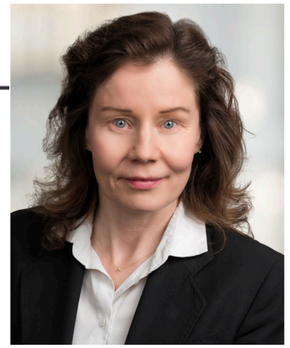
Quantum-Metric Correction to Kinetic Theory and Liouville's Theorem

Kazuya Mameda
Tokyo University of Science
with Naoki Yamamoto, arXiv: 2509.15731 (2025)

An Essay of “Quantum Geometry”

PHYSICAL REVIEW LETTERS **131,** 240001 (2023)



Essay: Where Can Quantum Geometry Lead Us?

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(Received 17 November 2023; published 11 December 2023)

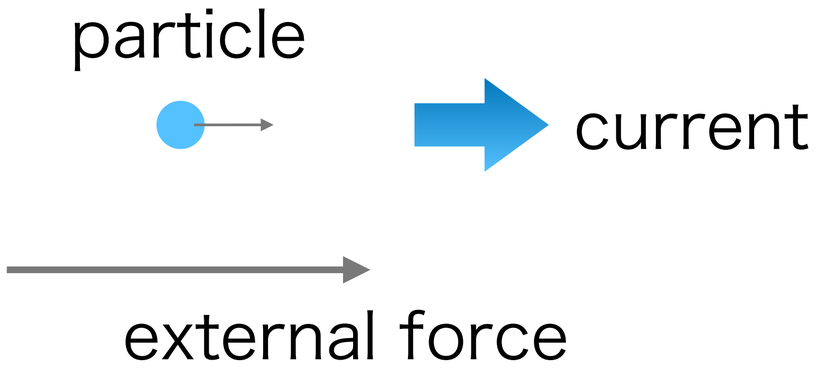
Quantum geometry defines the phase and amplitude distances between quantum states. The phase distance is characterized by the Berry curvature and thus relates to topological phenomena. The significance of the full quantum geometry, including the amplitude distance characterized by the quantum metric, has started to receive attention in the last few years. Various quantum transport and interaction phenomena have been found to be critically influenced by quantum geometry. For example, quantum geometry allows counterintuitive flow of supercurrent in a flat band where single electrons are immobile. In this Essay, I will discuss my view of the important open problems and future applications of this research topic and will try to inspire the reader to come up with further ideas. At its best, quantum geometry can open a new chapter in band theory and lead to breakthroughs as transformative as room-temperature superconductivity. However, first, more experiments directly showing the effect of quantum geometry are needed. We also have to integrate quantum geometry analysis in our most advanced numerical methods.

this talk

KM, Yamamoto (2025)

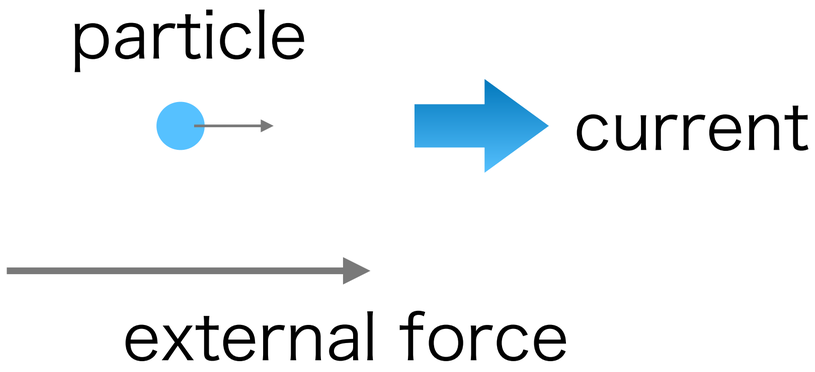
Sogabe’s talk (Oct. 27th) KM, Sogabe (2026)

Classical Transport Phenomena





Classical Transport Phenomena



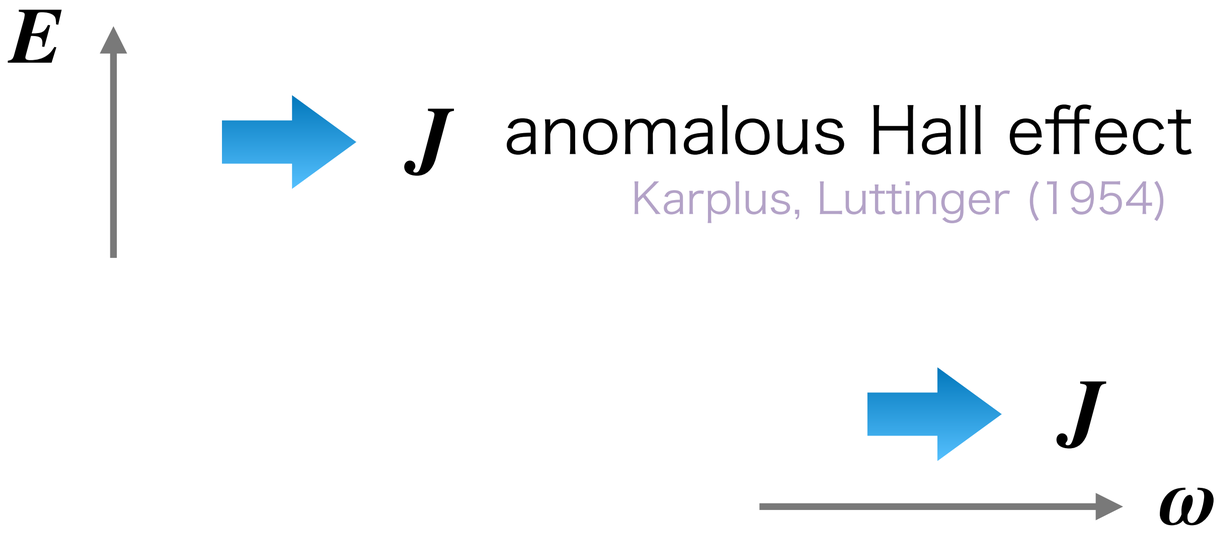
$\boldsymbol{J}$ $\boldsymbol{E}$

Ohm's law

$\boldsymbol{J}_{\epsilon}$ $-\nabla T$

Fourier's law

Quantum Transport Phenomena



anomalous Hall effect

Karplus, Luttinger (1954)

chiral magnetic effect

Vilenkin (1980)

Fukushima, Kharzeev, Warringa (2008)

chiral vortical effect

Vilenkin (1979), Son, Surowka (2009)

Landsteiner, Megias, Pena-Benitez (2011)

kinetically described by **Berry curvature** Berry (1984)

$$\boldsymbol{b}(\boldsymbol{p}) = \boldsymbol{\nabla}_{\boldsymbol{p}} \times \boldsymbol{a} \qquad \boldsymbol{a}(\boldsymbol{p}) = -\mathrm{i}\langle u_{\boldsymbol{p}} | \boldsymbol{\nabla}_{\boldsymbol{p}} u_{\boldsymbol{p}} \rangle$$

Berry Curvature Corrected Kinetic Theory

Xiao, Shi, Niu (2005)

$$(1+\hbar \boldsymbol{B}\cdot\boldsymbol{b})\partial_t f + [\boldsymbol{v}+\hbar \boldsymbol{E}\times\boldsymbol{b}+\hbar(\boldsymbol{v}\cdot\boldsymbol{b})\,\boldsymbol{B}]\cdot\boldsymbol{\nabla} f$$

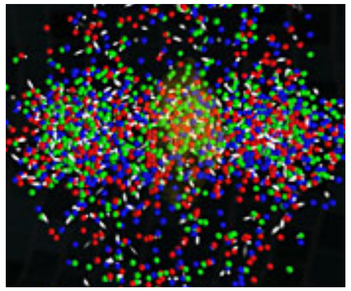
$$+[\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}+\hbar(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{b}]\cdot\boldsymbol{\nabla}_{\boldsymbol{p}} f = (1+\hbar\boldsymbol{B}\cdot\boldsymbol{b})c\,[f]$$

universal formulation up to Berry curvature (BC)

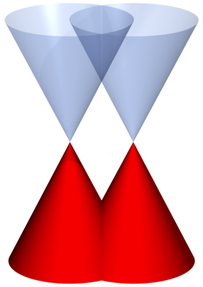
e.g. chiral kinetic theory

Son, Yamamoto (2012) Stephanov, Yin (2012)
Chen, Pu, Wang, Wang (2012)

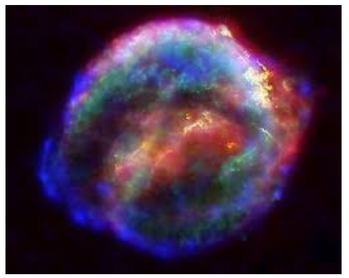
$$\boldsymbol{b} = \frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|^2}$$



QGP



Weyl SM



supernove

Modified Phase Space

Boltzmann kinetic theory

$$(n, \boldsymbol{j}) = \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}}) f \quad \Rightarrow \quad \partial_t n + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$$

solution of Boltzmann eq.

Modified Phase Space

BC corrected kinetic theory (without anomaly)

$$(n, \boldsymbol{j}) \neq \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}}) f \quad \Rightarrow \quad \partial_t n + \boldsymbol{\nabla} \cdot \boldsymbol{j} \neq 0$$

Modified Phase Space

BC corrected kinetic theory (without anomaly)

$$(n, \boldsymbol{j}) = \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}}) \sqrt{\omega} f \quad \Rightarrow \quad \partial_t n + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$$

$$= 1 + \hbar \boldsymbol{b} \cdot \boldsymbol{B}$$

Modified Phase Space

BC corrected kinetic theory (without anomaly)

$$(n, \boldsymbol{j}) = \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}}) \sqrt{\omega} f \quad \Rightarrow \quad \partial_t n + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$$

$$= 1 + \hbar \boldsymbol{b} \cdot \boldsymbol{B}$$

modified phase space measure $\sqrt{\omega}\, \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{p}$

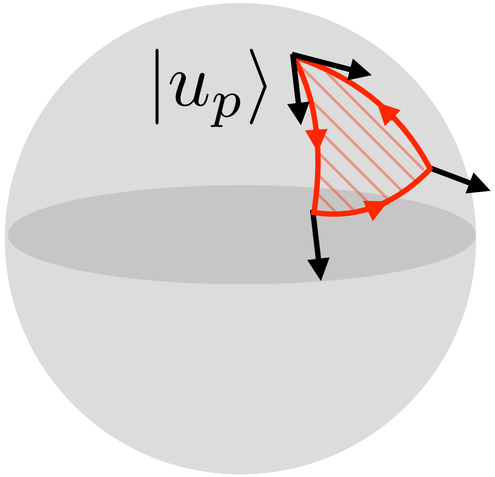
Xiao, Shi, Niu (2005)
Duval, Horvath, Horvathy, Martina, Stichel (2005)

obtained from Poisson bracket

$$\sqrt{\omega} := \det\{\xi^a, \xi^b\}^{-1} = 1 + \hbar \boldsymbol{b} \cdot \boldsymbol{B} \qquad \xi^a = (p_i, x_i)$$

special only for the BC effect?

Another Quantum-Geometric Quantity

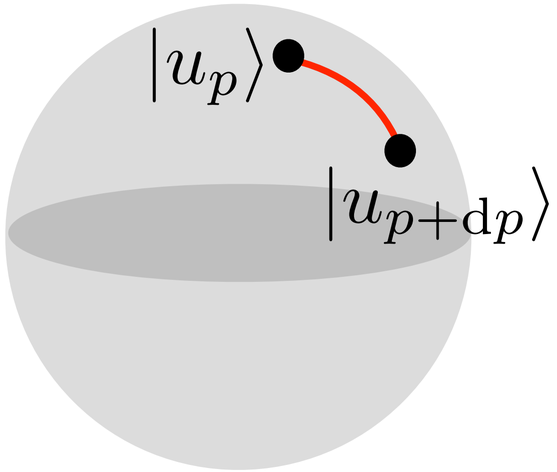


"angle" between two states

$$\alpha = \int_S \boldsymbol{b} \cdot \mathrm{d}\boldsymbol{S}$$

Berry curvature

Berry (1984)



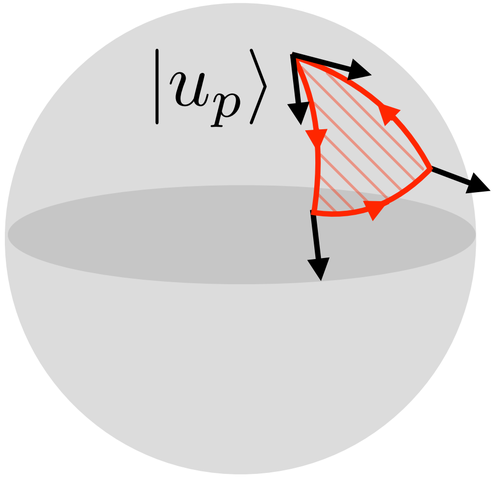
"distance" between nearby states

$$\mathrm{d}s^2 = 1 - |\langle u_p | u_{p+\mathrm{d}p} \rangle|^2 \approx g_{ij} \mathrm{d}p_i \mathrm{d}p_j$$

quantum metric

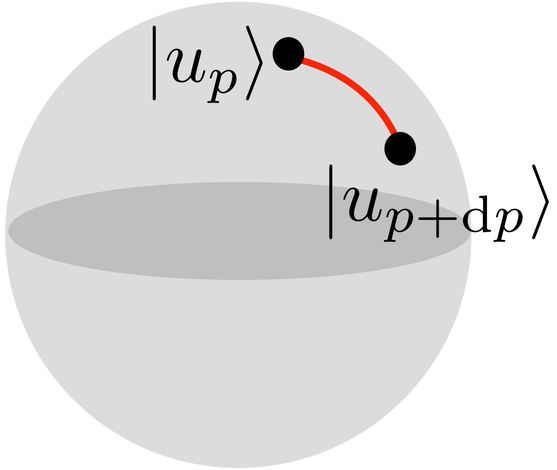
Provost, Vallee (1980)

Another Quantum-Geometric Quantity



ex.) anomalous Hall effect

$$j_i = \epsilon_{ijk} E_j \int_{\boldsymbol{p}} b_k f$$



ex.) nonlinear anomalous Hall effect

Gao, Yang, Niu (2014)

$$j_i = \hbar^2 E_j E_k \int_{\boldsymbol{p}} (\partial_i^p G_{jk} - \partial_j^p G_{ik}) f$$

$$G_{ij} \sim g_{ij}/\Delta\epsilon$$

Foundation Is Missing

	Berry curvature	Quantum metric
transport phenomena	AHE/CME/CVE etc.	nonlinear AHE Gao, Yang, Niu (2014)
definition of currents	$(n, \boldsymbol{j}) = \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}}) \sqrt{\omega} f$ $\sqrt{\omega} = 1 + \hbar \boldsymbol{b} \cdot \boldsymbol{B}$	?
conservation (up to anomaly)	$\partial_t n + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$	?

Berry Curvature Corrected Kinetic Theory

Dynamical System with Berry Connection

Sundaram, Niu (1999) Stephanov, Yin (2012)

$$L = \boldsymbol{p} \cdot \dot{\boldsymbol{x}} + \boldsymbol{A} \cdot \dot{\boldsymbol{x}} - \hbar \boldsymbol{a} \cdot \dot{\boldsymbol{p}} - \epsilon_{\boldsymbol{p}} - \Phi \qquad \xi^a = (p_i, x_i)$$

universal form up to $\boldsymbol{a}$ (not restricted in CKT)

Dynamical System with Berry Connection

Sundaram, Niu (1999)　Stephanov, Yin (2012)

$$L = \underbrace{\boldsymbol{p}\cdot\dot{\boldsymbol{x}} + \boldsymbol{A}\cdot\dot{\boldsymbol{x}} - \hbar\boldsymbol{a}\cdot\dot{\boldsymbol{p}}}_{\pi_a(\xi)\dot{\xi}^a} \underbrace{- \epsilon_{\boldsymbol{p}} - \Phi}_{-H} \qquad \xi^a = (p_i, x_i)$$

canonical eq. $\dot{\xi}^a = \{H, \xi^a\} = -\{\xi^a, \xi^b\}\dfrac{\partial H}{\partial \xi^b}$

matching with $\delta \int L\,\mathrm{d}t = 0$

Son, Yamamoto (2012)

Berry Curvature Corrections

modified Poisson bracket Duval, Horvath, Horvathy, Martina, Stichel (2005)

$$\{p_i, p_j\} = -(1 - \hbar \boldsymbol{b} \cdot \boldsymbol{B})\epsilon_{ijk} B_k$$
$$\{x_i, x_j\} = \hbar \epsilon_{ijk} b_k$$
$$\{p_i, x_j\} = (1 - \hbar \boldsymbol{b} \cdot \boldsymbol{B})\delta_{ij} + \hbar b_i B_j$$

phase space factor

$$\sqrt{\omega} := \det\{\xi^a, \xi^b\}^{-1} = 1 + \hbar \boldsymbol{b} \cdot \boldsymbol{B}$$ Xiao, Shi, Niu (2005)

Quantum Metric Corrected Kinetic Theory

Dynamical System with Quantum Metric

w/o mag. field

Ren (2025) Yoshida, Yokoyama (2025)

$$L = \boldsymbol{p} \cdot \dot{\boldsymbol{x}} - \hbar \boldsymbol{a} \cdot \dot{\boldsymbol{p}} - \epsilon_{\boldsymbol{p}} - \Phi - \frac{\hbar^2}{2} \dot{p}_i \dot{p}_j G_{ij}$$

universal form up to $\boldsymbol{a}$ and G_{ij}

KM, Yamamoto (2025)

We have derived it from the path-integral formalism

Stephanov, Yin (2012)

Dynamical System with Quantum Metric

w/o mag. field

Ren (2025) Yoshida, Yokoyama (2025)

$$L = \underbrace{\boldsymbol{p}\cdot\dot{\boldsymbol{x}} - \hbar\boldsymbol{a}\cdot\dot{\boldsymbol{p}}}_{\pi_a(\xi)\dot{\xi}^a} \underbrace{- \epsilon_{\boldsymbol{p}} - \Phi}_{-H} - \frac{\hbar^2}{2}\dot{p}_i\dot{p}_j G_{ij}$$

canonical eq. $\dot{\xi}^a = \{H, \xi^a\} = -\{\xi^a, \xi^b\}\dfrac{\partial H}{\partial \xi^b}$

matching with $\delta \int L\,\mathrm{d}t = 0$

Son, Yamamoto (2012)

Dynamical System with Quantum Metric

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$$L = \boldsymbol{p}\cdot\dot{\boldsymbol{x}} - \hbar\boldsymbol{a}\cdot\dot{\boldsymbol{p}} - \epsilon_{\boldsymbol{p}} - \Phi - \frac{\hbar^2}{2}\dot{p}_i\dot{p}_j G_{ij} \sim \dot{\xi}^a\dot{\xi}^b$$

$\pi_a(\xi)\dot{\xi}^a$ $-H$ different form

canonical eq. $\dot{\xi}^a = \{H, \xi^a\} = -\{\xi^a, \xi^b\}\dfrac{\partial H}{\partial \xi^b}$

matching with $\delta\int L\,\mathrm{d}t = 0$

Son, Yamamoto (2012)

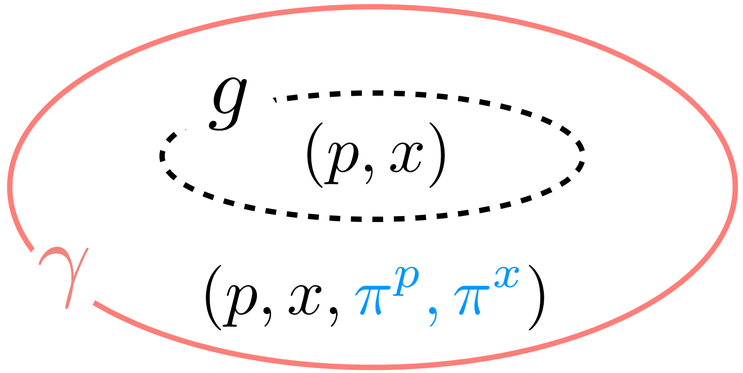
Dynamical Degrees of Freedom

Lagrangian

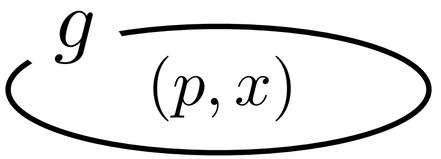
$$L(p, x, \dot{p}, \dot{x})$$

Hamiltonian

$$H(p, x, \pi^p, \pi^x)$$



$$\neq$$



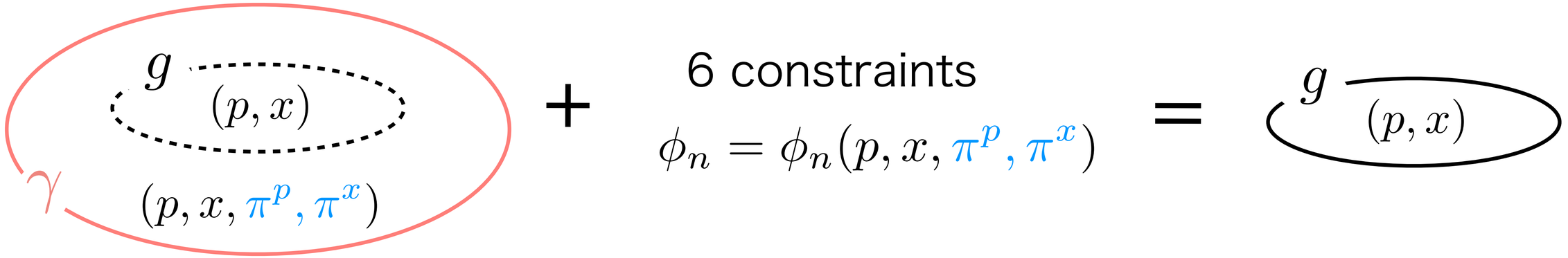
Dynamical Degrees of Freedom

Lagrangian

$$L(p, x, \dot{p}, \dot{x})$$

Hamiltonian

$$H(p, x, \pi^p, \pi^x)$$



Dirac (1950) Bergmann (1949)

"canonical formalism in constrained systems" with using Dirac bracket

$$\{X, Y\}_{\mathrm{D}} = \{X, Y\} - \{X, \phi_n\}\{\phi_n, \phi_m\}^{-1}\{\phi_m, Y\}$$

Quantum Metric Corrections

KM, Yamamoto (2025)

modified Poisson bracket (w/o mag. field)

$$\{p_i, p_j\}_{\mathrm{D}} = 0$$

$$\{x_i, x_j\}_{\mathrm{D}} = \hbar\epsilon_{ijk} b_k - \hbar^2(\partial_i^p G_{jk} - \partial_j^p G_{ik}) E_k$$

$$\{p_i, x_j\}_{\mathrm{D}} = \delta_{ij} - \hbar^2 G_{ik}\partial_j E_k$$

phase space factor

$$\sqrt{\omega} = \det\{\xi^a, \xi^b\}_{\mathrm{D}}^{-1} = 1 + \hbar^2 G_{ij}\partial_i E_j$$

QM-Corrected Kinetic Equation

modified Liouville's theorem

KM, Yamamoto (2025)

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \partial_t \rho + \{H, \rho\}_{\mathrm{D}} = 0 \qquad \rho = \sqrt{\omega} f$$

w/o mag. field and collisions

$$\left[\partial_t + \dot{\boldsymbol{x}} \cdot \boldsymbol{\nabla} + \dot{\boldsymbol{p}} \cdot \boldsymbol{\nabla}_{\boldsymbol{p}}\right] f = 0$$

$$\dot{\boldsymbol{x}} = \frac{\partial \epsilon'_{\boldsymbol{p}}}{\partial \boldsymbol{p}} + \hbar \boldsymbol{E} \times \boldsymbol{b}' + \hbar^2 \boldsymbol{V}$$

$$\dot{\boldsymbol{p}} = \boldsymbol{E}$$

$$\epsilon'_{\boldsymbol{p}} = \epsilon_{\boldsymbol{p}} - \frac{\hbar^2}{2} G_{ij} E_i E_j \qquad b'_i = \epsilon_{ijk} \partial^p_j (a_k + \hbar G_{kl} E_l) \qquad V_i = -G_{ij} \frac{\partial \epsilon_{\boldsymbol{p}}}{\partial p_k} \frac{\partial E_j}{\partial x_k}$$

valid for chiral fermions: we have derived the $\mathcal{O}(\hbar^2)$ chiral kinetic eq.

Conserved Charge/Energy Currents

KM, Yamamoto (2025)

$$(n, \boldsymbol{j}) = \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}}) \sqrt{\omega} f \qquad (\varepsilon, \boldsymbol{J}) = \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}}) \left[\epsilon_{\boldsymbol{p}} - \frac{\hbar^2}{2} E_i E_j G_{ij} \right] \sqrt{\omega} f$$

$$\partial_t n + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0 \qquad \partial_t \varepsilon + \boldsymbol{\nabla} \cdot \boldsymbol{J} = \boldsymbol{E} \cdot \boldsymbol{j}$$

ex.) nonlinear AHE is correctly reproduced

$$j_i = \hbar^2 E_j E_k \int_{\boldsymbol{p}} (\partial_i^p G_{jk} - \partial_j^p G_{ik}) f$$ Gao, Yang, Niu (2014)

first derivation meeting the conservation law

→ resolution of the tensions between many literatures

Application to Chiral/Dirac Fermions

KM, Yamamoto (2025)

$$\text{chiral fermion}: G_{ij} = \frac{1}{4|\boldsymbol{p}|^3}(\delta_{ij} - \hat{p}_i \hat{p}_j)$$

- equilibrium current: consistent with QFT

Yang, Gao, Liang, Wang (2020)
Mameda (2023) Yang, Gao, Pu (2025)

$$n^{(EE)} = -\frac{\hbar^2 C}{24\pi^2}|\boldsymbol{E}|^2 \qquad \boldsymbol{j}^{(EE)} = \frac{\hbar^2 C}{12\pi^2}\left[-\boldsymbol{E}(\boldsymbol{E}\cdot\boldsymbol{u}) + |\boldsymbol{E}|^2\boldsymbol{u}\right] \qquad \varepsilon^{(\partial E)} = \frac{\hbar^2 \mu}{12\pi^2}\boldsymbol{\nabla}\cdot\boldsymbol{E}$$

- straightforward application to Dirac (massive) fermions:
  QM does not require chirality imbalance, unlike Berry curvature

Summary

$$(n, \boldsymbol{j}) = \int_{\boldsymbol{p}} (1, \dot{\boldsymbol{x}}) \sqrt{\omega} f \quad \Rightarrow \quad \partial_t n + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$$

$\sqrt{\omega} = 1$ (classical)

$\sqrt{\omega} = 1 + \hbar \boldsymbol{b} \cdot \boldsymbol{B}$ (Berry curvature) Xiao, Shi, Niu (2005)

$\sqrt{\omega} = 1 + \hbar^2 G_{ij} \partial_i E_j$ (quantum metric) KM, Yamamoto (2025)

- Derived quantum metric correction to quantum kinetic theory
- Consistent with QFT → Validity of Dirac-bracket approach
- Lorentz invariance from magnetic-field contribution? Hayata, Hidaka, KM (2020) KM (2023)

Dynamical System with Quantum Metric

$$L = \boldsymbol{p} \cdot \dot{\boldsymbol{x}} - \hbar \boldsymbol{a} \cdot \dot{\boldsymbol{p}} - \epsilon_{\boldsymbol{p}} - \Phi - \frac{\hbar^2}{2} \dot{p}_i \dot{p}_j G_{ij}$$

Sundaram, Niu (1999)

Dirac (1950)
Bergmann (1949)

manner that cannot be expressed explicitly in general. This clearly shows the difficulty of the Hamiltonian formalism in the presence of Berry-phase terms. If one insists on using the Hamiltonian approach, one can employ the method of Lagrange multipliers, which allows the spurious degrees of freedom to be formally treated as independent,[32] and obtain the equations of motion (2.19). The ''requantization'' proce-

[32] P. A. M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science (Yeshiva University, New York 1964); Can. J. Math. **2**, 129 (1950).