A New Method for Locating a Tricritical Point Using Binder-Cumulant Derivatives and Its Application to Heavy-Quark QCD

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Collaborators : Masakiyo Kitazawa, Kazuyuki Kanaya (Univ. of Tsukuba)

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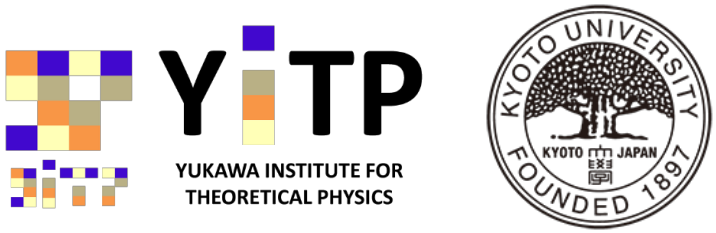
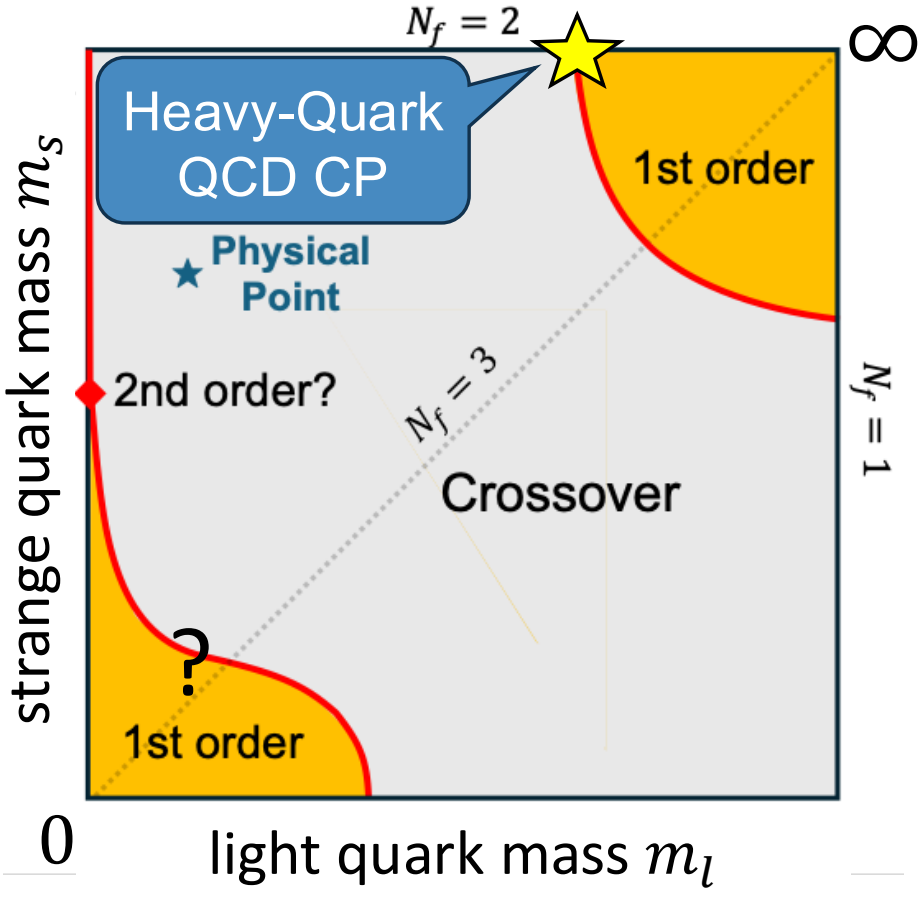


Columbia Plot



Goal: **QCD critical point at finite T and μ**

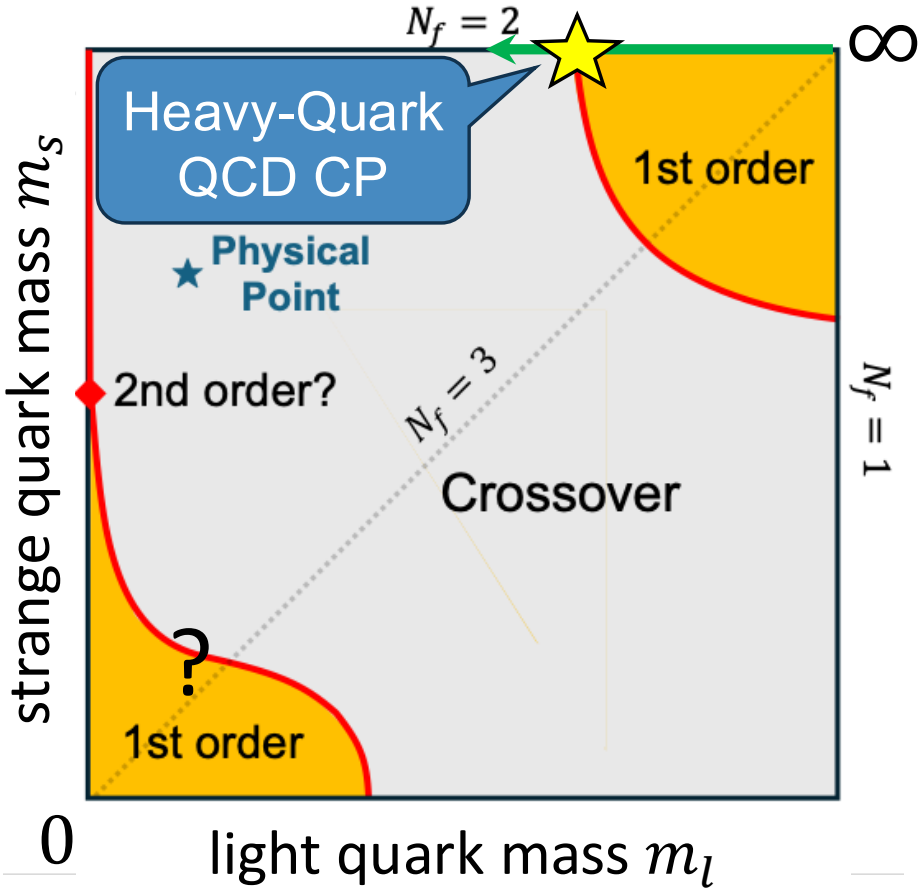
Problem: **sign problem(Monte Carlo)**

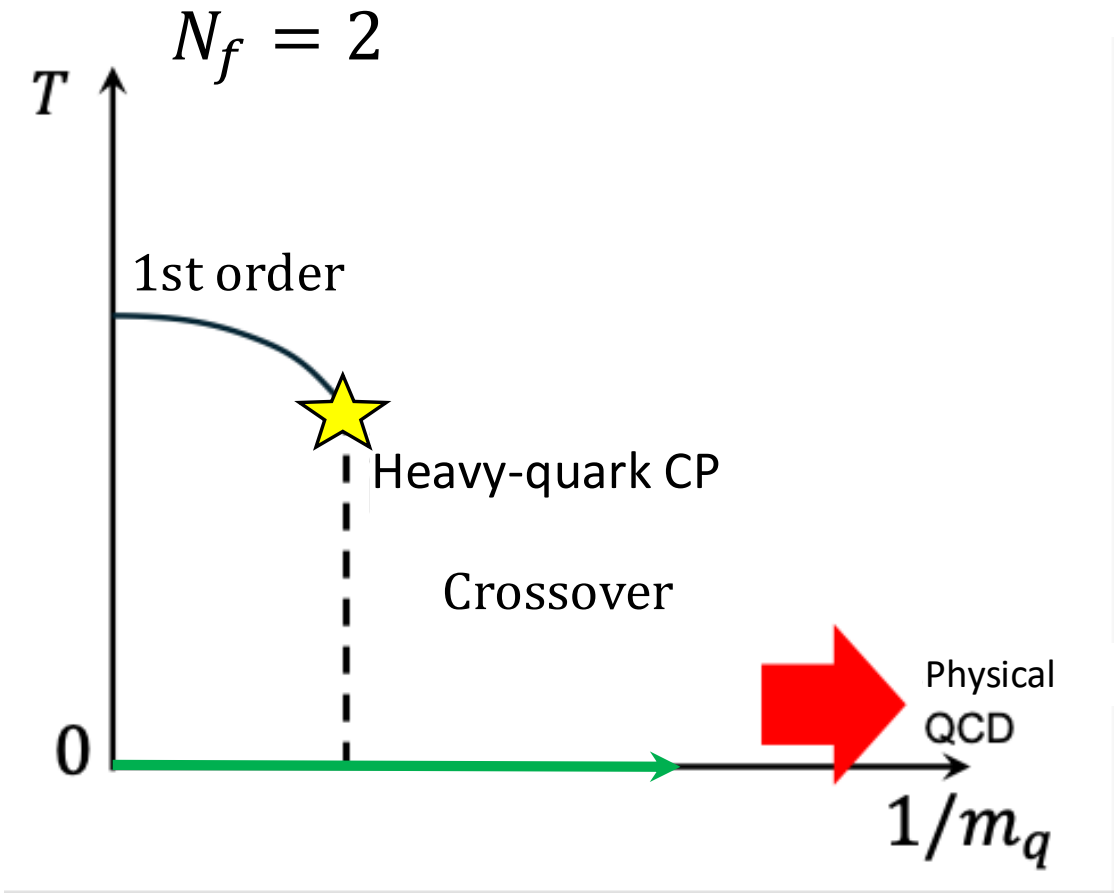
Columbia plot :

$\mu = 0$: order of the phase transition

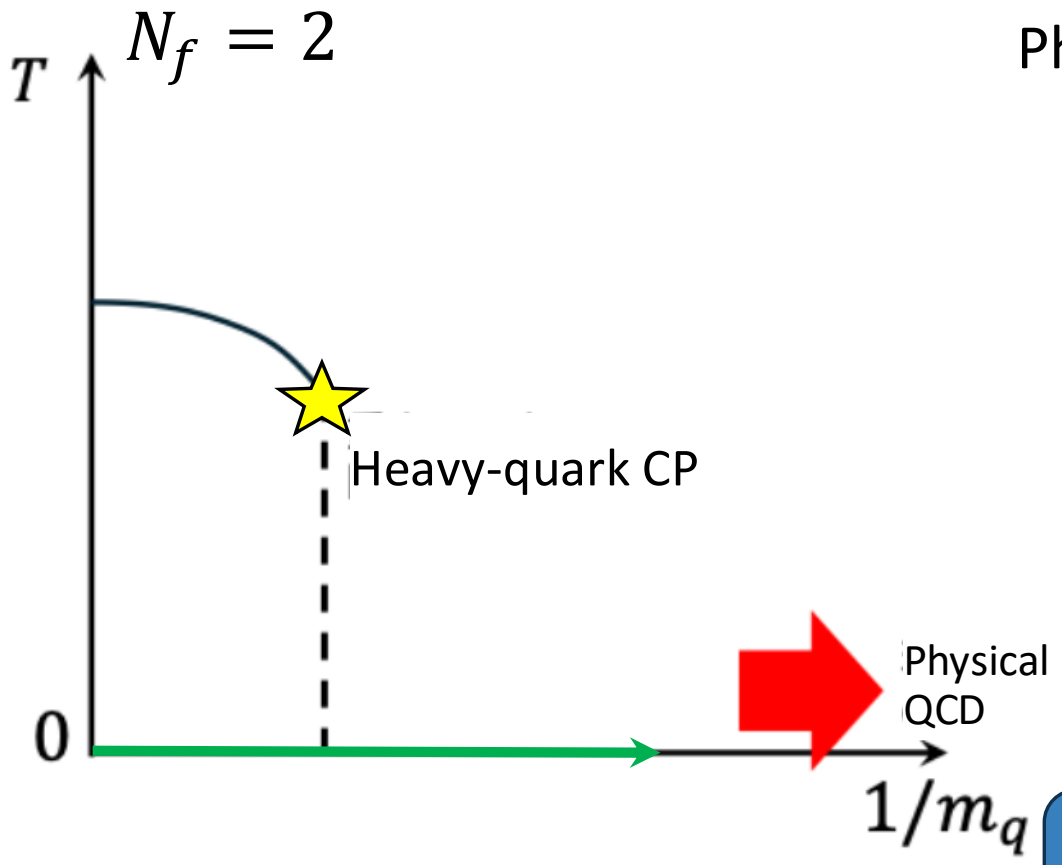
- $m \to m_{\mathrm{phys}}$: $N_f = 2 + 1$ QCD
  - → Crossover
- $m \sim 0$: Light quark sector
  - → 1st order or crossover? Unresolved (by Tsukuba group (∼2020))
- $m \to \infty$: Pure Yang-Mills theory
  - → 1st order (Kiyohara, *et al.* (2021))
  - HQQCD-CP (Ashikawa, *et al.* (2024))

Columbia Plot





Zero Chemical Potential



Kiyohara, *et al.* (2021)
Ashikawa, *et al.* (2024)

Phase structure at μ = 0

- $m_q \to \infty$: SU(3)Yang–Mills theory
  → 1st-order transition

Decreasing the quark mass weakens the 1st-order transition

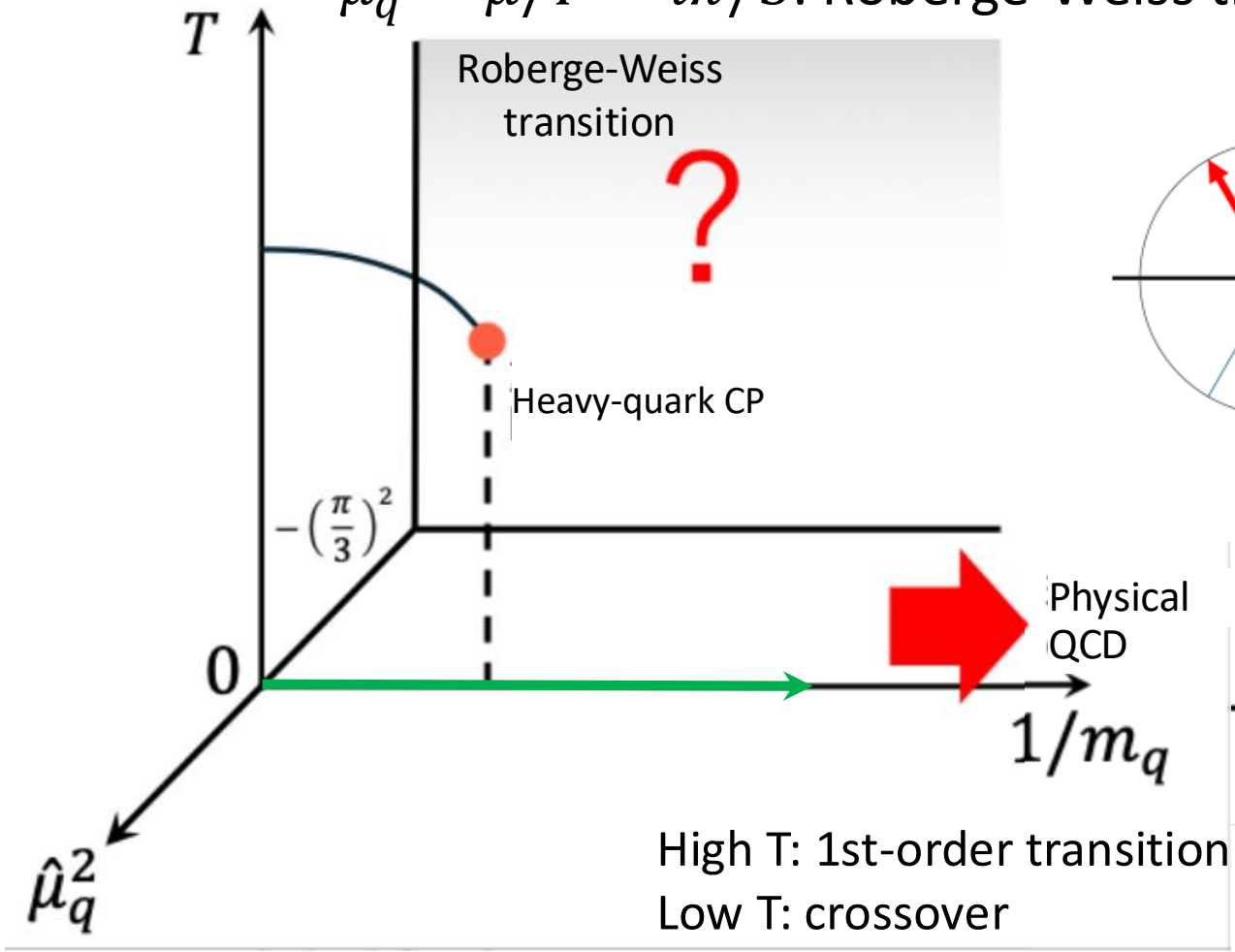
- m_q = Intermediate
  → Crossover

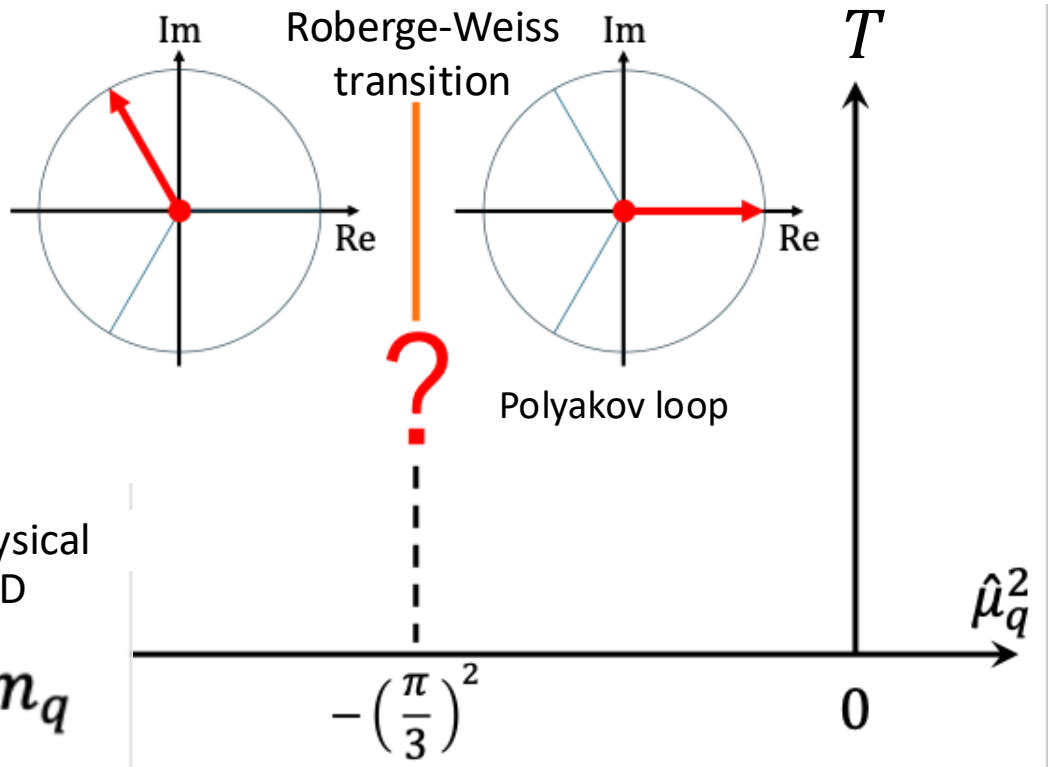
Boundary: 2nd-order transition
= **Heavy-Quark Critical Point**

At finite density $\mu \neq 0$,
what is the order of transition?

Roberge-Weiss Transition Plane

$\hat{\mu}_q = \mu/T = i\pi/3$: Roberge-Weiss transition(no sign problem)

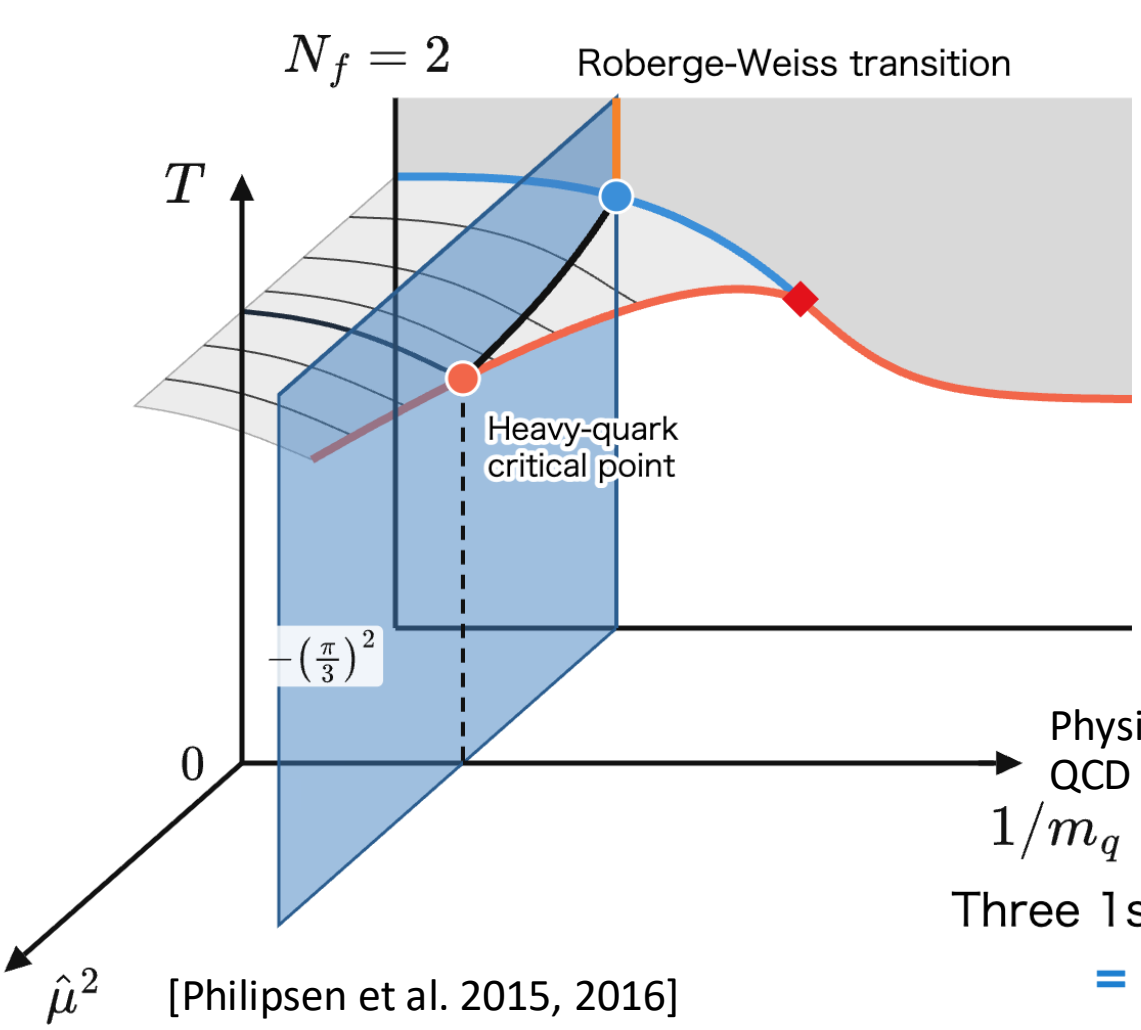




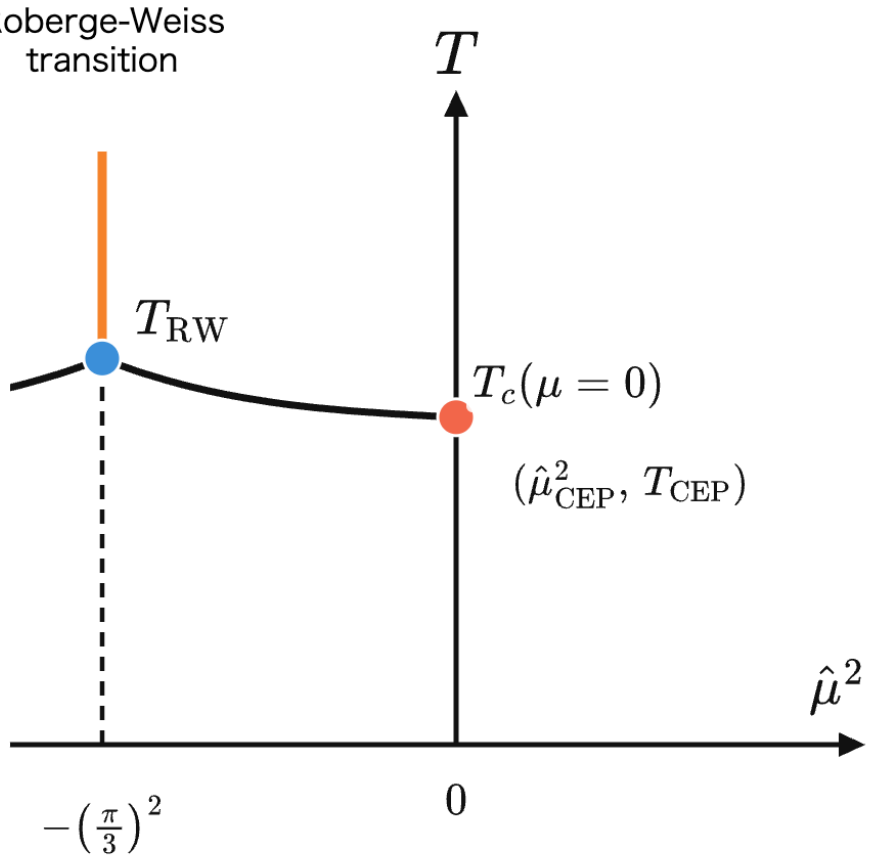
High T: 1st-order transition
Low T: crossover

» Order at intermediate T?

Phase Structure on the RW Plane



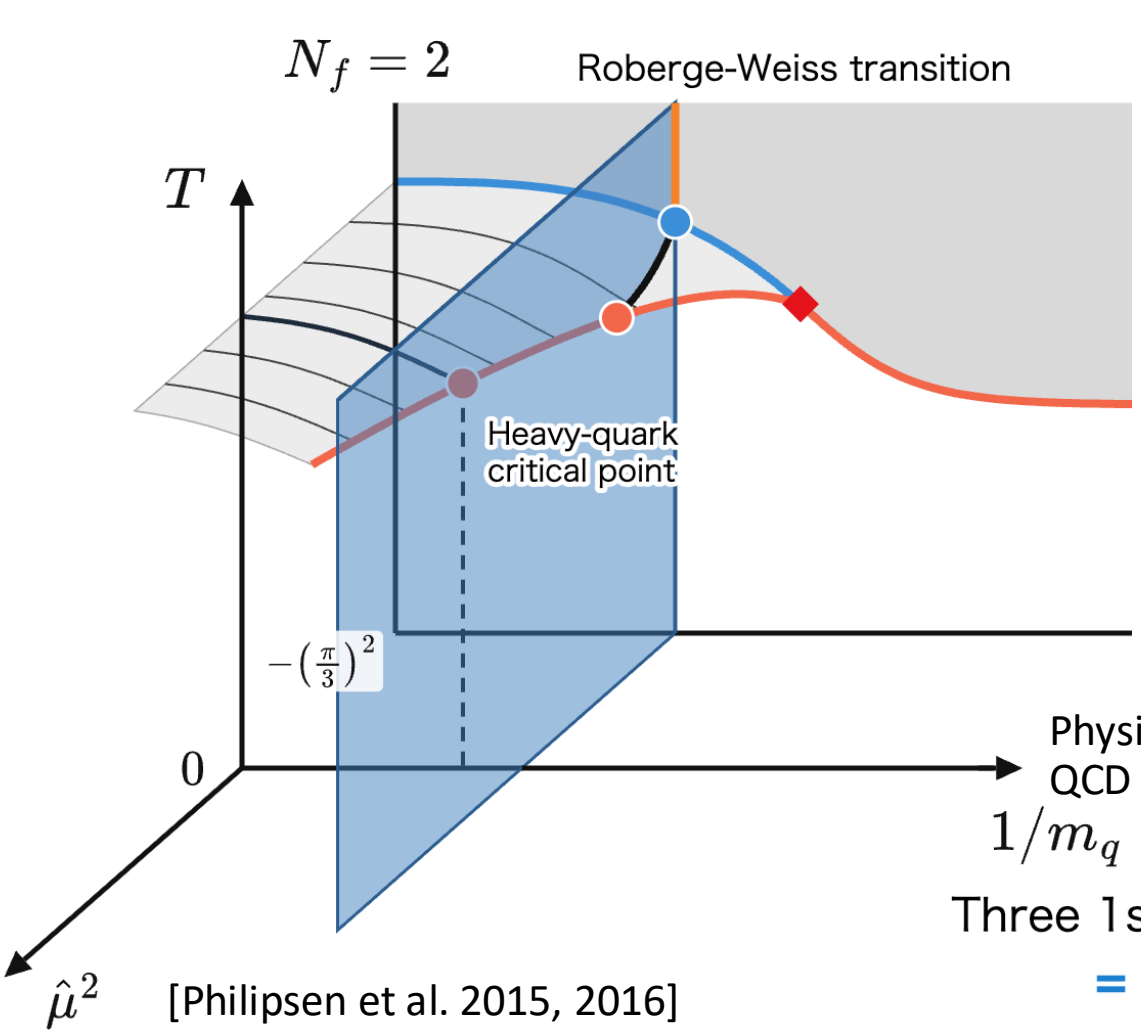
[Philipsen et al. 2015, 2016]



Three 1st-order lines (RW×1, deconf.×2) meet

= Triple point (1st order)

Phase Structure on the RW Plane



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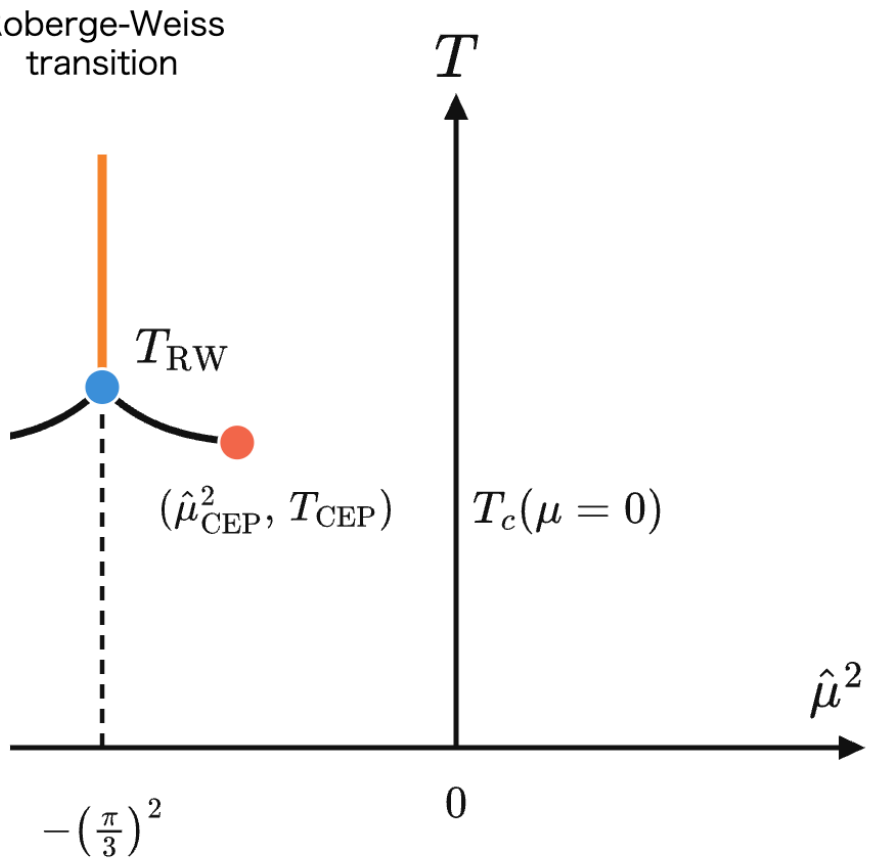

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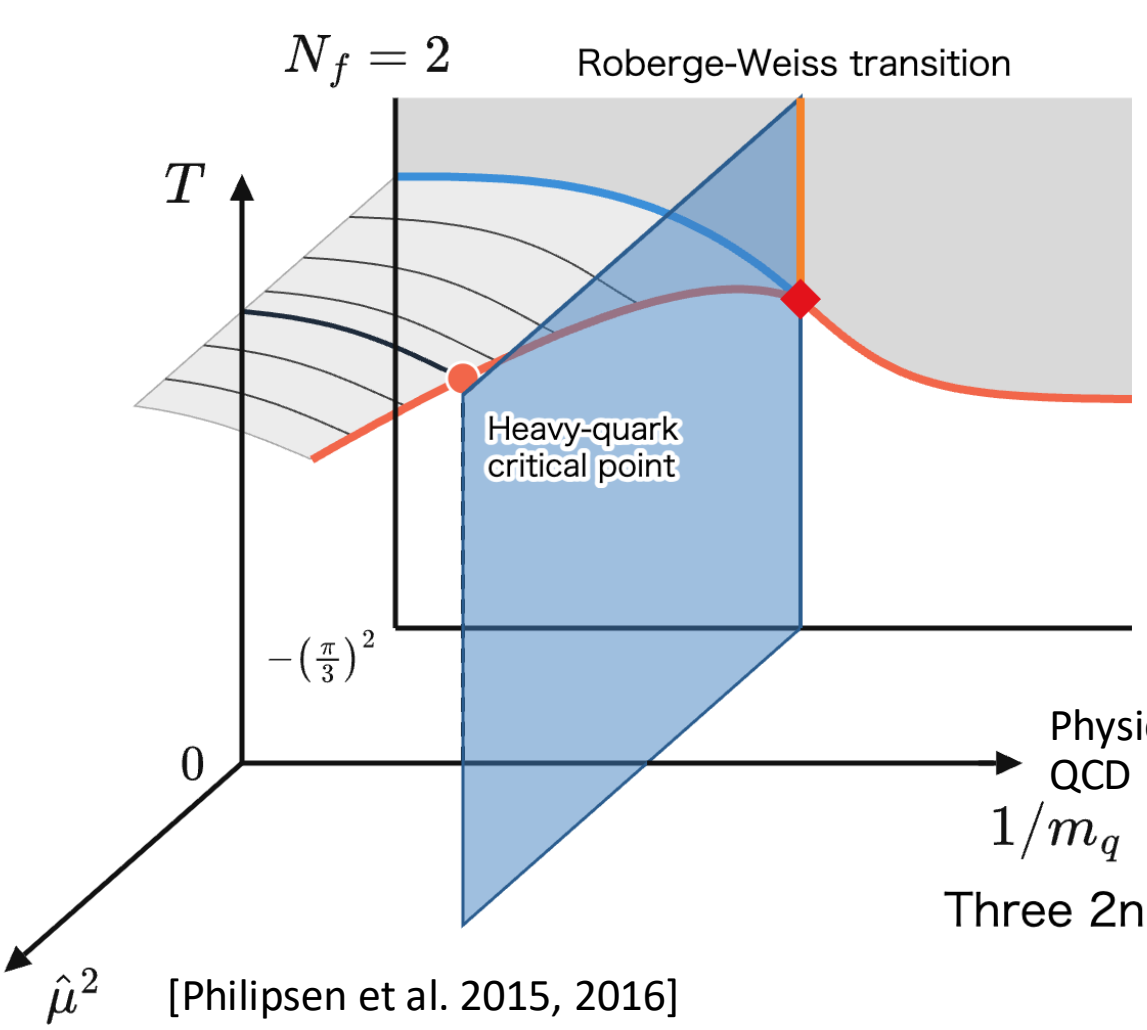
Phase Structure on the RW Plane



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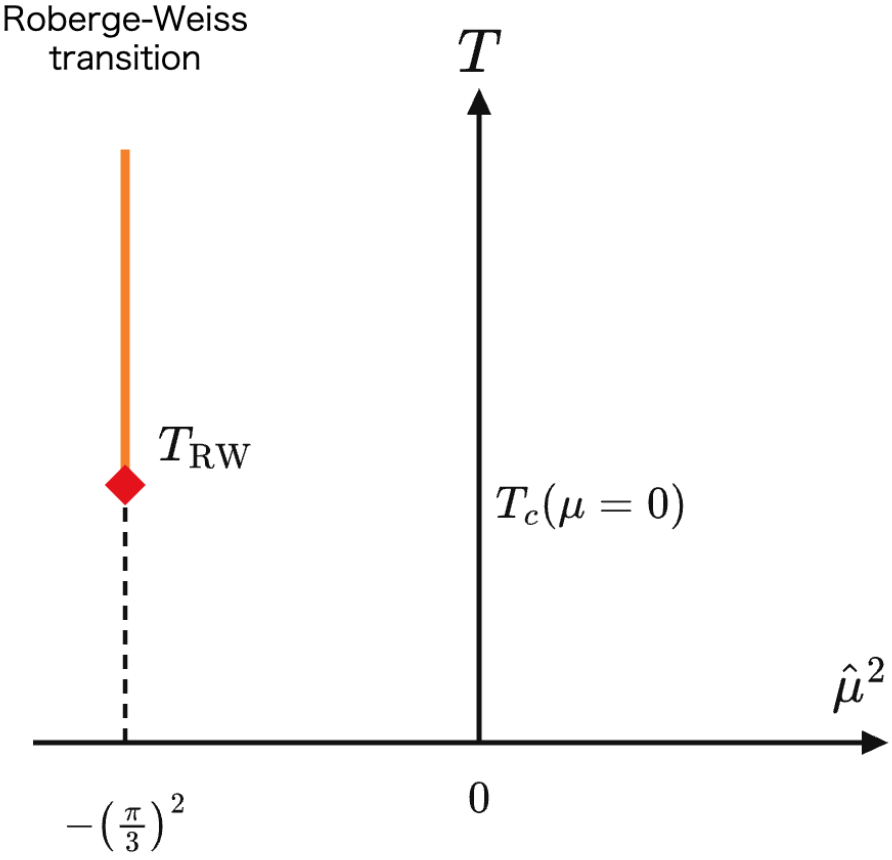
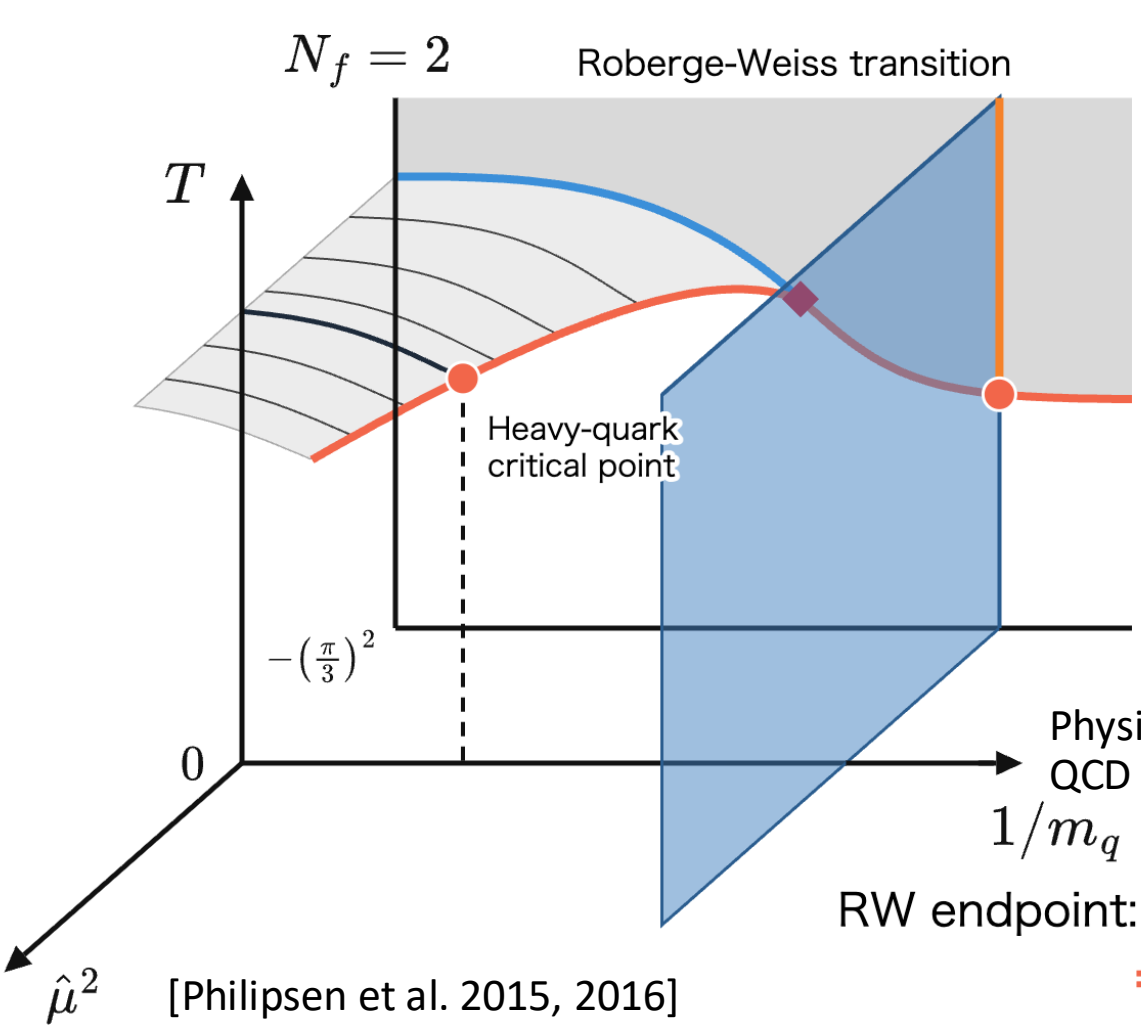


Three 2nd-order lines (RW×1, deconf.×2) meet

= Tricritical point

Phase Structure on the RW Plane



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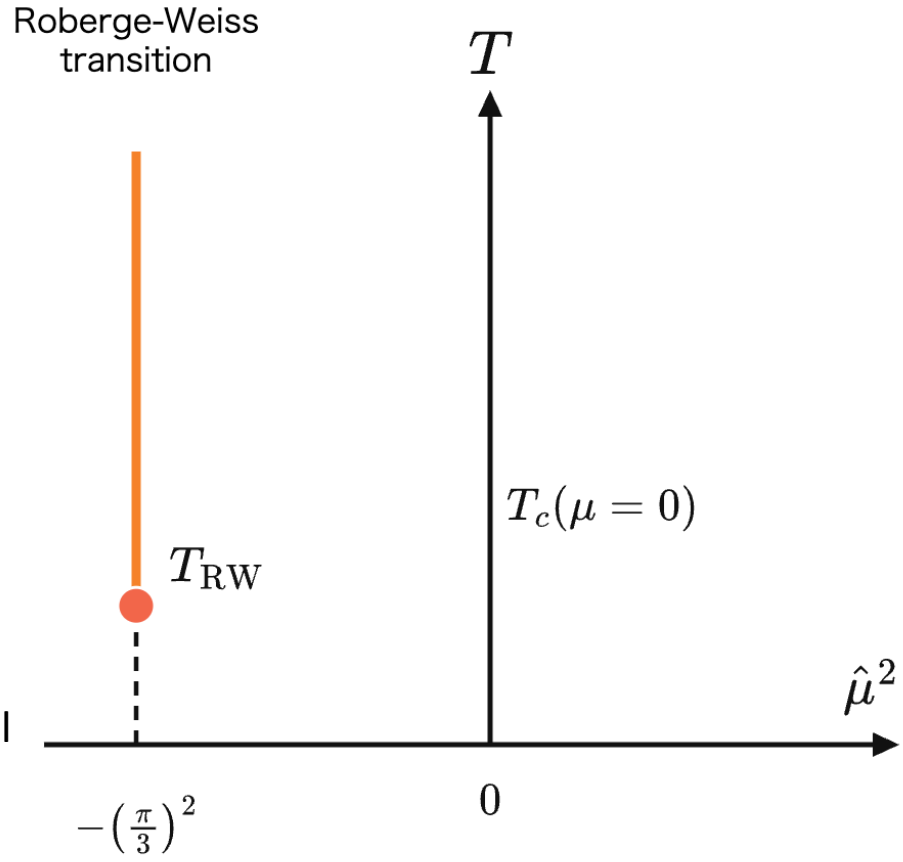


RW endpoint: 2nd order, deconfinement: crossover

= 2nd-order endpoint

Locating Critical Points: Binder Cumulant

Binder cumulant @$\mu/T = i\pi$

$$B_4(\beta, L) = \frac{\langle (\mathrm{Im}\,\Omega)^4 \rangle_c}{\langle (\mathrm{Im}\,\Omega)^2 \rangle_c} + 3$$

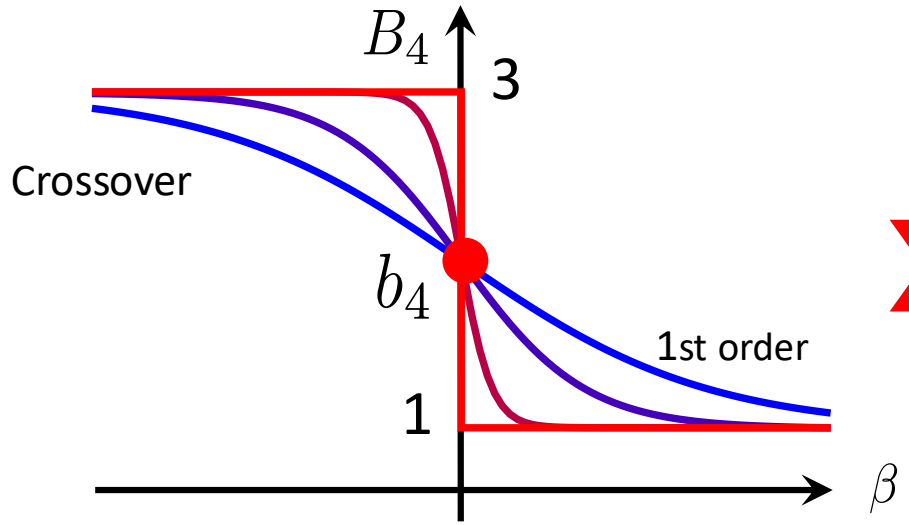
Cf. RW periodicity

$$Z(T, \mu) = Z(T, \mu \pm \frac{2\pi i}{3})$$

Ω : Polyakov loop
$\beta = 1/g^2$
L : lattice size

Near the critical point (finite-size scaling)

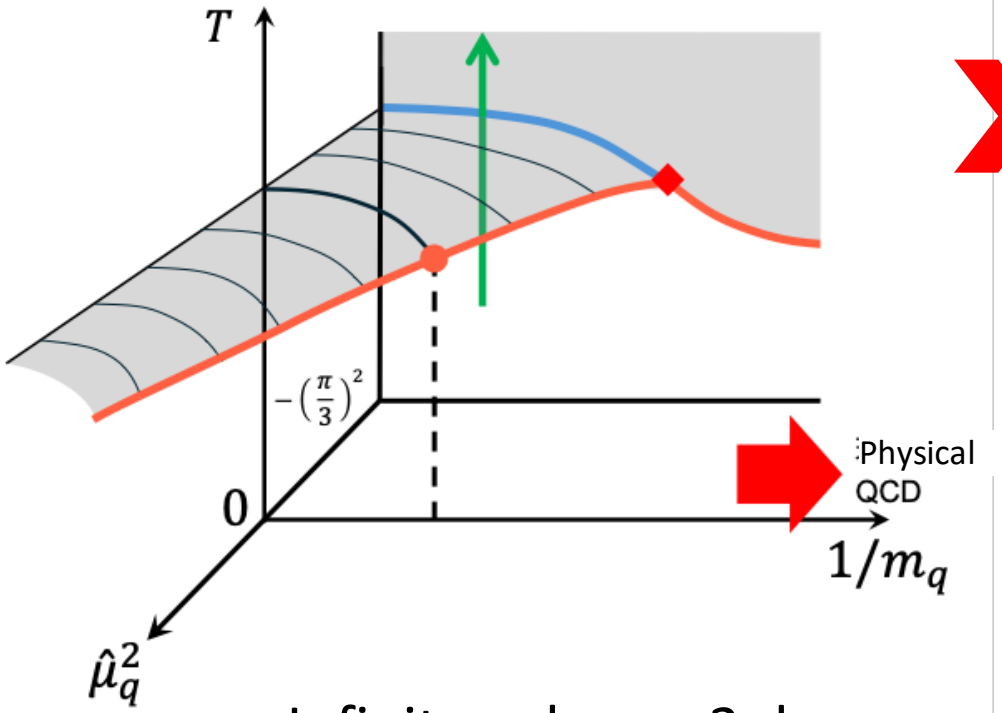
$$B_4(\beta, L) = b_4 + c_4(\beta - \beta_c)L^{y_t} + \mathcal{O}(L^{2y_t})$$

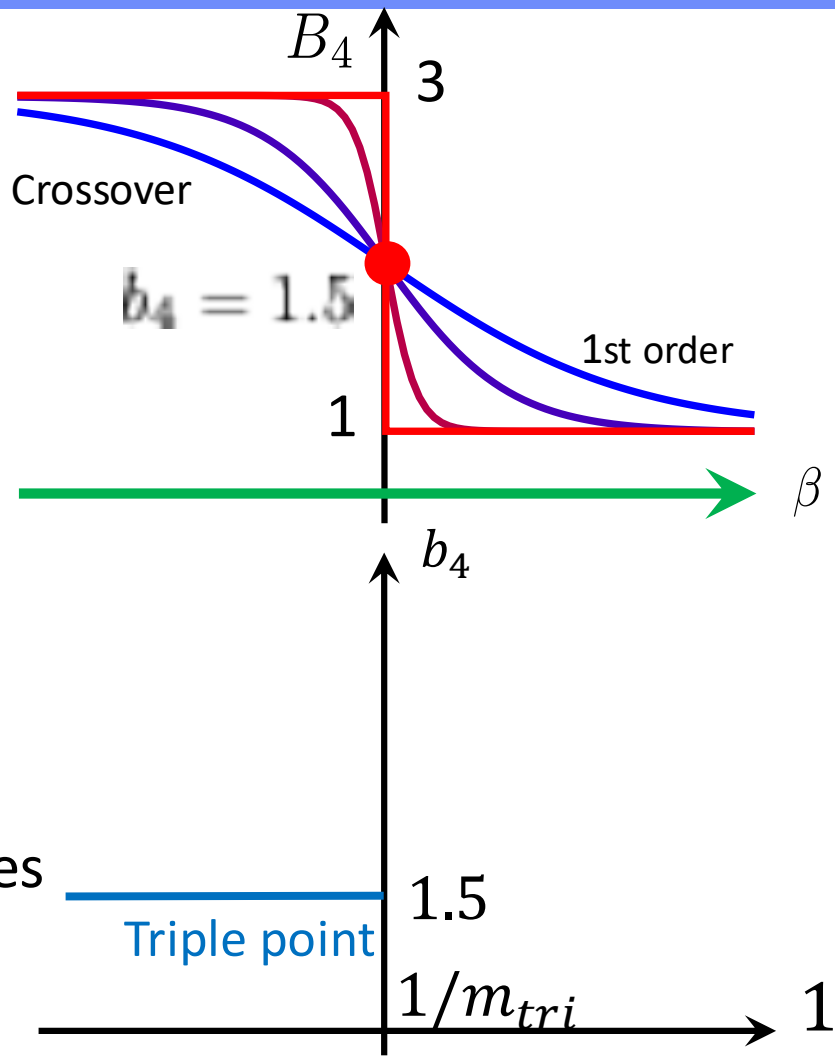


b_4 *and* y_t depend on the universality class

» Analytically expected values?

Behavior of the Binder Cumulant



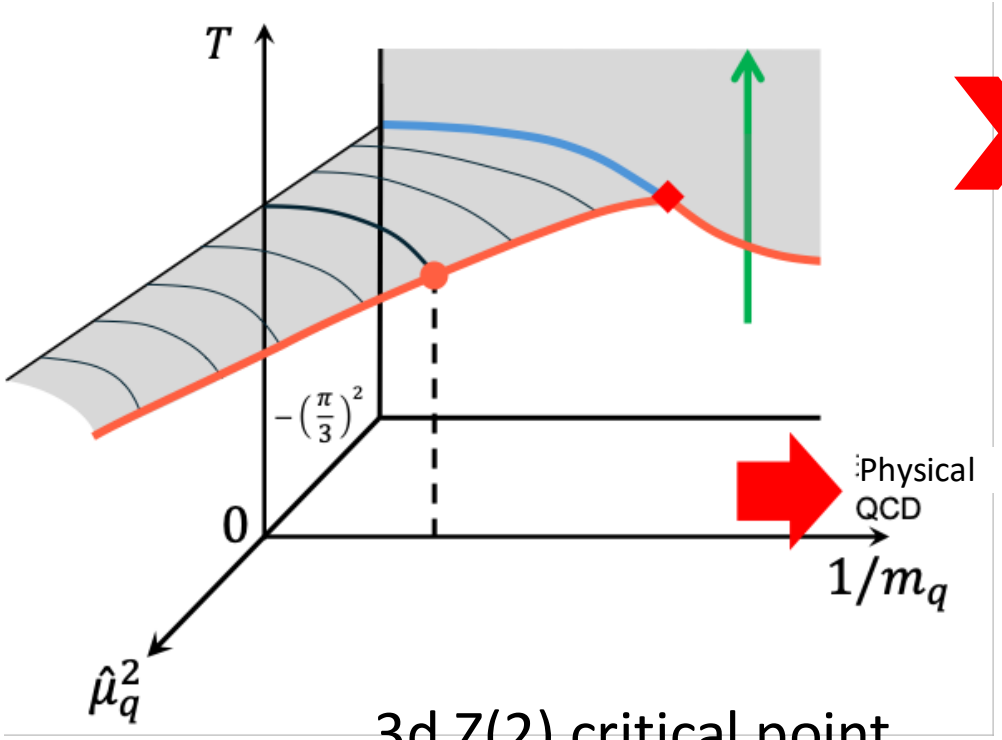


Infinite volume: 3 degenerate states

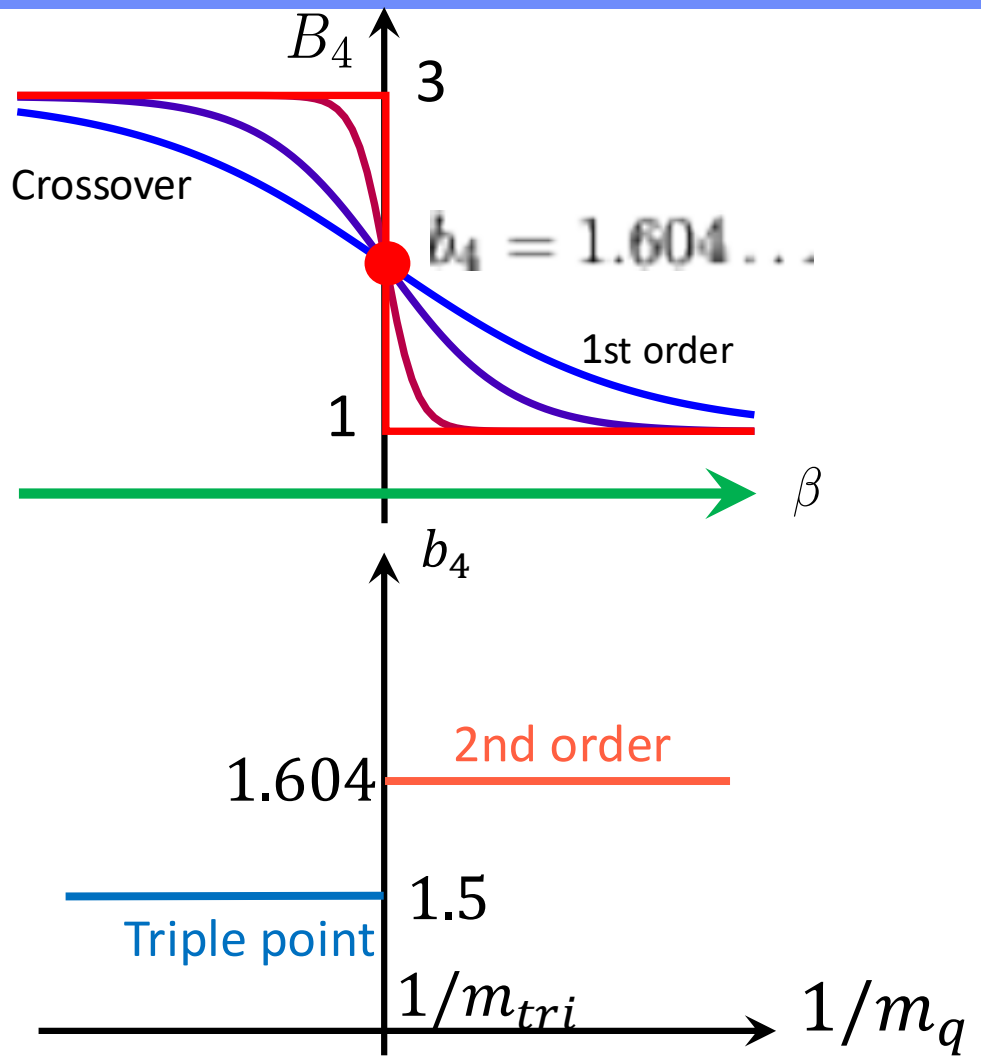
1st-order transitions

→analytically calculable: $b_4 = 1.5$

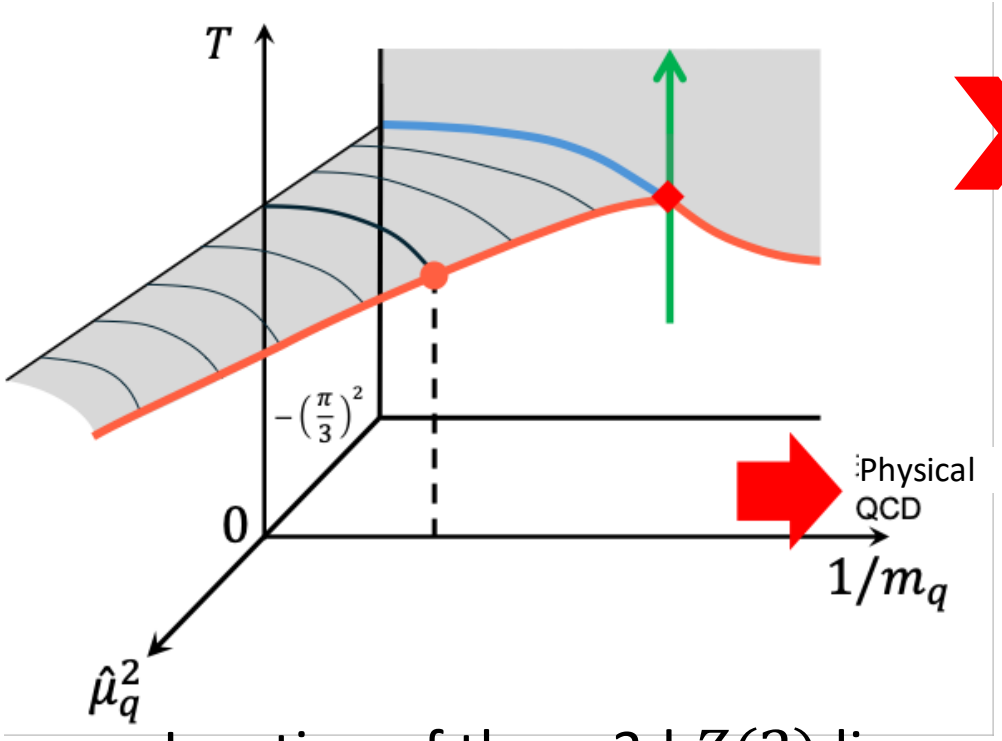
Behavior of the Binder Cumulant

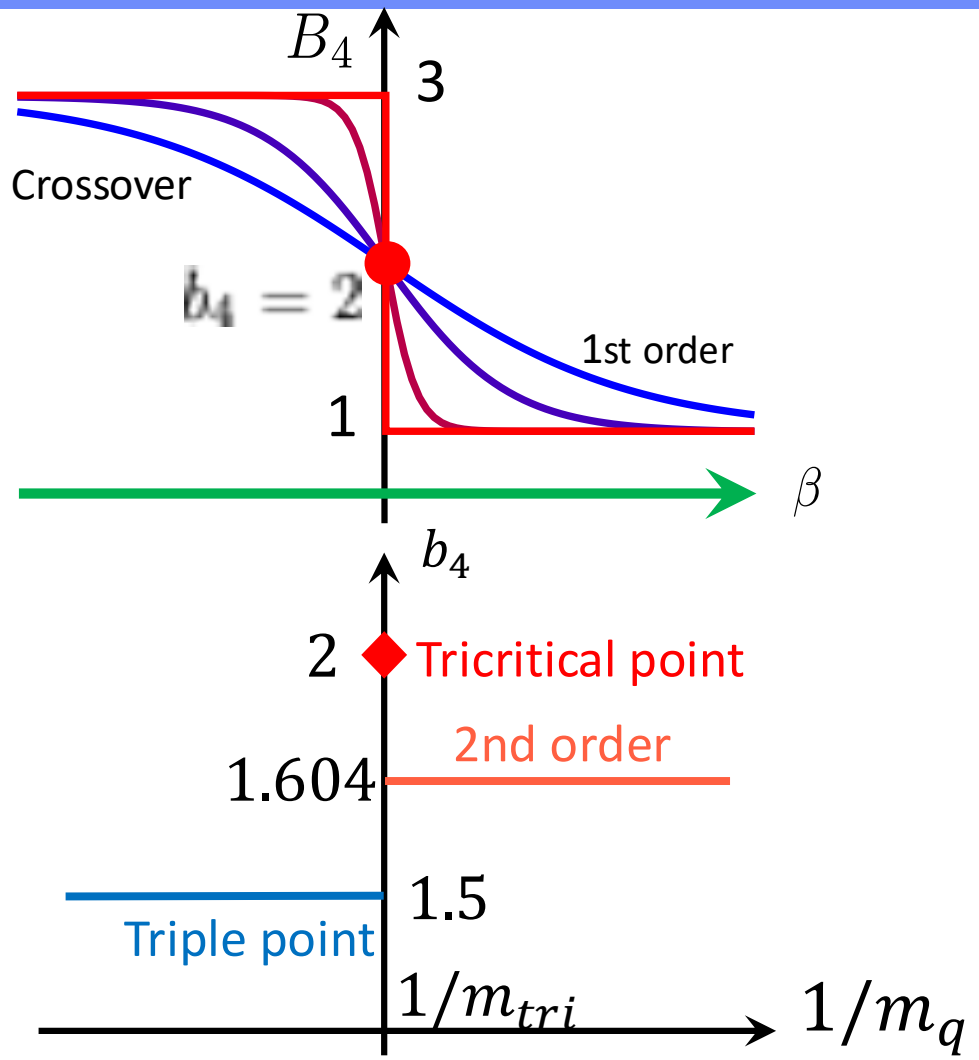


3d Z(2) critical point
Value known numerically
$\rightarrow b_4 = 1.604 \ldots$



Behavior of the Binder Cumulant





Junction of three 3d $Z(2)$ lines

ϕ^6-Landau potential (log corrections)

→ analytically calculable: $b_4 = 2$

Previous Study

Czaban, Cuteri, Philipsen et al, 2015, 2016

Wilson fermions, lattice QCD simulation

$\kappa \sim 1/2am_q$ Hopping parameter

Aspect ratio

$$LT = N_s/N_t = \begin{cases} 2, \ldots, 5 \, (N_t = 4) \\ 3, \ldots, 6 \, (N_t = 6) \end{cases}$$

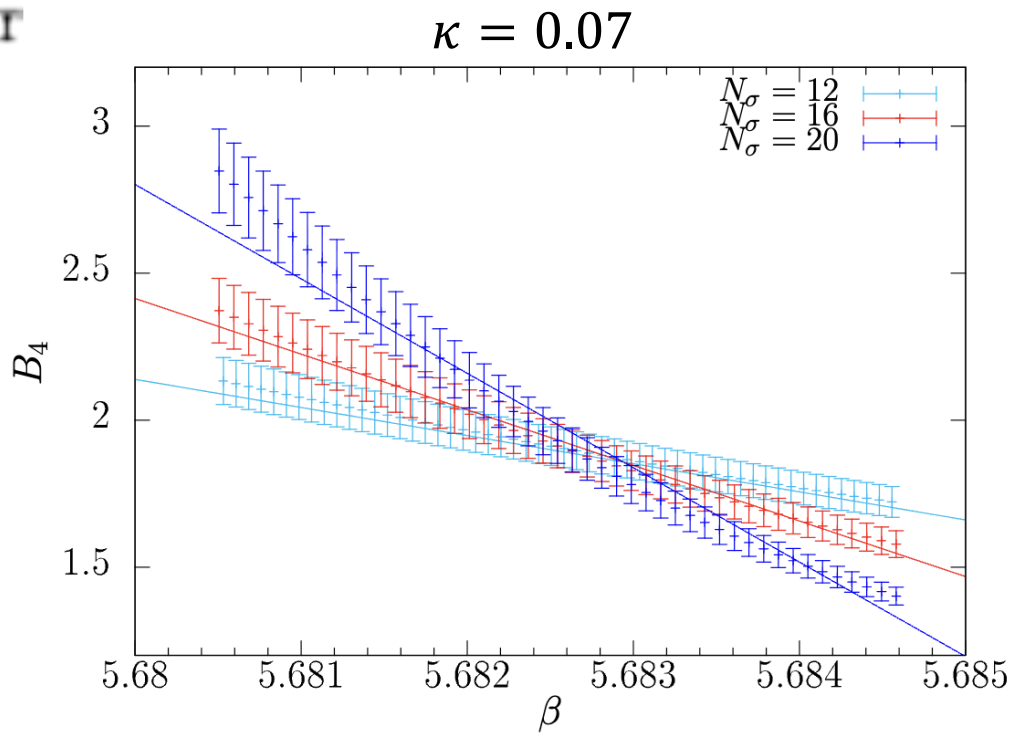
At most $LT = 6$

Binder Cumulant B_4

$$B_4(\beta, L) = b_4 + c_4(\beta - \beta_c)L^{y_t} + \mathcal{O}(L^{2y_t})$$

b_4 and critical exponent y_t fixed by universality class
→ fitting by b_4, c_4, β_c, and y_t

	3d-Z_2 2nd order	Triple point (1st order)	Tricritical
b_4	1.60359(4)	1.5	2
y_t	1/0.6301(4)	3	2



b_4 does not match the expected values
→ y_t instead

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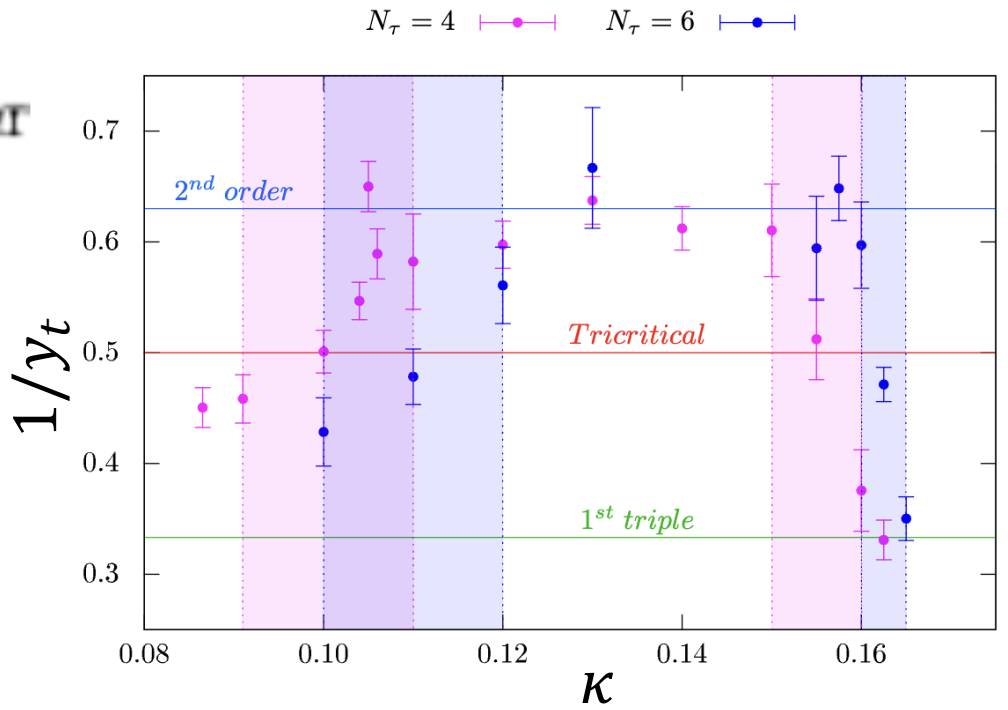


The approach based on critical exponents is promising.
Logarithmic corrections and finite-size effects must be taken into account.

Our Approach

Retain the previous approach using critical exponents.

[Philipsen et al. 2015, 2016]

1. New method: Binder cumulant Derivative
   - No fitting method
   - Incorporates finite-volume effects
2. Numerical: Large volume
   - Hopping parameter expansion
   - High precision
   - Finite-volume effects is under control.

New Method: Binder-Cumulant Derivative

With logarithmic scaling (y_t and $\hat{y}_t$ are critical exponent) TW+ in preparation

$$B_4(\beta, LT) = b_4 + c_4(\beta - \beta_c)(LT)^{y_t}(\ln LT)^{\hat{y}_t} + \mathcal{O}((LT)^{2y_t}(\ln LT)^{2\hat{y}_t})$$

β-derivative

$$\partial_\beta B_4(\beta, LT) = c_4(LT)^{y_t}(\ln LT)^{\hat{y}_t} + \mathcal{O}((LT)^{2y_t}(\ln LT)^{2\hat{y}_t})$$

Take the ratio of two volumes LT and $s\times LT$.

$$R_4(\beta, LT, s) \equiv \frac{\partial_\beta B_4(\beta, sLT)}{\partial_\beta B_4(\beta, LT)} = s^{y_t}\left(\frac{\ln(sLT)}{\ln(LT)}\right)^{\hat{y}_t}$$

Take the logarithm

$$\frac{\ln R_4(\beta, LT, s)}{\ln s} = y_t + \hat{y}_t \ln\left(\frac{\ln(sLT)}{\ln(LT)}\right)/\ln s \to y_t \quad (LT \to \infty)$$

Critical exponent extracted directly!
Critical β also from the peak



No artificial bias from fitting
Applicable to any critical point!

Lattice Simulation

TW+ in preparation

- Wilson fermions+ hopping-parameter expansion → heat bath

$\kappa \sim 1/2am_q$ Hopping parameter

$\lambda = 64N_f N_c \kappa^4$

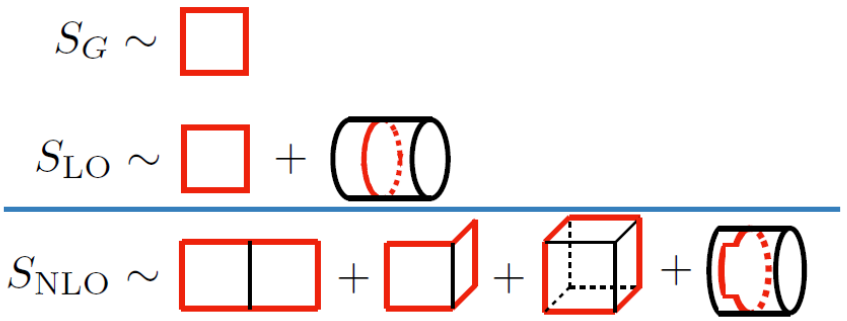
Cost ∼ Yang–Mills theory

Large volumes & high statistics!

- Lattice size: $N_t = 4, N_s = 24 \sim 60$

  Aspect ratio $\boldsymbol{LT = N_s/N_t = 6\sim15}$

- Statistics $\sim 10^6$
- Leading order for configuration generation
- NLO included by reweighting
- Multi-point reweighting → continuous scan in parameter space

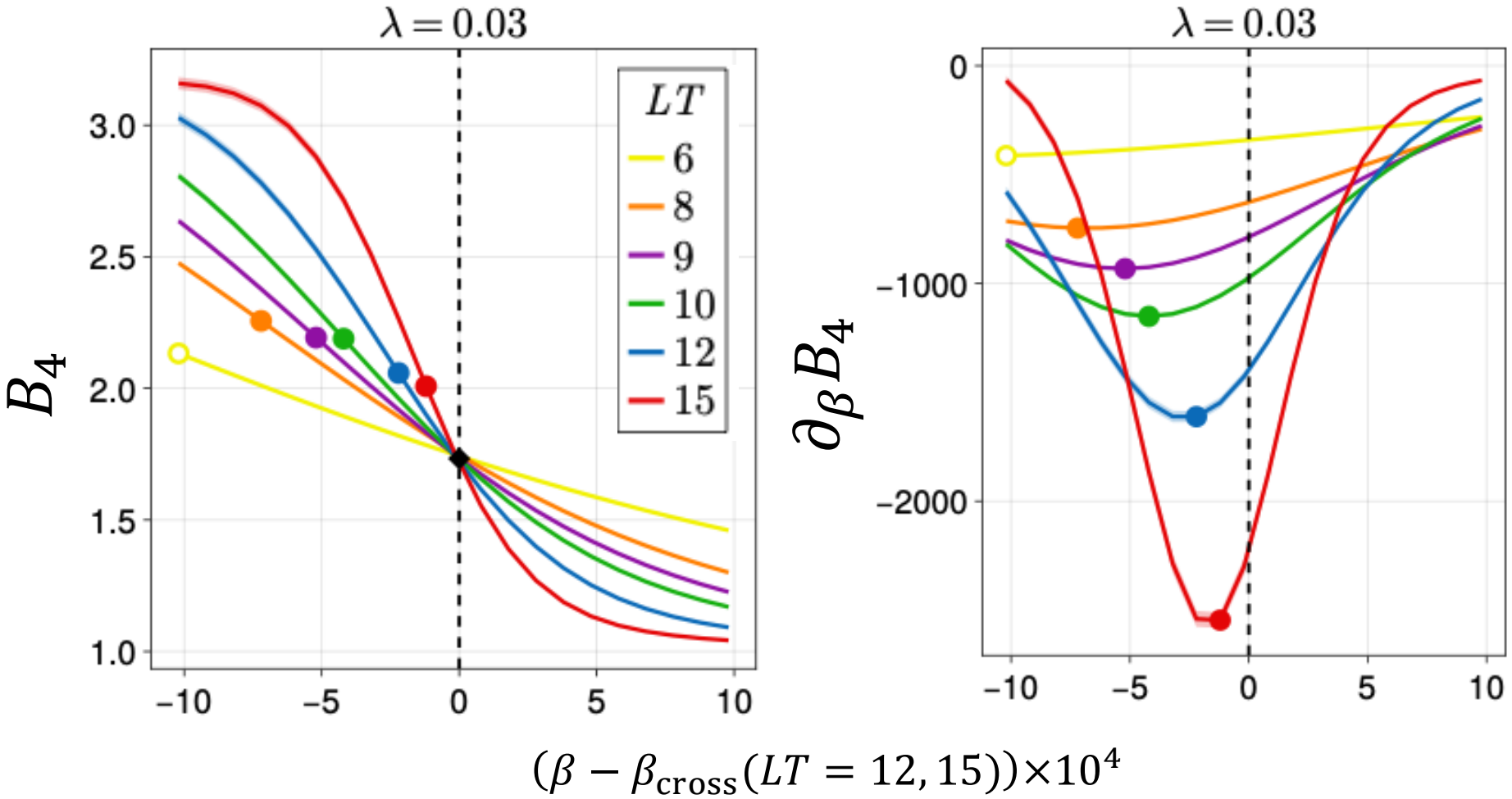


Previous study $(LT = 6)$ → now up to $LT = 15$

High precision, finite-volume effects under control!

Binder cumulant and derivative

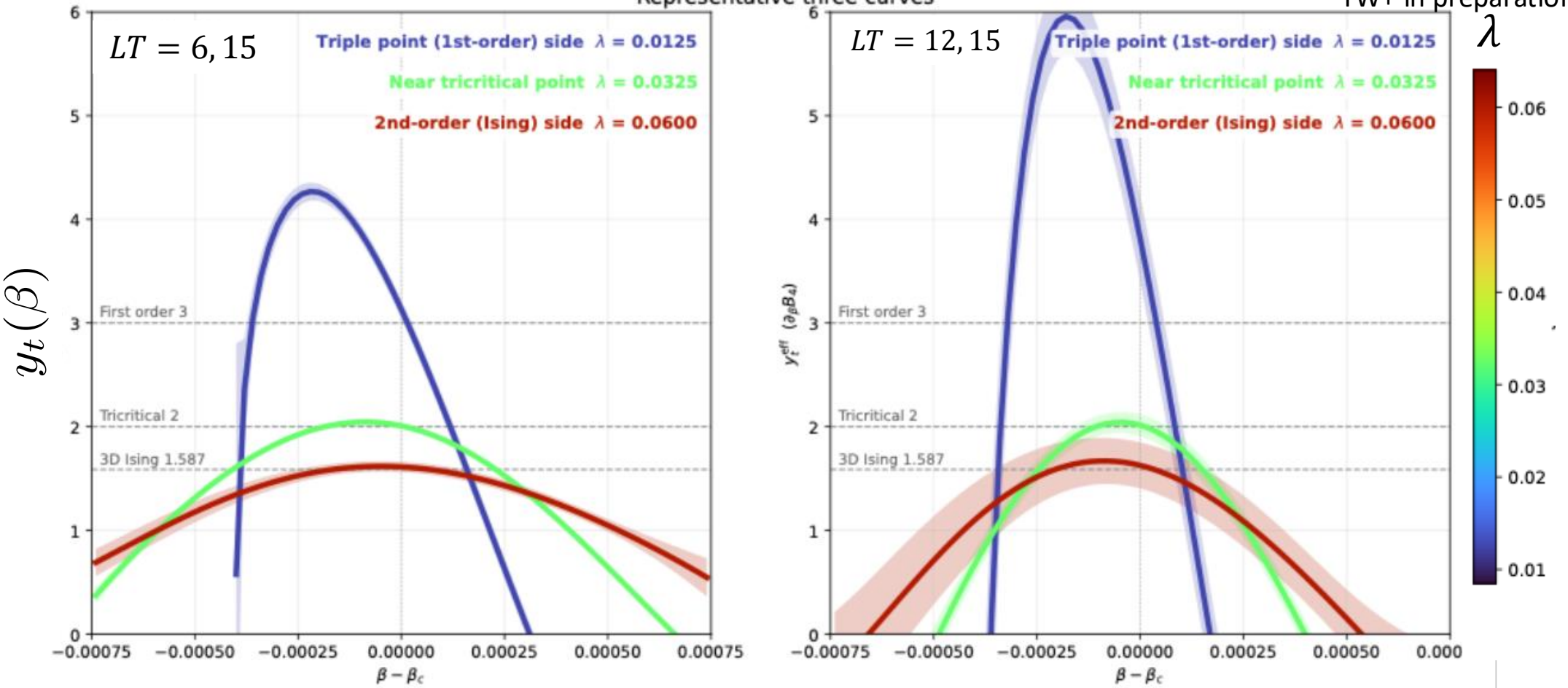
TW+ in preparation



Effective Exponent from the Derivative Ratio

Representative three curves

TW+ in preparation

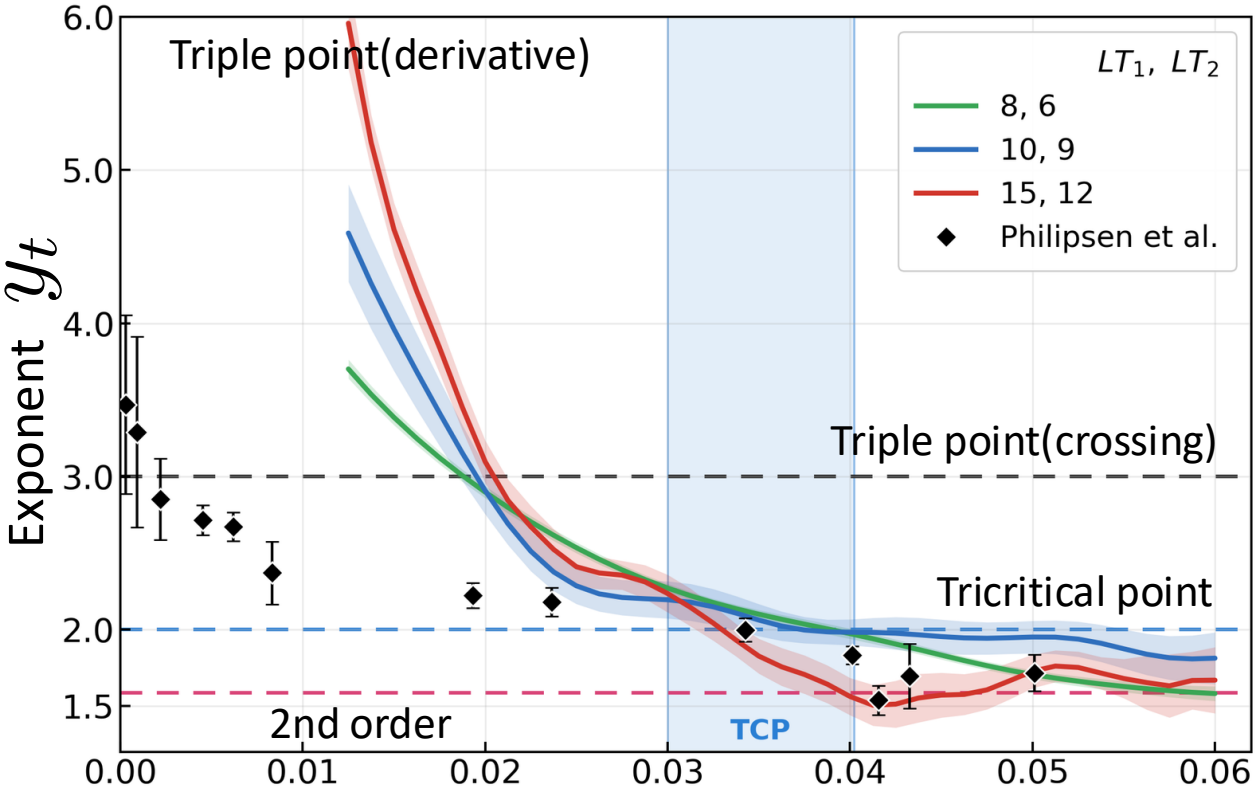


$\beta - \beta_{\text{cross}}(LT = 12{,}15)$

Numerical Results

TW+ in preparation

Ratio of two neighboring volumes



$$R_4(\beta, LT_1, LT_2) \equiv \frac{\partial_\beta B_4(\beta, LT_1)}{\partial_\beta B_4(\beta, LT_2)}$$

$$\frac{\ln R_4(\beta, LT_1, LT_2)}{\ln(LT_1/LT_2)} \to y_t \quad (LT_{1,2} \to \infty)$$

- Slope grows with volume
- Tricritical point at $\lambda_{tric} \sim 0.03$
- Consistent with previous study

Ratio of Binder-cumulant derivatives locates the tricritical point!

$$\lambda \sim (1/mass)^4 \qquad \lambda = 64 N_f N_c \kappa^4$$

Summary & Outlook

Summary

- ✓ **Derivative of the Binder cumulant**: A new method to locate critical points
  Extracts critical exponents directly; controlling logarithmic corrections
- ✓ **Applied to the Roberge-Weiss transition** in Heavy-Quark QCD
  → Tricritical point estimated
- ✓ Tricritical point is located at $\lambda_{tric} \sim 0.03$

Outlook

- ☐ Apply the method to the chiral limit
- ☐ Locate the tricritical point with **Lee-Yang zeros**
- ☐ Extend to finite real chemical potential