

Imaginary Rotation Breaks Charge Conjugation in Hot QCD

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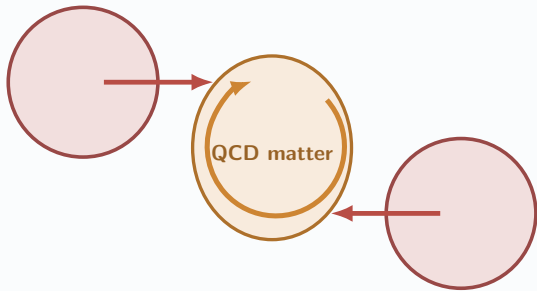
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Motivation for rotating QCD

In non-central heavy-ion collisions, hot QCD matter carries a large angular momentum.

The global polarization of Λ and $\bar{\Lambda}$ hyperons reveals an enormous vorticity.



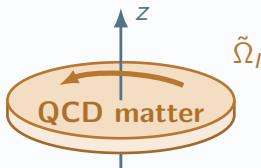
Characteristic
vorticity

$$\omega \sim 10^{22} \text{ s}^{-1}$$

Why imaginary rotation?

Real rotation gives a complex Euclidean path-integral weight. This sign problem obstructs standard lattice simulations.

Imaginary rotation avoids this problem. Here I study it as a theoretical setting in its own right.



Yamamoto-Hirono, PRL **111**, 081601 (2013); Chen-Fukushima-Shimada, PRL **129**, 242002 (2022).

A symmetry puzzle

Charge conjugation C exchanges quarks and antiquarks. Rotation couples to angular momentum, which is C even. At zero quark chemical potential, the theory still has C symmetry.

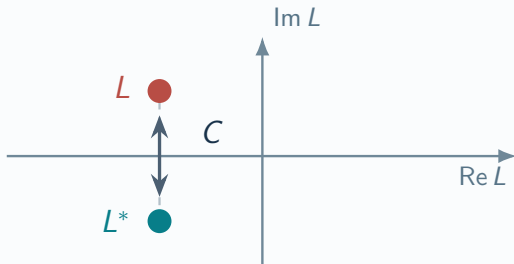


But must the equilibrium state respect C ?

The unexpected answer

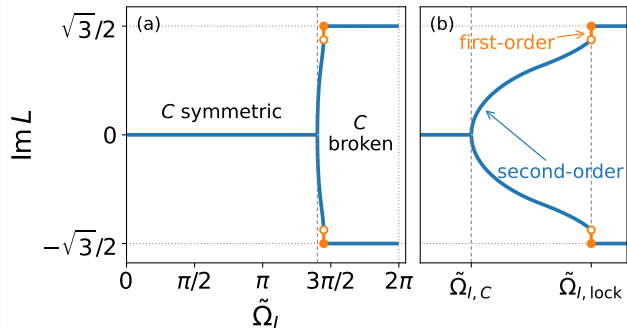
No. Imaginary rotation can spontaneously break C , even at zero quark chemical potential.

The Polyakov loop L can take two complex-conjugate values. These two equilibrium states are exchanged by $C : L \mapsto L^*$.



C breaks continuously

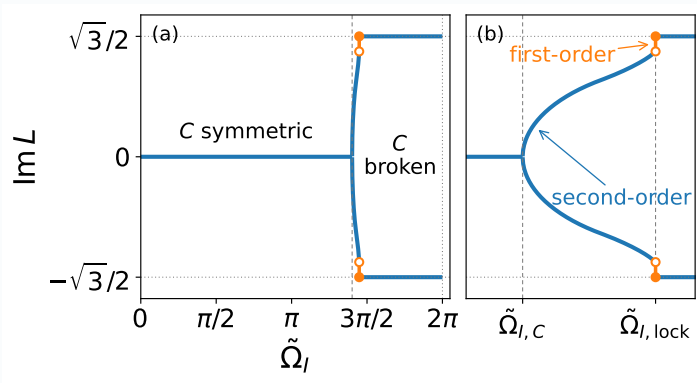
On the rotation axis, I minimize the one-loop potential. Panel (a) shows two C -conjugate branches of $\text{Im } L$ emerging continuously at a second-order transition.



R. Takada, arXiv:2608.25849 (2026), Fig. 3; 2 + 1 flavors, $m_s/T = 0.30$, on-axis one-loop.

First-order holonomy lock-in

Panel (b) shows a first-order jump within the C -broken phase. The preferred holonomy locks to a nontrivial center element.



R. Takada, arXiv:2608.25849 (2026), Fig. 3.

Two transition points

For three massless flavors, both transition points can be obtained analytically. C breaking begins before holonomy lock-in.

Second-order onset of C breaking

$$\frac{\tilde{\Omega}_{I,C}}{\pi} = \frac{22 - 2\sqrt{22}}{9} \simeq 1.4021$$

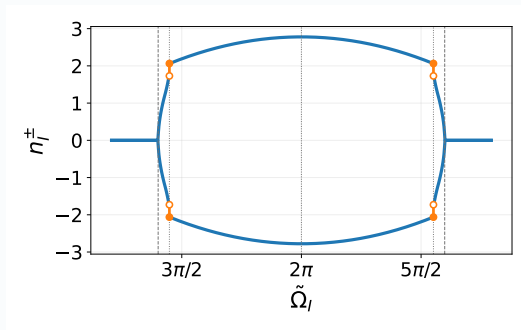
First-order holonomy lock-in

$$\frac{\tilde{\Omega}_{I,\text{lock}}}{\pi} = \frac{2}{3} + \frac{2\sqrt{111}}{27} \simeq 1.4471$$

The plotted 2 + 1-flavor benchmark instead gives 1.4013 and 1.4483 at $m_s/T = 0.30$.

A second signature: imaginary quark number

At zero quark chemical potential, two C -conjugate phases have opposite imaginary quark-number densities. This is an independent thermodynamic signal of broken C from a conserved charge.



R. Takada, arXiv:2608.25849 (2026), Fig. 4; $m_s/T = 0.30$, on-axis one-loop.

What can we say beyond one loop?

The finite-angle transitions are one-loop predictions. But a full 2π turn yields an exact identity of QCD.

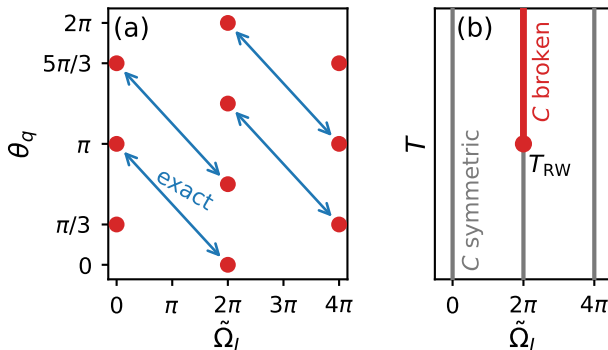
In QCD, a full 2π rotation equals fermion parity, and hence quark-number parity:

$$e^{2\pi i \hat{J}_z} = (-1)^{\hat{F}} = e^{i\pi \hat{Q}}$$

This identity holds on the physical state space, without relying on perturbation theory.

The RW point: an exact mapping

At $\tilde{\Omega}_I = 2\pi$ and $\theta_q = 0$, QCD is exactly at the RW point $\theta_q = \pi$. In (a), blue arrows link identical QCD ensembles; (b) shows the RW endpoint.



What is exact, and what is predicted?

The exact correspondence fixes the RW point; the finite-angle transitions are separate one-loop predictions.

Exact RW anchor

At $\tilde{\Omega}_I = 2\pi$: exact at any T and position. Above T_{RW} , C breaks.

One-loop window

At high T on the axis, one loop predicts a continuous C onset and first-order lock-in.

The RW identity alone fixes neither finite-angle transition.

What can lattice QCD test?

Do the predicted continuous C onset and first-order holonomy lock-in survive beyond one loop?

Track the C -odd observables $\text{Im } L$ and n_I as the imaginary angular velocity changes.

Dynamical quarks are essential.

The predicted C -breaking onset is driven by quark contributions absent in quenched QCD.

Summary

1. Imaginary rotation alone can spontaneously break C , even at zero quark chemical potential.
2. On the axis, one-loop theory predicts a continuous C onset and a separate first-order lock-in.
3. The RW correspondence gives an exact anchor at $\tilde{\Omega}_I = 2\pi$, but does not fix the finite-angle window.

Outlook

The exact RW mapping and the on-axis one-loop results raise new questions for QCD.

- ▶ Can lattice simulations with dynamical quarks resolve both transitions?
- ▶ How far from the axis does C breaking extend?
- ▶ What is the critical behavior near the RW endpoint?
- ▶ Does first-order lock-in survive beyond one loop?

Thank you for your attention.

Q&A index: exact mapping and one-loop setup

Topics follow the paper; click a B-number to jump.

Symmetry and exact relations

- [B01](#) Parameters and thermal trace
- [B02](#) Charge conjugation symmetry
- [B03](#) Spin–statistics identity
- [B04](#) Mixed and angular periodicity
- [B05](#) Why this is an RW point
- [B06](#) Polyakov loop and static probe
- [B07](#) Mirror relation about 2π
- [B08](#) No exact finite-width window
- [B09](#) Selecting a broken phase

Scope and one-loop ingredients

- [B10](#) Exact versus one-loop scope
- [B11](#) Rotating Euclidean geometry
- [B12](#) Tetrad and spin connection
- [B13](#) Angular-momentum modes on axis
- [B14](#) Weyl alcove and C reflection
- [B15](#) Bernoulli/Weiss building block
- [B16](#) Gluon and FP-ghost counting
- [B17](#) Massive-quark potential
- [B18](#) Half-angle and exact check

[Next: phase structure and tests](#)

[Back to summary](#)

Q&A index: phase structure and tests

Next in the paper: minima, transitions, densities, tests, and outlook.

Minima and phase transitions

B19 Minimum motion (original Fig. 2)

B20 Global versus edge minima

B21 Second-order C onset

B22 First-order holonomy lock-in

B23 Strange-quark mass dependence

B24 Imaginary quark-number density

B25 Massless accidental degeneracy

Comparison, tests and outlook

B26 Center symmetry versus C

B27 Concrete lattice observables

B28 Temperature, radius, universality

B29 Continuation to real rotation

B30 Selected references

B31 Source material and checks

[Previous: exact mapping and setup](#)

[Back to summary](#)

B01 What do the parameters mean?

The two imaginary parameters weight two conserved charges.

$$\begin{aligned}\Omega &= i\Omega_I, & \tilde{\Omega}_I &= \beta\Omega_I, & \beta &= 1/T, \\ \mu_q &= iT\theta_q, & t &= \tilde{\Omega}_I/\pi, & \tilde{r} &= rT, \\ \hat{\rho}(\tilde{\Omega}_I, \theta_q) &= e^{-\beta\hat{H}} e^{i\tilde{\Omega}_I\hat{J}_z} e^{i\theta_q\hat{Q}}, & Z &= \text{tr } \hat{\rho}.\end{aligned}$$

The trace is over the color-singlet physical state space satisfying the Gauss law. The paper uses an unnormalized statistical-weight operator. The high-temperature phase scan sets the quark chemical potential to zero.

$\hat{H}$: Hamiltonian; $\hat{J}_z$: total angular momentum; $\hat{Q}$: net quark number.
 $[\hat{H}, \hat{J}_z] = [\hat{H}, \hat{Q}] = 0$. The parameter t is an angle ratio, not a time.

R. Takada, RESCEU-22/26 (2026), Eq. (1) and definitions preceding Eq. (14).

B02 Why does imaginary rotation preserve C ?

Quark number is C odd. Energy and angular momentum are C even.

$$\hat{C}\hat{H}\hat{C}^{-1} = \hat{H}, \quad \hat{C}\hat{J}_z\hat{C}^{-1} = \hat{J}_z, \quad \hat{C}\hat{Q}\hat{C}^{-1} = -\hat{Q},$$

$$\hat{C}\hat{\rho}(\tilde{\Omega}_I, \theta_q)\hat{C}^{-1} = \hat{\rho}(\tilde{\Omega}_I, -\theta_q).$$

At zero quark chemical potential, the statistical-weight operator is invariant. Spontaneous breaking means that this symmetry exchanges two equilibrium states rather than leaving each state invariant.

$$C : L \mapsto L^*, \quad C : \text{Im } L \mapsto -\text{Im } L, \quad C : n_I \mapsto -n_I.$$

R. Takada, RESCEU-22/26 (2026), introduction and the C transformations before Eq. (7).

B03 Why is a 2π rotation a quark-number twist?

A full rotation gives fermion parity, which equals quark-number parity.

$$\begin{aligned} e^{2\pi i \hat{J}_z} &= (-1)^{\hat{F}}, \\ \hat{F} &= \hat{N}_q + \hat{N}_{\bar{q}} \equiv \hat{N}_q - \hat{N}_{\bar{q}} = \hat{Q} \pmod{2}, \\ \therefore e^{2\pi i \hat{J}_z} &= e^{i\pi \hat{Q}}. \end{aligned}$$

The relation is an operator identity on the physical state space. It uses neither a weak-coupling expansion nor a model for the effective potential. Quark and antiquark numbers need not be separately conserved; their parity relation is the relevant statement.

R. Takada, RESCEU-22/26 (2026), Eq. (2); Dumitrescu-Hsin, SciPost Phys. **17**, 093 (2024).

B04 Where do the angular periodicities come from?

Two full angular shifts restore the original quark-number twist.

$$\begin{aligned}\hat{\rho}(\tilde{\Omega}_I + 2\pi, \theta_q) &= \hat{\rho}(\tilde{\Omega}_I, \theta_q + \pi), \\ Z(\tilde{\Omega}_I + 4\pi, \theta_q) &= Z(\tilde{\Omega}_I, \theta_q).\end{aligned}$$

On color-singlet states, the net quark number is the number of colors times the integer baryon number. The parity of the number of colors then matters.

$$\hat{Q} = N_c \hat{B}, \quad e^{i\pi \hat{Q}} = \begin{cases} 1, & N_c \text{ even,} \\ (-1)^{\hat{B}}, & N_c \text{ odd.} \end{cases}$$

For $SU(3)$, a 2π angular shift is not generally a period at fixed θ_q .

The RW period $2\pi/3$ in θ_q is a separate relation for the partition function.

R. Takada, RESCEU-22/26 (2026), Eqs. (3), (4), (5).

B05 Why is the corresponding point an RW point?

The point at π belongs to the RW sequence and has ordinary C symmetry.

$$\begin{aligned}\hat{\rho}(2\pi, 0) &= \hat{\rho}(0, \pi), \\ Z(0, \theta_q) &= Z(0, \theta_q + 2\pi/3), \\ \theta_q^{\text{RW}} &= (2k + 1)\pi/3, \quad k \in \mathbb{Z}.\end{aligned}$$

Choosing the representative at π makes the symmetry transparent: changing the sign of the quark-number twist gives the same operator because quark number is integer.

$$e^{-i\pi\hat{Q}} = e^{i\pi\hat{Q}}, \quad \hat{C}\hat{\rho}(0, \pi)\hat{C}^{-1} = \hat{\rho}(0, \pi).$$

For physical $2 + 1$ -flavor QCD, Bonati et al. find $T_{\text{RW}} = 208(5) \text{ MeV}$ and evidence for 3D Ising criticality.

P4, Eq. (6); Roberge-Weiss, NPB **275**, 734 (1986); Bonati et al., PRD **93**, 074504 (2016).

B06 Why does the Polyakov loop map as well?

The same spinless on-axis probe obeys the mapping.

$$P(\mathbf{x}) = \mathcal{P} \exp \left(ig \int_0^\beta d\tau A_4(\tau, \mathbf{x}) \right),$$
$$L(\mathbf{x}) = \frac{1}{3} \text{tr} P(\mathbf{x}).$$

The Wilson line carries no spin. The paper extends the same parity identity to the static-source sector, giving the observable correspondence.

$$L|_{(2\pi,0)} = L|_{(0,\pi)}, \quad n_I|_{(2\pi,0)} = n_I|_{(0,\pi)}.$$

Use corresponding phases and the same probe normalization. One-loop curve heights retain their perturbative status.

R. Takada, RESCEU-22/26 (2026), discussion of the static fundamental source and Eq. (7).

B07 Why is the phase window mirrored about 2π ?

Spatial rotation reverses the angular momentum; periodicity fixes the reflection centers.

$$\hat{U} \hat{J}_z \hat{U}^{-1} = -\hat{J}_z,$$

$$[\hat{U}, \hat{H}] = [\hat{U}, \hat{Q}] = 0,$$

$$\hat{\rho}(2\pi n - \delta, 0) = \hat{U} \hat{\rho}(2\pi n + \delta, 0) \hat{U}^{-1}.$$

Here the spatial rotation is by π about an axis perpendicular to the rotation axis. The on-axis potential therefore has the same phase transitions at mirror-related angular velocities.

$$\begin{aligned}\tilde{\Omega}_{I,C} &\longleftrightarrow 4\pi - \tilde{\Omega}_{I,C}, \\ \tilde{\Omega}_{I,\text{lock}} &\longleftrightarrow 4\pi - \tilde{\Omega}_{I,\text{lock}}.\end{aligned}$$

R. Takada, RESCEU-22/26 (2026), Eq. (8) and the text following Eq. (18).

B08 Why does the exact point not prove a finite window?

A small angle is not uniformly perturbative at arbitrarily large radius.

$$\hat{\rho}(2\pi + \delta, 0) = \hat{\rho}(2\pi, 0) e^{i\delta \hat{J}_z},$$
$$|J_z| \sim rT, \quad |\delta|RT \lesssim 1.$$

Thermal modes far from the axis carry larger orbital angular momentum. The region in which an expansion in the angle is uniform shrinks with the system radius.

$$R \rightarrow \infty \quad \implies \quad \text{uniform expansion window} \rightarrow 0.$$

The finite window on the axis is an independent result of the effective-potential calculation.

R. Takada, RESCEU-22/26 (2026), text following Eq. (8); Chernodub (2022), Pătureanu et al. (2025).

B09 A symmetric ensemble versus a broken phase

A symmetric mixture averages the two broken phases together.

In a finite volume with symmetry-preserving conditions, conjugate states contribute symmetrically. The signed order parameter averages to zero.

$$\langle O_C \rangle_V = 0, \quad C : O_C \mapsto -O_C.$$

O_C can be $\text{Im } L$ or n_I ; h denotes a source coupled to it.

The plotted branches are values in the two selected phases. On a finite lattice one examines distributions and finite-size behavior.

A broken phase is selected before the symmetry-breaking field is removed. The two limits produce opposite signs.

$$\langle O_C \rangle_{\pm} = \lim_{h \rightarrow 0^{\pm}} \lim_{V \rightarrow \infty} \langle O_C \rangle_{V,h}.$$

P4: two conjugate equilibrium states. Phase-selection notation spells out spontaneous breaking; see also the RW finite-size analysis in Bonati et al. (2016).

B10 Which conclusions are exact?

Exact correspondence

At the special angle, the rotating and nonrotating statistical-weight operators coincide. The correspondence holds at every temperature and throughout space.

$$\hat{\rho}(2\pi, 0) = \hat{\rho}(0, \pi).$$

It transfers the known RW phase structure: above its endpoint, charge conjugation is broken.

On-axis one-loop predictions

The finite angular window, the two transition locations, and the displayed values of the order parameters come from minimizing the high-temperature potential on the axis.

$$\tilde{\Omega}_I/\pi \simeq 1.4021,$$

$$\tilde{\Omega}_I/\pi \simeq 1.4471.$$

These numerical locations are the massless $N_f = 3$ results.

The identity fixes a line; the one-loop calculation supplies a window.

R. Takada, RESCEU-22/26 (2026), exact correspondence, Eqs. (6), (16), (18), and conclusion.

B11 How does imaginary rotation enter the metric?

The thermal twist can be encoded in a rotating Euclidean metric.

$$\begin{aligned}\varphi_{\text{lab}} &= \varphi + \Omega_I \tau, \\ ds_E^2 &= d\tau^2 + dr^2 + r^2(d\varphi + \Omega_I d\tau)^2 + dz^2, \\ \sqrt{g} &= r.\end{aligned}$$

The scalar kinetic operator then contains the derivative along the Euclidean-time direction of the rotating frame.

$$-D_s^2 = -(D_\tau - \Omega_I \partial_\varphi)^2 - \frac{1}{r} \partial_r(r \partial_r) - \frac{1}{r^2} \partial_\varphi^2 - \partial_z^2.$$

D_τ : temporal covariant derivative with the holonomy background. The angular sign follows the stated coordinate map.

Additional derivation from the rotating-coordinate setup; Chen-Fukushima-Shimada, PLB **859**, 139107 (2024), Eqs. (3) to (6).

B12 Where does the fermion spin-rotation term come from?

A rotating orthonormal frame contributes to the spinor connection.

$$\begin{aligned} e^{\hat{\tau}} &= d\tau, & e^{\hat{r}} &= dr, & e^{\hat{\varphi}} &= r(d\varphi + \Omega_I d\tau), & e^{\hat{z}} &= dz, \\ e_{\hat{\tau}} &= \partial_{\tau} - \Omega_I \partial_{\varphi}, & e_{\hat{\varphi}} &= r^{-1} \partial_{\varphi}, \\ \omega^{\hat{\varphi}}_{\hat{r}} &= d\varphi + \Omega_I d\tau, & \nabla_{\mu} &= \partial_{\mu} + \frac{1}{4} \omega_{\hat{a}\hat{b}\mu} \gamma^{\hat{a}} \gamma^{\hat{b}}. \end{aligned}$$

The connection follows from the torsion-free Cartan equation. Together with the orbital derivative, it makes the thermal twist act on total angular momentum, whose fermionic eigenvalues are half-integer.

$\{\gamma^{\hat{a}}, \gamma^{\hat{b}}\} = 2\delta^{\hat{a}\hat{b}}$ in the Euclidean orthonormal frame. The coframe is a local cylindrical choice; its connection also contains the usual polar-coordinate term.

Coordinate-frame derivation using the metric on B11; rotating-spinor construction: Yamamoto-Hirono, PRL **111**, 081601 (2013); CFS, PLB **859**, 139107 (2024).

B13 Why do special angular-momentum modes survive on the axis?

The Bessel functions select the modes that remain nonzero at the origin.

$$J_m(0) = \delta_{m0},$$

$$\text{gluon weight: } J_{\ell-1}^2(k_\perp r) + J_{\ell+1}^2(k_\perp r) \implies \ell = \pm 1,$$

$$\text{quark weight: } J_\ell^2(k_\perp r) + J_{\ell+1}^2(k_\perp r) \implies \ell = 0, -1.$$

The surviving gluonic modes carry spin-one phases. For the quark modes, total angular momentum is the orbital label plus one half.

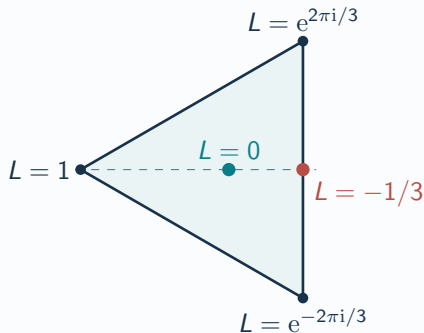
$$j_z = \ell + \frac{1}{2} = \pm \frac{1}{2}, \quad e^{i\tilde{\Omega}_I j_z} = e^{\pm i\tilde{\Omega}_I/2}.$$

P4, text preceding Eqs. (11), (12); mode weights from CFS, PLB **859**, 139107 (2024), Eqs. (7), (11).

B14 What exactly is the Weyl alcove?

It parametrizes inequivalent thermal holonomies.

$$\phi_1 + \phi_2 + \phi_3 = 0, \quad \phi_1 \geq \phi_2 \geq \phi_3, \quad \phi_1 - \phi_3 \leq 2\pi.$$



The vertices are center elements. At the two marked points,

barycenter: $P = \text{diag}(1, e^{2\pi i/3}, e^{-2\pi i/3})$,

edge midpoint: $P = \text{diag}(-1, 1, -1)$.

C reflects the triangle across the dashed line.

R. Takada, RESCEU-22/26 (2026), Eqs. (9), (14), and Fig. 2.

B15 How does the Bernoulli polynomial appear?

Expand the thermal logarithm, integrate the radial momentum, then sum the winding harmonics.

$$\begin{aligned}f_B(\varphi) &= T \int \frac{d^3k}{(2\pi)^3} \operatorname{Re} \ln(1 - e^{-k/T + i\varphi}), \\&= -\frac{T^4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(n\varphi)}{n^4}, \\&= \frac{\pi^2 T^4}{3} B_4\left(\left\{\frac{\varphi}{2\pi}\right\}\right), \\B_4(x) &= x^2(1-x)^2 - \frac{1}{30}, \quad f_B(0) = -\frac{\pi^2 T^4}{90}.\end{aligned}$$

Here $\{x\}$ is the fractional part. The result is per real bosonic degree of freedom; the zero-temperature part has been omitted.

P4, Eq. (10); Gross-Pisarski-Yaffe, RMP **53**, 43 (1981); Weiss, PRD **24**, 475 (1981).

B16 How are the gluon and FP-ghost degrees counted?

Two transverse polarizations survive FP-ghost cancellation.

$$\Gamma_g^{(1)} = \frac{1}{2} \text{Tr} \ln \Delta_1 - \text{Tr} \ln \Delta_0,$$

$$V_g = 4f_B(\tilde{\Omega}_I) + 2 \sum_{1 \leq a < b \leq 3} \sum_{s=\pm 1} f_B(\phi_a - \phi_b + s\tilde{\Omega}_I).$$

Two diagonal color generators give four degrees of freedom. Three off-diagonal color pairs give twelve: two conjugate charges times two polarizations per pair.

$$\tilde{\Omega}_I = 0, \quad P = \mathbf{1}_3 : \quad V_g = 16f_B(0) = -\frac{8\pi^2 T^4}{45}.$$

Δ_1 : gauge-fixed vector operator; Δ_0 : FP-ghost operator.

Tr: functional trace; tr: ordinary trace.

P4, Eq. (11); FP cancellation and rotating modes: CFS (2022, 2024).

B17 Where does the massive-quark factor come from?

The massive radial integral gives K_2 for each winding harmonic.

$$\int \frac{d^3k}{(2\pi)^3} e^{-n\sqrt{k^2+m_f^2}/T} = \frac{m_f^2 T}{2\pi^2 n} K_2(nm_f/T),$$
$$\frac{V_f}{T^4} = -\frac{2y_f^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} K_2(ny_f)}{n^2} \cos\left(\frac{n\tilde{\Omega}_I}{2}\right) \\ \times \sum_{a=1}^3 \cos[n(\phi_a + \theta_q)], \quad y_f = m_f/T.$$

The alternating sign comes from Fermi statistics; the half-angle comes from the two on-axis spin modes.

Massless check: $y_f^2 K_2(ny_f) \rightarrow 2/n^2$ and $V_f(0,0)/T^4 = -7N_c\pi^2/180$ at trivial holonomy.

R. Takada, RESCEU-22/26 (2026), Eq. (12), one Dirac flavor.

B18 Why does the one-loop result reproduce the exact identity?

Every quark winding harmonic transforms in the required way.

$$\cos\left[\frac{n(\tilde{\Omega}_I + 2\pi)}{2}\right] = (-1)^n \cos\left(\frac{n\tilde{\Omega}_I}{2}\right),$$
$$\cos[n(\phi_a + \theta_q + \pi)] = (-1)^n \cos[n(\phi_a + \theta_q)].$$

A full angular shift and a quark-number twist therefore have the same effect term by term. The gluon potential is already periodic under a full angular turn.

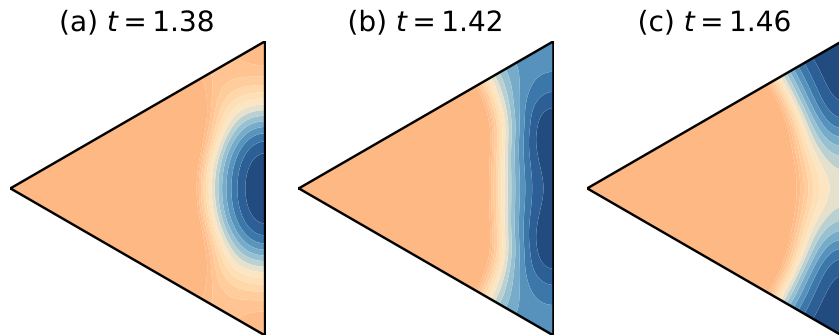
$$V(\tilde{\Omega}_I + 2\pi, \theta_q) = V(\tilde{\Omega}_I, \theta_q + \pi).$$

This is a consistency check on the one-loop expressions. The exact relation itself follows from the operator identity on B03.

R. Takada, RESCEU-22/26 (2026), paragraph following Eq. (13).

B19 How the equilibrium minimum moves

Two conjugate minima appear before the first-order jump to center elements.



Original P4 Fig. 2: three massless flavors; $t = \tilde{\Omega}_I/\pi$. Darker regions have lower potential.

B20 How do you know the edge minima are global?

The edge analysis follows a search over the full two-dimensional alcove.

Masked grid over the full alcove



Local minimization
from multiple seeds



All three edges and
all three vertices

Compare potential values and fold candidates into the same alcove. Repeat with a finer grid.

The analytic edge polynomials then locate the observed changes of the global minimum.

The code `alcove_global_scan.py` implements this numerical global-minimum check. The analytic derivation then uses the relevant edge.

R. Takada, RESCEU-22/26 (2026), global minimization and the numerical implementation.

B21 How is the continuous onset derived?

The symmetric point becomes unstable while the quartic term stays positive.

$$\begin{aligned}(\phi_1, \phi_2, \phi_3) &= (\pi u, \pi(2 - 2u), \pi(u - 2)), & x &= 1 - u, \\ \frac{V(x) - V(0)}{\pi^2 T^4} &= a(t)x^2 + 9x^4, & a(t) &= \frac{27t^2 - 132t + 132}{4}, \\ \partial_x V = 0 &\implies x = 0 \text{ or } x^2 = -a(t)/18, \\ a(t_C) = 0 &\implies t_C = \frac{22 - 2\sqrt{22}}{9}.\end{aligned}$$

Here $t = \tilde{\Omega}_I/\pi$ and $C : x \mapsto -x$. The polynomial applies within its Bernoulli patch: $|x| \leq \min(6 - 3t, 4t - 4)/12$. The full potential is used outside this patch.

The full-alcove search confirms that these new minima are global near onset. The order parameter then grows with the square root of the distance from the transition.

R. Takada, RESCEU-22/26 (2026), Eqs. (14) to (16).

B22 How is the first-order lock-in point derived?

Stationarity and equal free energies fix the discontinuous jump.

$$y = \frac{1}{3} - x,$$
$$\frac{V(x) - V(1/3)}{\pi^2 T^4} = b(t)y^2 - 2y^3 + 9y^4, \quad b(t) = \frac{27t^2 - 36t - 4}{4}.$$

For the nonzero minimum, impose both conditions:

$$\text{equal free energies: } b - 2y + 9y^2 = 0,$$

$$\partial_y V = 0 : \quad b - 3y + 18y^2 = 0,$$

$$\therefore \quad y = \frac{1}{9}, \quad b = \frac{1}{9} \quad \implies \quad t_{\text{lock}} = \frac{2}{3} + \frac{2\sqrt{111}}{27}.$$

The global minimum jumps from $x = 2/9$ to $x = 1/3$; $\text{Im } L$ jumps from $0.75679\dots$ to $\sqrt{3}/2$. The full-alcove comparison is on B20.

R. Takada, RESCEU-22/26 (2026), Eqs. (17), (18), within the stated center-element polynomial patch.

B23 How sensitive are the transitions to quark masses?

The strange-quark mass gives small shifts at the benchmark studied in the paper.

	$\tilde{\Omega}_{I,C}/\pi$	$\tilde{\Omega}_{I,\text{lock}}/\pi$
Three massless flavors	1.4021...	1.4471...
$m_u = m_d = 0, m_s/T = 0.30$	1.4013...	1.4483...

Both shifts are below one tenth of a percent at this benchmark. Over the range of strange-quark mass ratios studied in the paper, the largest quoted shifts remain small.

$$0 \leq m_s/T \leq 0.5 : \quad \text{onset shift} \leq 0.15\%, \quad \text{lock-in shift} \leq 0.21\%.$$

These are variations of the on-axis one-loop prediction, not an estimate of all higher-loop or nonperturbative corrections.

R. Takada, RESCEU-22/26 (2026), finite-quark-mass analysis after Eq. (18).

B24 How is the imaginary density obtained?

Differentiate the free energy of each selected phase.

$$\mu_q = iT\theta_q, \quad n_I = -\frac{in_q}{T^3},$$
$$n_I^\pm = \left. \frac{\partial}{\partial \theta_q} \frac{V(x, \theta_q)}{T^4} \right|_{\theta_q=0, x=\pm x_{\min}},$$

$$\tilde{\Omega}_I = 2\pi, \quad m_f = 0 : \quad L = e^{\pm 2\pi i/3}, \quad n_I^\pm = \mp \frac{8\pi N_f}{27}.$$

The displacement of the minimum does not contribute to the first derivative because the potential is stationary there. The two densities have opposite signs, as required by charge conjugation.

This is an equilibrium response in an imaginary-parameter ensemble. It is not a claim of real baryon production or of violation of quark-number conservation.

R. Takada, RESCEU-22/26 (2026), Eqs. (19), (20); branchwise derivative of the stated potential.

B25 What happens in the massless low-angle window?

An accidental circle of minima leaves the realization of C undetermined.

$$N_f = 3 : \quad \frac{2\pi}{3} < \tilde{\Omega}_I < \frac{12 - 2\sqrt{3}}{11}\pi.$$

The degenerate minima form a circle containing both charge-conjugation-symmetric and charge-conjugation-breaking configurations. At the finite strange-quark mass benchmark, this degeneracy is lifted and the minimum below the high-angle onset is symmetric.

This is why the full angular plot uses $2 + 1$ flavors with $m_s/T = 0.30$.

$$\text{pure } SU(3) : \quad \left(1 - \frac{1}{\sqrt{3}}\right)\pi < \tilde{\Omega}_I < \frac{12 - 2\sqrt{6}}{15}\pi.$$

The pure-gluon circle is distinct from the quark-driven high-angle broken phase.

R. Takada, RESCEU-22/26 (2026), Accidental degeneracy in the massless case.

B26 How is this different from perturbative confinement?

Center symmetry and charge conjugation act differently on the Polyakov loop.

$$\begin{aligned}\text{center: } L &\mapsto zL, & z &= e^{2\pi i k/3}, \\ \text{charge conjugation: } L &\mapsto L^*.\end{aligned}$$

Pure Yang-Mills

Center symmetry is exact. CFS identified a perturbatively confined phase under imaginary rotation, with zero traced holonomy.

The two nontrivial center elements in the present broken phase are exchanged by C . Their appearance is not the restoration of center symmetry.

P4: introduction, accidental-degeneracy discussion, lattice tests; CFS, PRL **129**, 242002 (2022).

QCD with dynamical quarks

Quarks explicitly break center symmetry. Charge conjugation remains exact at zero chemical potential and can break spontaneously.

B27 What would a lattice test actually measure?

Test the exact relations and the two C -odd observables.

At the exact point, compare corresponding ensembles and selected phases. The continuum predictions include the mixed periodicity, the reflection relation, and the observable mapping.

$$(\tilde{\Omega}_I, \theta_q) = (2\pi, 0) \longleftrightarrow (0, \pi),$$

$\text{Im } L \quad \text{and} \quad n_I.$

Near onset, compare dynamical and quenched calculations: the quark contribution is absent in the latter. Distributions and finite-size scaling distinguish phase selection from a symmetric average.

These are proposed tests. At finite lattice spacing, angular periodicity depends on the implementation; the exact identities are continuum targets.

R. Takada, RESCEU-22/26 (2026), Lattice tests; finite-size interpretation as in the cited RW literature.

B28 What remains open beyond the on-axis calculation?

The exact line and the on-axis window have distinct domains of validity.

At the exact angle, the RW endpoint temperature and its critical behavior carry over from nonrotating QCD. At generic angles, the present calculation finds the minima of the local potential on the rotation axis.

$$\begin{aligned}\tilde{\Omega}_I = 2\pi : & \text{ RW endpoint, evidence for 3D Ising,} \\ \tilde{\Omega}_I \simeq \tilde{\Omega}_{I,C} : & \text{ one-loop onset with mean-field exponent } \frac{1}{2}.\end{aligned}$$

Open questions include the radial extent of the broken region, its connection to the exact line at lower temperatures, and the possible endpoint of the lock-in transition.

The minimization is in holonomy space, not over radial profiles. Generic-angle universality requires a nonperturbative treatment.

R. Takada, RESCEU-22/26 (2026), conclusion and the stated on-axis scope; Bonati et al. (2016) for RW criticality.

B29 Can this be analytically continued to real rotation?

Continuation requires control of the boundary, analyticity, and the order of limits.

$$\begin{array}{ll}\text{real rotation:} & R|\Omega| < 1, \\ \text{imaginary rotation:} & \Omega = i\Omega_I.\end{array}$$

Rigid real rotation has a finite-radius causality bound. Imaginary rotation allows the thermodynamic limit used here. Taking that limit first does not provide a direct continuation to an infinite, rigidly rotating real system.

In addition, phase boundaries and the fractal angular dependence discussed in the cited work obstruct a naive continuation of an entire phase diagram. The present result concerns imaginary rotation itself.

P4: final paragraph; Chen-Fukushima-Shimada, PRL **129**, 242002 (2022); Chernodub, arXiv:2210.05651; Pătureanu et al., PRD **111**, 116004 (2025).

B30 Selected references

Project-4: Rin Takada, RESCEU-22/26 (2026). Background references:

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A. Yamamoto and Y. Hirono, *Phys. Rev. Lett.* **111**, 081601 (2013), arXiv:1303.6292.

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B31 Source material and numerical checks

The figures evaluate the Project-4 one-loop potential. The plotting code and numerical checks are included with the presentation source.

T. T. Dumitrescu and P.-S. Hsin, *SciPost Phys.* **17**, 093 (2024),
arXiv:2312.16898.

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T. Pătureanu et al., *Phys. Rev. D* **111**, 116004 (2025), arXiv:2502.09738.

K. Fukushima and Y. Shimada, *Phys. Lett. B* **868**, 139716 (2025),
arXiv:2506.03560.

Reproducible files in this package:

`scripts/generate_figures.py`: numerical figures and branch data.

`source_inputs/`: P4 TeX, bibliography, and original plotting code.

`checks/`: algebraic checks, sampled global searches, and build results.