & Yee hep-th/9803001
[2]Maldacena hep-th/9803002
[3]Drukker, Gross & Ooguri hep-th/9904191
[4]Alday & Maldacena 0710.1060
[5]Polchinski & Sully 1104.5077

BPS line vs the usual line

two Wilson loops in holography

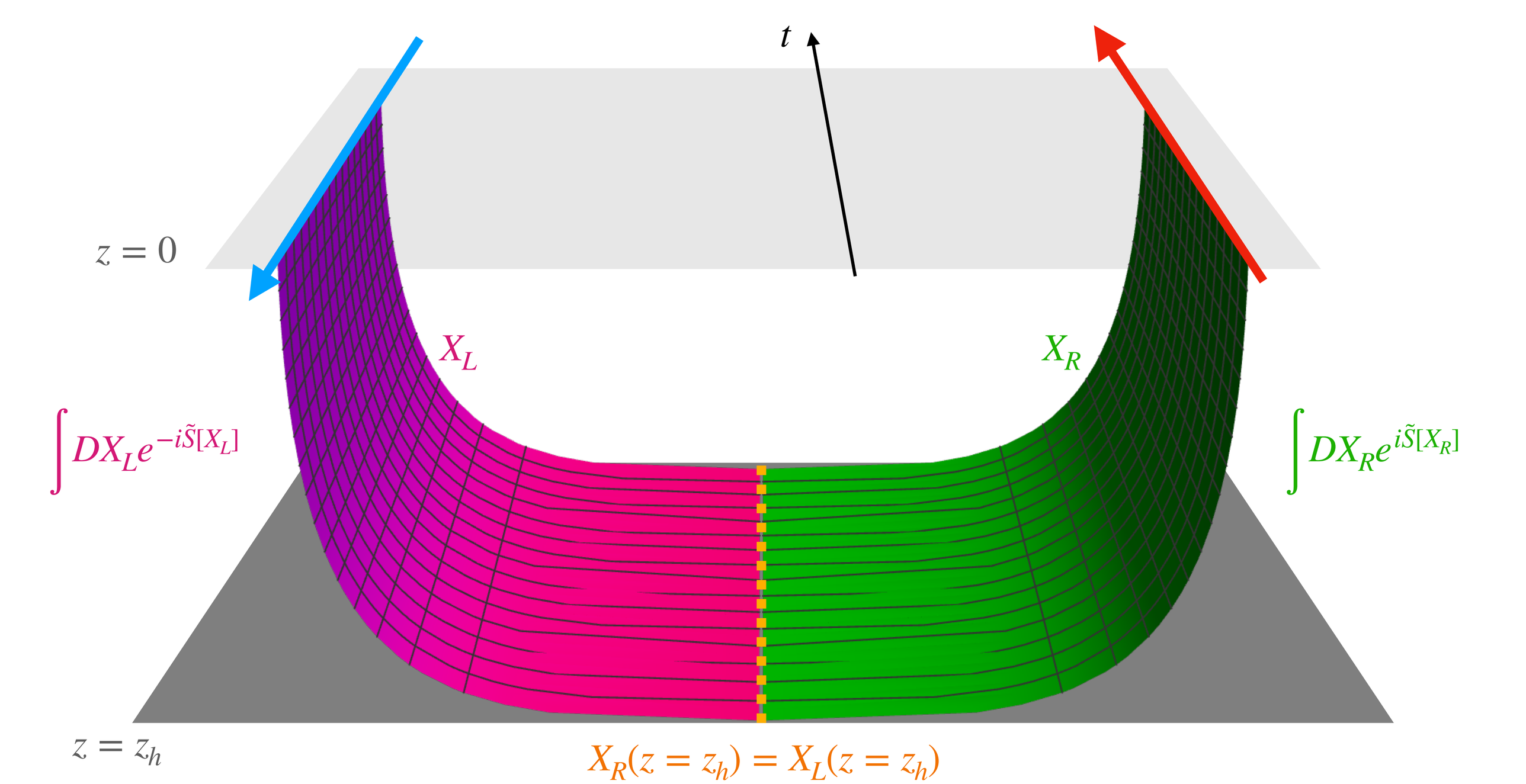
- The correspondence relates the following BPS loop to the string action[1,2,3]

$$W_S[\mathscr{C};\hat{n}] = \frac{1}{N_c}\mathrm{Tr}_{\mathrm{color}}\left[\mathscr{P}\exp\left(ig\oint_{\mathscr{C}} ds\, T^a\left[A^a_\mu\,\dot{x}^\mu + \hat{n}(s)\cdot\vec{\phi}^{\,a}\sqrt{\dot{x}^2}\right]\right)\right] .$$

- A prescription was given later for the usual Wilson loop[4,5]

$$W[\mathscr{C}] = \frac{1}{N_c}\mathrm{Tr}_{\mathrm{color}}\left[\mathscr{P}\exp\left(ig\oint_{\mathscr{C}} ds\, T^a\left[A^a_\mu\,\dot{x}^\mu\right]\right)\right] ,$$

by imposing Neumann b.c. for the worldsheet on the S_5. Therefore, if we find a worldsheet from $W_s[\mathscr{C};\hat{n}]$ with constant $\hat{n}\in S_5$, then at leading order in $1/\sqrt{\lambda}$ the result for $W[\mathscr{C}]$ will be the same.


t
$z = 0$
X_L
X_R
$\int DX_L e^{-i\tilde{S}[X_L]}$
$\int DX_R e^{i\tilde{S}[X_R]}$
$z = z_h$
$X_R(z = z_h) = X_L(z = z_h)$

Introducing Lagrange multipliers

to enforce the boundary conditions

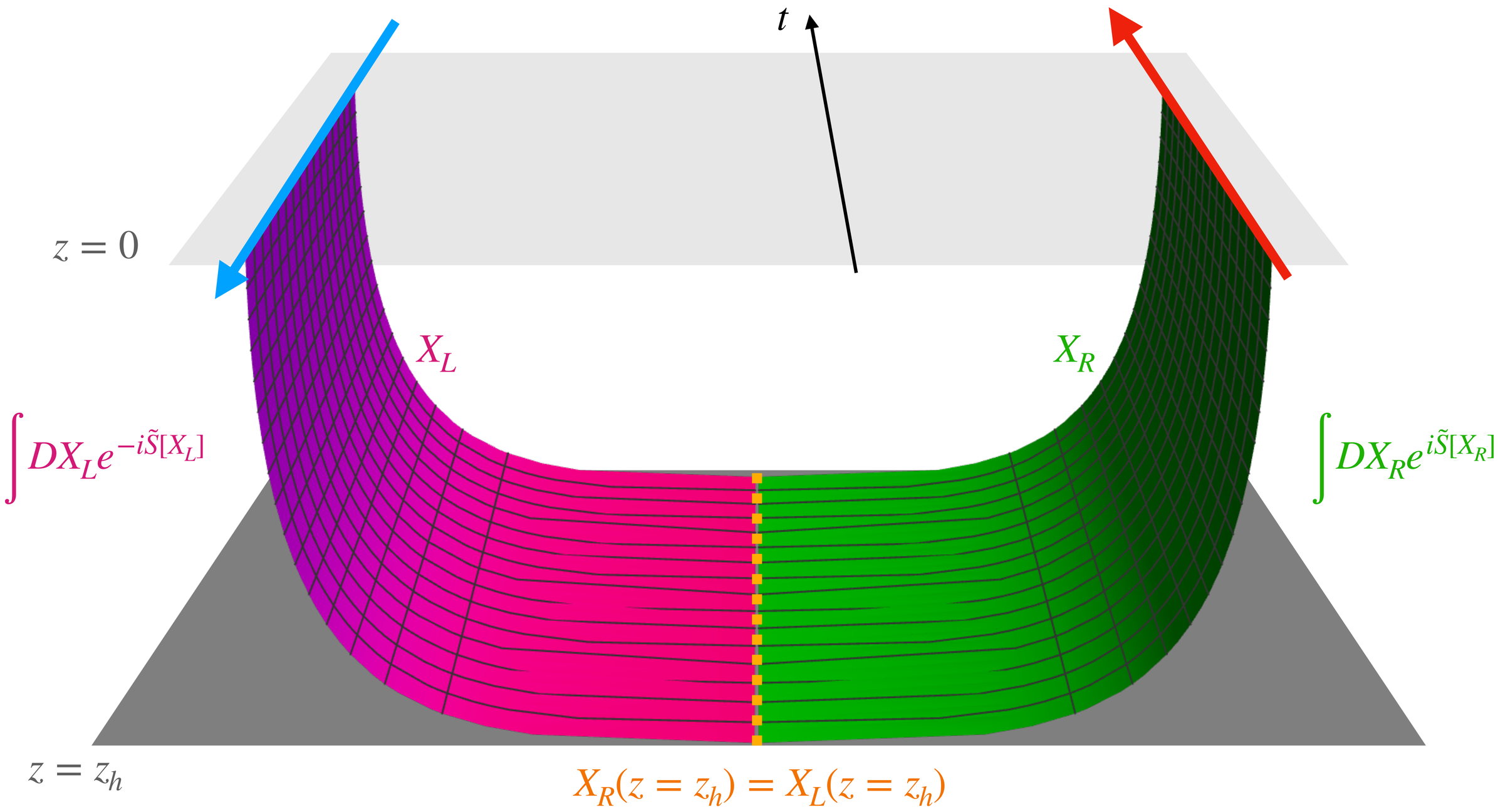
- We can parametrize the string configuration as

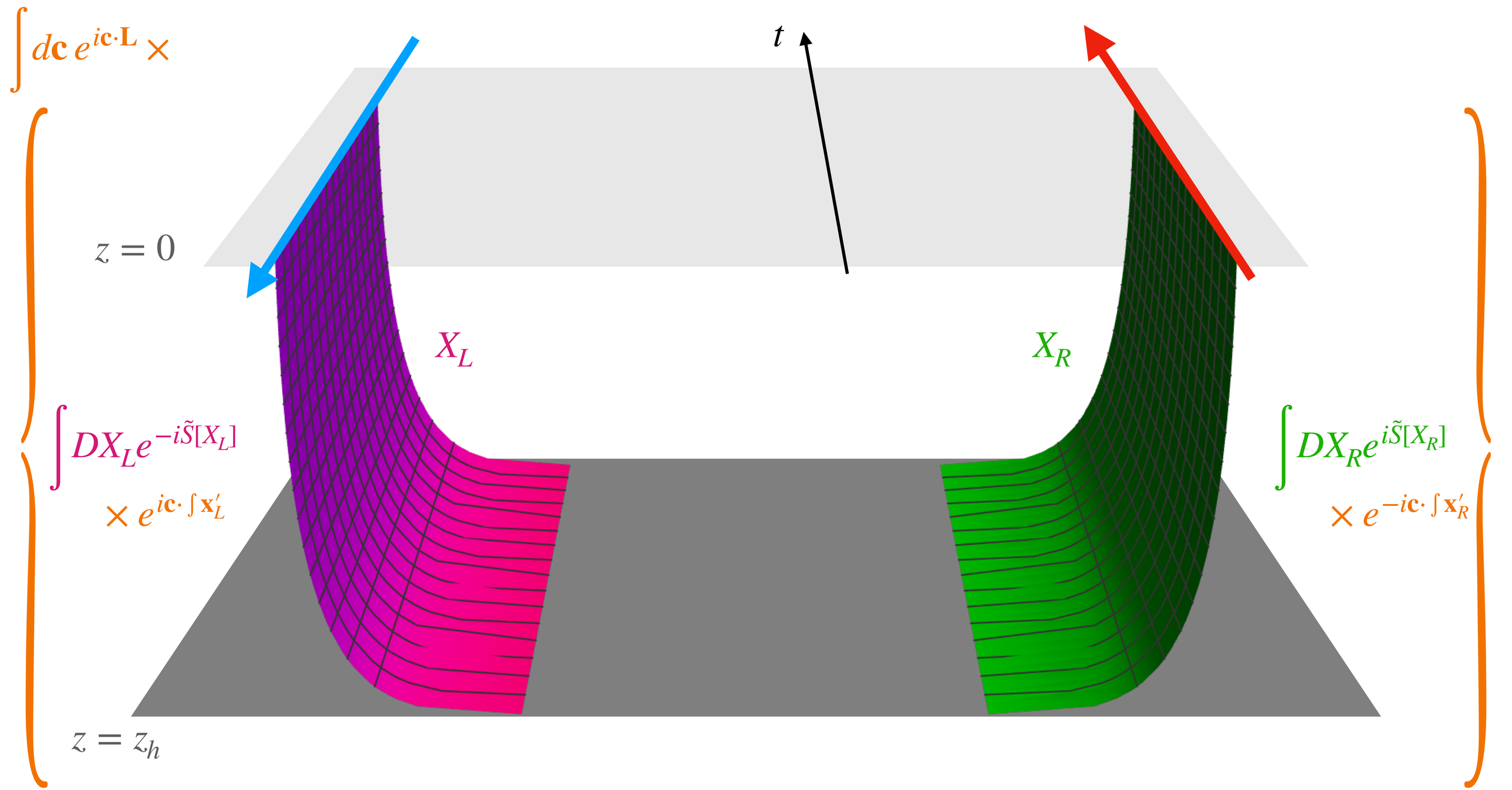
$$X_R(\tau, z) = \Big(\tau(1 - i\epsilon), x_{R1}(z), x_{R2}(z), v\tau + x_{R3}(z), z\Big)\,,$$
$$X_L(\tau, z) = \Big(\tau(1 + i\epsilon), x_{L1}(z), x_{L2}(z), v\tau + x_{L3}(z), z\Big)\,.$$

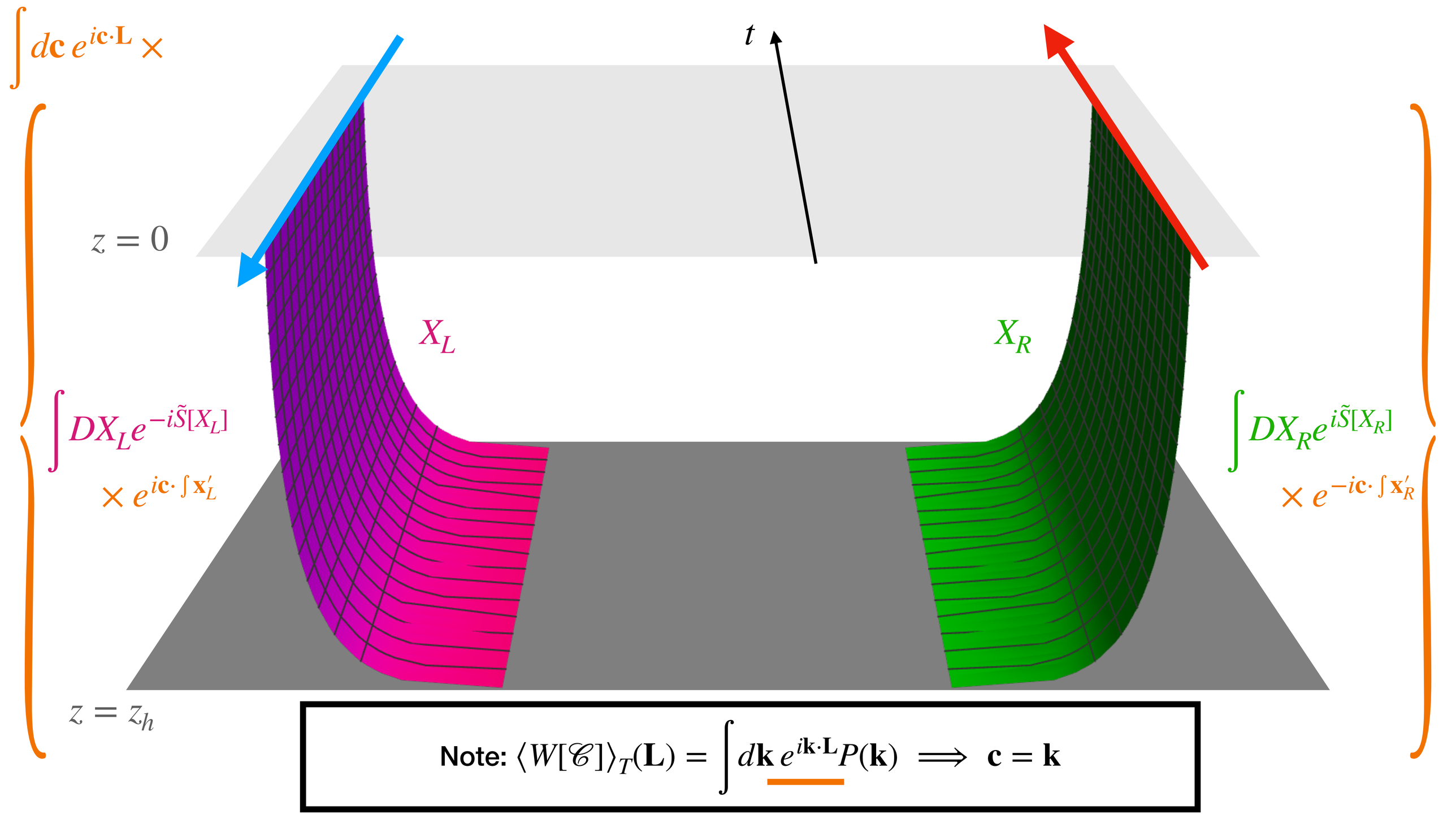
- The matching condition at $z = z_h = 1/(\pi T)$ can be enforced by adding Lagrange multipliers c_i to the action:

$$S_{\text{tot}} \longrightarrow S_{\text{tot}} - ic_j \left(\int_0^{z_h} dz\, x'_{Lj} - \int_0^{z_h} dz\, x'_{Rj} + L_j \right),$$

where L_i is the extent of the loop along each spatial direction.

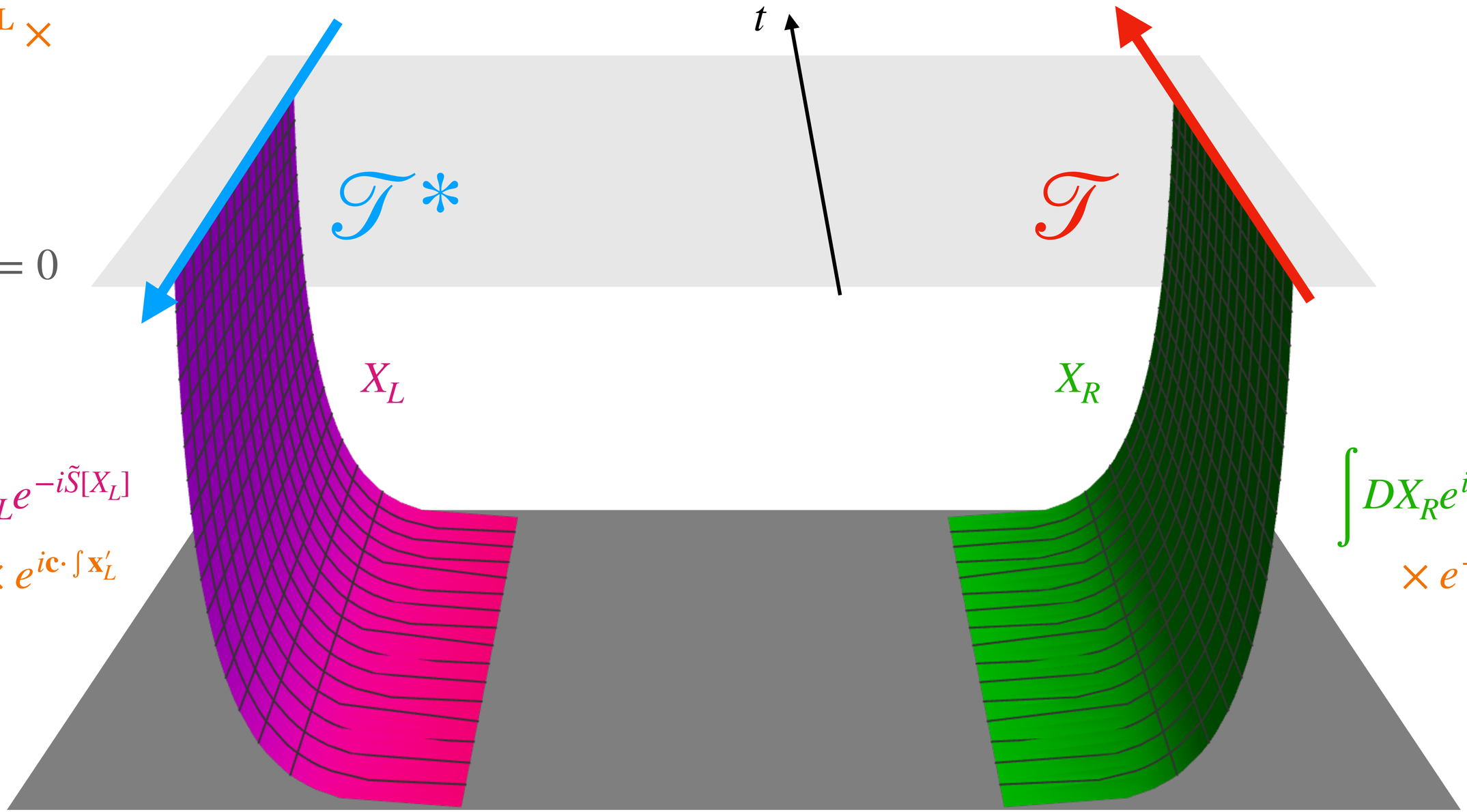

t
$z = 0$
X_L
X_R
$\int DX_L e^{-i\tilde{S}[X_L]}$
$\int DX_R e^{i\tilde{S}[X_R]}$
$z = z_h$
$X_R(z = z_h) = X_L(z = z_h)$


$\int d\mathbf{c}\, e^{i\mathbf{c}\cdot\mathbf{L}} \times$
t
$z = 0$
X_L
X_R
$\int DX_L e^{-i\tilde{S}[X_L]}$
$\times e^{i\mathbf{c}\cdot\int \mathbf{x}'_L}$
$\int DX_R e^{i\tilde{S}[X_R]}$
$\times e^{-i\mathbf{c}\cdot\int \mathbf{x}'_R}$
$z = z_h$

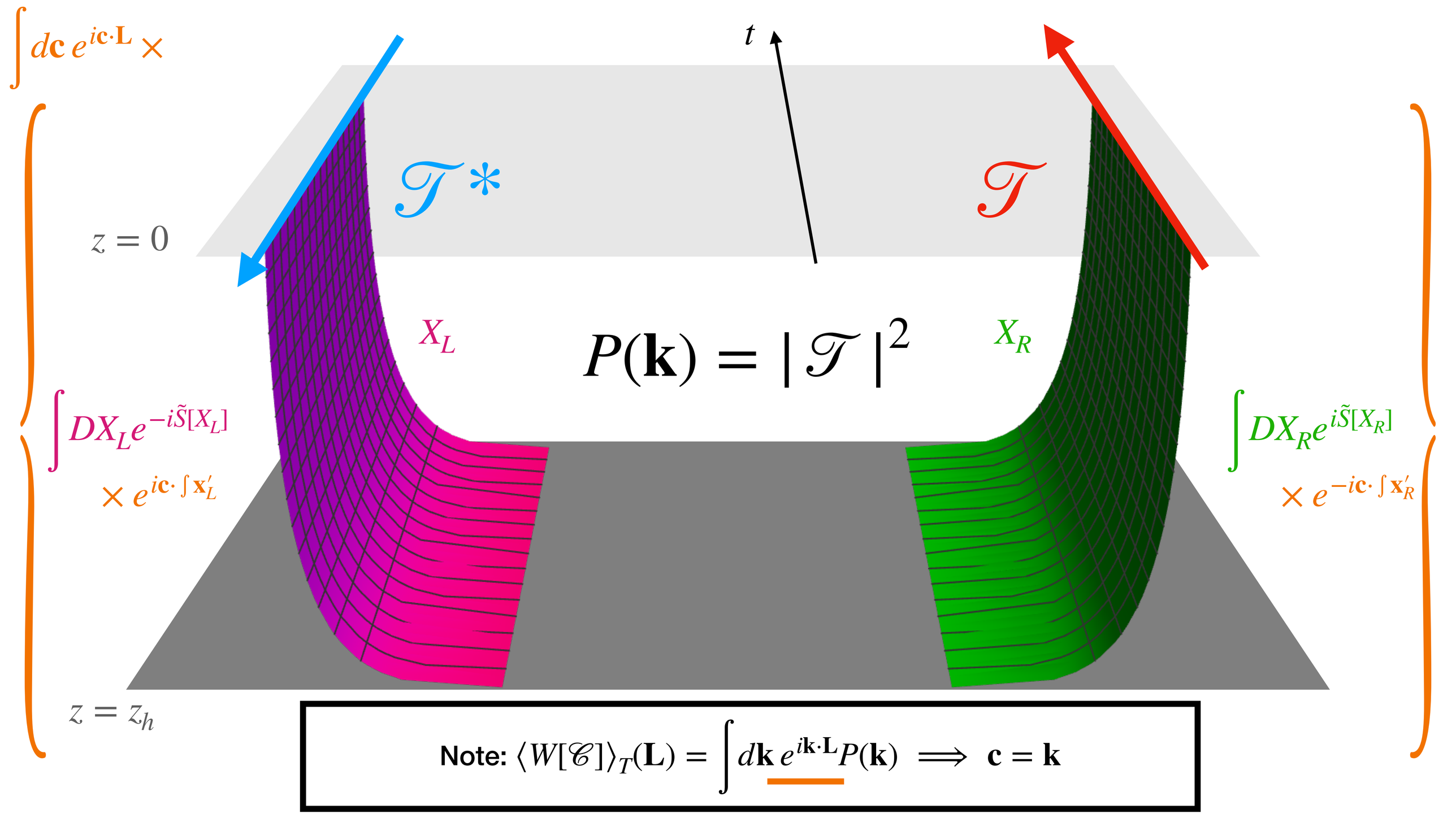


Note: $\langle W[\mathscr{C}]\rangle_T(\mathbf{L}) = \int d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{L}} P(\mathbf{k}) \implies \mathbf{c} = \mathbf{k}$

$$\int d\mathbf{c}\, e^{i\mathbf{c}\cdot\mathbf{L}} \times$$



Note: $\langle W[\mathscr{C}]\rangle_T(\mathbf{L}) = \int d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{L}} P(\mathbf{k}) \implies \mathbf{c} = \mathbf{k}$


$\int d\mathbf{c}\, e^{i\mathbf{c}\cdot\mathbf{L}} \times$
t
$\mathscr{T}^*$
$\mathscr{T}$
$z = 0$
X_L
$P(\mathbf{k}) = |\mathscr{T}|^2$
X_R
$\int DX_L e^{-i\tilde{S}[X_L]}$
$\times e^{i\mathbf{c}\cdot\int \mathbf{x}'_L}$
$\int DX_R e^{i\tilde{S}[X_R]}$
$\times e^{-i\mathbf{c}\cdot\int \mathbf{x}'_R}$
$z = z_h$
Note: $\langle W[\mathscr{C}]\rangle_T(\mathbf{L}) = \int d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{L}} P(\mathbf{k}) \implies \mathbf{c} = \mathbf{k}$

$$S[X_R] = \frac{\sqrt{\lambda}t}{2\pi}\int_0^{z_h}\frac{dz}{z^2}\sqrt{-\left(f(z){x'_{R3}}^2 + (f(z)-v^2)({x'_{R1}}^2 + {x'_{R2}}^2) + \frac{f(z)-v^2}{f}\right)}$$

- Because the Nambu-Goto action for these configurations satisfies $S[X] = S[\mathbf{x}']$, we have

$$\int D\mathbf{x}_R\, e^{i\tilde{S}[X_R]} e^{-i\mathbf{c}\cdot\int \mathbf{x}'_R} = \mathscr{J}\int D\mathbf{x}'_R\, e^{i\tilde{S}[\mathbf{x}'_R]} e^{-i\mathbf{c}\cdot\int \mathbf{x}'_R},$$

where $\mathscr{J}$ is an overall multiplicative factor.

In the saddle point approximation ($\sqrt{\lambda} \to \infty$), this can be evaluated by solving an algebraic equation. The result is that

$$\langle W[\mathscr{C}]\rangle_T = \int dC_j \exp\left(-\sqrt{\lambda}Tt\left[\tilde{S}_{\text{tot}}(\mathbf{C}) - \frac{i}{2}\mathbf{C}\cdot\mathbf{L}\right]\right),$$

and therefore

$$P(\mathbf{k}) \propto \exp\left[-\sqrt{\lambda}Tt\,\tilde{S}_{\text{tot}}\left(\frac{2\mathbf{k}}{\sqrt{\lambda}Tt}\right)\right].$$

$$f(z) = 1 - (\pi T z)^4$$

Result

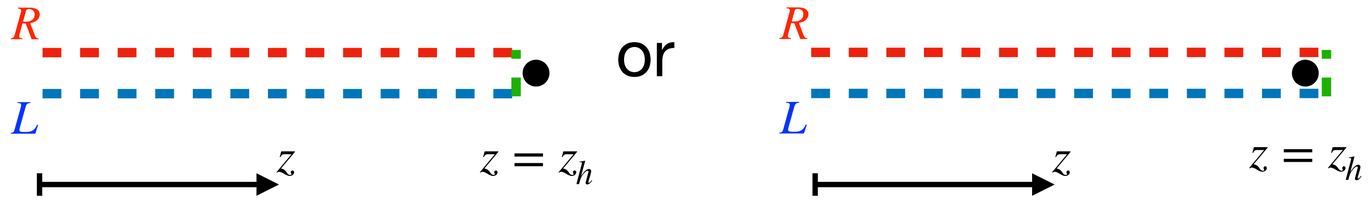
an intermediate step

- One obtains, in units where $z_h = 1$, and with $f_+(z) = (1 + i\epsilon) f(z)$,

$$\tilde{S}_{\rm tot}(\mathbf{C}) = \int_0^1 dz\, {\rm Re} \left\{ \frac{1}{z^2 f_+} \sqrt{-\frac{f_+}{f}\left[(f_+ - v^2)(f_+ - z^4 C_3^2) - C_\perp^2 z^4 f_+\right]} \right\} .$$

- Two contributions:

  1. The interval where the argument of the square root is positive
  2. The $1/f$ pole



Result

in its explicit form

- We obtain

$$\tilde{S}_{\rm tot}(\mathbf{C}) = \sqrt{1+\mathbf{C}^2}\,\frac{\pi}{32}\,\frac{(z_+^4 - z_-^4)^2}{z_-^5(1-z_-^4)}\,F_1\left(\frac{3}{2},\frac{5}{4},1;3;-\frac{z_+^4-z_-^4}{z_-^4},\frac{z_+^4-z_-^4}{1-z_-^4}\right)$$

$$+\frac{\pi}{2}\,|\,vC_3\,|\,\theta(-vC_3)\,,$$

where

$$z_\pm^4 = \frac{1+C_\perp^2+(1-v^2)(1+C_3^2)}{2(1+C_\perp^2+C_3^2)} \pm \frac{\sqrt{C_\perp^4+2C_\perp^2(v^2+C_3^2(1-v^2))+(v^2-(1-v^2)C_3^2)^2}}{2(1+C_\perp^2+C_3^2)}\,.$$

Result

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$$\tilde{S}_{\rm tot}(\mathbf{C}) = \sqrt{1+\mathbf{C}^2}\,\frac{\pi}{32}\,\frac{(z_+^4 - z_-^4)^2}{z_-^5(1-z_-^4)}\,F_1\left(\frac{3}{2},\frac{5}{4},1;3;-\frac{z_+^4-z_-^4}{z_-^4},\frac{z_+^4-z_-^4}{1-z_-^4}\right)$$

$$+\frac{\pi}{2}\,|vC_3|\,\theta(-vC_3)\,,$$

Only the contribution from the pole breaks the symmetry $k_3 \to -k_3$.

where

$$z_\pm^4 = \frac{1+C_\perp^2+(1-v^2)(1+C_3^2)}{2(1+C_\perp^2+C_3^2)} \pm \frac{\sqrt{C_\perp^4+2C_\perp^2(v^2+C_3^2(1-v^2))+(v^2-(1-v^2)C_3^2)^2}}{2(1+C_\perp^2+C_3^2)}\,.$$

$$f(z) = 1 - (\pi T z)^4$$

Result

In units where $z_h = 1$, and with $f_+(z) = (1 + i\epsilon)f(z)$,

$$\tilde{S}_{\rm tot}(\mathbf{C}) = \int_0^1 dz\, {\rm Re}\left\{ \frac{1}{z^2 f_+} \sqrt{-\frac{f_+}{f}\left[(f_+ - v^2)(f_+ - z^4 C_3^2) - C_\perp^2 z^4 f_+\right]} \right\} .$$

This integral can be evaluated in terms of an Appell Hypergeometric series:

$$\tilde{S}_{\rm tot}(\mathbf{C}) = \sqrt{1+\mathbf{C}^2}\,\frac{\pi}{32}\,\frac{(z_+^4 - z_-^4)^2}{z_-^5(1-z_-^4)} F_1\left(\frac{3}{2}, \frac{5}{4}, 1; 3; -\frac{z_+^4 - z_-^4}{z_-^4}, \frac{z_+^4 - z_-^4}{1 - z_-^4}\right) + \frac{\pi}{2}\,|\,vC_3\,|\,\theta(-vC_3)\,,$$

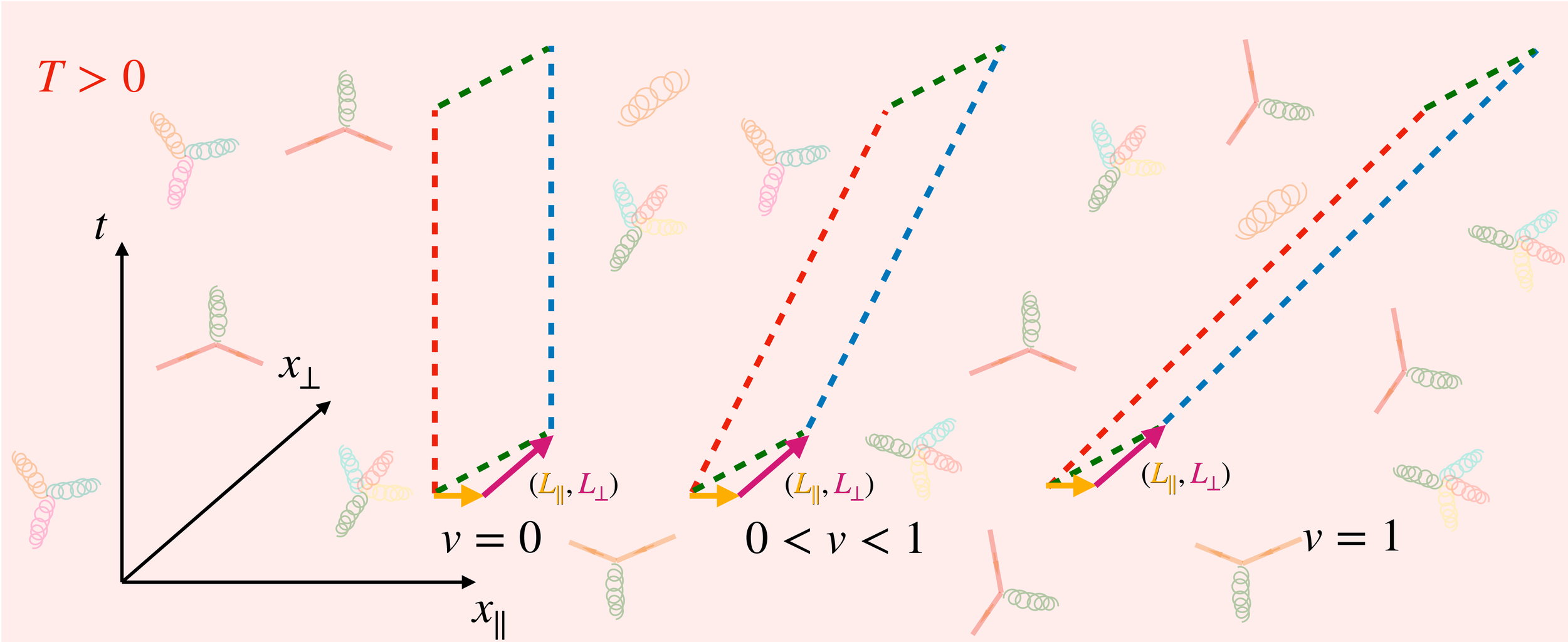
where

$$z_\pm^4 = \frac{1 + C_\perp^2 + (1-v^2)(1+C_3^2)}{2(1 + C_\perp^2 + C_3^2)} \pm \frac{\sqrt{C_\perp^4 + 2C_\perp^2(v^2 + C_3^2(1-v^2)) + (v^2 - (1-v^2)C_3^2)^2}}{2(1 + C_\perp^2 + C_3^2)} .$$

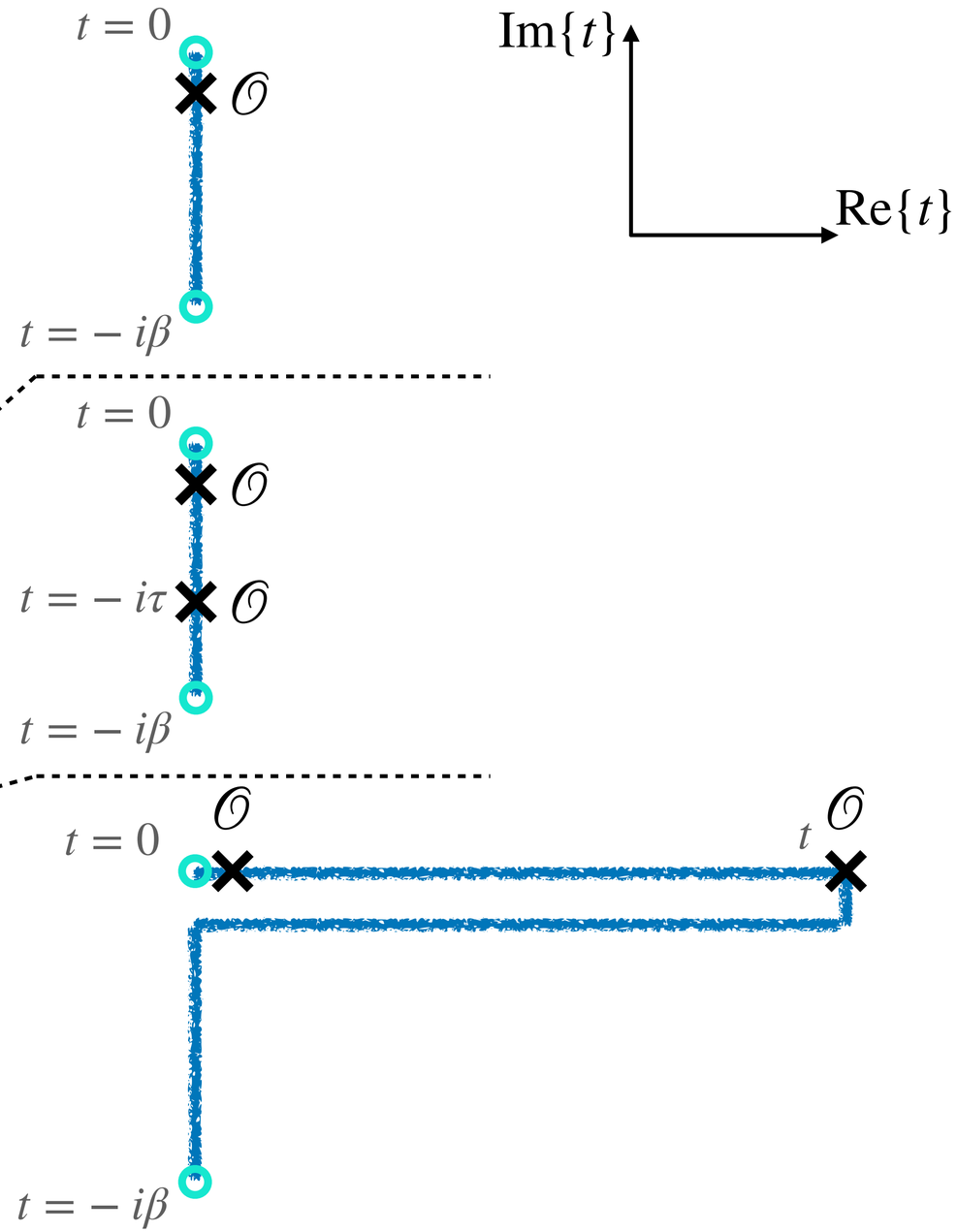
back

Wilson loops

the configurations of interest



What you can calculate
on the Schwinger-Keldysh contour



- Imaginary time calculations:

equilibrium #: $\langle \mathcal{O} \rangle = Z^{-1}\mathrm{Tr}\left[\mathcal{O}e^{-\beta H}\right]$,

and also two-point functions:

$$\langle \mathcal{O}(\tau)\mathcal{O}(0)\rangle = Z^{-1}\mathrm{Tr}\left[\mathcal{O}(0)e^{-\tau H}\mathcal{O}(0)e^{-(\beta-\tau)H}\right]$$

- Real time calculations:

$$\langle \mathcal{O}(t)\mathcal{O}(0)\rangle = Z^{-1}\mathrm{Tr}\left[e^{iHt}\mathcal{O}(0)e^{-iHt}\mathcal{O}(0)e^{-\beta H}\right]$$

What you can calculate
on the Schwinger-Keldysh contour

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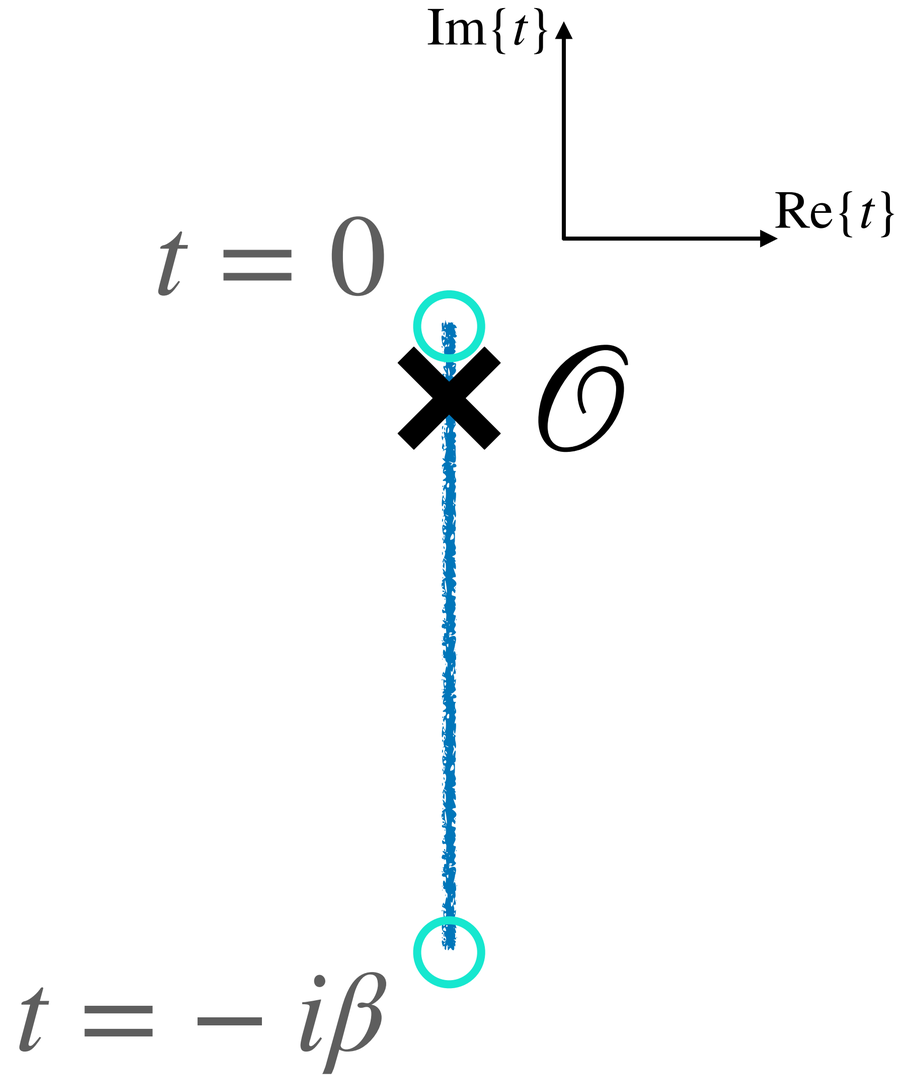
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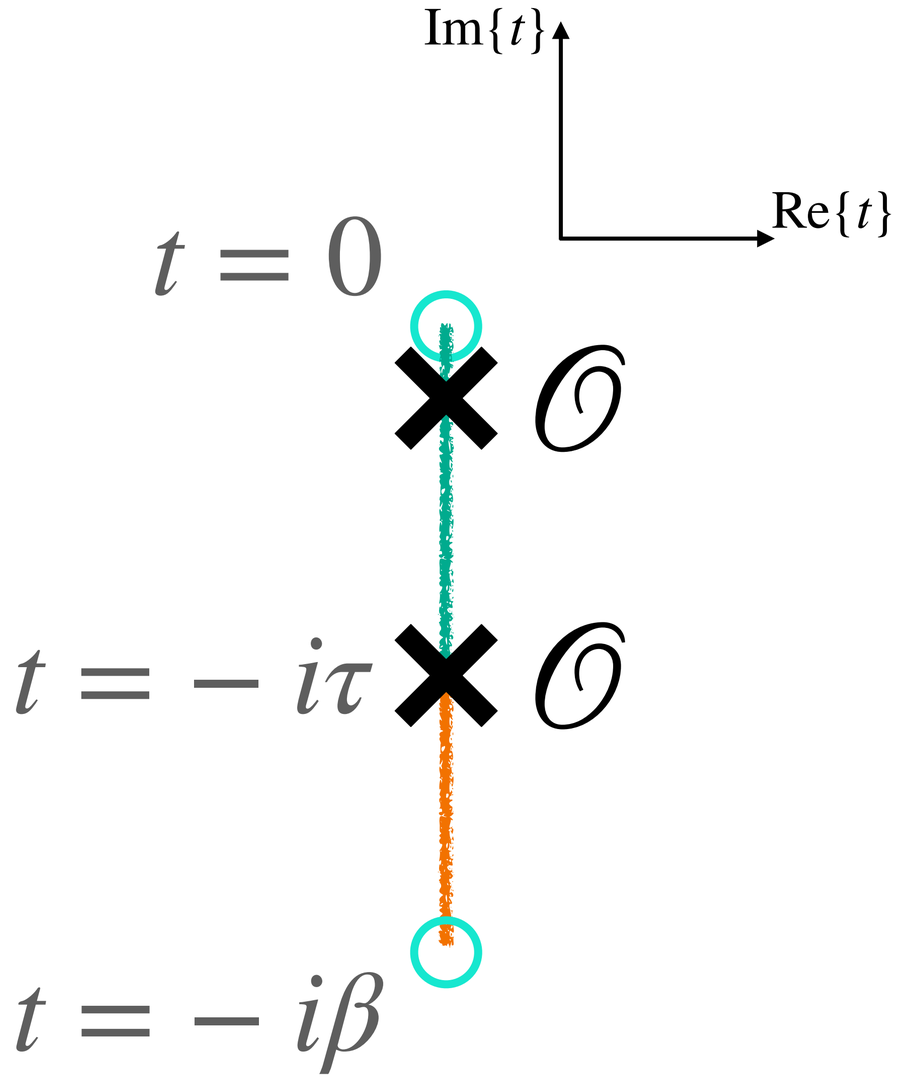
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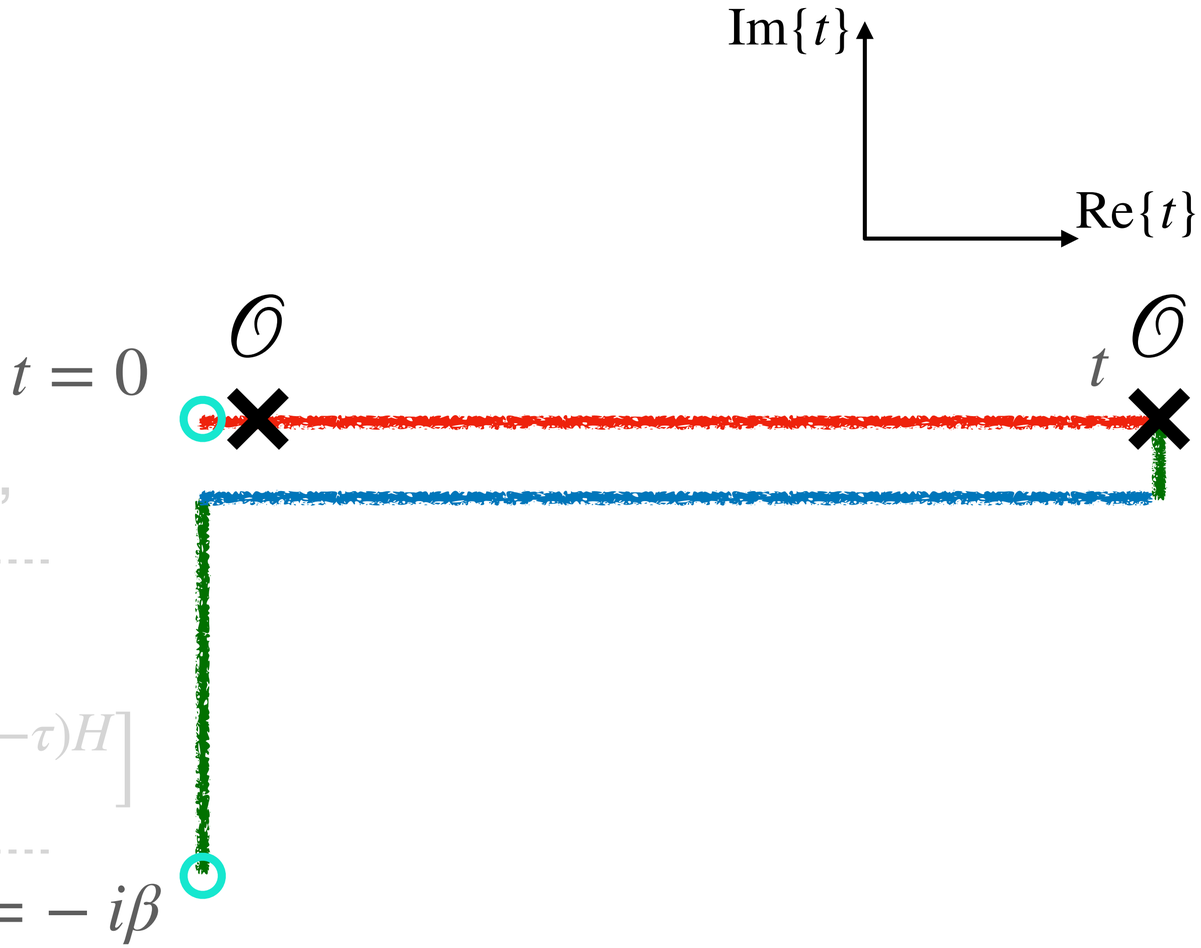
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What you can calculate
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back

The Wilson loop

as a moment generating functional

It follows from the definitions that

$$\langle W[\mathscr{C}]\rangle_T = \exp\left(-\sqrt{\lambda}Tt\, S_{\rm tot}(\mathbf{L})\right) = \int d^3\mathbf{k}\, e^{-i\mathbf{k}\cdot\mathbf{L}} P(\mathbf{k})$$

$$P(\mathbf{k}) \propto \exp\left[-\sqrt{\lambda}Tt\, \tilde{S}_{\rm tot}\left(\frac{2\mathbf{k}}{\sqrt{\lambda}Tt}\right)\right] \propto \int d^3\mathbf{L}\, e^{i\mathbf{k}\cdot\mathbf{L}} \langle W[\mathscr{C}]\rangle_T(\mathbf{L})$$

and therefore

$$\langle k_1^{n_1} k_2^{n_2} k_3^{n_3}\rangle_c = -\, i^{n_1+n_2+n_3}\sqrt{\lambda}Tt\, \frac{\partial^{n_1}}{\partial L_1^{n_1}} \frac{\partial^{n_2}}{\partial L_2^{n_2}} \frac{\partial^{n_3}}{\partial L_3^{n_3}} S_{\rm tot}(\mathbf{L})\Big|_{\mathbf{L}=0}.$$

Relating $S_{\rm tot}(\mathbf{L})$ and $\tilde{S}_{\rm tot}(\mathbf{C})$

how to calculate $S_{\rm tot}(\mathbf{L})$

(Always at leading order in $1/\sqrt{\lambda}$) one can show that

$$S_{\rm tot}(\mathbf{L}) = \left[\tilde{S}_{\rm tot}(\mathbf{C}) - \mathbf{C}\cdot\frac{\partial\tilde{S}_{\rm tot}}{\partial\mathbf{C}}\right]_{\mathbf{C}=\bar{\mathbf{C}}(\mathbf{L})},$$

where

$$L_j = 2i\,\frac{\partial\tilde{S}_{\rm tot}}{\partial C_j}\bigg|_{\mathbf{C}=\bar{\mathbf{C}}(\mathbf{L})}.$$

Skewness and drag-broadening correlations

the third order moments for HQ transport

- We find that the distribution has a non-zero skewness (which was manifest in the figures):

$$\langle (k_3 - \langle k_3 \rangle)^3 \rangle = \frac{9v}{2} \pi \sqrt{\lambda} t \gamma^4 T^4 \,,$$

- and a non-zero correlation between energy loss and momentum broadening:

$$\langle k_\perp^2 (k_3 - \langle k_3 \rangle) \rangle = 3v\pi \sqrt{\lambda} t \gamma^2 T^4 \,.$$

- Note that they can be significant even at modest speeds.

Kurtosis and longitudinal-transverse broadening correlation

the fourth order moments for HQ transport

- We find the kurtosis along the transverse direction

$$\langle k_1^4\rangle - 3\langle k_1^2\rangle^2 = \langle k_2^4\rangle - 3\langle k_2^2\rangle^2 = \frac{9(8-v^2)}{8}\pi\sqrt{\lambda}t\gamma^{3/2}T^5\,,$$

- the kurtosis along the longitudinal direction

$$\langle (k_3 - \langle k_3\rangle)^4\rangle - 3\langle (k_3 - \langle k_3\rangle)^2\rangle^2 = \frac{9(8+19v^2)}{8}\pi\sqrt{\lambda}t\gamma^{11/2}T^5\,,$$

- and the cross-correlation

$$\langle k_\perp^2 (k_3 - \langle k_3\rangle)^2\rangle - \langle k_\perp^2\rangle\langle (k_3 - \langle k_3\rangle)^2\rangle = \frac{3(8+9v^2)}{4}\pi\sqrt{\lambda}t\gamma^{7/2}T^5\,.$$

Non-Gaussian features

velocity scalings

- There is a pattern to all of the fully connected moments: at $\gamma \gg 1$, we have shown that

$$\langle k_\perp^{2m} k_3^n \rangle_c \propto \sqrt{\lambda} t (\sqrt{\gamma} T)^{n+2m+1} \times \gamma^{n-1} .$$

- This has an interesting interpretation in the rest frame of the heavy quark. Transforming the coordinates, one obtains

$$\langle k_\perp^{2m} k_3^n \rangle_{c,\mathrm{HQ}} \propto \sqrt{\lambda} t' (\sqrt{\gamma} T)^{n+2m+1}.$$

- That is to say, all of the moments are governed by a single scale $\sqrt{\gamma} T$.

Cherenkov radiation

- *Quantum and Thermal Fluctuations of Cherenkov Radiation from HQET*
  (2603.06784 with Joshua Lin)

The Wilson loop in pure Maxwell theory

Electromagnetism with no light fermions

- The simplest case for the medium is to have a free (noninteracting) theory. As a theoretical exercise, let us consider a medium where the speed of light in the medium c_{med} is less than 1.
- Then, calculate $\langle W_{\mathbf{v}} \rangle(\mathbf{L}) = \left\langle \exp\left(-ign^{\mu} \int_0^t A_{\mu}(t', \mathbf{v}t' + \mathbf{L})\, dt' \right) \exp\left(ign^{\mu} \int_0^t A_{\mu}(t', \mathbf{v}t')\, dt' \right) \right\rangle$, with $n^{\mu} = (1, \mathbf{v})$ and $v = |\mathbf{v}|$.
  - If $v < c_{\text{med}}$, then $\langle W_{\mathbf{v}} \rangle(\mathbf{L}) = 1$, and therefore $P(\mathbf{k}; \mathbf{v}) = \delta(\mathbf{k})$.
  - If $v > c_{\text{med}}$, then (at zero temperature)

$$\langle W_{\mathbf{v}} \rangle_{T=0}(\mathbf{L}) = \exp\left(-t \frac{g^2 (c_{\text{med}}^2 - v^2)}{4\pi v c_{\text{med}}} \int_0^{\infty} \mathrm{d}\omega \left[J_0(L_{\perp} k_{\perp}) e^{i\omega L_{\parallel}/v} - 1 \right] \right).$$

Recovering known dE/dx physics

- Moments of the momentum change probability can be obtained by taking derivatives of $\langle W_{\mathbf{v}} \rangle(\mathbf{L})$:

$$\langle W_{\mathbf{v}} \rangle(\mathbf{L}) = \int d^3\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{L}} P(\mathbf{k};\mathbf{v}) \,.$$

- In particular,

$$\langle k_L \rangle = -\, i \frac{\partial}{\partial L_\parallel} \langle W_{\mathbf{v}} \rangle(\mathbf{L}) \bigg|_{\mathbf{L}=\mathbf{0}} = \frac{tg^2}{4\pi} \int d\omega \frac{\omega}{c_{\rm med}} \left(1 - \frac{c_{\rm med}^2}{v^2} \right),$$

from where, using $d\langle k_L \rangle / dt = dE/dx$, one obtains the Frank-Tamm formula:

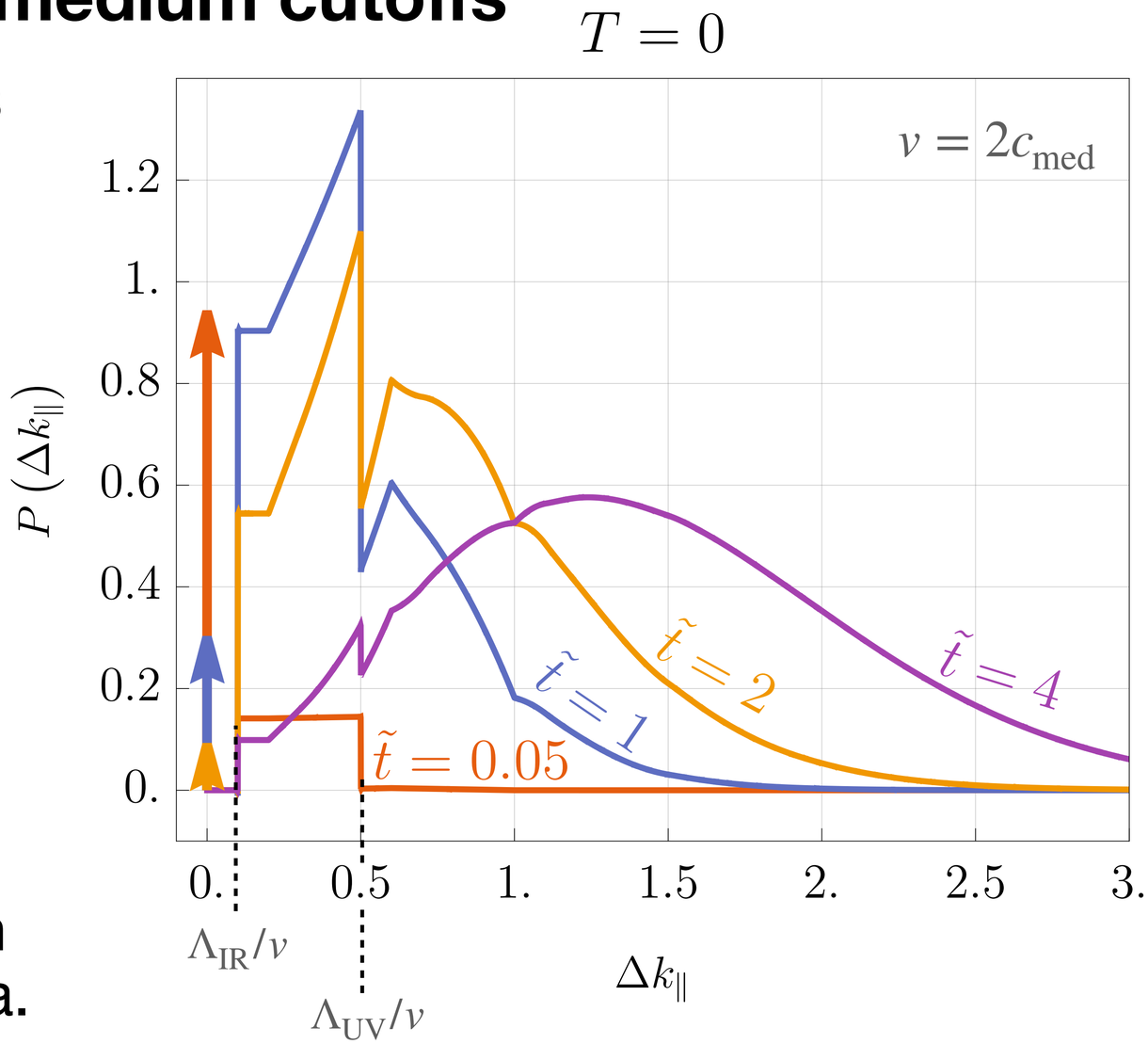
$$\frac{\partial^2 E}{\partial x\, \partial \omega} = \frac{g^2 \omega}{4\pi c_{\rm med}} \left(1 - \frac{c_{\rm med}^2}{v^2} \right) \theta(v - c_{\rm med}) \,.$$

Visualizing the full $P(k_L)$ in an idealized case

for a specific choice of UV and IR medium cutoffs

- One can use the preceding expressions to *directly calculate* the **longitudinal momentum change probability** $P(k_L)$ for an electron.

- The distribution is quite nontrivial:
  - It has a $\delta(k_L)$ piece (no radiation probability)
  - It has sharp features determined by the fact that photon emissions are discrete $\Longleftrightarrow$ *quantum*
  - It becomes smooth at late times, with a mean given by Frank-Tamm formula.



Remarks

1. The scale of the momentum transfer for real applications is set by the frequency band of the medium with $c_{\mathrm{med}}(\omega) < 1$. These frequencies are much smaller than $m_e \approx 511\ \mathrm{keV}$, which justifies the use of HQET.

2. What about thermal physics/equilibration?

   - Electrons never reach thermal equilibrium in this way, because they can't lose energy if $v < c_{\mathrm{med}}$.

   - However, they would if we set $c_{\mathrm{med}} = 0$, in which case the electrons interact with a bath of spatially decoupled harmonic oscillators.

     - This can be traced back to the KMS condition for the Wightman function

$$D^{>}(t - i\beta) = D^{>}(-t).$$