QCD Matter in de Sitter Spacetime

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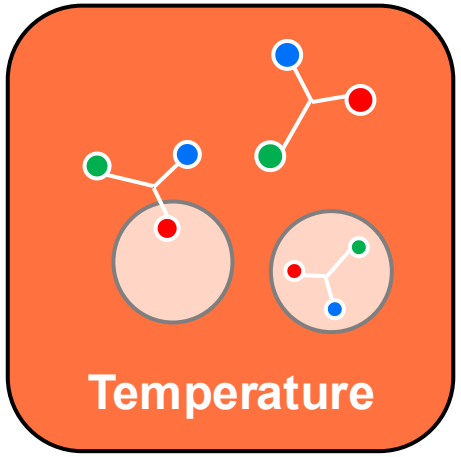
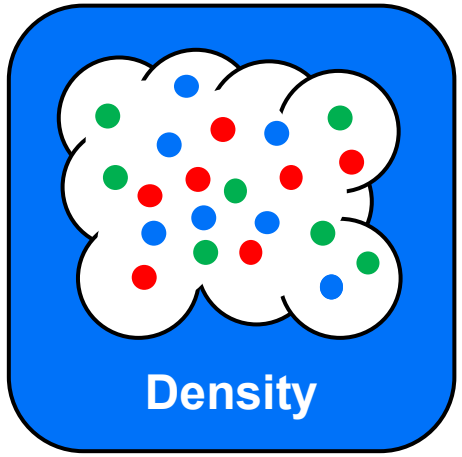
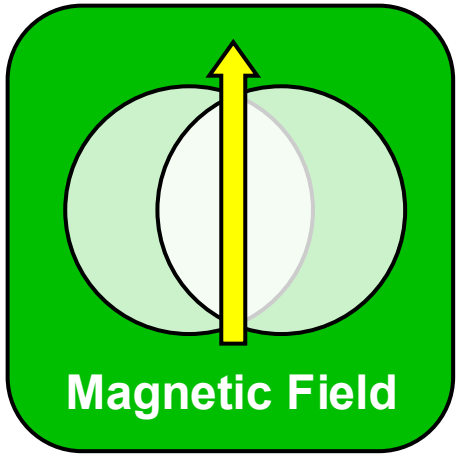
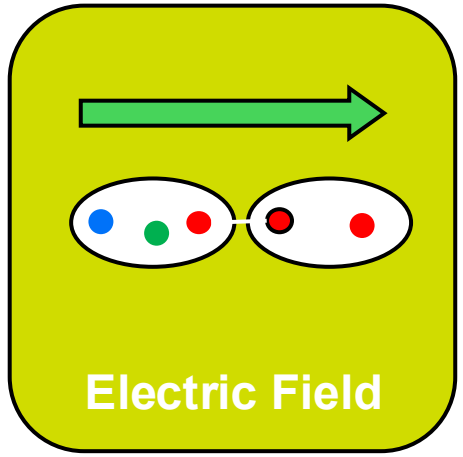
Introduction

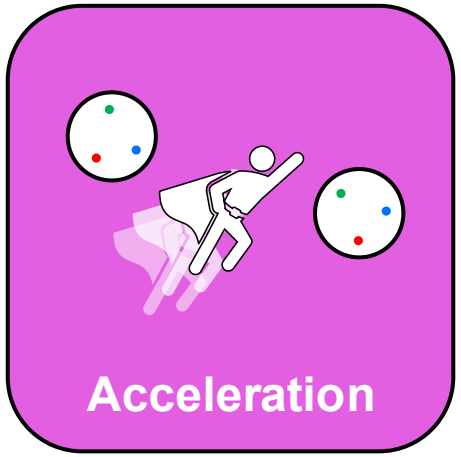
QCD under extreme conditions

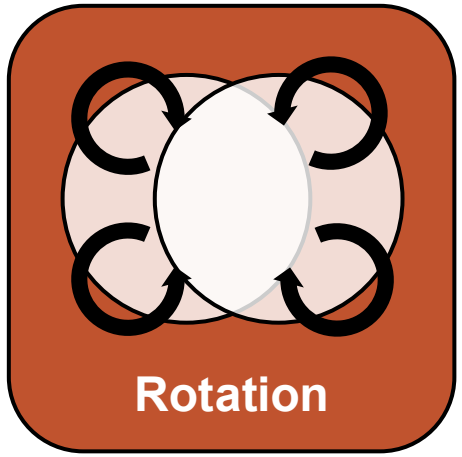








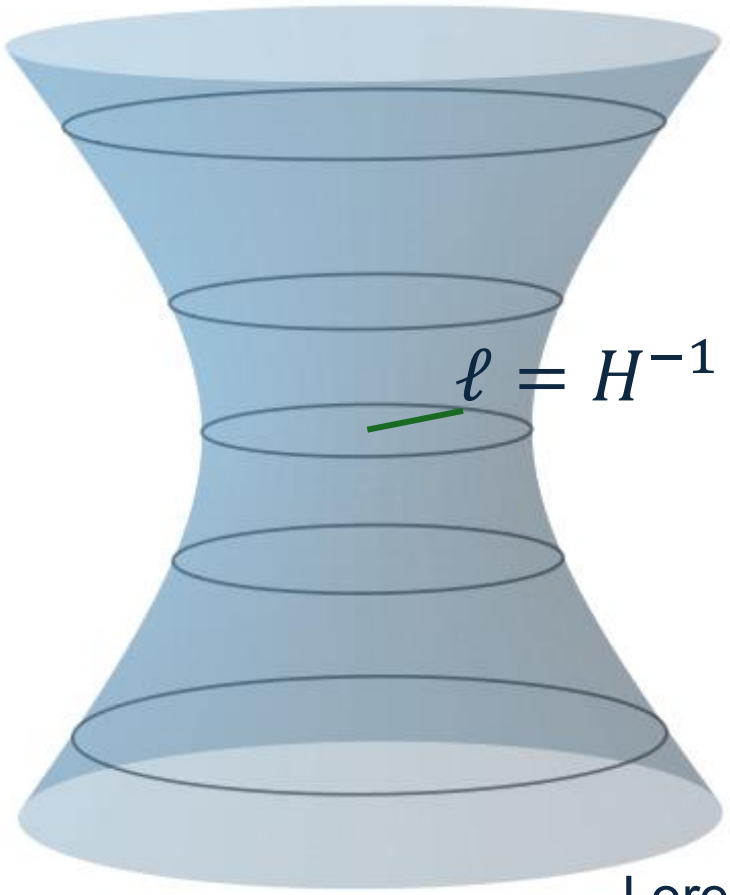






Why de Sitter Spacetime?

- Maximally symmetric
$$R = 12H^2$$
- Characteristic curvature scale
$$\ell = H^{-1}$$
- Thermal interpretation in the static patch
$$T_{dS} = \frac{H}{2\pi}$$
- Euclidean de Sitter: S^4



Lorentzian dS

A simple setting to explore curvature effects on QCD

Curvature can modify QCD-like phase structures

NJL-type models

[Inagaki, et al. (1997), Flachi, et al. (2014)]

Positive curvature tends to restore the chiral symmetry.

$$T_{R>0}^{\text{chiral}} < T_{R=0}^{\text{chiral}}$$

Holographic confining gauge theory

[Marolf, et al. (2011), Blackman, et al. (2011)]

Deconfinement can occur as the dS curvature increases. (model-dependent)

$$T_{\text{dS}_d}^{\text{deconf}} \sim T_{\text{flat}}^{\text{deconf}}/d$$

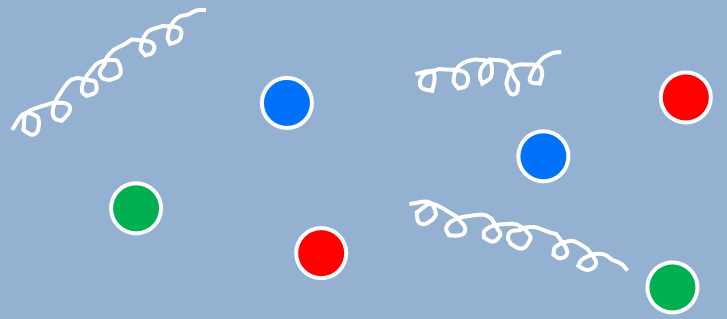
But what happens to the **hadronic** and **quark-gluon** descriptions of QCD?

Can strong curvature favor quarks over hadrons?



Hadronic description

Hadron Resonance Gas (HRG)



Quark-gluon description

Massless quarks and gluons (QG)

Our approach: compare their sphere free energy densities on S^4

$$f(H) = -\frac{1}{V_4} \ln Z\,(H)$$

$$V_4 \equiv \mathrm{Vol}\left(S^4_{H^{-1}}\right) = \frac{8\pi^2 H^{-4}}{3}$$

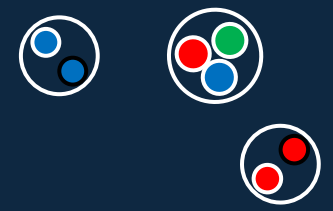
Sphere Partition Function on S^4

$$\ln Z_{\text{total}} = -S_{\text{gravity}} + \ln Z_{\text{matter}}$$

Sphere partition function for hadrons

Hadron Resonance Gas

$$\ln Z_{\mathrm{HRG}}(H) = \sum_{i \in \mathrm{PDG}} \ln Z_i^{\mathrm{free}}(H; m_i, s_i)$$



- One-loop functional determinant (scalar example)

$$\ln Z_i^{\mathrm{free}}(H) = -\frac{1}{2} \ln \det\left(-\Delta_{S^4}^{(0)} + m_i^2 + 12\xi_i H^2\right)$$

- **Decoupling renormalization condition**

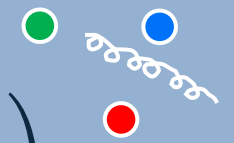
$$\lim_{H/m_i \to 0} \ln Z_{i,\mathrm{dec}}^{\mathrm{free}}(H) = 0$$

- Light-flavor (u, d, s) hadrons from the Particle Data Group
- Minimal coupling with curvature (e.g., $\xi_i = 0$)

Sphere partition function for quarks and gluons

Perturbative QCD

$$\ln Z_{\mathrm{pQCD}}(H) = \ln\left(\frac{C_{\mathrm{norm}}}{\mathrm{Vol}\big(SU(N_c)\big)}\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A\mathcal{D}\bar{c}'\mathcal{D}c' e^{-S_{\mathrm{QCD}}[\bar{\psi},\psi,A]-S_{\mathrm{GF+gh}}[A,\bar{c}',c']}\right)$$



- Quarks are massless.
- One-loop contribution + **Minimal subtraction scheme**

$$\ln Z_{\mathrm{pQCD}}(H) = \frac{N_c^2-1}{2}\ln\frac{g_s^2}{4\pi} - \left(\frac{11}{90}N_cN_f + \frac{31}{45}(N_c^2-1)\right)\ln\frac{\mu}{H} + \mathrm{const.}$$

$$\frac{g_s^2}{4\pi} = \frac{4\pi}{\beta_0 \ln(\mu_s^2/\Lambda_{\mathrm{QCD}}^2)}$$

- Set $N_c = N_f = 3, \quad \mu_s/\Lambda_{\mathrm{QCD}} = 10, \quad \mu/\Lambda_{\mathrm{QCD}} = \mu_0/\Lambda_{\mathrm{QCD}} = 5.$

Renormalization

Gravity

(Einstein-Hilbert + R^2 + E)

$$\Omega_4 \equiv \mathrm{vol}(S_1^4) = \frac{8\pi^2}{3}$$

$$-S^{\mathrm{bare}}_{\mathrm{gravity}} = \left(-\frac{\Lambda_{\mathrm{bare}}}{8\pi G_{\mathrm{bare}}} H^{-4} + \frac{3}{4\pi G_{\mathrm{bare}}} H^{-2} + c_{4,\mathrm{bare}} \right) \Omega_4$$

$$-\frac{\Omega_4}{8\pi}\frac{\Lambda_{\mathrm{bare}}}{G_{\mathrm{bare}}} + a_0 = -\frac{\Omega_4}{8\pi}\frac{\Lambda_{\mathrm{ren}}}{G_{\mathrm{ren}}}$$

Matter

$$\ln Z_{\mathrm{bare}} = a_0 H^{-4} + a_2 H^{-2} + a_4 + \ln Z_{\mathrm{ren}}$$

Phenomenological matching of HRG and QG

$$\Delta \ln Z_{\text{total}}^{\text{IR}-\text{UV}}(H) \equiv \ln Z_{\text{total}}^{\text{IR}}(H) - \ln Z_{\text{total}}^{\text{UV}}(H)$$

$$= -S_{\text{gravity}}^{\text{dec}} + S_{\text{gravity}}^{\text{MS}} + \ln Z_{\text{HRG}}(H) - \ln Z_{\text{pQCD}}(H)$$

Parametrize the mismatch of local gravitational terms as

$$-S_{\text{gravity}}^{\text{dec}} + S_{\text{gravity}}^{\text{MS}} = BV_4$$

Curvature-dependent bag parameter

$$B = B_0 + B_2 R + B_4 R^2$$

Modified QG sphere partition function

$$\ln Z_{QG}(H) \equiv \ln Z_{\text{pQCD}}(H) - BV_4$$

$$\Delta \ln Z_{\text{total}}^{\text{IR}-\text{UV}}(H) = \ln Z_{\text{HRG}}(H) - \ln Z_{\text{QG}}(H)$$

Comparing the sphere free energy densities

Sphere Free Energy Density

$$f = -\frac{1}{V_4}\ln Z$$

Hadrons

$$f_{\mathrm{HRG}}(H) = -\frac{1}{V_4}\sum_i \ln Z_{i,\mathrm{dec}}^{\mathrm{free}}(H)$$

$m_i < 10\mathrm{GeV}$

Coupling with curvature neglected

Quarks and gluons

$$f_{\mathrm{QG}}(H) = -\frac{1}{V_4}\ln Z_{\mathrm{pQCD}}(H) + B$$

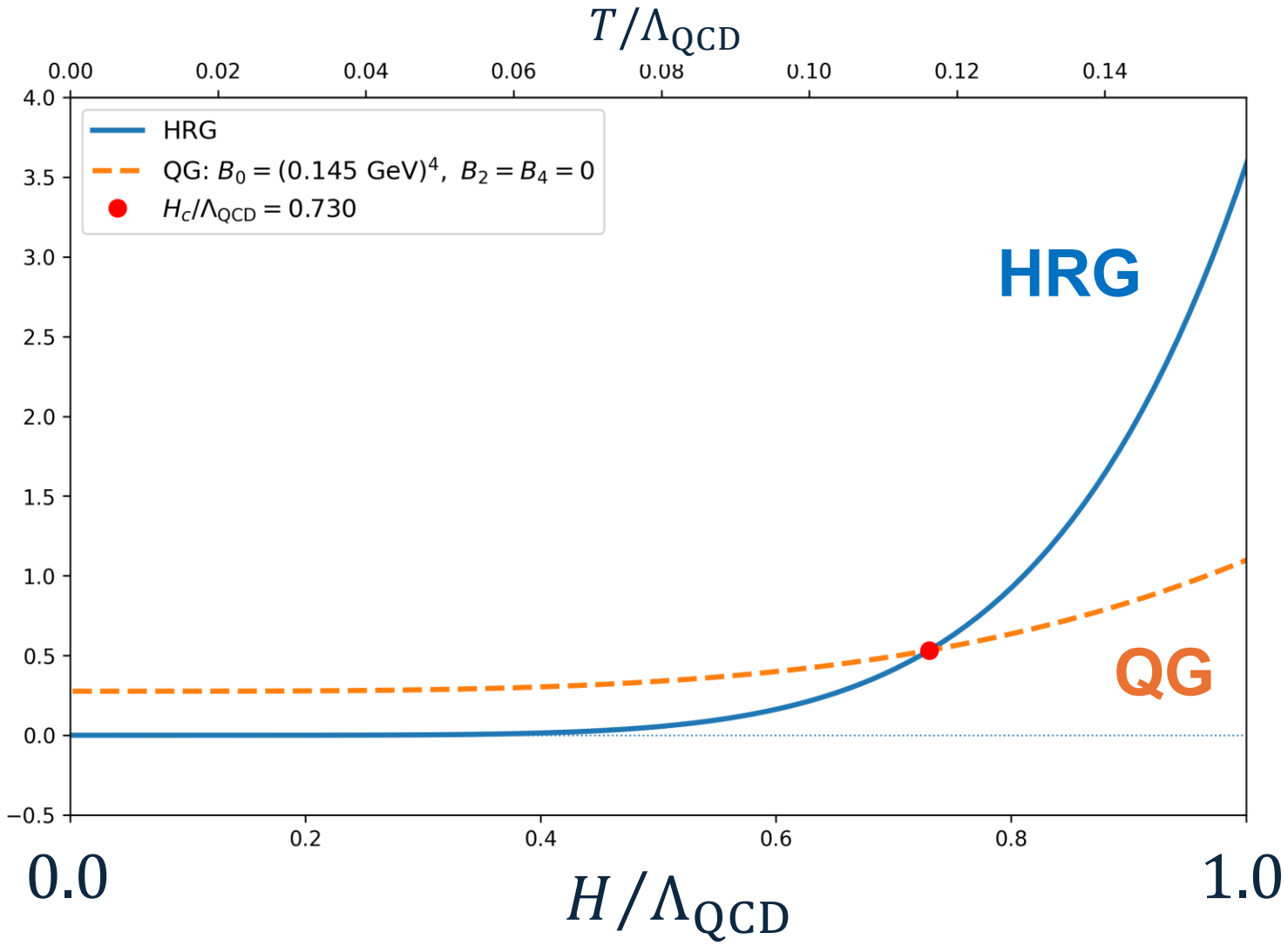
$B_0 =$ flat-space bag constant,

Dependence of $B_2, B_4(\mu_0)$ estimated later

Result

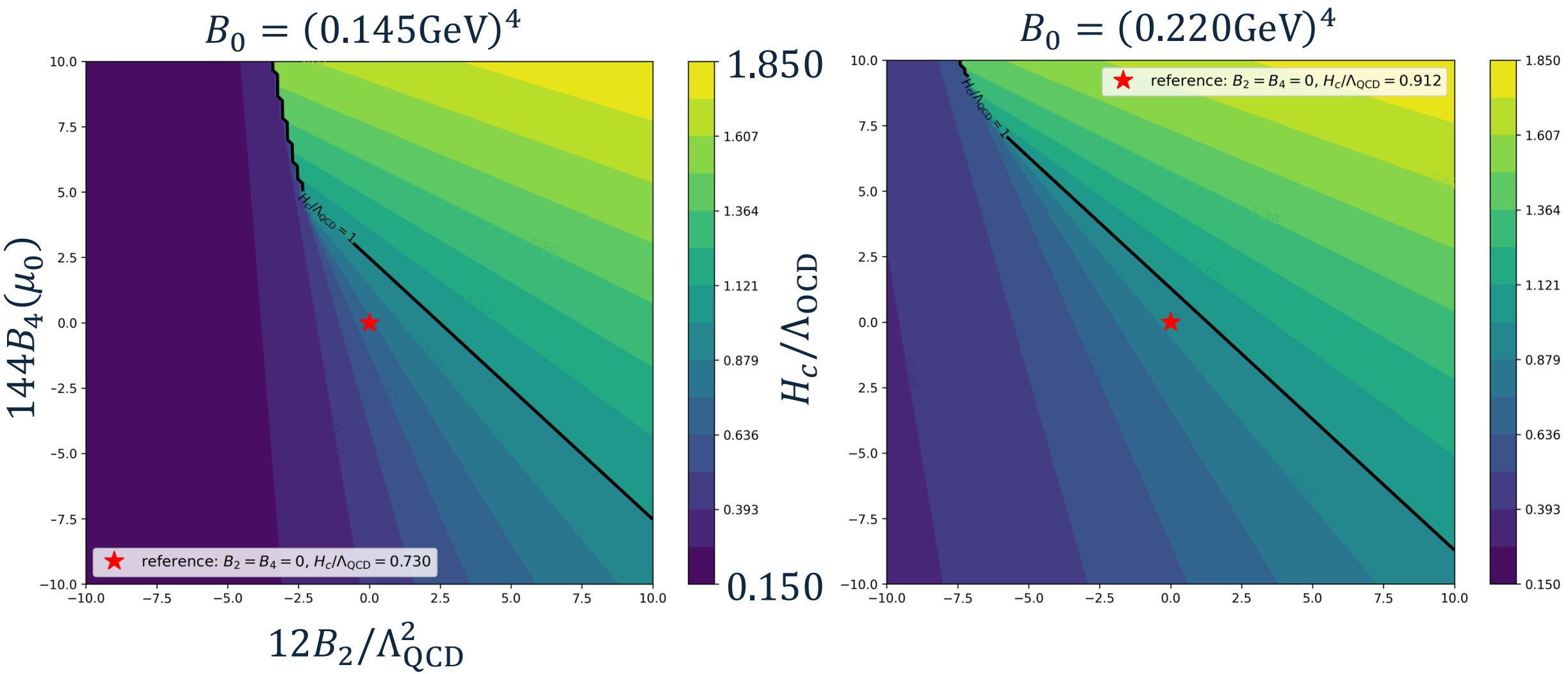
Result in a reference parameter set

Reference parameter set: $B_0 = (0.145\mathrm{GeV})^4,\ B_2 = 0,\ B_4 = 0$



Crossing occurs.

Crossing points for different parameter sets

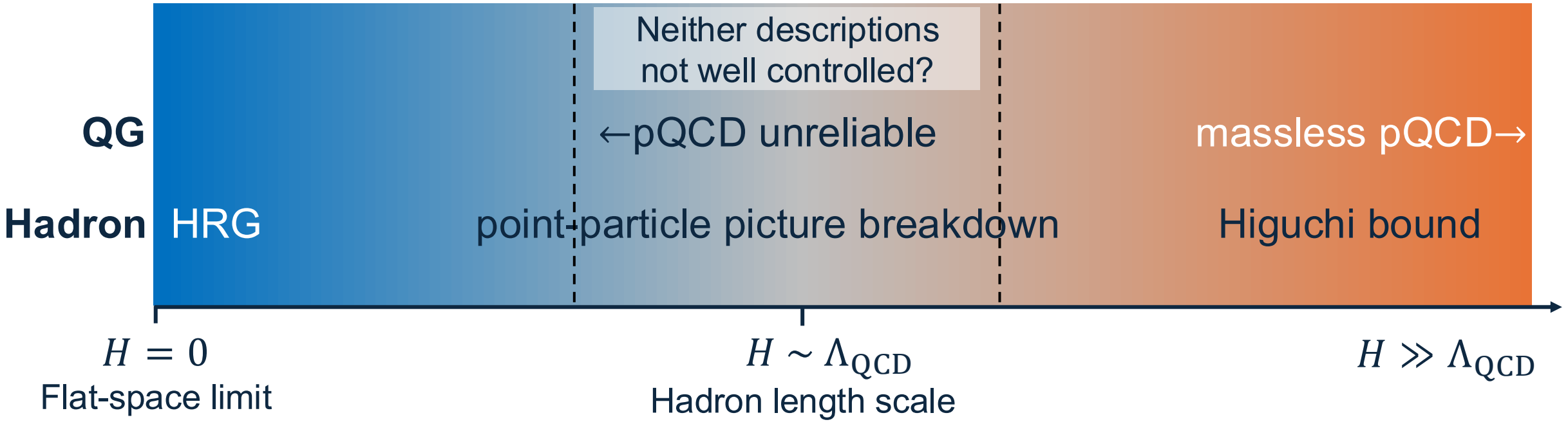


When B_0~commonly-used value and $12B_2/\Lambda_{\text{QCD}} \sim 144B_4 \sim \mathcal{O}(1)$, $H_c = \boldsymbol{\mathcal{O}(\Lambda_{\text{QCD}})}$

Discussion

What does the crossing tell us?

Schematic illustration



HRG and pQCD are expected to describe QCD at low and sufficiently high curvature, respectively.

Crossing is a possible indicator of the curvature-induced change from hadrons to quarks as H increases.

Summary and Outlook

Our finding:

- Found a crossing of the HRG and QG sphere free energies at

$$H_c = \mathcal{O}\left(\Lambda_{\mathrm{QCD}}\right),$$

suggesting a possible change in the effective degrees of freedom of QCD.

Limitations:

- The crossing does not establish a physical phase transition or crossover, nor determine its precise location.

Outlook:

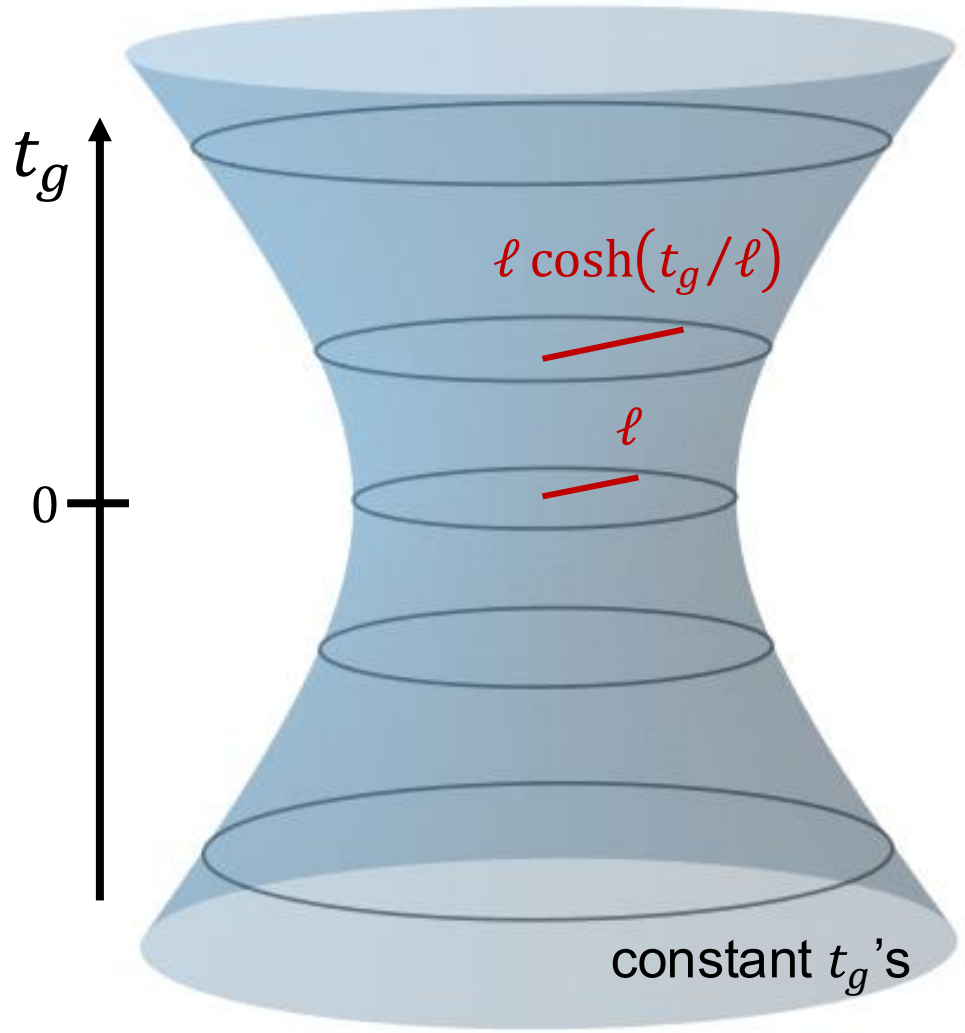
- Clarify curvature-hadron couplings.
- Compute the curvature-dependent hadron spectrum and density of states.

Thank you.

Supplemental Slides

Global coordinate system:

$$ds^2 = -dt_g^2 + \ell^2 \cosh^2\left(\frac{t_g}{\ell}\right) d\Omega_3^2$$



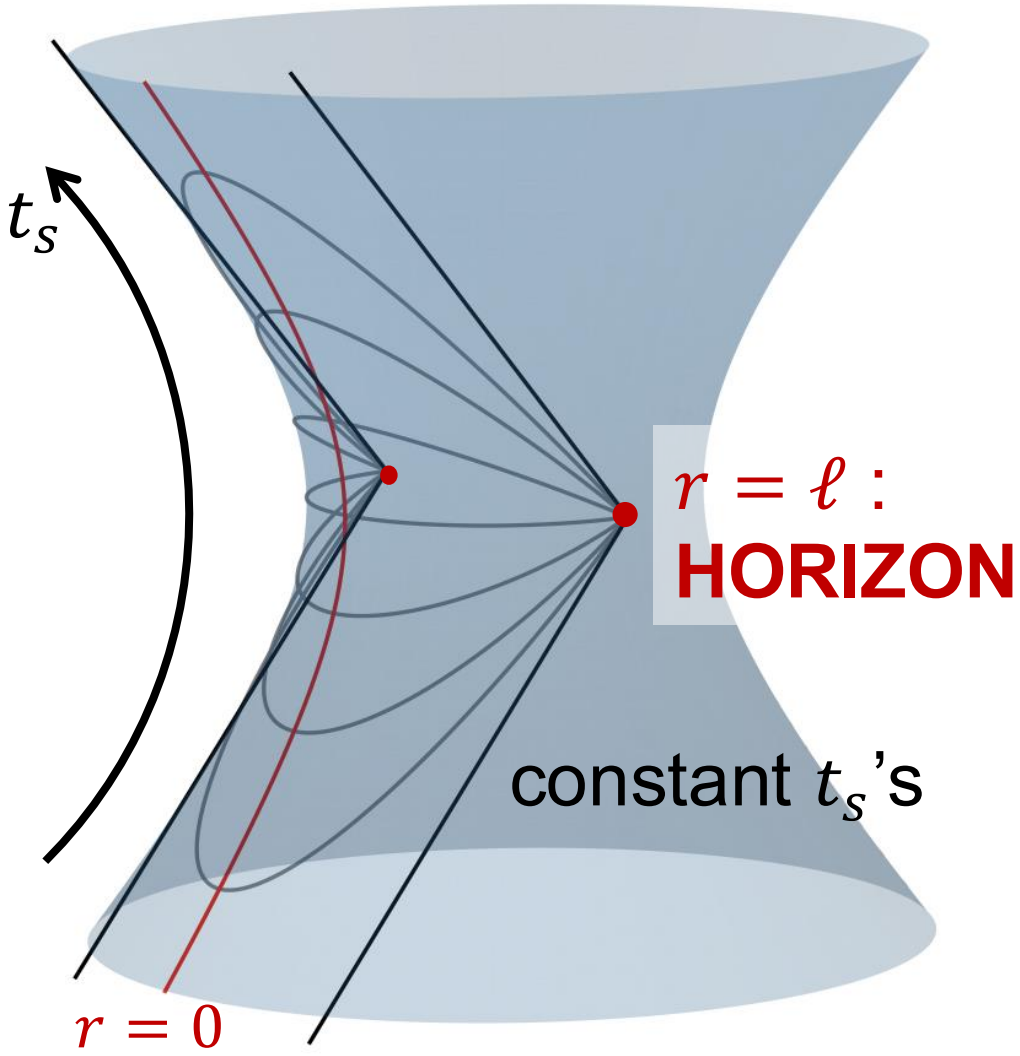
Static coordinate system:

$$ds^2 = -f(r)dt_s^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2$$

$$f(r) = 1 - \frac{r^2}{\ell^2}$$

Gibbons-Hawking Temperature

$$T = \frac{1}{2\pi\ell}$$

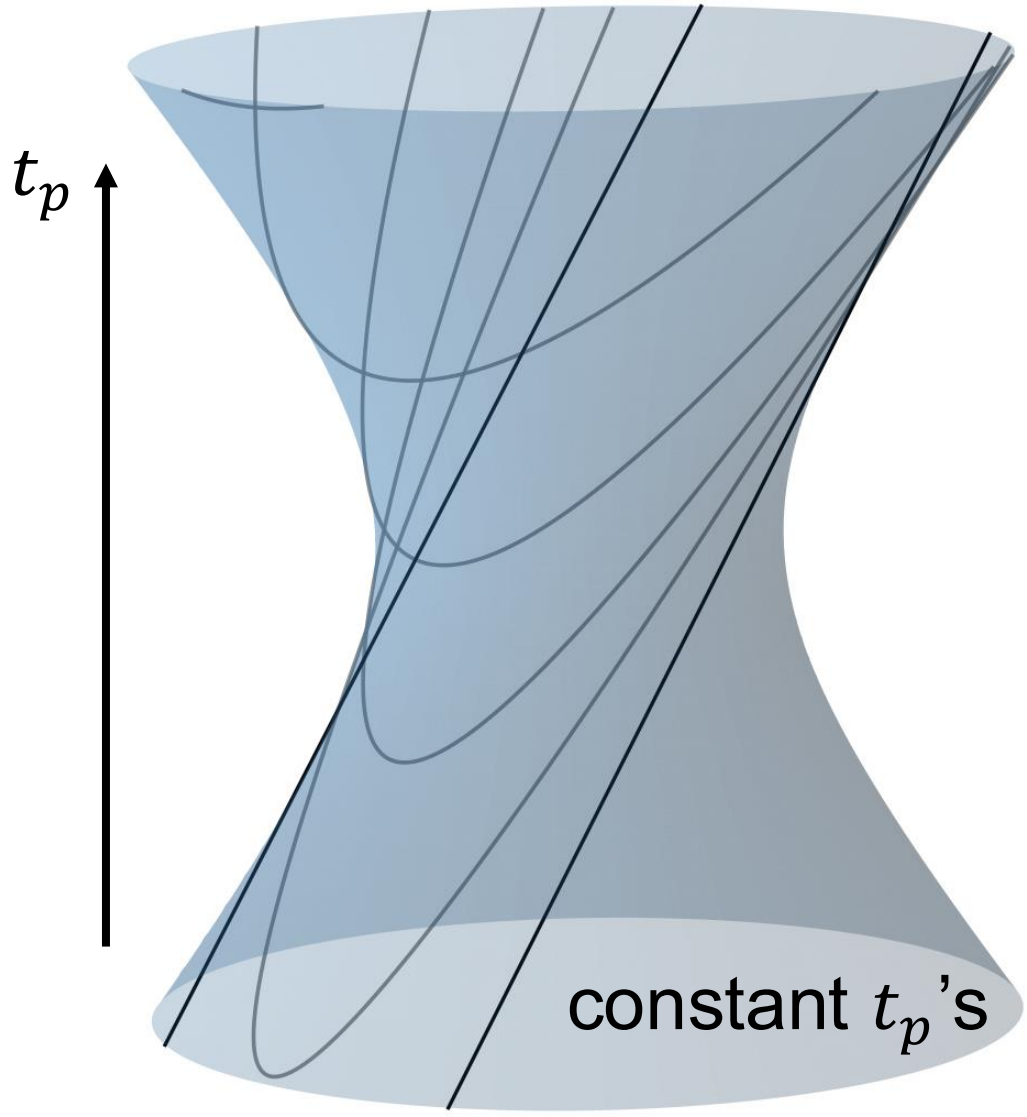


Planar coordinate system:

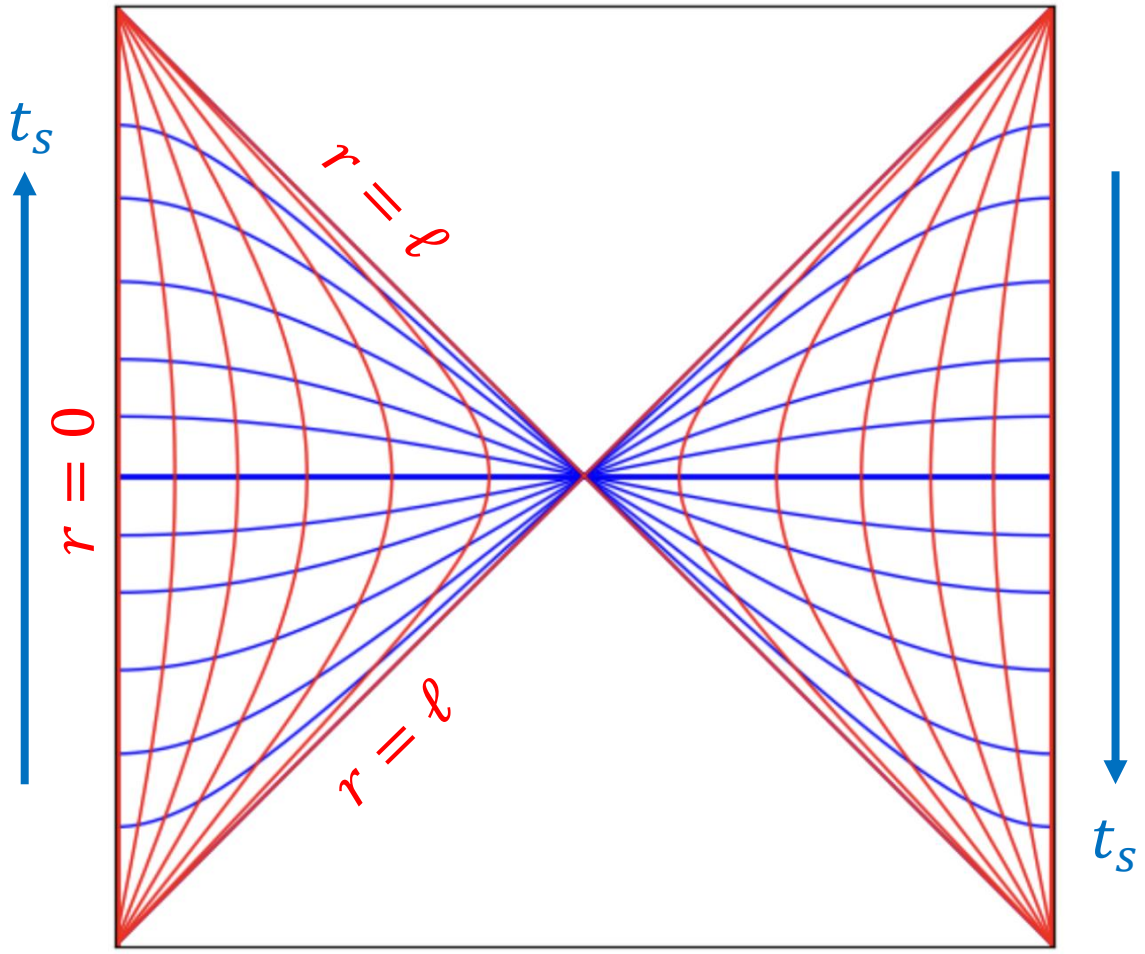
$$ds^2 = -dt_p^2 + e^{2t_p/\ell} d\vec{x}^2$$

Scale factor: $a(t_p) = e^{t_p/\ell}$

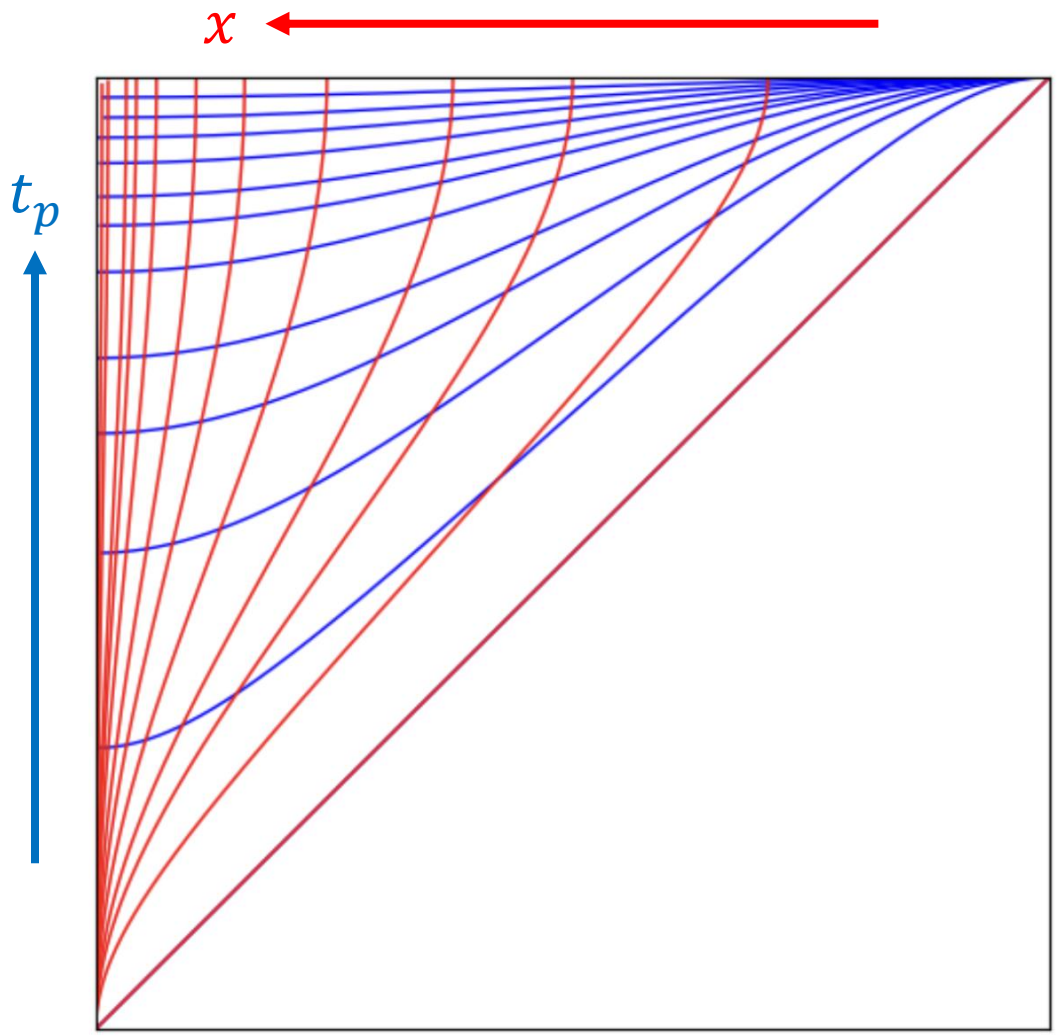
Hubble para.: $H = 1/\ell$



Penrose Diagram



Static Patch

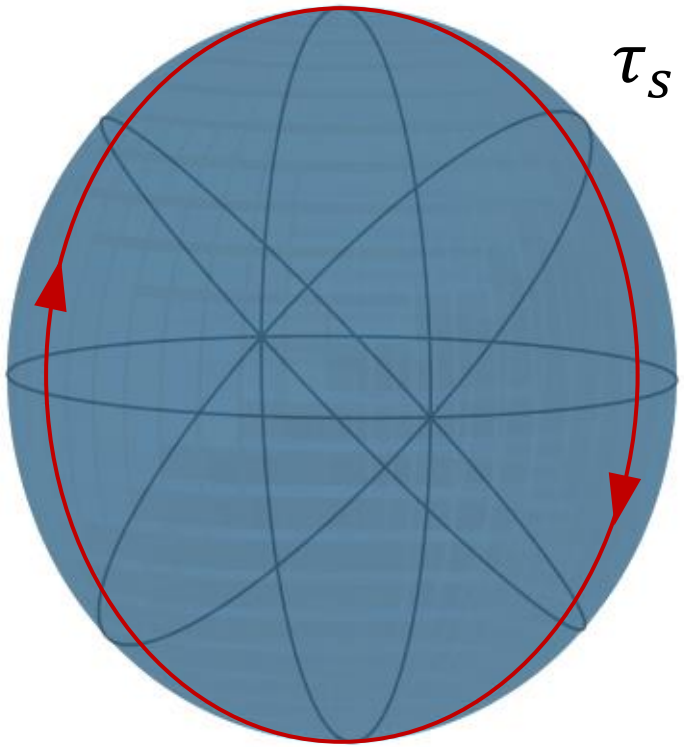


Planar Patch

Thermal Equilibrium

When $\beta = 2\pi\ell$,

Euclidean de Sitter space is regular everywhere.



Euclidean two-point function

$$G_E(\tau_{s,1}, x_1; \tau_{s,2}, x_2) = \langle T_\tau \mathcal{O}(\tau_{s,1}, x_1)\mathcal{O}(\tau_{s,2}, x_2)\rangle_E$$

$$G_E(\tau_{s,1} + \beta, x_1; \tau_{s,2}, x_2) = \pm G_E(\tau_{s,1}, x_1; \tau_{s,2}, x_2).$$

Thermal Equilibrium

With the Lorentzian Wightman functions

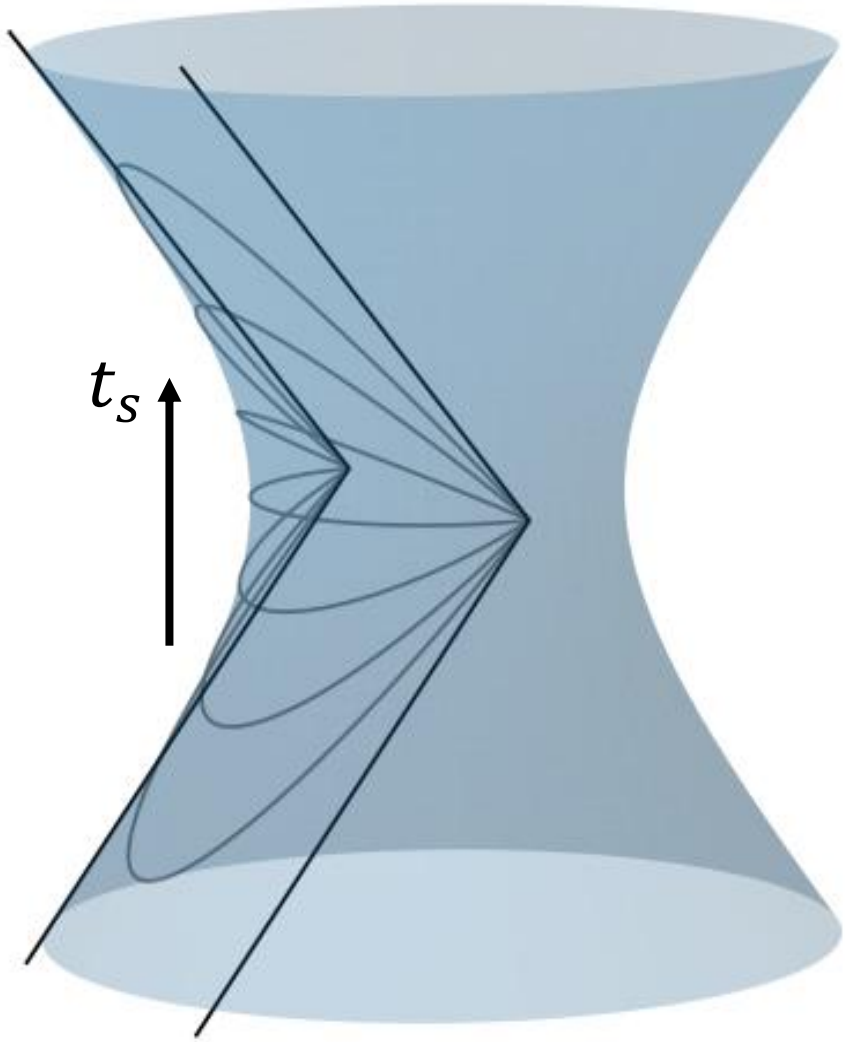
$$G^+(t_s, x_1; 0, x_2) = \langle \mathcal{O}(t_s, x_1)\mathcal{O}(0, x_2)\rangle$$
$$G^-(t_s, x_1; 0, x_2) = \langle \mathcal{O}(0, x_2)\mathcal{O}(t_s, x_1)\rangle,$$

$$G_E(\tau_s, x_1; 0, x_2) = G^+(-i\tau_s, x_1; 0, x_2).$$

Under the analytic continuation

$$\tau = it + \epsilon,$$

$$G^+(t - i\beta, x_1; 0, x_2) = \pm G^-(t, x_1; 0, x_2)$$



Kubo-Martin-Schwinger relation

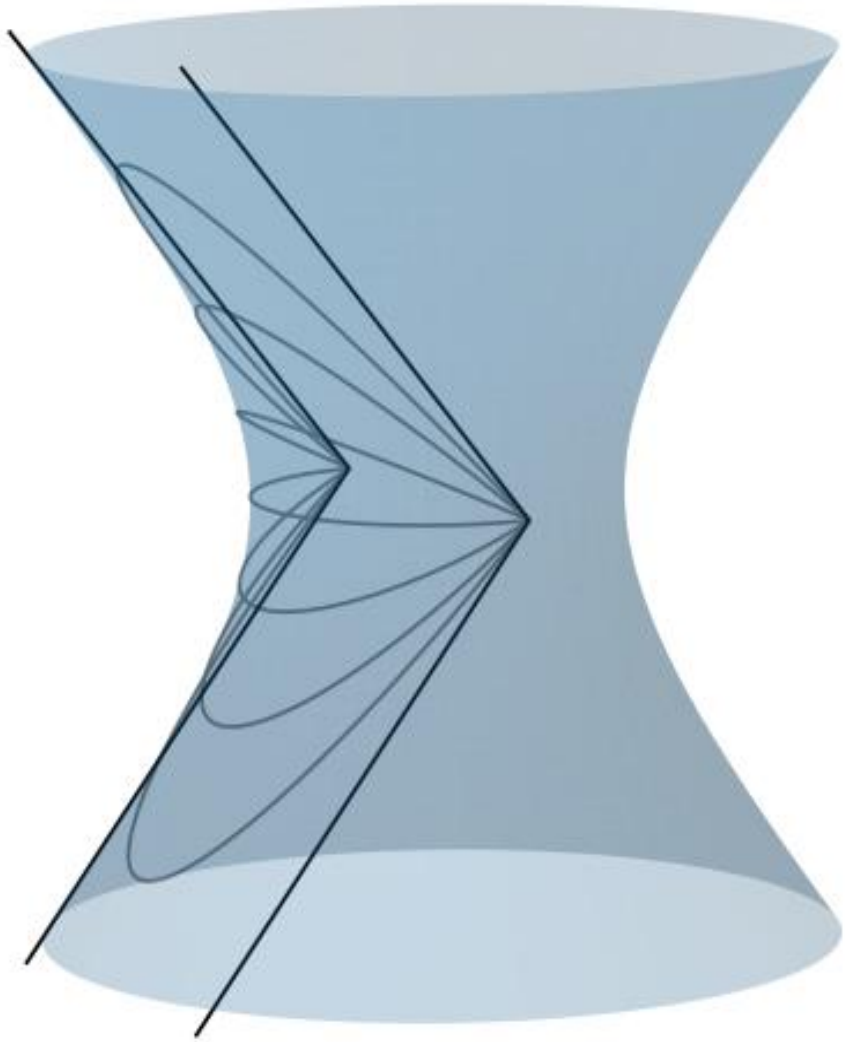
Thermal Equilibrium

Kubo-Martin-Schwinger relation

II

The static patch is in **thermal equilibrium** with temperature

$$T = \frac{1}{2\pi\ell}.$$



Euclidean Quantities

Euclidean Energy Density:

$$\langle T_{\mu\nu} \rangle = -\rho \, g_{\mu\nu} \quad \Rightarrow \quad \rho = -\frac{\ell}{4 \, \mathrm{vol}\left(S^4_\ell\right)} \frac{d}{d\ell} \ln Z_{\mathrm{ren}}$$

Euclidean Entropy:

$$S = \ln Z_{\mathrm{ren}} + \rho \, \mathrm{vol}\left(S^4_\ell\right)$$

Euclidean Weyl Susceptibility:

$$C = -\ell \frac{dS}{d\ell} = -\frac{d\rho}{d\ell} \ell \, \mathrm{vol}\left(\mathrm{S}^4_\ell\right)$$

de Sitter Thermodynamics

Internal energy in the static patch:

$$U_{\text{static}} = \frac{\rho\ \text{vol}\left(S^4_\ell\right)}{\beta}, \qquad \beta = 2\pi\ell$$

Entropy:

$$S = \ln Z + \rho\ \text{vol}\left(S^4_\ell\right) = \ln Z + \beta U_{\text{static}}$$

First law of thermodynamics:

$$d\ln Z = -\rho d\left(\text{vol}\left(S^4_\ell\right)\right),$$

$$dS = \text{vol}\left(S^4_\ell\right)d\rho = \beta dU_{\text{static}} - \beta\rho\, d\left(\text{vol}\left(B^3_\ell\right)\right)$$

de Sitter Symmetry

dS isometry = SO($1, d+1$)

Generators:

$$J_{MN} = X_M \partial_N - X_N \partial_M, \ (M, N = 0, \cdots, d+1)$$

$$[J_{KL}, J_{MN}] = -\eta_{KM} J_{LN} - \eta_{LN} J_{KM} + \eta_{KN} J_{LM} + \eta_{LM} J_{KN}$$

Eigenvalue of quadratic Casimir operator:

$$C_2^{\mathrm{Bos}}(\Delta, s) = \Delta(d-\Delta) - s(s+d-2), \qquad \Delta = 3/2 + i\nu$$

$$C_2^{\mathrm{Fer}}(\Delta, s) = \Delta(d-\Delta) - \frac{(d-3)(d-2)}{8} - s(s+d-2)$$

Irreducible Representation of SO(1,4)

Bosons [Higuchi(1987), Basile, et al.(2017)] :

Family of UIR	Weight and Mass	Correspondence
Principal series	$\Delta = \frac{d}{2} + i\nu, \quad \nu \in \mathbb{R},$ $\tilde{m}^2\ell^2 = \begin{cases} \Delta(d-\Delta) & \text{for } s=0 \\ (\Delta+s-2)(d-\Delta+s-2) & \text{for } s \geq 1 \end{cases}$	**Massive Field**
Complementary series	$\Delta = \frac{d}{2} + \mu, \begin{cases} -d/2 < \mu < d/2 & \text{for } s=0 \\ -(d/2-1) < \mu < d/2-1 & \text{for } s \geq 1 \end{cases}$ $0 < \tilde{m}\ell < d/2, \quad \text{for } s=0,$ $\sqrt{(d+s-3)(s-1)} < \tilde{m}\ell < d/2+s-2, \quad \text{for } s \geq 1.$	**(Light) Massive Field**
Exceptional series	$\tilde{m}^2\ell^2 = (s-1-t)(d+s-3+t), \quad t=0,1,\cdots,s-1$	**(Partially) Massless Field**
Discrete series		**No propagating mode in bulk**

Higuchi Bound

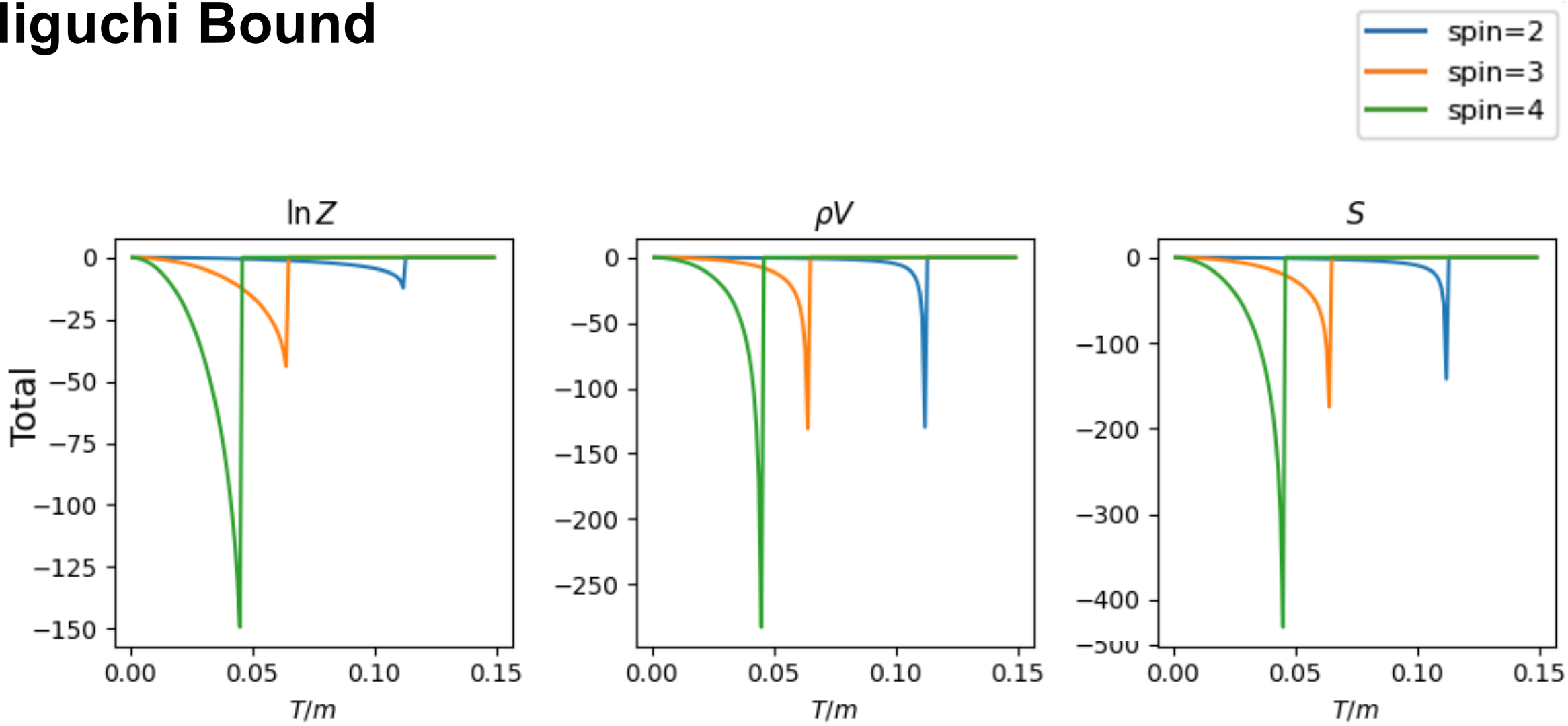
In order for irreps. to be unitary,

Higuchi Bound

$$\sqrt{(d+s-3)(s-1)} < m\ell$$

Otherwise, they have some ghost modes.

Higuchi Bound



Irreducible Representation of Spin(1,4)

Fermions [Letsios(2023)] :

Family of UIR	Weight and Mass	Correspondence
Principal series	$\Delta = \frac{d}{2} + i\nu, \quad \nu = \tilde{m}\ell \in \mathbb{R}$	**Massive Field**
Discrete series	$\tilde{m}\ell = i\left(s - \frac{1}{2} - t + \frac{d-1}{2}\right) \quad t = 0, 1, \cdots, s-1$	**(Partially) Massless Field**

Sphere partition function for free massive fields

Heat kernel-type integral

Scalar

$$\ln Z_i^{\text{free}}(H) = -\frac{1}{2}\ln\det\left(-\Delta_{S^4}^{(0)} + m_i^2 + 12\xi_i H^2\right)$$

$$= \int_0^\infty \frac{dt}{2t}\sum_{n=0}^{\infty} D_{n,0}^5\left(e^{-(n+\Delta_+)Ht} + e^{-(n+\Delta_-)Ht}\right)$$

$$-\Delta_{S^4}^{(0)} = n(n+3)H^2 \quad (n = 0,1,\cdots),$$

$$\Delta_\pm = \frac{3}{2} \pm i\sqrt{\frac{m_i^2}{H^2} + 12\xi_i - \frac{9}{4}}, \qquad D_{n,0}^5\text{: degeneracy}$$

Spin-1/2 Dirac

$$\ln Z_i^{\text{free}}(H) = \ln\det(-\not{\nabla}_{S^4} + m_i)$$

$$= -\int_0^\infty \frac{dt}{t}\sum_{n=1/2}^{\infty} D_{n,\frac{1}{2}}^5\left(e^{-(n+\Delta_+)Ht} + e^{-(n+\Delta_-)Ht}\right)$$

$$-\not{\nabla}_{S^4}^2 = \left(n+\frac{3}{2}\right)^2 H^2 \quad \left(n = \frac{1}{2},\frac{3}{2},\cdots\right),$$

$$\Delta_\pm = \frac{3}{2} \pm i\frac{m_i}{H}, \qquad D_{n,\frac{1}{2}}^5\text{: degeneracy}$$

Sphere partition function for free massive fields

Free massive spin-s_i fields on S^4

[Anninos, Denef, Law, Sun (2022)]

$$\ln Z_i^{\text{free}}(H) = (-2)^{F_s} \int_0^\infty \frac{dt}{2t} \sum_{n=-1+\frac{F_s}{2}}^{\infty} D^5_{n,s_i} \left(e^{-(n+\Delta_+)Ht} + e^{-(n+\Delta_-)Ht} \right)$$

$F_s = 0(1)$ for bosons (fermions), $\Delta_\pm = \frac{3}{2} \pm i\nu$, $\nu = \begin{cases} \sqrt{m_i^2/H^2 + 12\xi_i - 9/4} & \text{for scalar} \\ \sqrt{m_i^2/H^2 - (s_i - 1/2)^2} & \text{for higher spin bosons} \\ m_i/H & \text{for fermions} \end{cases}$

Sphere partition function for free massive fields

For bosons

$$\ln Z(\ell) = \ln Z_{\zeta'}(\ell) + b_4(\nu)\ln(\mu\ell) - \frac{(2s+1)^3}{24}\nu^2 - \frac{2s+1}{9}\nu^4$$
$$- b_4(\nu)\ln(\mu\bar{\epsilon}) - \frac{(2s+1)((2s+1)^2+4\nu^2)}{12}\left(\frac{\ell}{\epsilon}\right)^2 + \frac{4(2s+1)}{3}\left(\frac{\ell}{\epsilon}\right)^4$$

$$\ln Z_{\zeta'}(\ell)$$
$$= (2s+1)\left[\frac{1}{6}\left(\zeta'(-3,\frac{3}{2}+i\nu)+\zeta'(-3,\frac{3}{2}-i\nu)\right) - \frac{i\nu}{2}\left(\zeta'(-2,\frac{3}{2}+i\nu)-\zeta'(-2,\frac{3}{2}-i\nu)\right)\right.$$
$$\left. - \frac{1}{2}\left(\nu^2+\frac{1}{12}\right)\left(\zeta'(-1,\frac{3}{2}+i\nu)+\zeta'(-1,\frac{3}{2}-i\nu)\right) + \frac{i\nu}{6}\left(\nu^2+\frac{1}{4}\right)\left(\zeta'(0,\frac{3}{2}+i\nu)-\zeta'(0,\frac{3}{2}-i\nu)\right)\right]$$
$$- \frac{s(s+1)(2s+1)}{6}\left[\left(\zeta'(-1,\frac{1}{2}+i\nu)+\zeta'(-1,\frac{1}{2}-i\nu)\right) - i\nu\left(\zeta'(0,\frac{1}{2}+i\nu)-\zeta'(0,\frac{1}{2}-i\nu)\right)\right],$$

$$b_4(\nu) = -\frac{(2s+1)(17+40s+40s^2)}{2880} + \frac{(2s+1)^3}{24}\nu^2 + \frac{2s+1}{12}\nu^4.$$

Sphere partition function for free massive fields

For bosons

$$\ln Z^{\text{free}}_{\text{dec}}(\ell) = \ln Z_{\zeta'}(\ell) - \ln Z_{\text{asym}}(\ell)$$

For scalars

$$\ln Z_{\text{asym}}(m\ell) = \Big(-\frac{(m\ell)^4}{12} + \frac{(1-6\xi)(m\ell)^2}{3} - \frac{232 - 2880\xi + 8640\xi^2}{720} \Big) \ln(m\ell) + \frac{25(m\ell)^4}{144} + \frac{(-15+88\xi)(m\ell)^2}{24} + \frac{675 - 7920\xi + 23040\xi^2}{1440}.$$

For higher-spin bosons

$$\begin{aligned} &\ln Z_{\text{asym}}(m\ell) \\ &= (2s+1)\Bigg[\Big(-\frac{(m\ell)^4}{12} + \frac{s(s-1)(m\ell)^2}{6} + \frac{2 - 15s^2(s-1)^2}{180} \Big) \ln(m\ell) \\ &\qquad + \frac{25(m\ell)^4}{144} + \frac{(-1+22s-22s^2)(m\ell)^2}{72} + \frac{(2s-1)^2(8s^2-8s-1)}{288} \Bigg] \\ &- \frac{s(s+1)(2s+1)}{6} \Bigg[\Big((m\ell)^2 + \frac{-1+3s-3s^2}{3} \Big) \ln(m\ell) - \frac{3(m\ell)^2}{2} + \frac{1}{12}(3 - 12s + 12s^2) \Bigg]. \end{aligned}$$

Sphere partition function for free massive fields

For fermions

$$\ln Z(\ell) = \ln Z_{\zeta'}(\ell) + b_4(\nu)\ln(\mu\ell) + \frac{(2s+1)^3}{12}\nu^2 + \frac{2(2s+1)}{9}\nu^4$$
$$- b_4(\nu)\ln(\mu\bar{\epsilon}) + \frac{(2s+1)((2s+1)^2+4\nu^2)}{6}\left(\frac{\ell}{\epsilon}\right)^2 - \frac{8(2s+1)}{3}\left(\frac{\ell}{\epsilon}\right)^4,$$

$$\ln Z_{\zeta'}(\ell)$$
$$= -\frac{2s+1}{3}\Big[(\zeta'(-3,2+i\nu)+\zeta'(-3,2-i\nu)) - 3i\nu(\zeta'(-2,2+i\nu)-\zeta'(-2,2-i\nu))$$
$$-(3\nu^2+1)(\zeta'(-1,2+i\nu)+\zeta'(-1,2-i\nu)) + i\nu(\nu^2+1)(\zeta'(0,2+i\nu)-\zeta'(0,2-i\nu))\Big]$$
$$+\frac{(2s-1)(2s+1)(2s+3)}{12}\Big[(\zeta'(-1,1+i\nu)+\zeta'(-1,1-i\nu)) - i\nu(\zeta'(0,1+i\nu)-\zeta'(0,1-i\nu))\Big],$$

$$b_4(\nu) = -\frac{(2s+1)(7+20s+20s^2)}{360} - \frac{(2s+1)^3}{12}\nu^2 - \frac{2s+1}{6}\nu^4.$$

Sphere partition function for free massive fields

For fermions

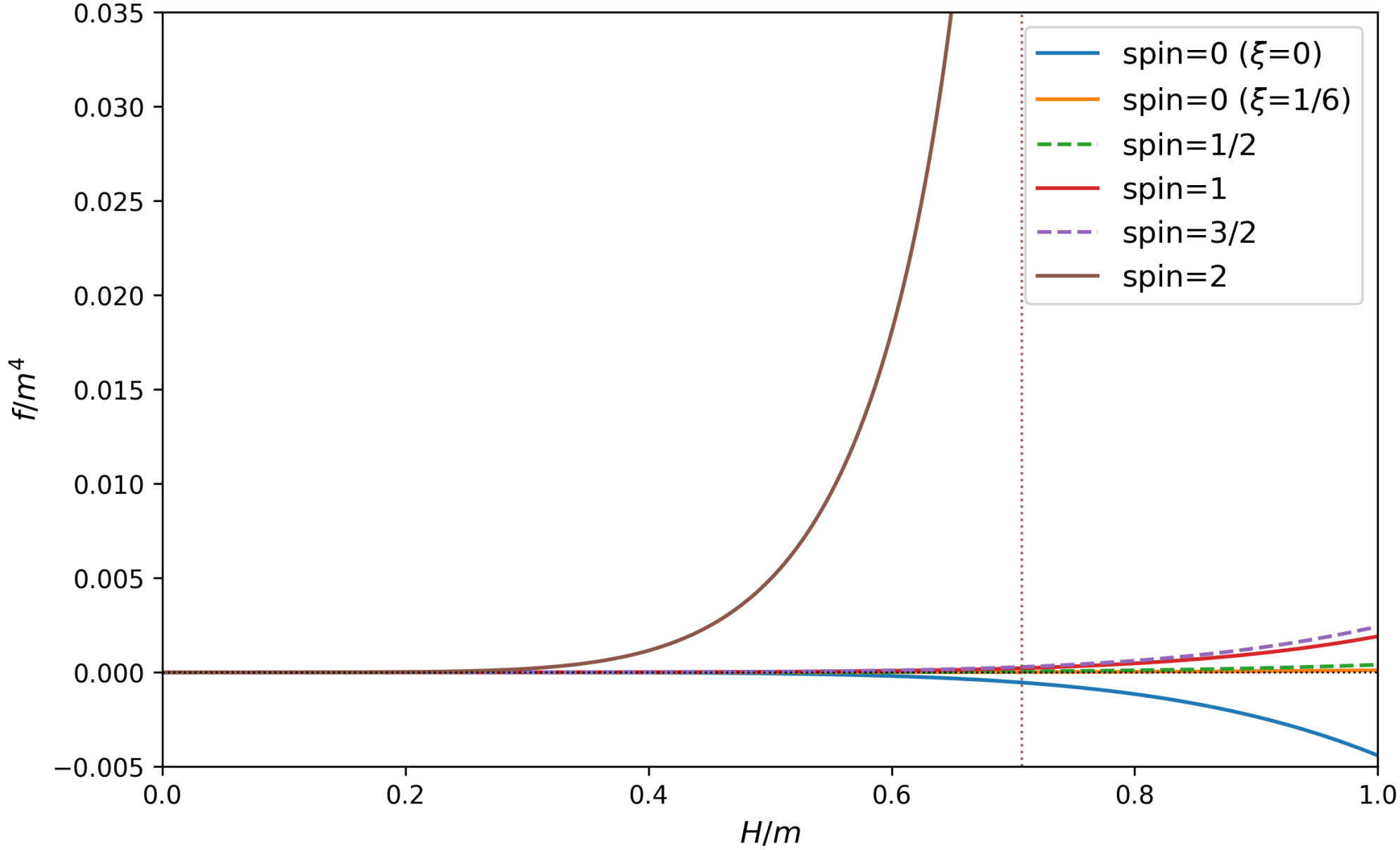
$$\ln Z_{\text{dec}}^{\text{free}}(\ell) = \ln Z_{\zeta'}(\ell) - \ln Z_{\text{asym}}(\ell)$$

$$\ln Z_{\text{asym}}(m\ell) = -\frac{2s+1}{3}\left[\left(-\frac{(m\ell)^4}{2} - (m\ell)^2 - \frac{11}{60}\right)\ln(m\ell) + \frac{25(m\ell)^4}{24} + \frac{3(m\ell)^2}{2}\right]$$
$$+ \frac{(2s-1)(2s+1)(2s+3)}{12}\left[\left((m\ell)^2 + \frac{1}{6}\right)\ln(m\ell) - \frac{3(m\ell)^2}{2}\right].$$

Spin Dependence

Higuchi bound

$m^2 = (s-1)(s+d-3)H^2$



Matching of HRG and QG

$$\Delta \ln Z_{\text{total}}^{\text{IR}-\text{UV}}(H) \equiv \ln Z_{\text{total}}^{\text{IR}}(H) - \ln Z_{\text{total}}^{\text{UV}}(H)$$

$$= -\frac{\Omega_4}{8\pi}\left(\frac{\Lambda_{\text{dec}}}{G_{\text{dec}}} - \frac{\Lambda_{\text{MS}}}{G_{\text{MS}}}\right) H^{-4} + \frac{3\Omega_4}{4\pi}\left(\frac{1}{G_{\text{dec}}} - \frac{1}{G_{\text{MS}}}\right) H^{-2} + \Omega_4\left(c_{4,\text{dec}} - c_{4,\text{MS}}(\mu)\right)$$

$$+ \ln Z_{\text{HRG}}(H) - \ln Z_{\text{pQCD}}(H)$$

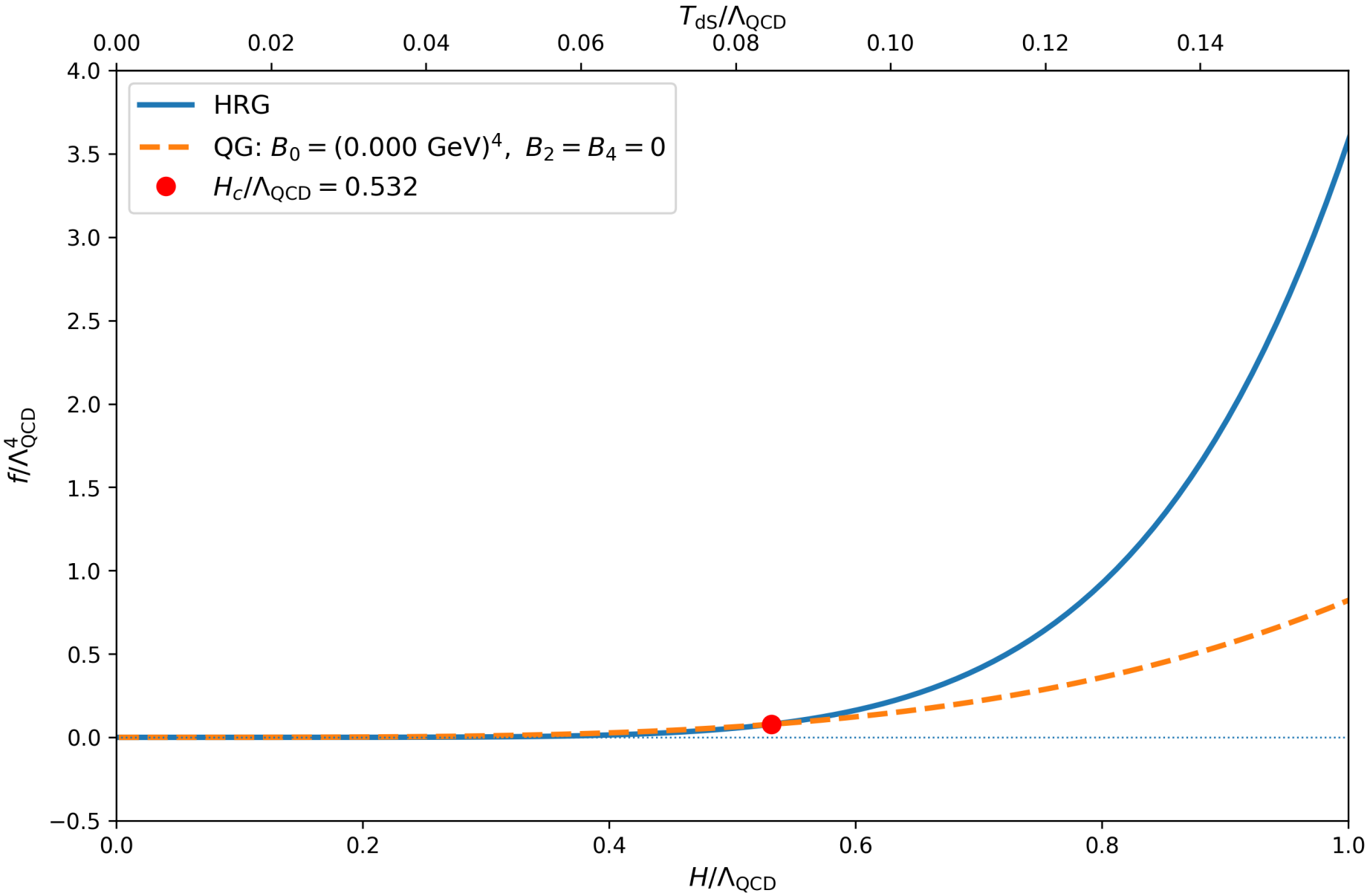
Bag parameter: $B = B_0 + B_2 R + B_4 R^2$

$$B_0 \equiv -\frac{1}{8\pi}\left(\frac{\Lambda_{\text{dec}}}{G_{\text{dec}}} - \frac{\Lambda_{\text{MS}}}{G_{\text{MS}}}\right), \qquad B_2 \equiv \frac{1}{16\pi}\left(\frac{1}{G_{\text{dec}}} - \frac{1}{G_{\text{MS}}}\right), \qquad B_4(\mu) \equiv \frac{1}{144}\left(c_{4,\text{dec}} - c_{4,\text{MS}}(\mu)\right)$$

Modified QG sphere partition function: $\ln Z_{QG}(H) \equiv \ln Z_{\text{pQCD}}(H) - BV_4$

$$\Delta \ln Z_{\text{total}}^{\text{IR}-\text{UV}}(H) = \ln Z_{\text{HRG}}(H) - \ln Z_{\text{QG}}(H)$$

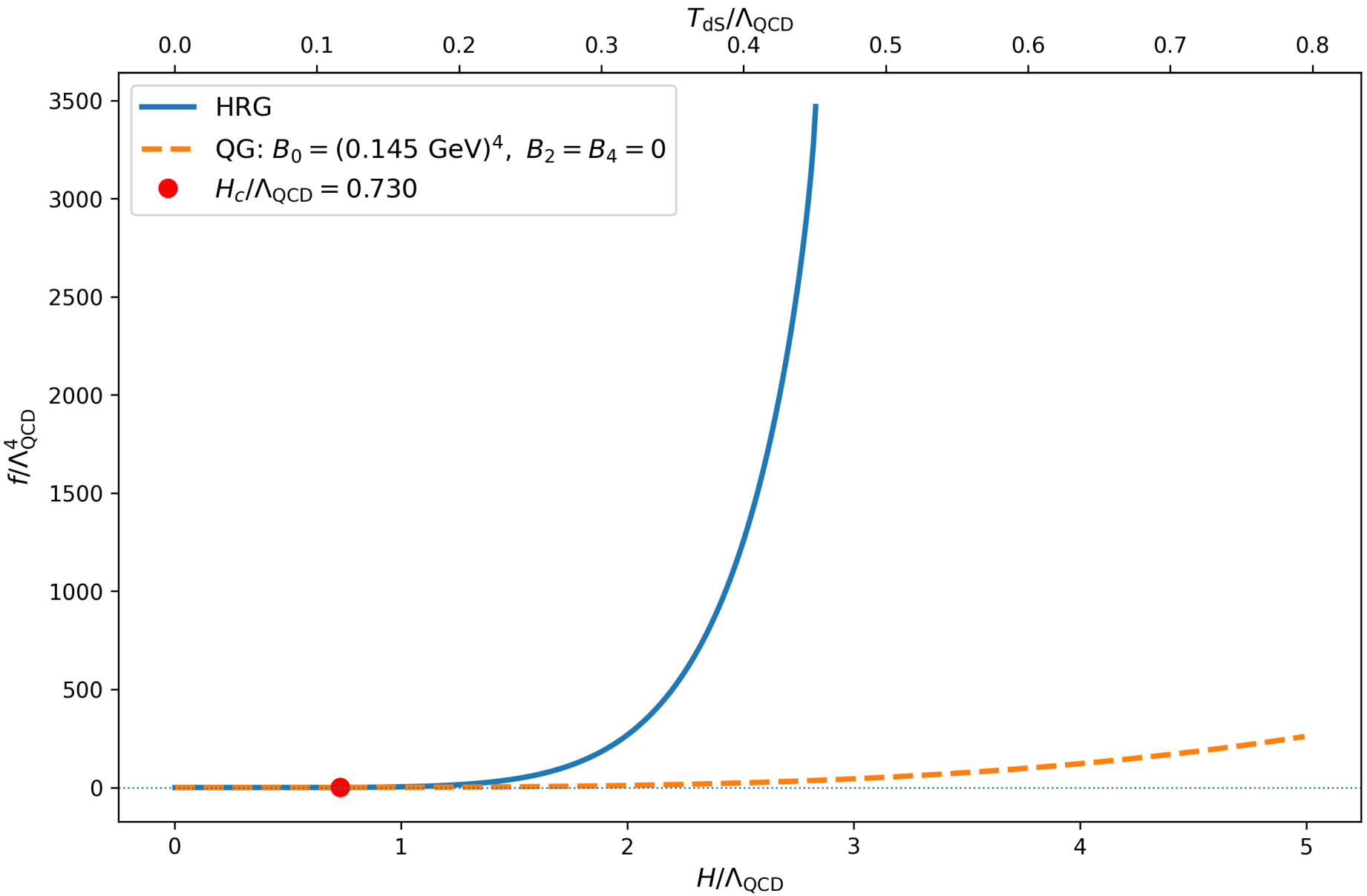
Crossing occurs even when bag constant is zero.



Higuchi bound in HRG

Higuchi bound for $a_4(1970)$

$H/\Lambda_{\mathrm{QCD}} = 2.839$



Trace Anomaly

$$\boxed{\langle T^{\mu}_{\mu} \rangle = \frac{1}{(4\pi)^2}(cW^2 - aE_4) + \sum_i \beta_i \mathcal{O}_i + \text{mass}}$$

Weyl tensor : $W^2 = W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$

Euler's density : $E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$

For maximally symmetric spacetime, $W^2 = 0$

For conformal fields, $720a = 2N_0 + 11N_{1/2}^{\text{Weyl}} + 124N_1 + \cdots$

➔ $a_q = 2N_c N_f \frac{11}{720}, \quad a_g = (N_c^2 - 1)\frac{31}{180}$