Loïc Fernandez

HELSINGIN YLIOPISTO

LABORATOIRE CHARLES COULOMB MONTPELLIER

CENTRAL CHINA NORMAL UNIVERSITY

Cold Quark Matter

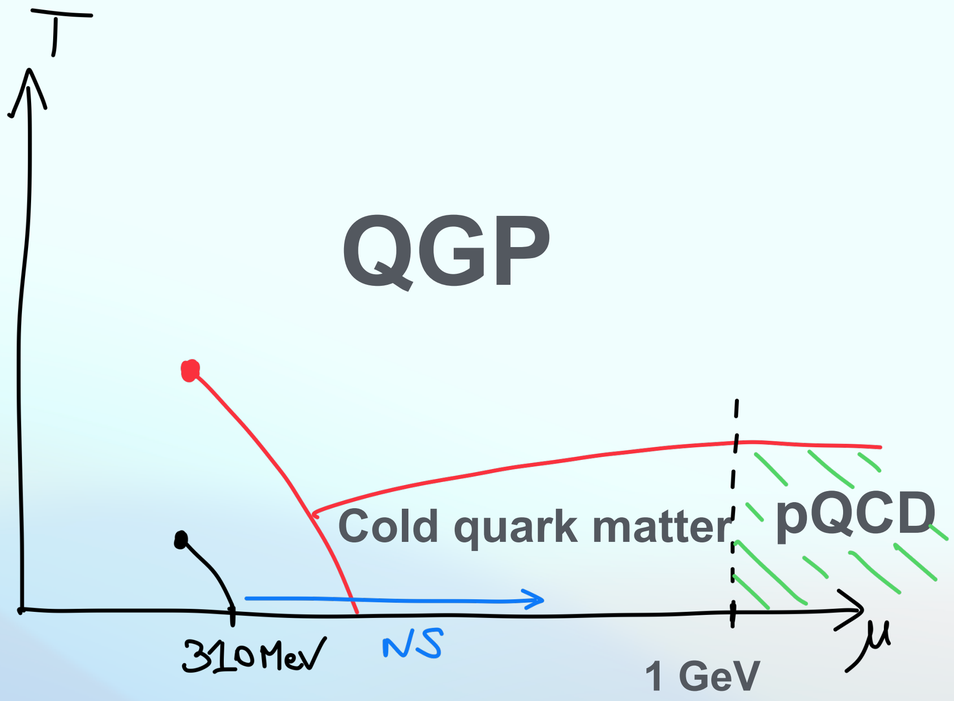
Renormalization group improvement of the perturbative series

New frontiers in QCD



Context

Context

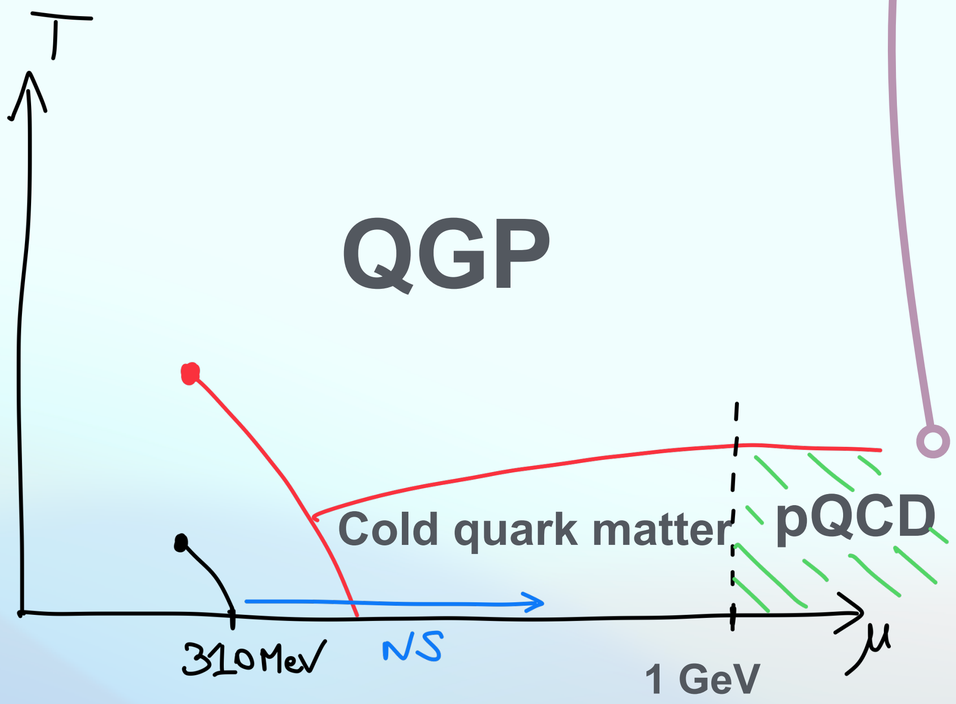


Large uncertainties in pQCD

Context

$$\frac{P_{QCD}(T=0,\mu)}{P_f} = 1 + \alpha_s A_1 + \alpha_s^2(A_2 + B_1 \ln(\alpha_s)) + \ldots$$

known



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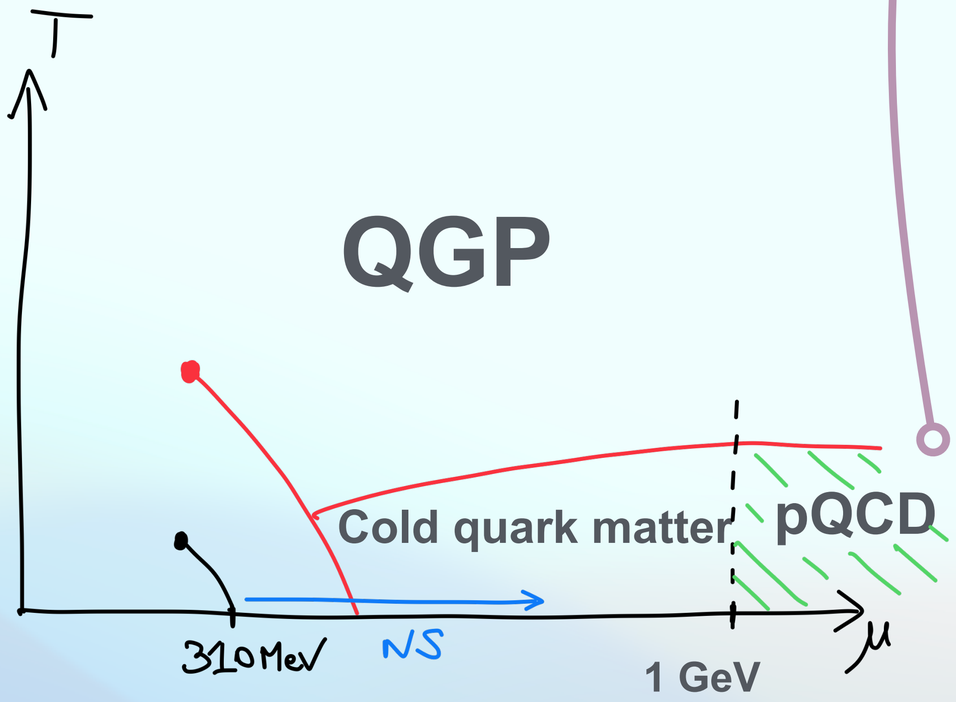

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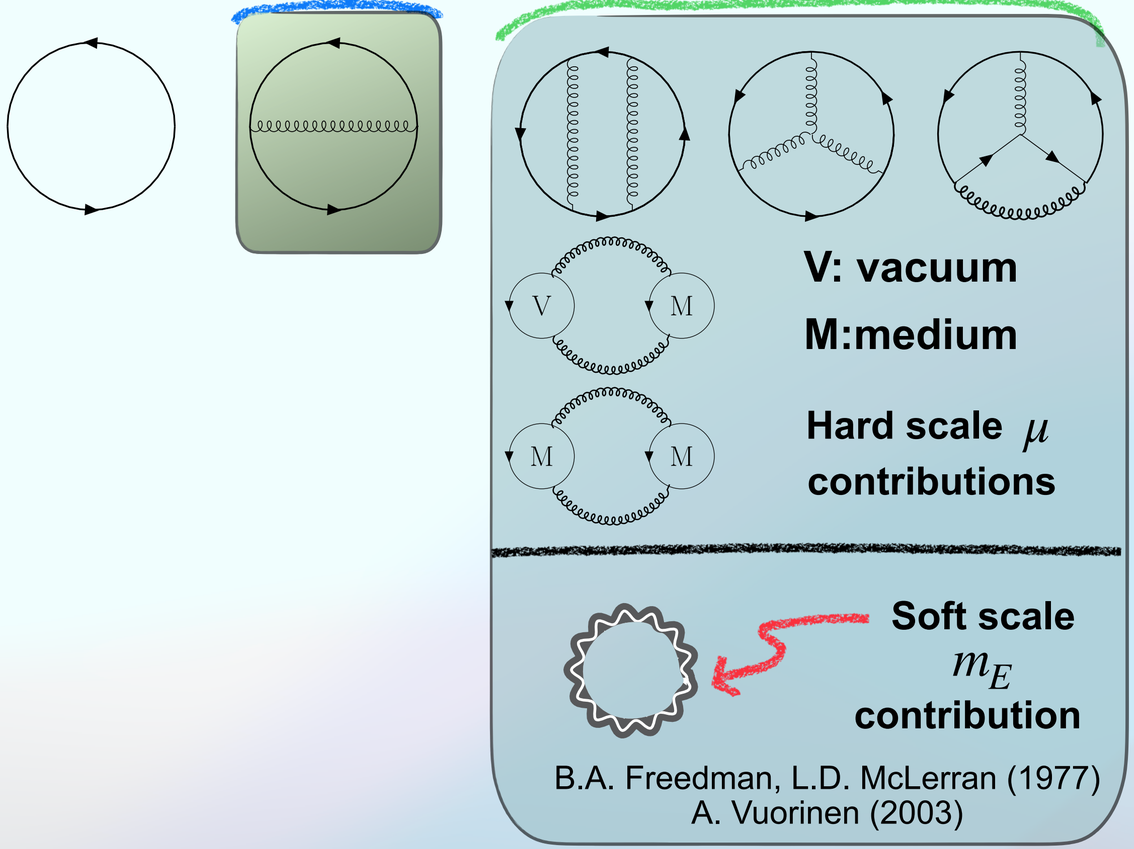
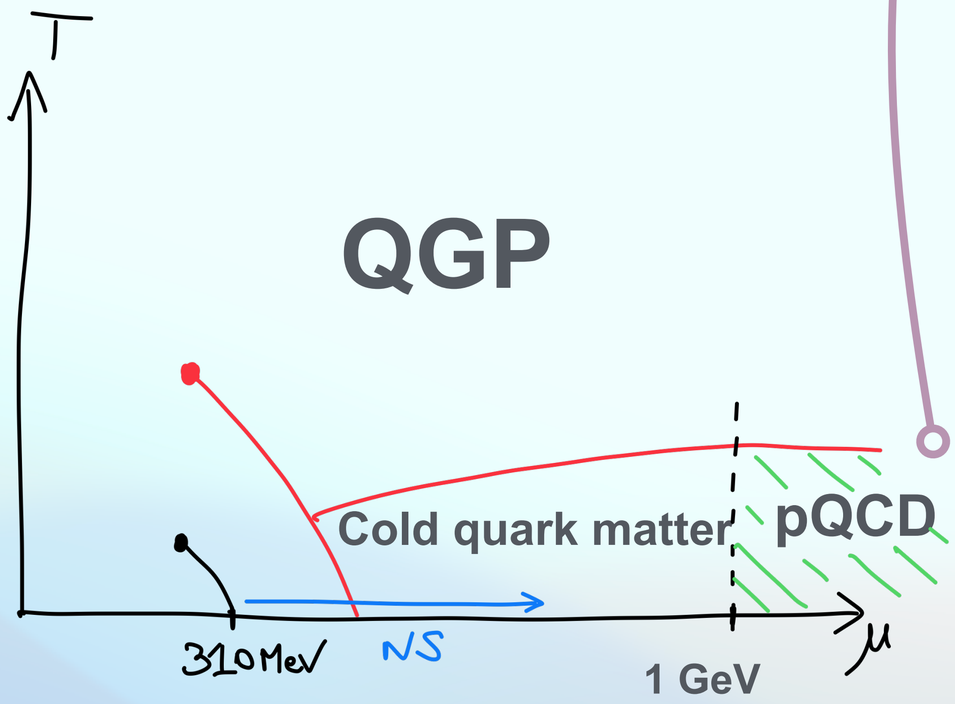


B.A. Freedman, L.D. McLerran (1977)
A. Vuorinen (2003)

Context



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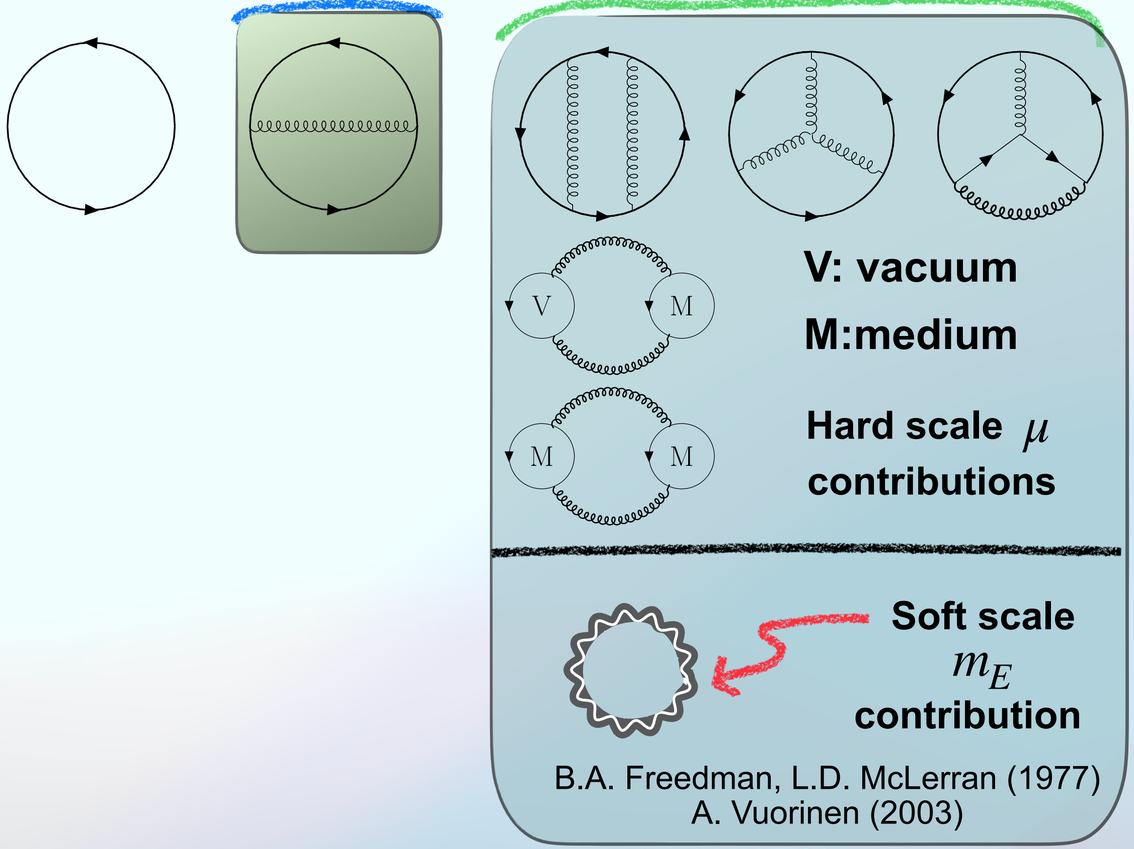

V: vacuum

M:medium

Hard scale μ contributions

Soft scale m_E contribution

B.A. Freedman, L.D. McLerran (1977)
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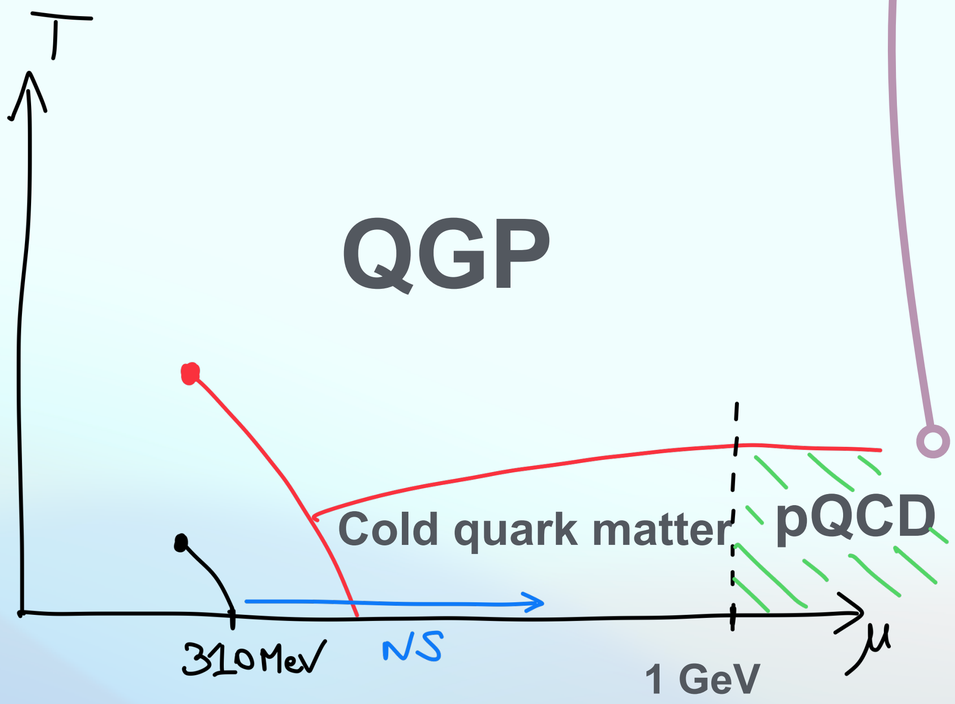
$$m_E \sim \sqrt{\alpha_s \mu}$$

Context



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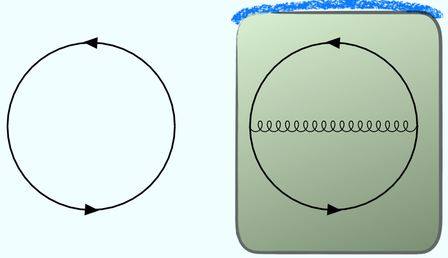
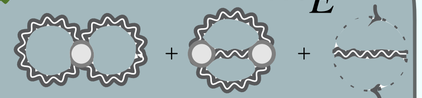


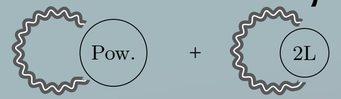
α_s^3

Soft sector m_E



Gorda et al., PRL 127, 162003 (2021)

Mixed sector μ, m_E

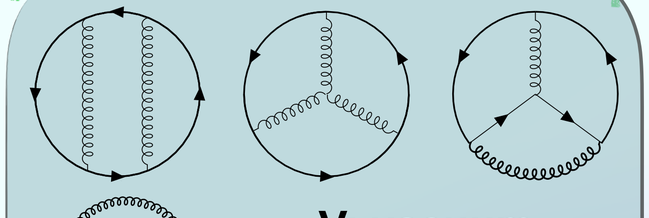


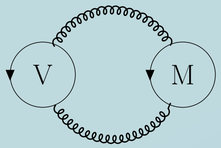
Gorda et al., PRL 131, 181902 (2023)

Hard sector μ

Kärkkäinen et al. 250117921

2026?





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Going beyond the perturbative expansion

3

Going beyond the perturbative expansion

$$\frac{P_{QCD}(T=0,\mu)}{P_f} = \underbrace{1 + \alpha_s\, A_1 + \alpha_s^2(\, A_2 + \, B_1 \ln(\alpha_s)) + \ldots}_{\textit{Renormalization scale dependent and missing higher orders (correlated)}}$$

Uncertainty estimate for missing higher orders <=> renormalization scale variation

$$\alpha_s(\Lambda) \to \Lambda \in \{\mu, 4\mu\}$$

$$g^2 = 4\pi\alpha_s$$

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↓ **Resummation**

Perturbative renormalization group

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Resummation

Perturbative renormalization group

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Soft scale

Resummation of logarithms in HTL effective theory

LF & J-L. Kneur, PRL 129 (2022)
And on-going work

Going beyond the perturbative expansion

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Resummation

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Hard scale

Renormalization group optimized perturbation theory (RGOPT)

Main topic of this talk

LF & J-L. Kneur, PRD 111, 034020 (2025)

1

Soft scale

Resummation of logarithms in HTL effective theory

LF & J-L. Kneur, PRL 129 (2022)

And on-going work

Resummation of logarithms in HTL effective theory

With massless quarks for simplicity

4

Physics at the soft scale

$$m_E \sim \sqrt{\alpha_s \mu}$$

Physics at the soft scale

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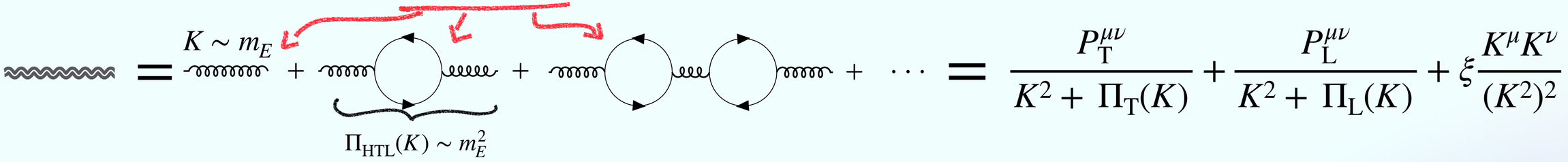
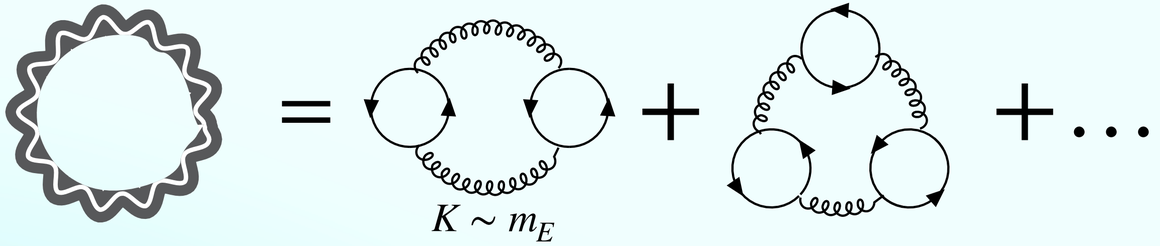
Parametrically of the same order



$$= \frac{P_{\mathrm{T}}^{\mu\nu}}{K^2 + \Pi_{\mathrm{T}}(K)} + \frac{P_{\mathrm{L}}^{\mu\nu}}{K^2 + \Pi_{\mathrm{L}}(K)} + \xi \frac{K^\mu K^\nu}{(K^2)^2}$$



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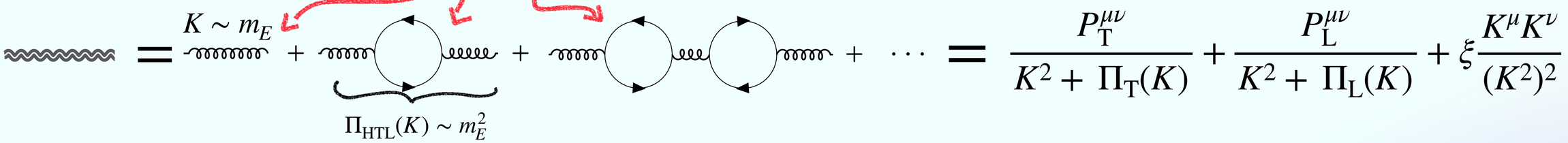

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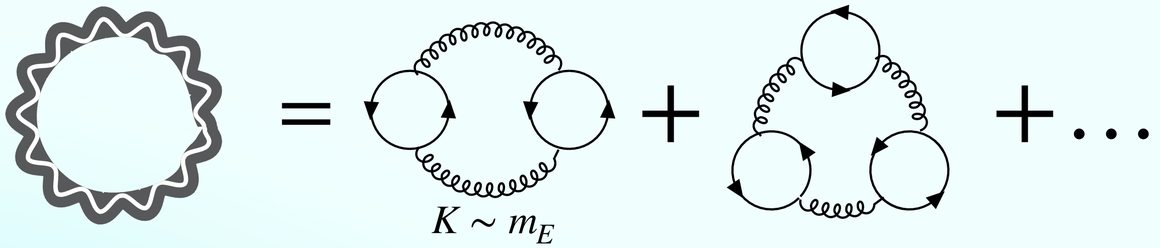


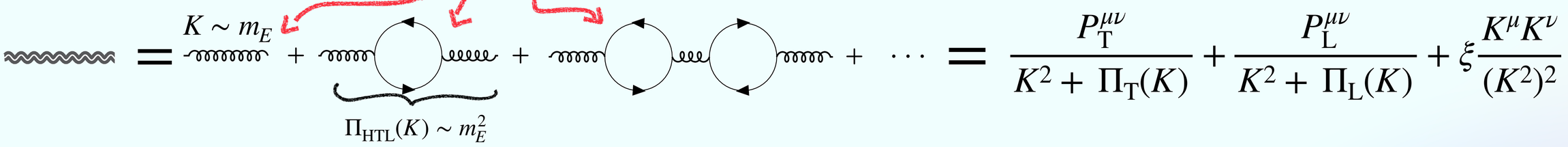


Hard Thermal Loop resummation.

Physics at the soft scale

$$m_E \sim \sqrt{\alpha_s}\mu$$

Parametrically of the same order



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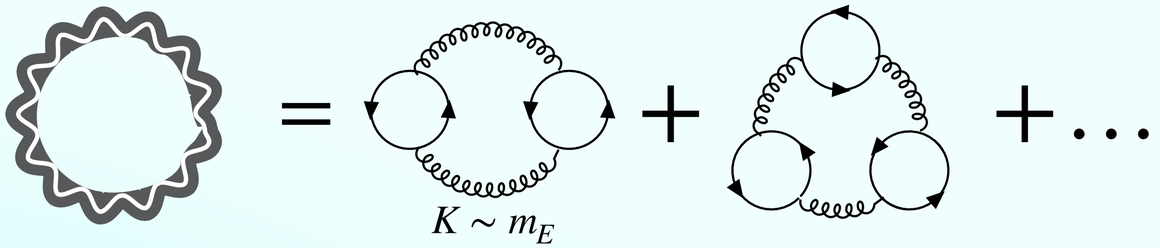




Hard Thermal Loop resummation.

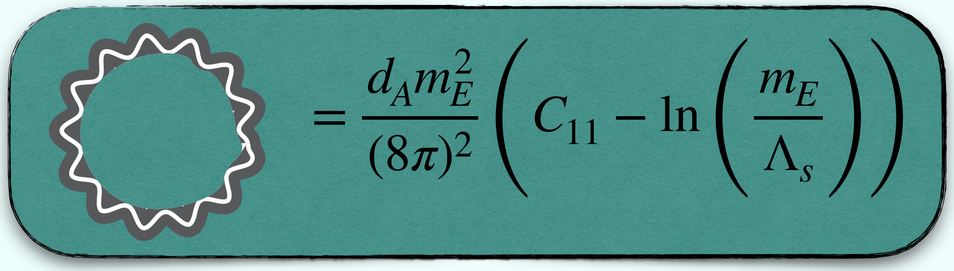
The soft contributions are encoded in the HTL effective Lagrangian !

$$L_{\mathrm{YM}}^{\mathrm{HTL}} = \frac{m_E^2}{2}\mathrm{Tr}\int_{\hat{v}} F^{\alpha\beta}\frac{v_\beta v^\gamma}{(v\cdot D_{\mathrm{adj.}})^2}F_{\gamma\alpha}$$

E. Braaten, R.D. Pisarski, PRD 45(1992)

HTL effective theory and resummation of logs

$$\frac{P_{QCD}(T=0,\mu)}{P_f} = 1 + \alpha_s\, A_1 + \alpha_s^2\left(A_2 + B_1 \ln(\alpha_s)\right) + \dots$$



$$= \frac{d_A m_E^2}{(8\pi)^2}\left(C_{11} - \ln\left(\frac{m_E}{\Lambda_s}\right)\right)$$

$$\Lambda_s \sim \mu \to \ln\frac{m_E}{\Lambda_s} \sim \ln\alpha_s$$

In this EFT: $L_{\rm YM} + L_{\rm YM}^{\rm HTL}$ **effective gluon mass.**

Anomalous mass dimension : γ_m^g

m_E **interpreted at first as blind to its physical origin.**

Therefore the RG operator reads:

$$\Lambda_s \frac{d}{d\Lambda_s} = \Lambda_s \frac{\partial}{\partial \Lambda_s} + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s} - m_E \gamma_m^g \frac{\partial}{\partial m_E}$$

6

HTL effective theory and resummation of logs

$$P^{\text{soft}} = m_E^4 \sum_i (\alpha_s)^{i-1} \sum_{k=0}^{i} a_{i,k} \ln^{i-k}\left(\frac{m_E}{\Lambda_s}\right)$$

$$\frac{P^{soft}}{m_E^4} = \begin{array}{llll} a_{1,0} L & +\alpha_s a_{2,0} L^2 & +\ldots & \text{LL} \\ +a_{1,1} & +\alpha_s a_{2,1} L & +\ldots & \text{NLL} \\ & +\alpha_s a_{2,2} & +\ldots & \end{array}$$

$$L = \ln\left(\frac{m_E}{\Lambda_s}\right)$$

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$$L = \ln\left(\frac{m_E}{\Lambda_s}\right)$$

Requiring $\Lambda_s \dfrac{dP^{\text{soft}}}{d\Lambda_s} = 0$ ↪

$$-p\, a_{i,0} = \left(4\gamma_0^g + 2b_0^g(i-2)\right) a_{i-1,0}$$

recurrence relation between leading soft logarithms (LL)

HTL effective theory and resummation of logs

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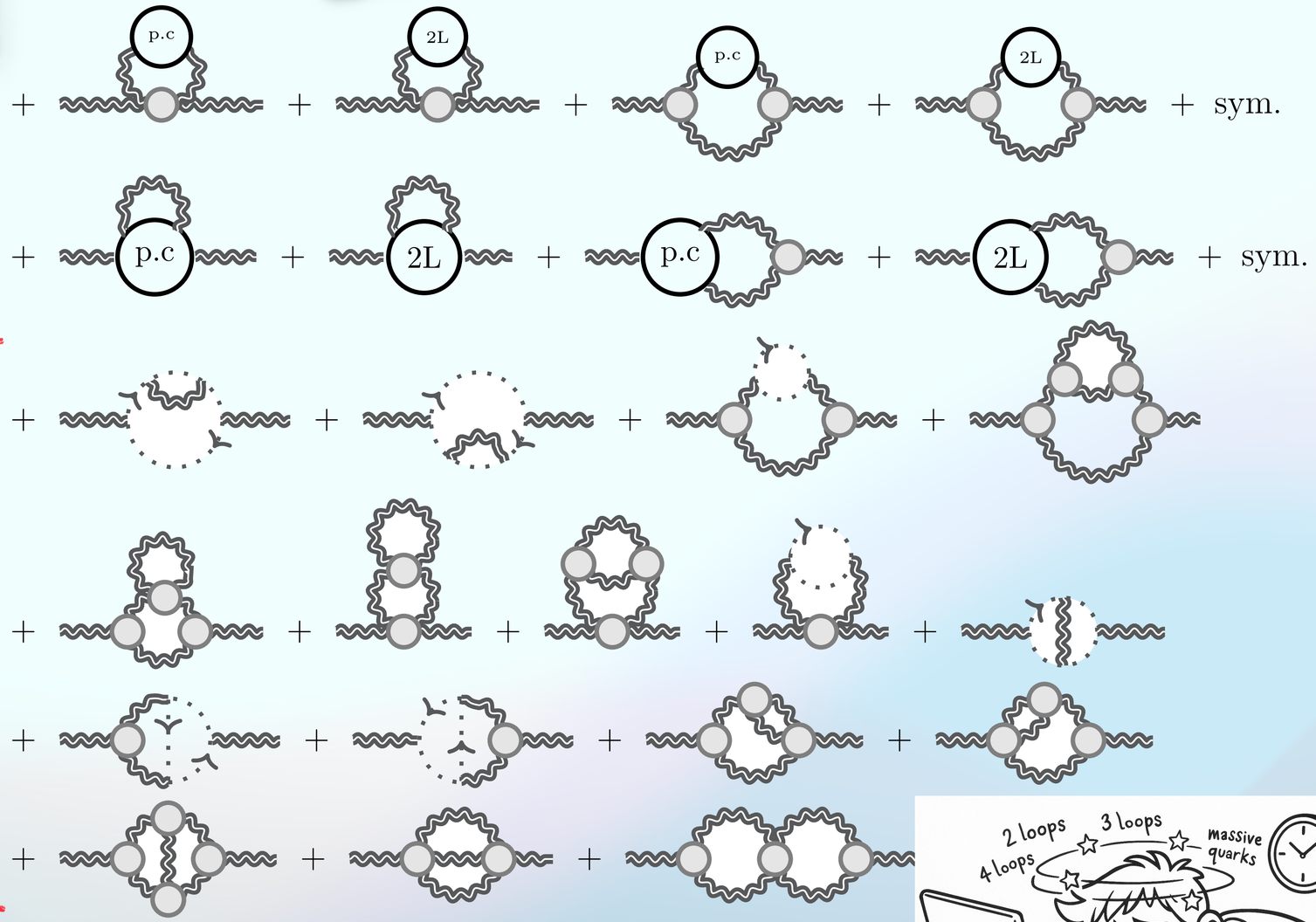
“Resummation of a resummation” (or resummation in two steps)

Glimpse of the next step

WORK IN PROGRESS

Full resummation of the NLL series



Need new anomalous dimension coefficients. A lot of fun ahead. :)

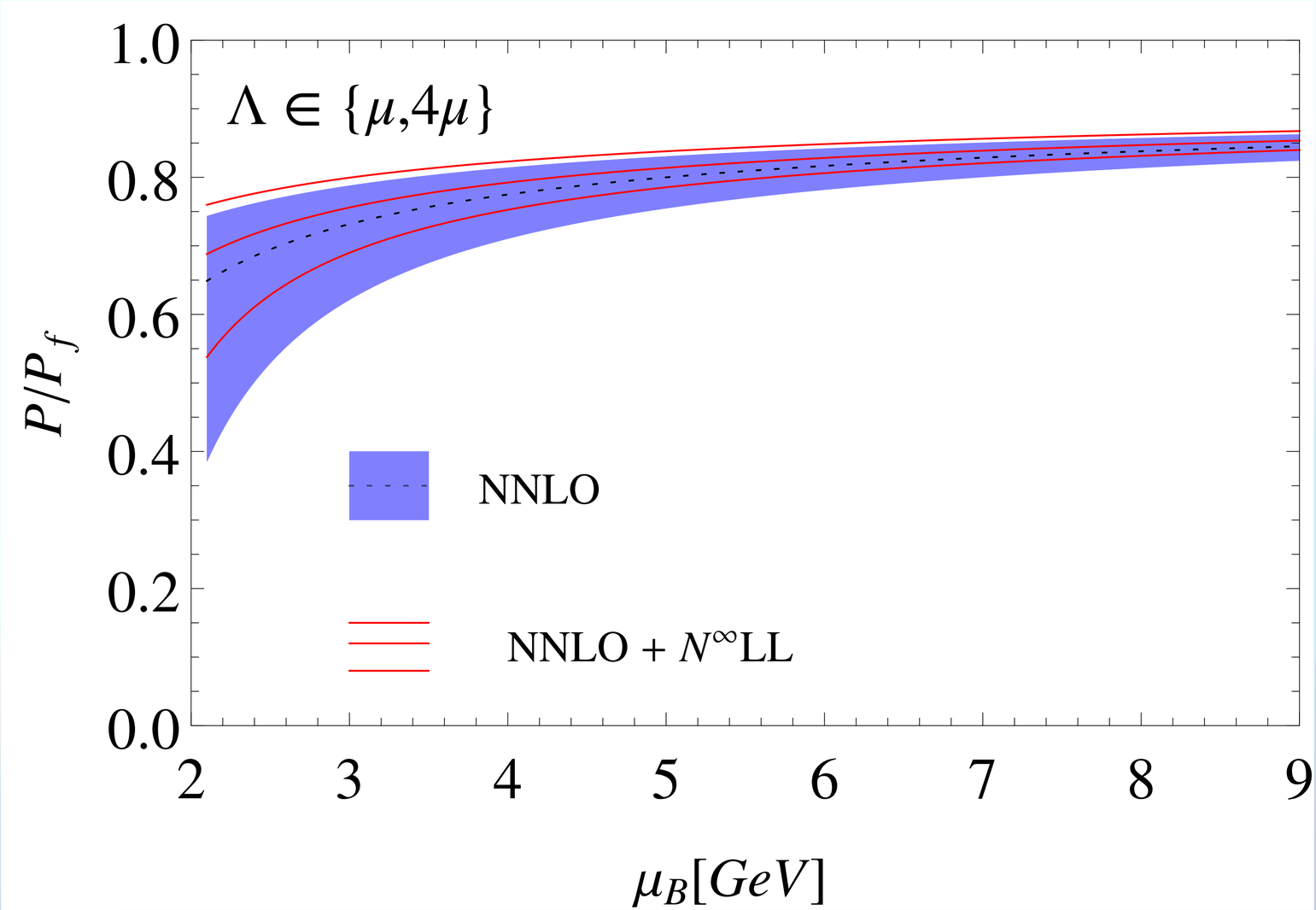
Completion of soft ones within reach

In collaboration with R. Paatelainen, A.Vuorinen, K. Seppänen, S. Säppi, J-L. Kneur



Resuming the LL

Significant reduction in renormalization scale dependence from resumming the leading soft logarithms.



Renormalization Group Optimized Perturbation Theory For Cold Quark matter at NNLO

Now we will focus on the hard sector with masses for the quarks !

(No Hard thermal Loops effective theory anymore)

Previous work at NLO: Kneur, Pinto, Restrepo 2019

10

Optimized/Screened Perturbation Theory (OPT/SPT)

In hot/dense QCD, the fields develop medium-dressed masses.
It seems best to start from a massive action. OPT does so and optimize the parameter (mass).

$$\mathcal{L}(\alpha_s) \to \mathcal{L}(\delta\alpha_s, m(1-\delta))$$

variational parameter

Interpolation

and fix m through local minima

$$\left.\frac{\partial P}{\partial m}\right|_{m=\tilde{m}} = 0$$

OPT eq.

SPT

Karsch, Patkos, Petreczky (1997)

Parwani (1992)

HTLpt = OPT + HTL

Andersen, Braaten, Petitgirard, Strickland (2002)

Andersen, Petitgirard, Strickland (2004)

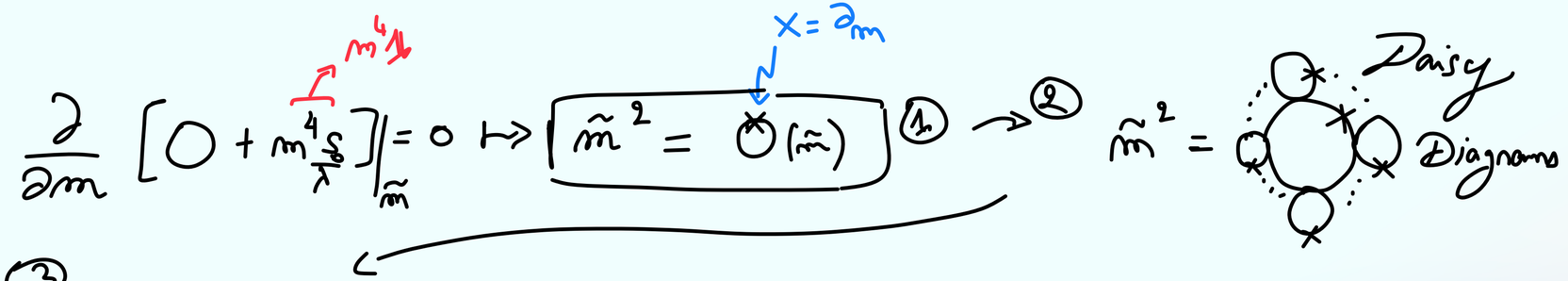
Andersen, Leganger, Strickland, Su, (2011)

Haque, Andersen, Mustafa, Strickland, Su,(2014)

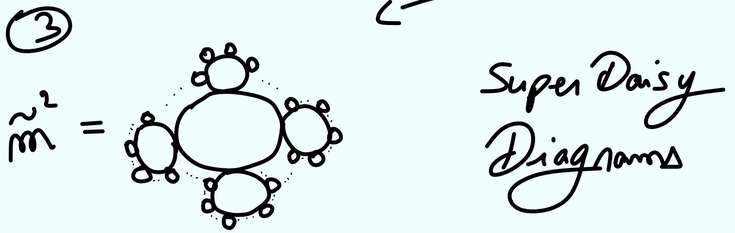
11

Visual representation of the resummation in RGOPT

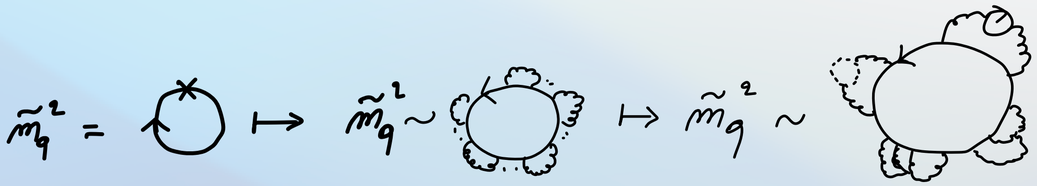
In $\lambda\phi^4$ with simplified picture:



Only a certain subclass of diagrams are resumed



In QCD it is more complicated to visualize:



12

Renormalization Group Optimized Perturbation Theory (RGOPT)



1. Starting from the massless action, introduce a *fictitious* mass for the fields *(here, the quarks)*.

Keep the vacuum contributions !

RGOPT

Kneur, Neveu, QCD T=0, (2013)

Kneur, Neveu , QCD T=0, (2015)

LF, Kneur, $\lambda\phi^4, \mathrm{NNLO},$ (2021)

Kneur, Pinto, Restrepo Hot QCD(2021)

Very close to Lattice results for $T \gtrsim T_c$

Renormalization Group Optimized Perturbation Theory (RGOPT)



① Starting from the massless action, introduce a *fictitious* mass for the fields *(here, the quarks)*.

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Vacuum energy anomalous mass dimension

② $\mathcal{L}(\alpha_s, m) \to \mathcal{L}(\alpha_s, m) + m^4\,\mathbb{1}$

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④ Evaluate your observable at $O(\alpha_s^k)$ then expand in δ at the same order and take $\delta \to 1$

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RG eq. $\left[\Lambda\frac{\partial}{\partial\Lambda}\mathcal{P} + \beta(g^2)\frac{\partial}{\partial g^2}\mathcal{P}\right] = 0$

Solved once and for all !

Solve at leading order for any m

$a = \frac{\gamma_0}{2b_0}$, **Universal**

RGOPT

Kneur, Neveu, QCD T=0, (2013)

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Very close to Lattice results for $T \gtrsim T_c$

Renormalization Group Optimized Perturbation Theory (OPT)

⑤ Last step:

Solve “m” through OPT *or* RG equation which incorporates the *all order resummation.*

$$\left[\Lambda\frac{\partial}{\partial\Lambda}\mathcal{P}+\beta(g^2)\frac{\partial}{\partial g^2}\mathcal{P}\right]_{\overline{m}_{RG}}=0$$

$$\left.\frac{\partial P}{\partial m}\right|_{m=\tilde{m}}=0$$

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Multiple solutions, but *only one* satisfies asymptotic freedom thanks to “a”.

NB: at NNLO only RG gives physically consistent screening mass

$$\begin{pmatrix} \overline{m}_{\mathrm{RG},1} & \tilde{m}_1 \\ \overline{m}_{\mathrm{RG},2} & \tilde{m}_2 \\ \overline{m}_{\mathrm{RG},3} & \tilde{m}_3 \\ \dots & \dots \end{pmatrix}$$

$$\mathcal{L}(\alpha_s, m) \to \mathcal{L}(\delta\alpha_s, m(1-\delta)^a)$$

Renormalization Group Optimized Perturbation Theory (OPT)

⑤ Last step:

Solve “m” through OPT *or* RG equation which incorporates the *all order resummation.*

$$\left[\Lambda\frac{\partial}{\partial\Lambda}\mathcal{P}+\beta(g^2)\frac{\partial}{\partial g^2}\mathcal{P}\right]_{\overline{m}_{RG}}=0 \qquad \left.\frac{\partial P}{\partial m}\right|_{m=\tilde{m}}=0$$

Multiple solutions, but *only one* satisfies asymptotic freedom thanks to “a”.

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What if this solution is not real ?

Use the arbitrariness of the renormalization scheme to change to a new one where the coefficients in the pressure lead to a real solution.

Renormalization scheme change (RSC)

$$\text{2bis}\quad m\to m\left(1+B_2\ g^4\right)$$

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Det. eq. $$\frac{\partial f_{\mathrm{RG}}}{\partial g^2}\frac{\partial f_{\mathrm{OPT}}}{\partial m}-\frac{\partial f_{\mathrm{RG}}}{\partial m}\frac{\partial f_{\mathrm{OPT}}}{\partial g^2}=0$$

Solving RG & Det. : $(\overline{m}_{\mathrm{RG}},\tilde{B}_2)$

For Cold Quark Matter at NNLO, the prescription is unique !

Cold Quark Matter At NNLO

Kurkela, Romatschke, Vuorinen, PRD 81, 105021 (2010)

Setup:



$$(0,\mu_u)\quad (0,\mu_d)\quad (m_s,\mu_s)$$

massive quark pressure

$$P(m,\mu) = P_{\rm LO}(m,\mu) + P_{\rm NLO}(m,\mu) + P_{\rm NNLO}(m,\mu)$$

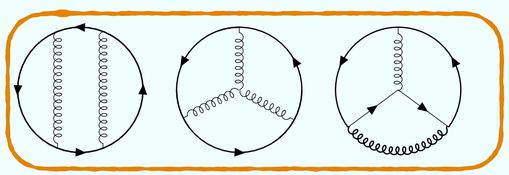
$$P_{\rm LO}(m,\mu) = \Theta(\mu^2-m^2)\frac{N_c}{12\pi^2}\left[\mu\, p_F\left(\mu^2-\frac{5}{2}m^2\right)+\frac{3}{2}m^4\ln\left(\frac{\mu+p_F}{m}\right)\right]$$

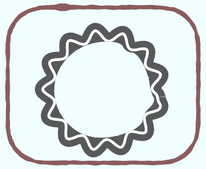
$$P_{\rm NLO}(m,\mu) = -\,\Theta(\mu^2-m^2)\frac{d_A g^2}{4(2\pi)^4}\left\{3\left[m^2\ln\left(\frac{\mu+p_F}{m}\right)-\mu\, p_F\right]^2-2p_F^4\right\}$$

$$-\,\Theta(\mu^2-m^2)\frac{d_A g^2}{4(2\pi)^4}m^2\left(4-6\ln\frac{m}{\Lambda}\right)\left[\mu\, p_F - m^2\ln\left(\frac{\mu+p_F}{m}\right)\right]$$

α_s

Flashing expressions





Generalized for arbitrary massive quarks

$$\mathcal{M}_3^{N_f=2+1^*} = \frac{d_A\mu^4}{2\pi^2}\Big\{-\hat{m}^2[(11C_A-2N_f)z+18C_F(2z-\hat{u})](L_m)^2$$
$$-\frac{1}{3}\Big[C_A\left(22\hat{u}^4-\frac{185}{2}z\hat{m}^2-33z^2\right)+\frac{9C_F}{2}(16\hat{m}^2\hat{u}(1-\hat{u})-3(7\hat{m}^2-8\hat{u})z-24z^2)$$
$$-N_f(4\hat{u}^4-13z\hat{m}^2-6z^2)\Big]L_m+C_A\left(-\frac{11}{3}\ln\frac{\hat{m}}{2}-\frac{71}{9}+G_1(\hat{m})\right)+C_F\left(\frac{17}{4}+G_2(\hat{m})\right)$$
$$+N_f\left(\frac{2}{3}\ln\frac{\hat{m}}{2}+\frac{11}{9}+G_3(\hat{m})\right)+G_4(\hat{m})\Big\},\quad \alpha_s^2$$

$$\Omega_{\rm Ring}^{N_f=2+1^*} = \frac{d_A\, g^4}{512\pi^6}\Big\{(\vec{\mu}^2)^2\left[\left(2\ln\left(\frac{g}{4\pi}\right)-\frac{1}{2}\right)+\frac{1}{2}\left(-\frac{19}{3}+\frac{2\pi^2}{3}+\frac{I_{15}(\vec{\mu})}{(\vec{\mu}^2)^2}+\frac{16}{3}(1-\ln 2)\ln(2)+I_{16}(\hat{m},\vec{\mu}^2)\right)\right]$$
$$+2\mu^2\sum_{i=1}^{N_l=2}\mu_i^2\left[I_{14}\left(2\ln\left(\frac{g}{4\pi}\right)-\frac{1}{2}\right)+\frac{1}{2}\left(I_{17}(\hat{m},\mu_i)+\frac{16}{3}(1-\ln 2)\ln(2)\,I_{18}+I_{19}(\hat{m},\vec{\mu}^2)\right)\right]$$
$$+\mu^4\left[I_{13}\left(2\ln\left(\frac{g}{4\pi}\right)-\frac{1}{2}\right)+\frac{1}{2}\left(I_{20}(\hat{m})+\frac{16}{3}(1-\ln 2)\ln(2)I_{21}+I_{22}(\hat{m},\vec{\mu}^2)\right)\right]\Big\},\quad \vec{\mu}=(\mu_1,\mu_2)$$

$$\Omega_{{\rm VM},x}^{N_f=2+1^*} = d_A\frac{g^4}{(4\pi)^2}\frac{m^4}{12\pi^2}\sum_{i=1}^{N_l=2}I_x\left(\frac{\mu_i}{m+\mu_i}\right)\alpha_s^2\ln\alpha_s$$

(Without exhaustive detail)

15

Cold Quark Matter At NNLO

With RGOPT

Setup:



(m,μ) (m,μ) $(m+m_s,\mu)$

m_3

Need generalization for three massive quarks with different masses !

We reproduced* existing results based on their detailed calculation and derived the generalization to 3 massive quarks including mixing terms

*We identified a mistake in G2 but numerically irrelevant for physical quark masses.

LF & J-L. Kneur, PRD 111, 034020 (2025)

Flashing expressions for the curious

Vacuum contributions with two masses $m_3 = m + m_s$

$$P_{\rm NNLO}^{v,N_f=2^*+1^*} = \left(\frac{g^4}{\pi^2\,(4\pi)^4}\right)\Big((N_f-1)m^4\left(-156.833+253.775L_m-243L_m^2+100L_m^3\right)$$
$$+2(N_f-1)m^2m_3^2\left(-4.75802-6\left(L_m+L_{m_3}\right)\right)$$
$$+\,m_3^4\left(-152.075+265.775L_{m_3}-243L_{m_3}^2+100L_{m_3}^3\right)\Big),$$

Contribution from the vac. anomalous dimension

$$P_{\rm sub}^{N_f=2^*+1^*} \equiv -\,(N_f-1)\frac{m^4}{g^2}\left(s_0+s_1\,g^2\right)-\frac{m_3^4}{g^2}\left(s_0+s_1\,g^2\right)$$
$$-\,(N_f-1)s_{2,1}m^4\,g^2-s_{2,3}\,m_3^4\,g^2-2(N_f-1)s_2^{nd}\,m^2m_3^2\,g^2.$$

Modify $\mathcal{M}_3^{N_f=2+1^*}$ *to account for multiple masses*

$$N_f\left(\frac{2}{3}\ln\frac{\hat m}{2}+\frac{11}{9}+G_3(\hat m)+G_4(\hat m)\right)\to N_f\left(\frac{2}{3}\ln\frac{\hat m_i}{2}+\frac{11}{9}+G_3^{\rm bis}(\hat m_i)\right)+\hat m_i^2 z(m_i)\sum_j^{N_f}\Big\{\mathrm{Li}_2(\nu_{ij})\left(1+\nu_{ij}^{-2}\right)$$
$$-\,\Phi\left(\nu_{ij},2,\frac{3}{2}\right)(1+\nu_{ij})-\nu_{ij}^{-2}+\mathcal{F}(\nu_{ij})\Big\}-\sum_j^{N_f}\frac{4}{3}I_{12}^{\rm bis}(\hat m_i,\nu_{ij}),$$

(Without exhaustive detail)

16

Cold Quark Matter At NNLO

With RGOPT

Setup:

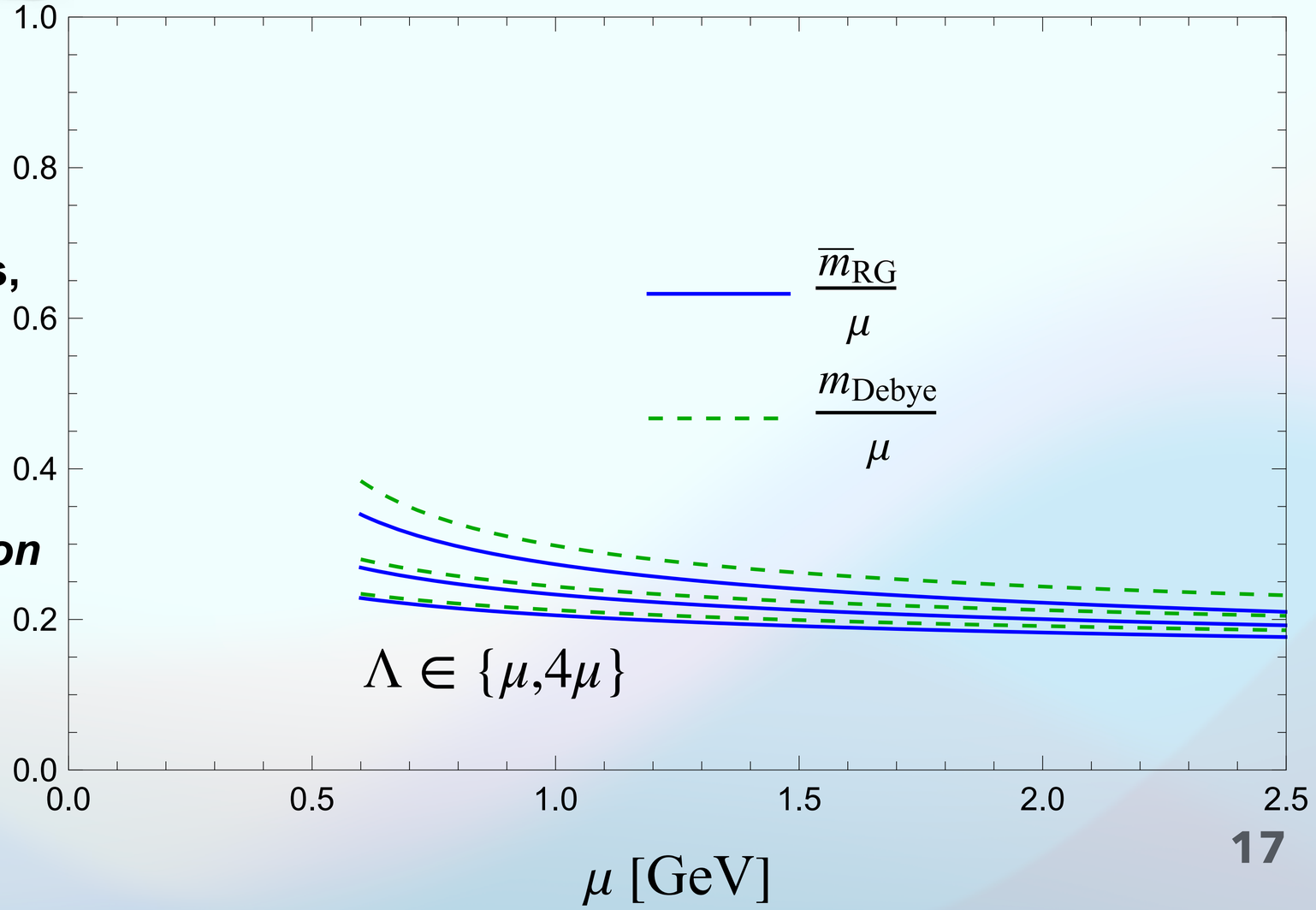


(m,μ) (m,μ) $(m+m_s,\mu)$

Solving the RG (and det.) equations, we get $\overline{m}_{\rm RG}$

This solution agrees with the Debye mass at large densities (small α_s **) but encodes an *all order resummation***

$$m_{\rm D}^2 = \frac{g^2\, C_F}{8\pi^2}\mu^2.$$



17

Cold Quark Matter At NNLO

With RGOPT

Setup:



(m, μ) (m, μ) $(m + m_s, \mu)$

Implementing the solutions

Deviation from $\overline{\text{MS}}$

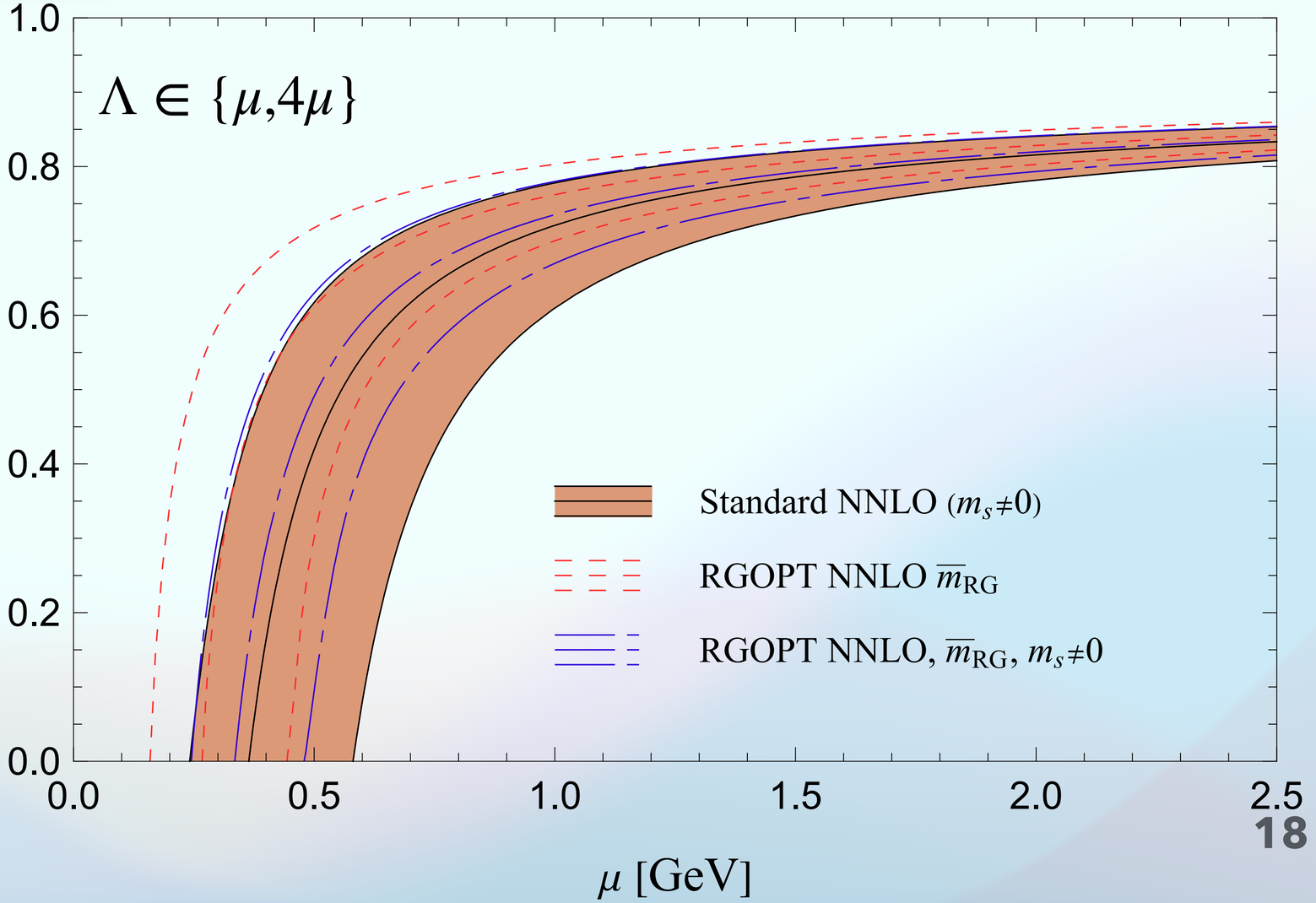
$$\{\overline{m}_{\text{RG}}, \tilde{B}_2\}$$

$$\frac{P}{P_f}$$

Noticeably reduced dependence with respect to renormalization scale variations.

$$\frac{\mathcal{P}^{N_f=2^*+1^*}_{\text{RGOPT}}(\mu, \Lambda = X\,\mu)}{\mathcal{P}_f(\mu, N_f = 3)} = (\quad + c_2 X^{\nu_3}) - \frac{d_1 (3\tilde{\mu})^{\alpha_1} X^{\nu_1}}{(3\tilde{\mu} - d_2\, X^{-\nu_2})},$$

Pocket formula



Cold Quark Matter At NNLO

With RGOPT

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Implementing the solutions

Deviation from $\overline{\text{MS}}$

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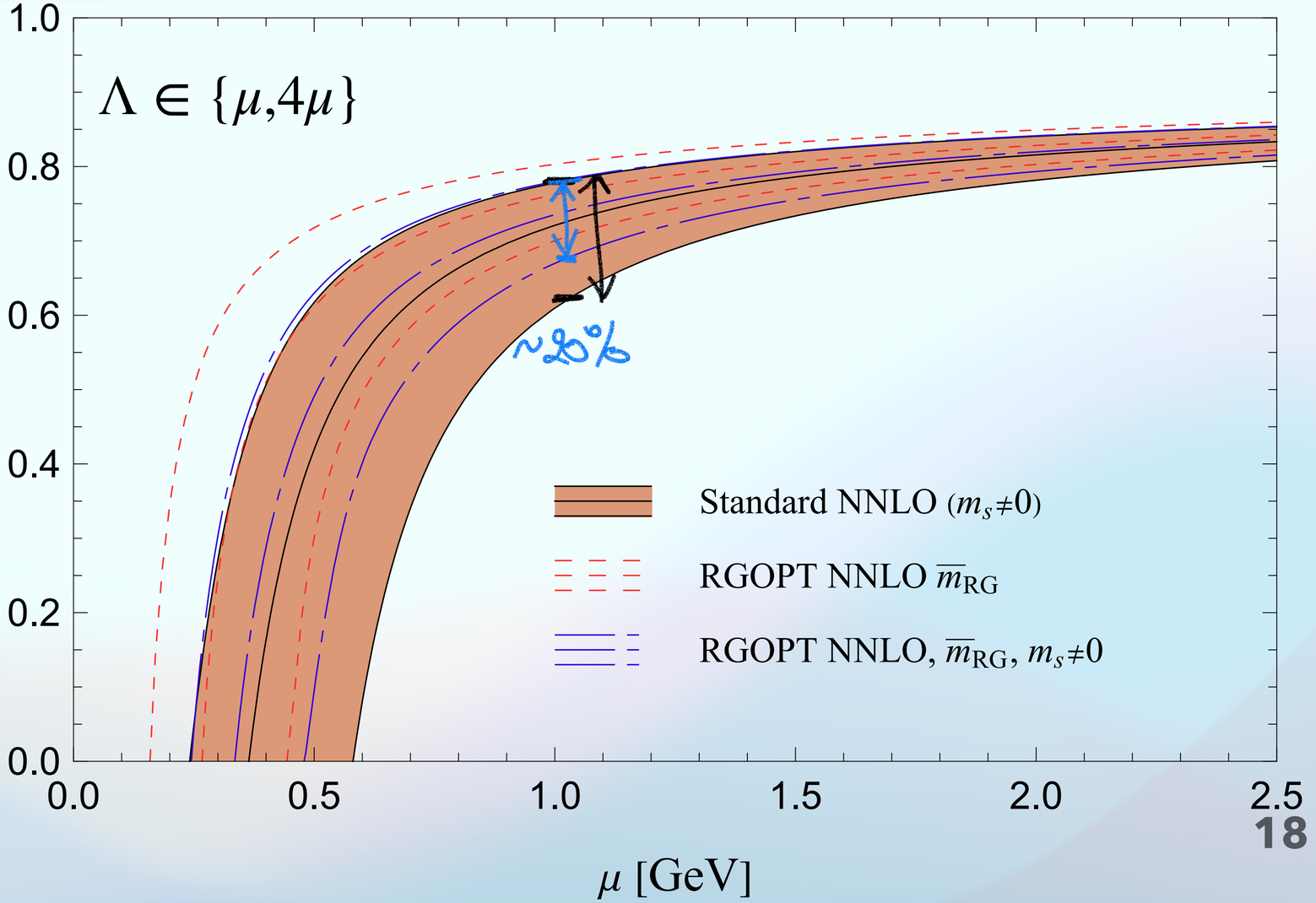


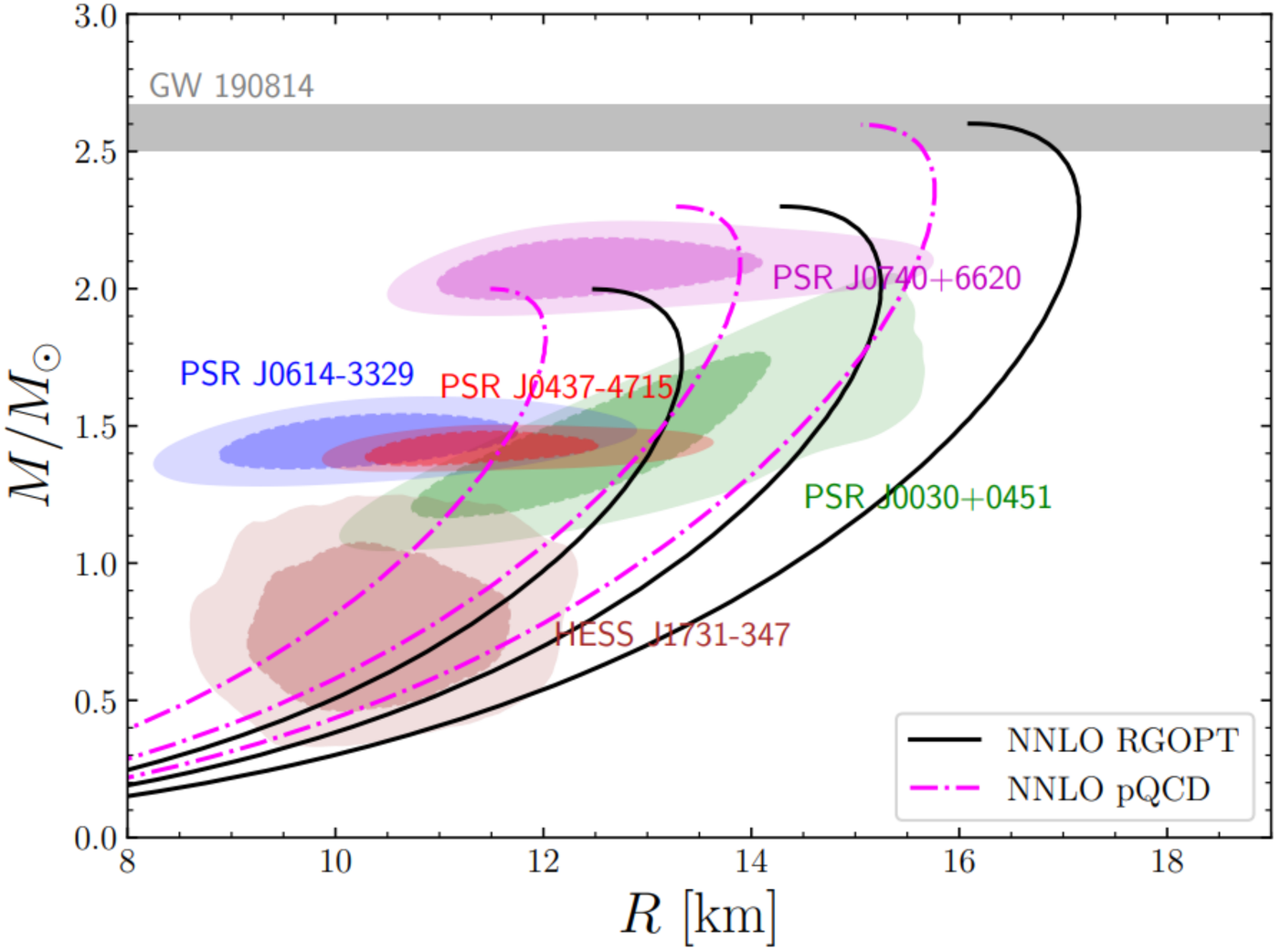
Application to quark stars

Arxiv:2606.29576

Setup:



(m, μ_u) (m, μ_d) $(m+m_s, \mu_s)$



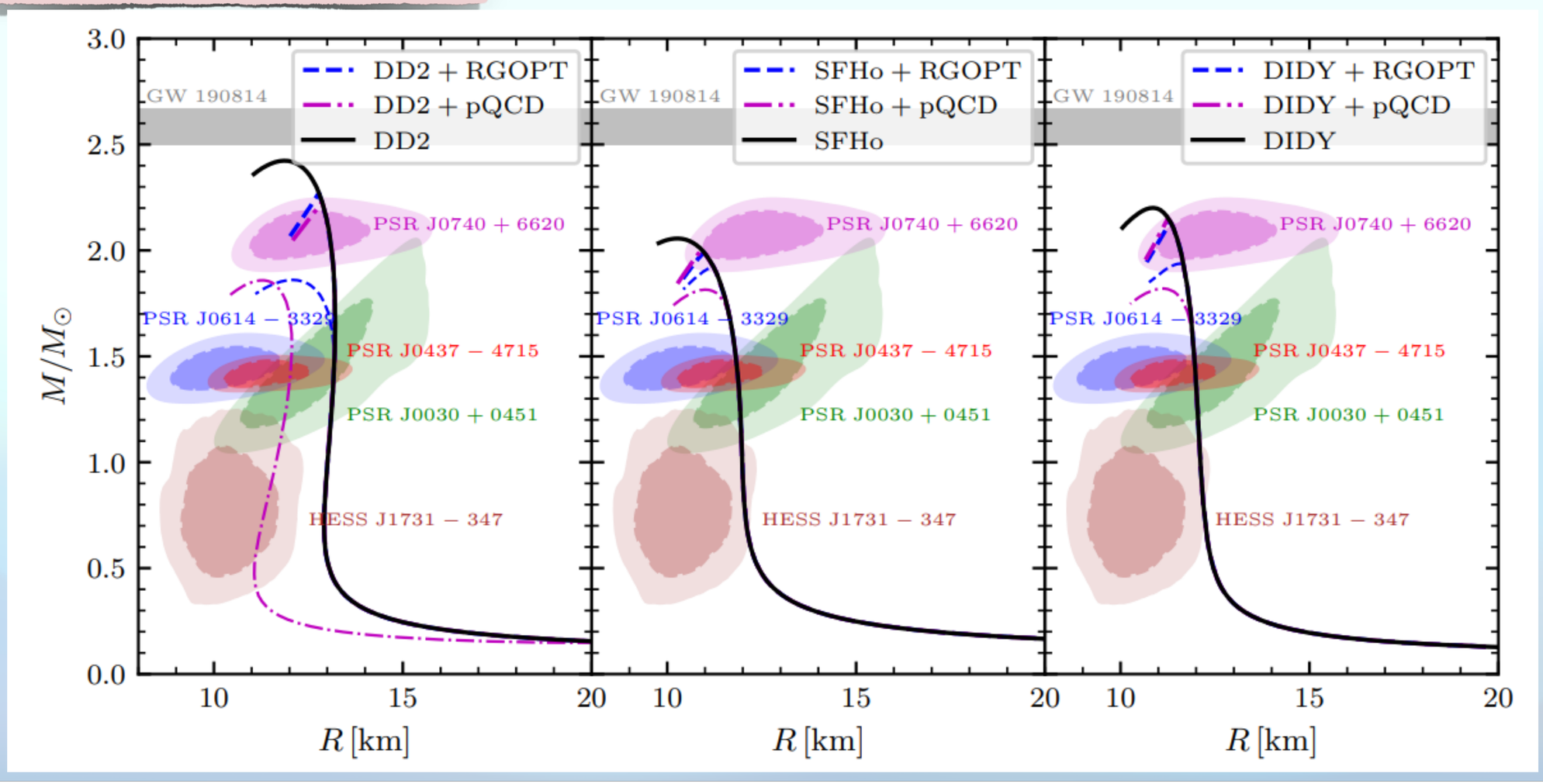
Application to neutron stars

Arxiv:2606.29576

Setup:



(m, μ_u) (m, μ_d) $(m+m_s, \mu_s)$



20

Summary

1. RGOPT has been quite effective at zero temperature and at high temperature.

2. At finite densities and zero temperature the improvements are less efficient but still robust as the NNLO application demonstrated.

3. The apparent “freedom” in choosing a prescription for “m” is in practice not one since only one prescription is physically relevant.

4. Resummation of soft logarithms is also a powerful improvement to Cold and Dense QCD. The combination of the two approaches (RGOPT + Logs) remains however complicated. ***Can it also be used in the soft sector ?***

21

Backup

Cut results

$$\mathcal{M}_3^{1c} = \frac{d_A \mu^4}{(2\pi)^2}\Bigg\{\Bigg(- C_A\Big[\Big(22\ln\frac{\Lambda}{m} + \frac{185}{3}\Big)\ln\frac{\Lambda}{m} + \frac{1111}{24} - \frac{4\pi^2}{3} + 4\pi^2\ln 2 - 6\zeta(3)\Big]$$
$$- C_F\Big[3\Big(12\ln\frac{\Lambda}{m} + 5\Big)\ln\frac{\Lambda}{m} + \frac{313}{8} + \frac{35\pi^2}{6} - 8\pi^2\ln 2 + 12\zeta(3)\Big] + N_f\Big[\frac{2}{3}\Big(6\ln\frac{\Lambda}{m} + 13\Big)\ln\frac{\Lambda}{m} + \frac{71}{12} + \frac{2\pi^2}{3}\Big]$$
$$+ 6 - 2\pi^2\Bigg)\hat{m}^2 z + 4\hat{m}^2 C_F\Big[3\Big(3\ln\frac{\Lambda}{m} + 4\Big)\ln\frac{\Lambda}{m} + 4\Big](\hat{u} - z)\Bigg\},$$

$$\frac{\mathcal{M}_3^{2c}}{(4\pi)^2} = d_A\Bigg\{C_A\Big(- \frac{16}{9}I_1^2 + \frac{62}{9}m^2 I_2 + \frac{5}{3}I_{1c} - \frac{10}{3}m^2 I_{2c} + I_{10} - \frac{22}{3}[I_1^2 - 2m^2 I_2]\ln\frac{\Lambda}{m}\Big)$$
$$+ C_F\Big(I_{11} + 24m^2[I_2 - I_{1b}I_1 + 2m^2 I_{2b} + 2m^2 I_8]\ln\frac{\Lambda}{m}\Big) + N_f\Big(\frac{10}{9}I_1^2 - \frac{20}{9}m^2 I_2 - \frac{2}{3}I_{1c} + \frac{4}{3}m^2 I_{2c}$$
$$+ [\frac{4}{3}I_1^2 - \frac{8}{3}m^2 I_2]\ln\frac{\Lambda}{m}\Big) - \frac{2}{3}I_{12}\Bigg\},$$

$$\frac{\mathcal{M}_3^{3c}}{(4\pi)^4} = - d_A\Bigg\{C_A\Big[2I_1 I_2 - 4(I_5 + I_7 - 2m^4 I_6)\Big] + C_F\Big[2I_1^2 I_{1b} - 4I_1 I_2 - 8m^2 I_1 I_{2b} + 8m^2 I_3 + 8m^4 I_{3b} - 2I_4$$
$$+ 8(I_5 + I_7 - 2m^4 I_6) - 8m^2 I_1 I_8 + 8m^4 I_9\Big]\Bigg\},$$

22

Backup

Renormalization group material

$$\Lambda_{\overline{\rm MS}} = \Lambda e^{-\frac{1}{2b_0 g^2}} \left(\frac{b_0 g^2}{1+\frac{b_1}{b_0} g^2}\right)^{-\frac{b_1}{2b_0^2}} \qquad m_s(\Lambda) = m_s(\Lambda_0) \left(\frac{g^2(\Lambda)}{g^2(\Lambda_0)}\right)^{\frac{\gamma_0}{2b_0}} \left(\frac{1+\frac{b_1}{b_0} g^2(\Lambda)}{1+\frac{b_1}{b_0} g^2(\Lambda_0)}\right)^{\frac{\gamma_1}{2b_1}-\frac{\gamma_0}{2b_0}} \qquad \Lambda\frac{d}{d\Lambda} = \Lambda\frac{\partial}{\partial\Lambda} + \beta(g^2)\frac{\partial}{\partial g^2} - \gamma_m(g^2) m\frac{\partial}{\partial m} ,$$

$$\beta\left(g^2 \equiv 4\pi\alpha_s\right) = -2b_0 g^4 - 2b_1 g^6 + \cdots , \qquad \gamma_m\left(g^2\right) = \gamma_0 g^2 + \gamma_1 g^4 + \cdots , \qquad \Lambda\frac{d}{d\Lambda}\left[S(m,g)\right] \equiv \hat{\Gamma}^0(g) m^4 = m^4 \sum_{k\geq 0} \Gamma_k^0 g^{2k}$$

$$s_0 = \frac{\Gamma_0^0}{2\left(b_0 - 2\gamma_0\right)} = \frac{3}{7} \quad , \quad s_1 = \frac{1}{2\gamma_0}\left((b_1 - 2\gamma_1)s_0 - \frac{\Gamma_1^0}{2}\right) = -\frac{53}{224\pi^2}$$

$$s_2(N_h, N_l) = \frac{1}{b_0 + 2\gamma_0}\left((b_2 - 2\gamma_2)s_0 - 2\gamma_1 s_1 - \frac{\Gamma_2^0}{2}\right) \underset{N_h=3, N_l=0}{=} -0.00040082$$

$$s_3(N_h, N_l) = \frac{1}{2(b_0 + \gamma_0)}\left((b_3 - 2\gamma_3)s_0 - 2\gamma_2 s_1 - (b_1 + 2\gamma_1)s_2 - \frac{\Gamma_3^0}{2}\right) \underset{N_h=3, N_l=0}{=} -0.00008304.$$

$$s_{2,1} = s_2(N_h = 2, N_l = 1) = -0.00024683, \quad s_{2,3} = s_2(N_h = 1, N_l = 2) = -0.00009284, \quad s_2^{nd} = \frac{9}{8\pi^4(2N_f - 81)} \tag{A10}$$

23

Backup

Renormalization scheme change (RSC)

$$\gamma_2 \to \gamma_2' = \gamma_2 - 4b_0\, B_2$$
$$\gamma_3 \to \gamma_3' = \gamma_3 - 4b_1\, B_2$$

$$s_2 \to s_2' = s_2 + \frac{8b_0 s_0}{b_0 + 2\gamma_0} B_2 = -0.00040082 + \frac{216}{175} B_2,$$
$$s_{2,i} \to s_{2,i}' = s_{2,i} + \frac{216}{175} B_2,\ i = \{1, 3\}$$
$$(s_2^{nd})' = s_2^{nd}$$

$$s_3 \to s_3' = s_3 + \frac{4B_2}{b_0 + \gamma_0}\left(b_0 s_1 - \frac{b_0}{4}(a_{10} + 4a_{11}) + 2\frac{s_0}{b_0 + 2\gamma_0}(b_1\gamma_0 - b_0\gamma_1)\right) = -0.00008304 - \frac{111}{280\pi^2} B_2$$

where $a_{10} = -2(b_0 - 2\gamma_0)s_0$, $a_{11} = -(3/4)a_{10}$.

24

Backup

Fitting functions

$$G_1(\hat{m}) = 32\pi^4\hat{m}^2\Big(0.001715 - 0.000339\,\hat{u} + 0.002818\,\hat{u}^2 - 0.002282\,\hat{u}^3 + 0.005854\ln\hat{m} - 0.018427\,\hat{m}^2\ln^3\hat{m} + 0.000444\,\hat{m}^2\ln\left(\frac{1+\hat{u}}{\hat{m}}\right)\Big)$$

$$G_2(\hat{m}) = 32\pi^4\hat{m}^2\Big(-0.001363 + 0.000401\,\hat{u} + 0.003454\,\hat{u}^2 - 0.002983\hat{u}^3 + 0.021502\hat{u}^4 + 0.017914\ln\hat{m} - 0.032789\,\hat{m}^2\ln^2\hat{m} + 0.002067\,\hat{m}^2\ln\left(\frac{1+\hat{u}}{\hat{m}}\right)\Big)$$

$$G_3(\hat{m}) = 32\pi^4\hat{m}^2\Big(-0.000244 - 0.002192\,\hat{u} + 0.000086\,\hat{u}^2 + 0.001895\,\hat{u}^3 + 0.000054\ln\hat{m} + 0.000521\ln^2\hat{m} + 0.002176\,\hat{m}^2\ln\left(\frac{1+\hat{u}}{\hat{m}}\right)\Big)$$

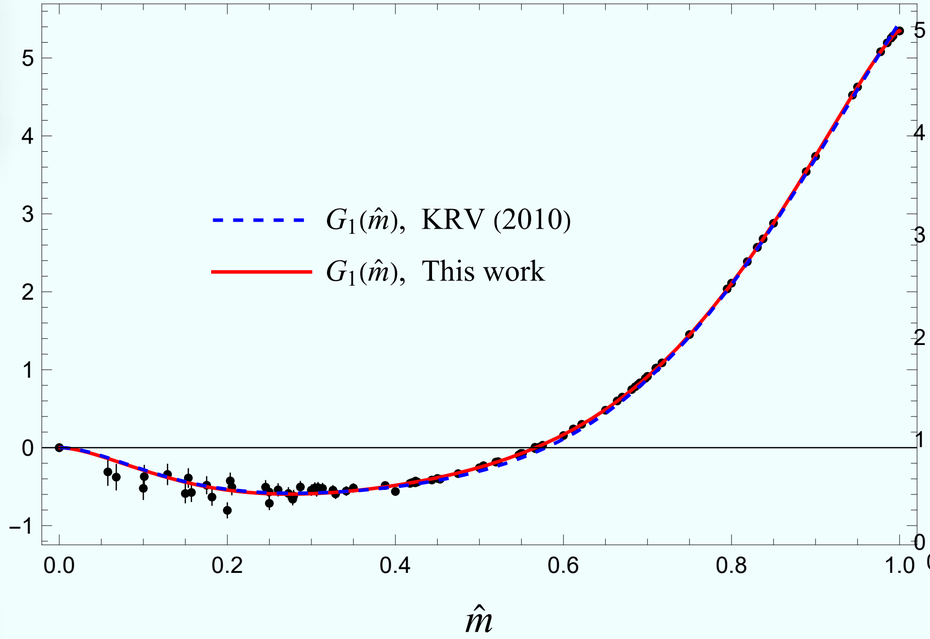
$$G_4(\hat{m}) = 32\pi^4\hat{m}^2\Big(-0.0020405\,\hat{u} + 0.0003254\,\hat{u}^3 + 0.0001777\,\hat{u}^4 + 0.0002580\hat{u}^5 - 0.0003811\ln\hat{m} - 0.0003289\ln^2\hat{m} + 0.0005292\hat{m}^2\ln\hat{m} + 0.0004012\hat{m}^2\ln^2\hat{m} + 0.0020462\,\hat{m}^2\ln\left(\frac{1+\hat{u}}{\hat{m}}\right)\Big)$$

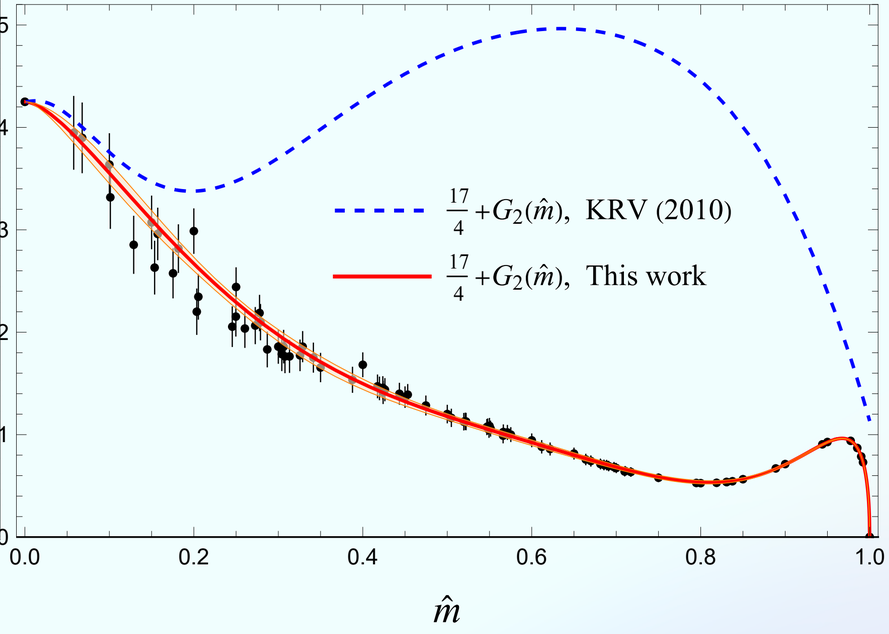
$$G_3^{\rm bis}(\hat{m}) = 32\pi^4\hat{m}^2\Big(-0.000244 - 0.003777\hat{u} + 0.000319\hat{u}^2 + 0.001263\hat{u}^3 + 0.000322\ln\hat{m} + 0.000572\ln^2\hat{m} + 0.003743\hat{m}^2\ln\left(\frac{1+\hat{u}}{\hat{m}}\right)\Big).$$

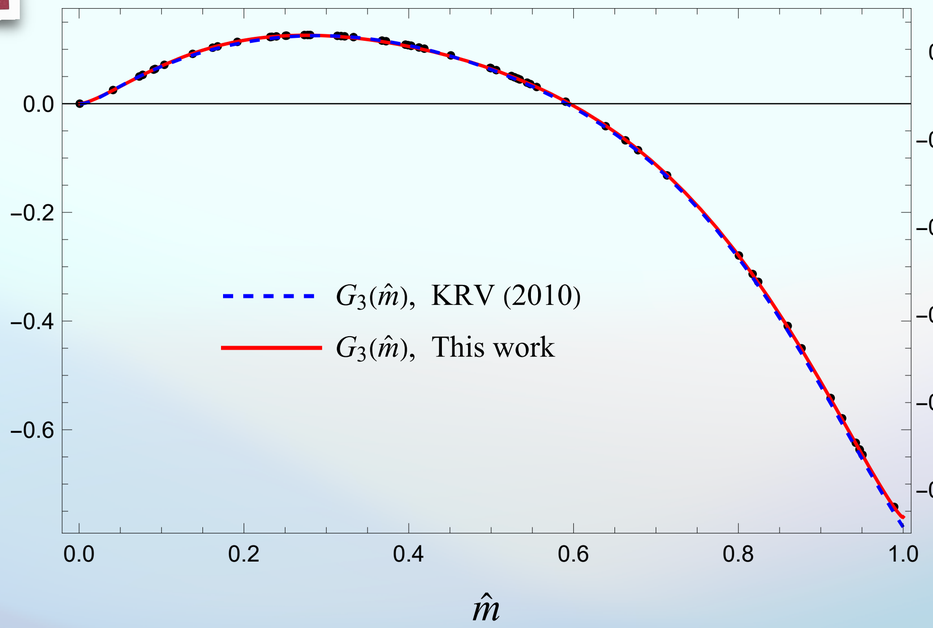
25

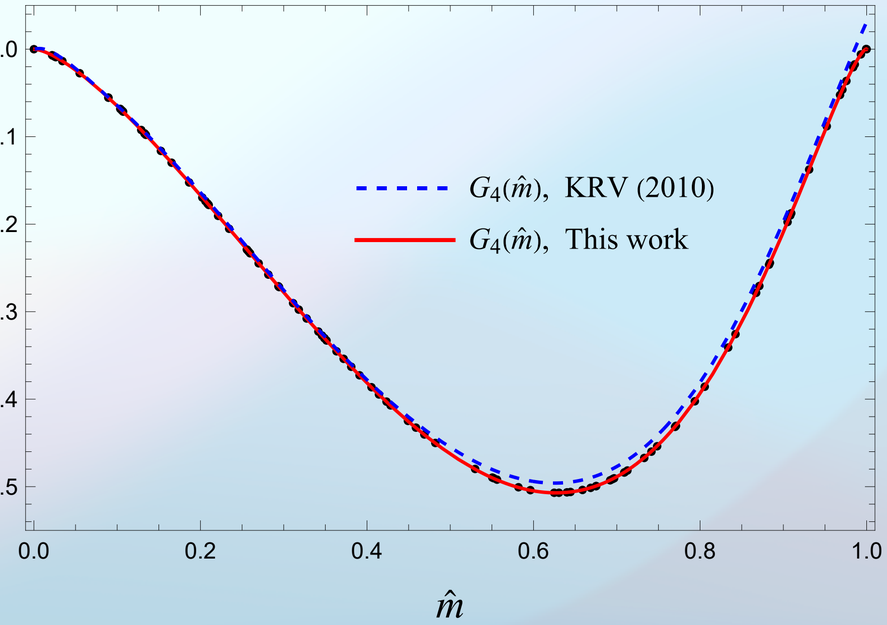
Backup

G_i comparisons

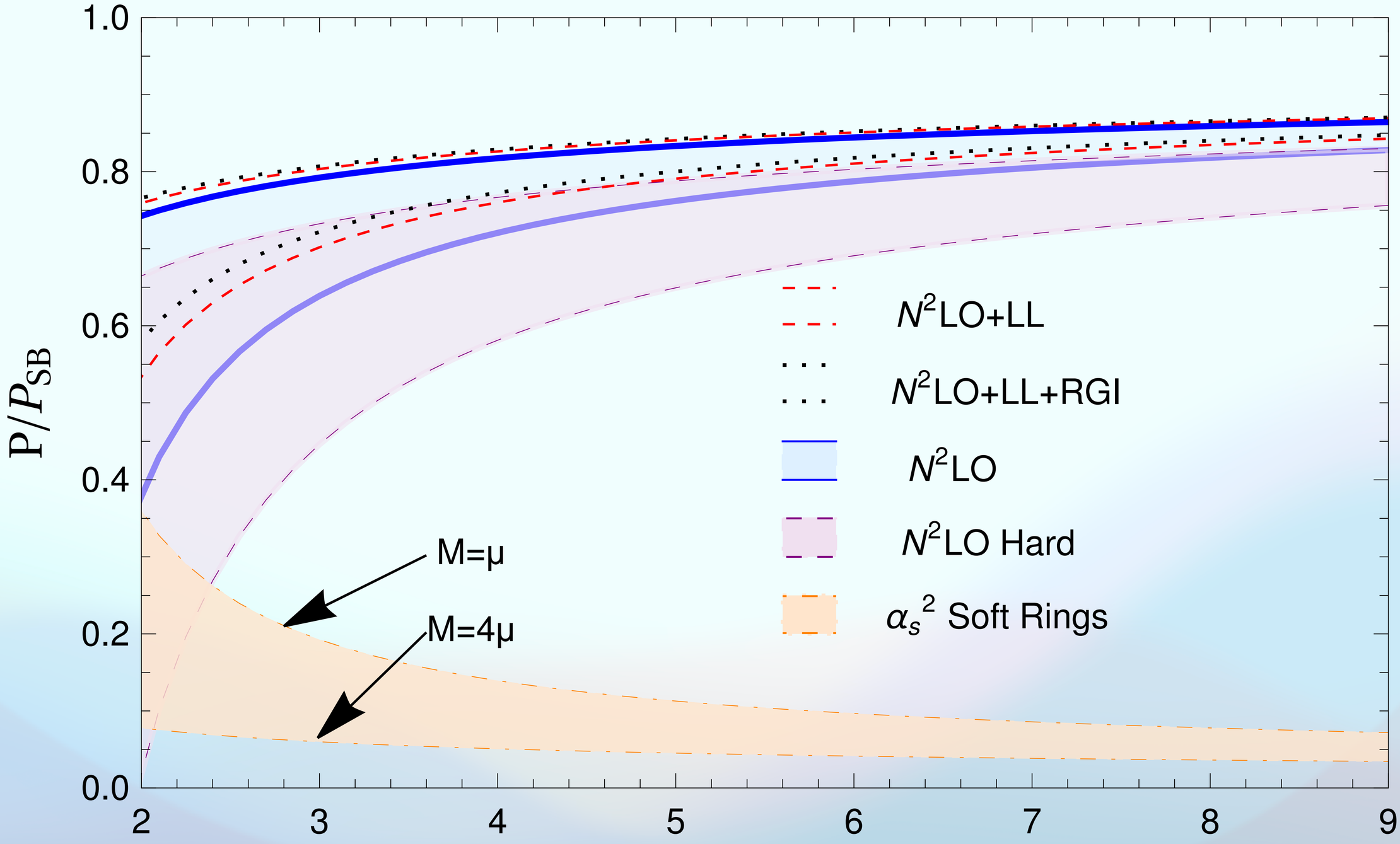








26


1.0
0.8
0.6
0.4
0.2
0.0
P/P_SB
2
3
4
5
6
7
8
9
N²LO+LL
N²LO+LL+RGI
N²LO
N²LO Hard
α_s² Soft Rings
M=μ
M=4μ

27