C. P. T symmetry and operation

define operation U_c, U_p, U_T are the operation operator for C, P, T

Then

- $\boxed{\text{P operation}}$

P-invariance : $|\alpha_p\rangle = U_p|\alpha\rangle \Rightarrow \langle\alpha|\beta\rangle = \langle\alpha_p|\beta_p\rangle$

$J_p = \gamma^0$

$U_p \psi(x_0, \vec{x}) U_p^{-1} = \gamma_0 \psi(x_0, -\vec{x})$

$J_p \gamma^\mu J_p^{-1} = -(-1)^{\delta_{\mu,0}} \gamma^\mu$

$x^\mu \xrightarrow{P} x_\mu$

$U_p A_\mu(x_0, \vec{x}) U_p^{-1} = A^\mu(x_0, -\vec{x})$

$p^\mu \xrightarrow{P} p_\mu$

$\partial^\mu \xrightarrow{P} \partial_\mu$

$\vec{p} \xrightarrow{U_p} -\vec{p}$

$\vec{s} \xrightarrow{U_p} \vec{s}$ (spin)

$U_p|\vec{p}, \vec{s}\rangle = |-\vec{p}, \vec{s}\rangle$ $\qquad U_p^\dagger = U_p^{-1}$

$F^{\mu\nu} \xrightarrow{P} F_{\mu\nu}$

- $\boxed{\text{T operation}}$

T invariance: $|\alpha_T\rangle = U_T|\alpha\rangle \Rightarrow \langle\alpha|\beta\rangle = \langle\beta_T|\alpha_T\rangle$

$U_T \psi(x_0, \vec{x}) U_T^{-1} = J^\dagger \psi(-x_0, \vec{x})$

$J^\dagger = J = J^{-1} = i\gamma^1\gamma^3$

$U_T A_\mu(x_0, \vec{x}) U_T^{-1} = A^\mu(-x_0, \vec{x})$

$J(\gamma^\mu)^* J^{-1} = \gamma_\mu$

$U_T|\vec{p}, \vec{s}\rangle = |-\vec{p}, -\vec{s}\rangle$

$x^\mu \xrightarrow{T} -x_\mu$

$\partial^\mu \xrightarrow{T} -\partial_\mu$

$p^\mu \xrightarrow{T} p_\mu$

$\vec{p} \xrightarrow{U_T} -\vec{p}$

$\vec{s} \xrightarrow{U_T} -\vec{s}$ (spin)

$U_T (c\#) U_T^{-1} = (c\#)^*$

$F^{\mu\nu} \xrightarrow{T} -F_{\mu\nu}$

- $\boxed{\text{C operation}}$

$U_c^\dagger = U_c^{-1}$

$J_c = i\gamma^2\gamma^0$ $\qquad$ transpose $\qquad$ $[J_c \gamma_\mu J_c^{-1} = -(\gamma_\mu)^t$

$U_c \bar{\psi}(x_0, \vec{x}) U_c^{-1} = -\psi(x_0, \vec{x}) J_c^{-1}$

$J_c^{-1} = J_c^\dagger = J_c^t = -J_c$

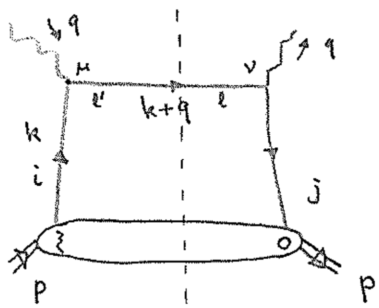
$U_c \psi(x_0, \vec{x}) U_c^{-1} = J_c \bar{\psi}(x_0, \vec{x})$

$U_c A_\mu(x_0, \vec{x}) U_c^{-1} = -A_\mu(x_0, \vec{x})$

$U_c|(\vec{p}, \vec{s})^{(-)}\rangle = |(\vec{p}, \vec{s})^{(+)}\rangle$

[The operation of charge conjugation
reverses particle and anti particle states,
while leaving spins and momenta unchanged]

How many correlation function / parton distribution function do we need to characterize the structure of a spin-½ proton?



$$\sim \langle PS|\ \tilde{\bar{\psi}}_j\ \gamma^{\nu}_{jl}\ (\not{k}+\not{q})_{ll'}\ \gamma^{\mu}_{l'i}\ \tilde{\psi}_i\ |PS\rangle$$

$$\sim \boxed{\langle PS|\ \tilde{\bar{\psi}}_j\ \tilde{\psi}_i\ |PS\rangle} * \left[\gamma^{\nu}(\not{k}+\not{q})\gamma^{\mu}\right]_{ji}$$

$$\Downarrow$$

- generic two-quark correlation function to characterize the nucleon structure
- So far in momentum space
- in coordinate space we have

$$\Phi_{ij}(k, P, S) = \int \frac{d^4\xi}{(2\pi)^4}\ e^{ik\cdot\xi}\ \langle PS|\ \bar{\psi}_j(0)\ \psi_i(\xi)\ |PS\rangle$$

NOTE: In general we also need gauge link to render the above definition gauge invariant, we'll talk about that later

Q: how is Φ_{ij} related to the parton distribution function, as well as many other quantities like transversity, Sivers function, etc ?

To proceed, let's study what requirements do we have from QCD

- Hermiticity

$$\Phi^{\dagger}(k, P, S) = \gamma^0 \, \Phi(k, P, S) \, \gamma^0$$

- Parity

$$\Phi(k, P, S) = \gamma^0 \, \Phi(\bar{k}, \bar{P}, -\bar{S}) \, \gamma^0$$

- Time reversal

$$\Phi^{*}(k, P, S) = (-i\gamma^5 C) \, \Phi(\bar{k}, \bar{P}, \bar{S}) \, (-i\gamma^5 C)$$

where $C = i\gamma^2\gamma^0$, $-i\gamma^5 C = i\gamma^1\gamma^3$, and $\bar{k} = (k^0, -\vec{k})$

To show this, we need some background how fields transform under "C, P, T", which you might find some limited information from any standard textbook on Quantum Field Theory; for extended discussion, see

CP violation by Branco, Lavoura, Silva

- Hermiticity

$$(\Phi^\dagger)_{ij} = \Phi^*_{ji} = \frac{1}{(2\pi)^4}\int d^4\xi\, e^{-ik\cdot\xi} \langle PS|\,\bar\psi_i(0)\,\psi_j(\xi)\,|PS\rangle^*$$

$$= \frac{1}{(2\pi)^4}\int d^4\xi\, e^{-ik\cdot\xi} \langle PS|\,\psi^\dagger_\ell(0)\,\gamma^0_{\ell i}\,\psi_j(\xi)\,|PS\rangle^*$$

$$= \frac{1}{(2\pi)^4}\int d^4\xi\, e^{-ik\cdot\xi} \langle PS|\,\psi^\dagger_j(\xi)\,\gamma^0_{i\ell}\,\psi_\ell(0)\,|PS\rangle$$

$\Downarrow$ Note

$$\gamma^0_{j'\ell'}\gamma^0_{\ell' j} = \delta_{jj'}$$

$$\psi^\dagger_j(\xi) = \psi^\dagger_{j'}(\xi)\,\delta_{jj'}$$

$$= \underbrace{\psi^\dagger_{j'}(\xi)\,\gamma^0_{j'\ell'}}\,\gamma^0_{\ell' j}$$

$$= \bar\psi_{\ell'}(\xi)\quad\gamma^0_{\ell' j}$$

$$= \frac{1}{(2\pi)^4}\int d^4\xi\, e^{-ik\cdot\xi} \langle PS|\,\bar\psi_{\ell'}(\xi)\,\gamma^0_{\ell' j}\,\gamma^0_{i\ell}\,\psi_\ell(0)\,|PS\rangle$$

$\Downarrow$ Change variable $\xi \to -\xi$

$$d^4\xi = d^4(-\xi)$$

$$= \frac{1}{(2\pi)^4}\int d^4\xi\, e^{ik\cdot\xi} \langle PS|\,\bar\psi_{\ell'}(-\xi)\,\gamma^0_{\ell' j}\,\gamma^0_{i\ell}\,\psi_\ell(0)\,|PS\rangle$$

translational invariance

$\Downarrow$

$$\langle PS|\,\bar\psi(-\xi)\cdots\psi(0)\,|PS\rangle = \langle PS|\,\bar\psi(0)\cdots\psi(\xi)\,|PS\rangle$$

$$= \frac{1}{(2\pi)^4}\int d^4\xi\, e^{ik\cdot\xi} \langle PS|\,\bar\psi_{\ell'}(0)\,\gamma^0_{\ell' j}\,\gamma^0_{i\ell}\,\psi_\ell(\xi)\,|PS\rangle$$

$$= \gamma^0_{i\ell}\,\Phi_{\ell\ell'}\,\gamma^0_{\ell' j} = [\,\gamma^0\,\Phi\,\gamma^0\,]_{ij}$$

$$\Rightarrow\quad \boxed{\Phi^\dagger(k,P,S) = \gamma^0\,\Phi(k,P,S)\,\gamma^0}$$

• Parity

① under parity $\qquad p^\mu \longrightarrow p_\mu$

$$p^\mu = (p^0, \vec{p}) \qquad p_\mu = (p^0, -\vec{p}) \Rightarrow \bar{p}$$

• momentum change

• spin does not change

$$S^\mu = (0, \vec{S}) \longrightarrow S^\mu = (0, \vec{S})$$

use notation $\quad \bar{S} \Rightarrow S_\mu = (0, -\vec{S})$

$$-\bar{S} = (0, \vec{S})$$

$$(K, P, S) \rightarrow (\bar{K}, \bar{P}, -\bar{S})$$

$$\Phi_{ij}(K, P, S) = \frac{1}{(2\pi)^4} \int d^4\xi \; e^{iK\cdot\xi} \langle PS | \bar{\psi}_j(0) \; \psi_i(\xi) | PS \rangle$$

$$\Downarrow \quad U_P U_P^{-1} = \mathbb{1}$$

$$= \frac{1}{(2\pi)^4} \int d^4\xi \; e^{iK\cdot\xi} \langle PS | U_P^{-1} U_P \bar{\psi}_j(0) U_P^{-1} U_P \psi_i(\xi) U_P^{-1} U_P | PS \rangle$$

$$U_P |PS\rangle = |\bar{P}, -\bar{S}\rangle$$

$$\langle PS | U_P^{-1} = \langle \bar{P}, -\bar{S} |$$

$$U_P \psi(\xi) U_P^{-1} = \gamma^0 \psi(\bar{\xi})$$

$$= \frac{1}{(2\pi)^4} \int d^4\xi \; e^{iK\cdot\xi} \langle \bar{P}, -\bar{S} | \bar{\psi}_l(0) \gamma^0_{lj} \quad \gamma^0_{ii'} \psi_{i'}(\bar{\xi}) | \bar{P}, -\bar{S} \rangle$$

NOTE:

$$k \cdot \xi = k^0 \xi^0 - \vec{k} \cdot \vec{\xi} \qquad k = (k^0, \vec{k}) \qquad \xi = (\xi^0, \vec{\xi})$$

$$\bar{k} \cdot \bar{\xi} = k^0 \xi^0 - \vec{k} \cdot \vec{\xi} \qquad \bar{k} = (k^0, -\vec{k}) \qquad \bar{\xi} = (\xi^0, -\vec{\xi})$$

Thus $k \cdot \xi = \bar{k} \cdot \bar{\xi}$

$$\Phi_{ij}(k, P, S) = \frac{1}{(2\pi)^4} \int d^4\xi \; e^{i \bar{k} \cdot \bar{\xi}} \langle \bar{P}, -\bar{S} | \bar{\psi}_l(0) \gamma^0_{lj} \; \gamma^0_{ii'} \psi_{i'}(\xi) | \bar{P}, -\bar{S} \rangle$$

$$\Downarrow \quad d^4\xi = d^4\bar{\xi}$$

$$= \frac{1}{(2\pi)^4} \int d^4\bar{\xi} \; e^{i \bar{k} \cdot \bar{\xi}} \langle \bar{P}, -\bar{S} | \bar{\psi}_l(0) \gamma^0_{lj} \; \gamma^0_{ii'} \psi_{i'}(\bar{\xi}) | \bar{P}, -\bar{S} \rangle$$

$$= \gamma^0_{ii'} \; \Phi_{i'l}(\bar{k}, \bar{P}, -\bar{S}) \; \gamma^0_{lj}$$

$$\boxed{\Phi(k, P, S) = \gamma^0 \, \Phi(\bar{k}, \bar{P}, -\bar{S}) \, \gamma^0}$$

- Time reversal

anti-unitary operator

$\langle\alpha| = \langle p,s|\,\hat{O}^\dagger \Rightarrow |\alpha\rangle = \hat{O}\,|p,s\rangle$

$|\beta\rangle = |p,s\rangle \Rightarrow \langle\beta| = \langle p,s|$

under time

$p \to \bar{p}$

$s \to \bar{s}$

$\xi \to -\bar{\xi}$

$\hookrightarrow (t, \vec{\xi})$

Then time-reversal invariance indicates

$$\langle\alpha|\beta\rangle = \underset{\substack{\uparrow \\ \text{"state" after T-operation}}}{\langle\beta_T|\alpha_T\rangle}$$

$|\beta_T\rangle = |\bar{p},\bar{s}\rangle \Rightarrow \langle\beta_T| = \langle\bar{p},\bar{s}|$

$|\alpha_T\rangle = U_T\,\hat{O}\,U_T^{-1}\,|\bar{p},\bar{s}\rangle$

Thus we have

$$\langle p\,s|\left[\bar{\psi}_j(0)\,\psi_i(\xi)\right]^\dagger|p\,s\rangle = \langle\bar{p}\,\bar{s}|\,U_T\left[\bar{\psi}_j(0)\,\psi_i(\xi)\right]U_T^{-1}\,|\bar{p}\,\bar{s}\rangle$$

NOTE:

$$\Phi^*_{ij}(k,p,s) = \frac{1}{(2\pi)^4}\int d^4\xi\; e^{-ik\cdot\xi}\,\langle p\,s|\left[\bar{\psi}_j(0)\,\psi_i(\xi)\right]^\dagger|p\,s\rangle$$

$$= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{-ik\cdot\xi}\,\langle\bar{p}\,\bar{s}|\,U_T\left[\bar{\psi}_j(0)\,\psi_i(\xi)\right]U_T^{-1}\,|\bar{p}\,\bar{s}\rangle$$

$$= \frac{1}{(2\pi)^4}\int d^4\xi\; e^{-ik\cdot\xi}\,\langle\bar{p}\,\bar{s}|\,U_T\,\bar{\psi}_j(0)\,U_T^{-1}\,U_T\,\psi_i(\xi)\,U_T^{-1}\,|\bar{p}\,\bar{s}\rangle$$

$$U_T \psi(\xi) U_T^{-1} = -i\gamma^5 C \psi(-\bar{\xi})$$

$$(-i\gamma^5 C)^\dagger = -i\gamma^5 C$$

$$\Phi^*_{ij}(K,P,S) = \frac{1}{(2\pi)^4}\int d^4\xi \; e^{-iK\cdot\xi} \langle \bar{P}\bar{S}|\, \bar{\psi}_\ell(0)(-i\gamma^5 C)_{\ell j} \;\; (-i\gamma^5 C)_{ii'} \; \psi_{i'}(-\bar{\xi})|\bar{P}\bar{S}\rangle$$

$$\Downarrow \qquad K\cdot\xi = \bar{K}\cdot\bar{\xi}$$

$$= \frac{1}{(2\pi)^4}\int d^4(-\bar{\xi}) \; e^{i\bar{K}\cdot(-\bar{\xi})} \langle \bar{P}\bar{S}|\, \bar{\psi}_\ell(0)(-i\gamma^5 C)_{\ell j}(-i\gamma^5 C)_{ii'}\psi_{i'}(-\bar{\xi})|\bar{P}$$

$$= \frac{1}{(2\pi)^4}\int d^4\xi \; e^{i\bar{K}\cdot\xi} \langle \bar{P}\bar{S}|\, \bar{\psi}_\ell(0)(-i\gamma^5 C)_{\ell j}(-i\gamma^5 C)_{ii'}\psi_{i'}(\xi)|\bar{P}\bar{S}\rangle$$

$$= (-i\gamma^5 C)_{ii'} \; \Phi_{i'\ell}(\bar{K},\bar{P},\bar{S}) \;\; (-i\gamma^5 C)_{\ell j}$$

Independent 4X4 matrix basis

$$1$$

$$\gamma^\mu$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

$$\gamma^\mu \gamma 5$$

$$i\gamma 5$$

Now perform expansion

assume proton is moving in +z direction

$$p^\mu = [p^0, 0, 0, p^z]$$

$$v^\pm = \frac{1}{\sqrt{2}}(v^0 \pm v^z)$$

$$p^\mu = p^+ \bar{n}^\mu$$

$$\bar{n}^\mu = [1^+, 0^-, 0_\perp]$$

$$n^\mu = [0^+, 1^-, 0_\perp]$$

spin of proton

$$S^\mu = \lambda \frac{p^+}{m} \bar{n}^\mu + S_T^\mu$$

helicity

$$\Phi_{ij}(k, P, S) = \int \frac{d^4\xi}{(2\pi)^4} \, e^{ik\cdot\xi} \langle PS| \bar{\psi}_j(0) \, \psi_i(\xi) |PS\rangle$$

- consider purely collinear case

In other words, integrate over k_T, k^- components and set $k^+ = xp^+$

$$\Phi_{ij}(x) = \int d^2k_T \, dk^- \, \Phi_{ij}(k, P, S)\Big|_{k^+ = xp^+}$$

$$= \int \frac{d\xi^-}{2\pi} \, e^{ik\cdot\xi} \langle PS| \bar{\psi}_j(0) \, \psi_i(\xi) |PS\rangle_{\xi^+ = \xi_T = 0}$$

In other words $\Phi_{ij}(x)$ should depend on $\not{P}$ only (as well as spin "S" vector) since $k \approx xP$

- What about TMD — k_T-dependent parton distribution

$$\Phi_{ij}(x, k_T) = \int dk^- \, \Phi_{ij}(k, P, S)\Big|_{k^+ = xp^+}$$

$$= \int \frac{d\xi^-}{2\pi} \frac{d^2\xi_T}{(2\pi)^2} \, e^{ik\cdot\xi} \langle PS| \bar{\psi}_j(0) \psi_i(\xi) |PS\rangle_{\xi^+ = 0}$$

- Start with a case when spin "S" is not observed (or spin-0 particle like pions)

Simpler

$$\Gamma = \{1,\ \gamma^\mu,\ \gamma^\mu\gamma^5,\ \sigma^{\mu\nu},\ i\gamma^5,\ \sigma^{\mu\nu} i\gamma^5\}$$

show $\gamma^0\gamma^\mu\gamma^0 = \gamma_\mu$

$$\mu = 0:\ \gamma^0\gamma^0\gamma^0 = \gamma^0$$

$$\mu \neq 0:\ \gamma^0\gamma^i\gamma^0 = \gamma^0(-\gamma^0\gamma^i) = -\gamma^i$$

$$\Big\}\quad (\gamma^0, -\gamma^i) = \gamma_\mu$$

$$\gamma^0\{1,\ \gamma^\mu,\ \gamma^\mu\gamma^5,\ \sigma^{\mu\nu},\ i\gamma^5,\ \sigma^{\mu\nu} i\gamma^5\}\gamma^0$$

$$= \{1,\ \gamma_\mu,\ -\gamma_\mu\gamma^5,\ \sigma_{\mu\nu},\ -i\gamma^5,\ -\sigma_{\mu\nu} i\gamma^5\}$$

$$\boxed{\begin{aligned} \Phi^\dagger(P,k) &= \gamma^0\,\Phi(P,k)\gamma^0 \\ \Phi(P,k) &= \gamma^0\,\Phi(\bar{P},\bar{k})\gamma^0 \\ \Phi^*(P,k) &= J\,\Phi(\bar{P},\bar{k})J \qquad \text{where } J = -i\gamma^5 C = i\gamma^1\gamma^3 \end{aligned}}$$

$$k^\mu = x P^\mu + k_T^\mu$$

usually, the γ-matrix basis is given by $\{\mathbb{1}, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}, i\gamma^5\}$

we can then contract with all momentum + $g^{\mu\nu}$, $\epsilon^{\mu\nu\rho\sigma}$

However, since $\sigma^{\mu\nu}\gamma^5 = \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}$

Thus for $\sigma^{\mu\nu}$ contribution, if we include $\sigma^{\mu\nu} i\gamma^5$, we do not need $\epsilon^{\mu\nu\rho\sigma}$ to contract with $\sigma_{\rho\sigma}$ term!

Two possible momenta : P, K

$\mathbb{1}$: A_1

γ^μ : $\{P_\mu, K_\nu\}$ $\Rightarrow$ A_2, A_3

$\gamma^\mu\gamma^5$: $\{P_\mu, K_\nu\}$ $\Rightarrow$ A_5, A_6

$\sigma^{\mu\nu}$: $K_\mu P_\nu$ $\Rightarrow$ A_4

$i\gamma^5$: A_7

$\sigma^{\mu\nu} i\gamma^5$: $K_\mu P_\nu$ $\Rightarrow$ A_8

$$\Phi(P,K) = \left[M A_1 + A_2 \not{P} + A_3 \not{K} + A_4 \frac{\sigma^{\mu\nu} K_\mu P_\nu}{M} \right]$$
$$+ \left[A_5 \not{P}\gamma^5 + A_6 \not{K}\gamma^5 + M A_7 (i\gamma^5) + A_8 \frac{\sigma^{\mu\nu} i\gamma^5 K_\mu P_\nu}{M} \right]$$

- Hermiticity $\Rightarrow$ All "A" should be real

e.g. $A_5 \not{P}\gamma^5$

$$(A_5 \not{P}\gamma^5)^\dagger = \gamma^0 (A_5 \not{P}\gamma^5)\gamma^0$$

Note $(\gamma^0)^2 = \mathbb{1}$
$\gamma^0 (\gamma^\mu)^\dagger \gamma^0 = \gamma^\mu$

$\Rightarrow$ LHS $= A_5^* (\gamma^\mu\gamma^5)^\dagger P_\mu = A_5^* \gamma^5 (\gamma^\mu)^\dagger P_\mu = A_5^* \gamma^5 \gamma^0\gamma^0 (\gamma^\mu)^\dagger \gamma^0\gamma^0 P_\mu$

$= A_5^* \gamma^5 \gamma^0 \gamma^\mu \gamma^0 P_\mu$

$= A_5^* (-\gamma^0 \gamma^5 \gamma^\mu \gamma^0) P_\mu$

$= A_5^* (\gamma^0 \gamma^\mu \gamma^5 \gamma^0) P_\mu$

used $\{\gamma^\mu, \gamma^5\} = 0$

RHS $= A_5 P_\mu \gamma^0 (\gamma^\mu\gamma^5)\gamma^0$

$\Rightarrow$ $\boxed{A_5^* = A_5}$

• parity $\Rightarrow$ $A_5 = A_6 = A_7 = A_8 = 0$

$$\Phi(P, K) = \gamma^0 \, \Phi(\bar{P}, \bar{K}) \, \gamma^0$$

e.g. $A_8 \dfrac{\sigma^{\mu\nu} i\gamma^5 K_\mu P_\nu}{M}$

$$\text{LHS} = A_8 \frac{\sigma^{\mu\nu} i\gamma^5 K_\mu P_\nu}{M}$$

$$\text{RHS} = \gamma^0 \left[A_8 \frac{\sigma^{\mu\nu} i\gamma^5 \bar{K}_\mu \bar{P}_\nu}{M} \right] \gamma^0$$

NOTE: $\gamma^0 (\sigma^{\mu\nu} i\gamma^5) \gamma^0 = -\sigma_{\mu\nu} i\gamma^5$

$\bar{P}_\mu = P^\mu$

$\bar{K}_\mu = K^\mu$

$$= A_8 \left[\frac{-\sigma_{\mu\nu} i\gamma^5 K^\mu P^\nu}{M} \right]$$

$$= -A_8 \frac{\sigma_{\mu\nu} i\gamma^5 K^\mu P^\nu}{M}$$

$$= -A_8 \frac{\sigma^{\mu\nu} i\gamma^5 K_\mu P_\nu}{M}$$

LHS = RHS $\Rightarrow$ $A_8 = -A_8$

$\Rightarrow$ $\boxed{A_8 = 0}$

$p^\mu = p^+ \bar{n}^\mu$ $\qquad\qquad$ $\bar{n}^\mu = [1^+, 0^-, 0_\perp]$

$k^\mu = x p^\mu + k_T^\mu$

$$\Phi(P,k) = M A_1 + A_2 \not{P} + A_3 (x \not{P} + \not{k}_T) + A_4 \frac{\sigma^{\mu\nu}(x P_\mu + k_{T\mu}) P_\nu}{M}$$

$\Downarrow \quad \sigma^{\mu\nu} P_\mu P_\nu = 0$

$$= M A_1 + A_3 \not{k}_T$$

$$+ (A_2 + A_3 x) \not{P}$$

$$+ A_4 \frac{\sigma^{\mu\nu} k_{T\mu} P_\nu}{M}$$

$\Downarrow$ NOTE $\quad P \gg k_T \sim M \quad$ Thus we can drop terms like $M A_1$, $A_3 \not{k}_T$

keep only the largest contribution (twist analysis)

$$= (A_2 + A_3 x) \not{P}$$

$$+ A_4 \frac{\sigma^{\mu\nu} k_{T\mu} P_\nu}{M}$$

$\Downarrow$ give them better names

$$= \frac{1}{2} \left[f_1(x, k_T^2) \not{P} + h_1^\perp(x, k_T^2) \frac{\sigma^{\mu\nu} k_{T\mu} P_\nu}{M} \right]$$

f_1: unpolarized PDFs

$h_1^\perp$: Boer-Mulders function

transversely polarized quark inside unpolarized nucleon/pion

- let's consider the spin dependent case

$\Phi(P, K, S)$ $\qquad\qquad S^\mu = \lambda \frac{P^+}{M} \bar{n}^\mu + S_T^\mu$

$\mathbb{1}$: A_1

γ^μ : $\{P_\mu, K_\mu\} \Rightarrow A_2, A_3$

$S_\mu \qquad B_1$

$\epsilon^{\mu\nu\rho\sigma} \gamma_\mu P_\nu k_\rho S_\sigma \Rightarrow A_{12}$

$\sigma^{\mu\nu}$: $\{P_\mu, K_\nu\} \Rightarrow A_4$

$i\gamma^5$: $K \cdot S \Rightarrow A_5$

$\gamma^\mu \gamma^5$: $\{P_\mu, K_\mu\} \Rightarrow$ Vanish because of parity

$S_\mu \Rightarrow A_6$

$\frac{K \cdot S}{M} P_\mu \Rightarrow A_7$

$\frac{K \cdot S}{M} K_\mu \Rightarrow A_8$

$\sigma^{\mu\nu} i\gamma^5$: $P_\mu S_\nu \Rightarrow A_9$

$K_\mu S_\nu \Rightarrow A_{10}$

$(K \cdot S) P_\mu K_\nu \Rightarrow A_{11}$

$$
\begin{aligned}
\Phi(P,K,S) = {} & M A_1 \mathbb{1} + \not{P} A_2 + \not{K} A_3 + \epsilon^{\mu\nu\rho\sigma} \gamma_\mu P_\nu K_\rho S_\sigma A_{12} \\
& + \frac{1}{M} \sigma^{\mu\nu} P_\mu K_\nu A_4 + (K\cdot S) i\gamma^5 A_5 + \not{S} \gamma^5 M A_6 \\
& + \frac{K\cdot S}{M} \not{P} \gamma^5 A_7 + \frac{K\cdot S}{M} \not{K} \gamma^5 A_8 \\
& + \sigma^{\mu\nu} i\gamma^5 P_\mu S_\nu A_9 \\
& + \sigma^{\mu\nu} i\gamma^5 K_\mu S_\nu A_{10} \\
& + \sigma^{\mu\nu} i\gamma^5 \frac{K\cdot S}{M^2} P_\mu K_\nu A_{11}
\end{aligned}
$$

check parity

$$\Phi(P,K,S) = \gamma^0 \Phi(\bar{P}, \bar{K}, -\bar{S}) \gamma^0$$

$$(K\cdot S) i\gamma^5 A_5 \to \gamma^0 \left[\bar{K}\cdot(-\bar{S}) \, i\gamma^5 A_5 \right] \gamma^0$$

$$\Big\Downarrow \; \gamma^0\gamma^5\gamma^0 = -\gamma^5$$

$$= \bar{K}\cdot\bar{S} \; i\gamma^5 A_5$$

$$\Big\Downarrow \; \bar{K} = (K^0, -\vec{K}) \quad \bar{S} = (0, -\vec{S}), \quad \bar{K}\cdot\bar{S} = K\cdot S$$

$$= K\cdot S \; i\gamma^5 A_5$$

Thus such term can exist!

Eventually you'll get

$$\Phi^{[\gamma^+]} = f_1(x, k_T^2) - \frac{\epsilon_T^{\rho\sigma} k_{T\rho} S_{T\sigma}}{M} f_{1T}^{\perp}(x, k_T^2)$$

$$\Phi^{[\gamma^+\gamma^5]} = \lambda\, g_1(x, k_T^2) - \frac{k_T \cdot S_T}{M} g_{1T}^{\perp}(x, k_T^2)$$

$$\Phi^{[i\sigma^{\alpha +}\gamma^5]} = S_T^{\alpha} h_{1T}(x, k_T^2) + \frac{\lambda k_T^{\alpha}}{M} h_{1L}^{\perp}(x, k_T^2)$$

$$- \frac{\vec{k}_T^{\,2}}{M^2}\left(\frac{1}{2} g_T^{\alpha\rho} + \frac{k_T^{\alpha} k_T^{\rho}}{\vec{k}_T^{\,2}}\right) S_{T\rho}\, h_{1T}^{\perp}(x, k_T^2)$$

$$- \frac{\epsilon_T^{\alpha\rho} k_{T\rho}}{M} h_1^{\perp}(x, k_T^2)$$

famous mistake — Sivers function vanishes ?!

$$f_{q/p\uparrow}(x, k_T, S_T) = f_{q/p}(x, k_T^2) + \vec{S}_T \cdot (\vec{k}_T \times \hat{p}) \frac{1}{M} f_{1T}^{\perp}(x, k_T)$$

$$\to \int \frac{d\xi^-}{2\pi} \frac{d^2\xi_T}{(2\pi)^2} e^{ik\cdot\xi} \langle PS| \bar{\psi}(0) \frac{\gamma^+}{2} \psi(\xi) |PS\rangle_{\xi^+=0}$$

$$f_{1T}^{\perp}(x, k_T) \propto \boxed{f_{q/p\uparrow}(x, k_T, S_T) - f_{q/p\uparrow}(x, k_T, -S_T)}$$

Apply both P and T invariance, see what happens

you'll find

$$f_{q/p\uparrow}(x, k_T, S_T) = f_{q/p\uparrow}(x, k_T, -S_T)$$

$\Rightarrow$ Vanish ?!

$\Rightarrow$ gauge link !

$$\langle \alpha | = \langle \vec{p}, \vec{s} | \hat{O}$$

$$|\beta\rangle = |\vec{p}, \vec{s}\rangle$$

T-invariance $\Rightarrow$ $\langle \alpha | \beta \rangle = \langle \beta_T | \alpha_T \rangle$

$$\langle \vec{p}, \vec{s} | \hat{O} | \vec{p}, \vec{s} \rangle = \langle -\vec{p}, -\vec{s} | \, U_T \, \hat{O}^+ \, U_T^{-1} \, | -\vec{p}, -\vec{s} \rangle$$

$$= \langle -\vec{p}, -\vec{s} | \, U_P^{-1} U_P \, U_T \, \hat{O}^+ \, U_T^{-1} \, U_P^{-1} U_P \, | -\vec{p}, -\vec{s} \rangle$$

$$= \langle \vec{p}, -\vec{s} | \, U_P \, U_T \, \hat{O}^+ \, U_T^{-1} \, U_P^{-1} \, | \vec{p}, -\vec{s} \rangle$$

$$\hat{O} = \bar{\Psi}(0) \, \Gamma \, \psi(\xi) \qquad \text{with} \quad \Gamma = \frac{\gamma^+}{2}$$

$$\hat{O}^+ = \psi^+(\xi) \, \Gamma^+ \, \gamma^0 \, \psi(0)$$

$$U_P U_T \hat{O}^+ U_T^{-1} U_P^{-1} = U_P \, U_T (\psi^+(\xi)) U_T^{-1} U_P^{-1} \; \Gamma^+ \gamma^0 \, U_P U_T \psi(0) U_T^{-1} U_P^{-1}$$

$$= U_P \, \psi^+(-\xi^0, \vec{\xi}) J U_P^{-1} \;\; \Gamma^+ \gamma^0 \;\; U_P \, J \, \psi(0) \, U_P^{-1}$$

$$= \psi^+(-\xi^0, -\vec{\xi}) \, \gamma^0 J \;\; \Gamma^+ \gamma^0 \;\; J \gamma^0 \psi(0)$$

$$= \bar{\Psi}(-\xi) \;\; J \; \Gamma^+ \; \gamma^0 J \; \gamma^0 \;\; \psi(0)$$

$$J = -i \gamma^5 C = i \gamma^1 \gamma^3$$

$$\Downarrow \qquad \gamma^0 J = J \gamma^0$$

$$= \bar{\Psi}(-\xi) \;\; J \, \Gamma^+ \, J \;\; \psi(0)$$

$$= \bar{\Psi}(-\xi) \;\; \Gamma^+ \;\; \psi(0)$$

Thus with translational invariance well have

$$f_{q/p\uparrow}(x, k_T, \vec{S}_T) = f_{q/p\uparrow}(x, k_T, -\vec{S}_T)$$

$\Rightarrow$ Sivers function vanish ?!

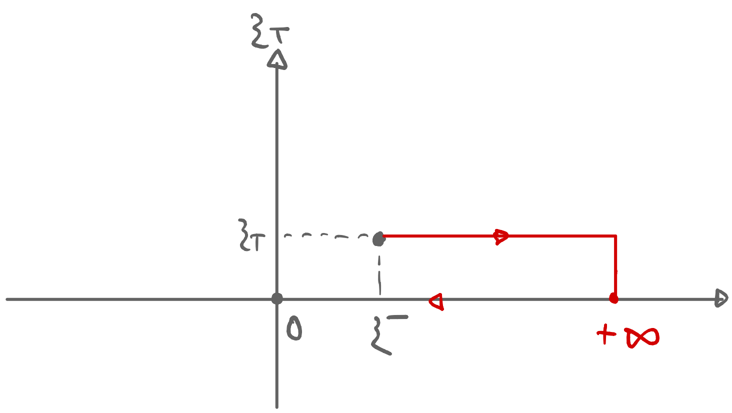
Not Really

$$f_{q/p\uparrow}(x, k_T) = \int \frac{d\xi^-}{2\pi} \frac{d^2\xi_T}{(2\pi)^2} e^{ik\cdot\xi}$$

$$\langle ps | \bar{\psi}(0)\, W_{[0,\xi]} \frac{\gamma^+}{2} \psi(\xi) | ps \rangle_{\xi^+=0}$$

for SIDIS

$$W_{[0,\xi]} = V^\dagger(\infty, 0; 0_T) \; \bar{V}(\infty, \xi^-; \xi_T)$$



$$V^\dagger(\infty, 0; 0_T) = p \exp\left[ig \int_0^\infty d\lambda^- \; A^+(\lambda^-, 0^+, 0_\perp) \right]$$

$$\lambda^- A^+ = \lambda^\mu A_\mu$$

where $\lambda^\mu = \lambda^- n^\mu$ $\qquad n^\mu = [0^+, 1^-, 0_\perp]$

$$\lambda^\mu A_\mu = \lambda^- n^\mu A_\mu = \lambda^- A^+$$

now

$$V^{\dagger}(\infty, 0; 0_T) = p\exp\left[ig \int_{0n^\mu}^{\infty n^\mu} d\lambda^\mu \, A_\mu(\lambda) \right]$$

$\Downarrow$ under $\lambda^\mu = (\lambda^0, \vec{\lambda}) \xrightarrow{P} (\lambda^0, -\vec{\lambda}) \xrightarrow{T} (-\lambda^0, -\vec{\lambda}) = -\lambda^\mu$

under P and T (anti-unitary)

$$V^{\dagger}(\infty, 0; 0_T) \xrightarrow{PT} P\exp\left[-ig \int_{0n^\mu}^{\infty n^\mu} d(-\lambda^\mu) \, A_\mu(-\lambda) \right]$$

The integration limits are given by $\lambda^\mu = 0n^\mu \to +\infty n^\mu$

now change variable $-\lambda^\mu = y^\mu$ then integration limit

becomes $y^\mu = 0n^\mu \to -\infty n^\mu$

$$\Rightarrow V^{\dagger}(\infty, 0; 0_T) \to p\exp\left[-ig \int_{0n^\mu}^{-\infty n^\mu} dy^\mu \, A_\mu(y) \right]$$

$$= p\exp\left[-ig \int_{0}^{-\infty} dy^- \, A^+(y^-, 0^+, 0_\perp) \right]$$

$$= V(-\infty, 0; 0_T)$$

$$V(\infty, \xi^-; \xi_T) = p\exp\left[-ig\int_{\xi^-}^{\infty} d\lambda^- A^+(\lambda^-, 0^+; \xi_T)\right]$$

$$\lambda^- = \frac{(\lambda^0 - \lambda^z)}{\sqrt{2}} \xrightarrow{P} \frac{\lambda^0 + \lambda^z}{\sqrt{2}} \xrightarrow{T} \frac{-\lambda^0 + \lambda^z}{\sqrt{2}} = -\lambda^-$$

$$A^+ = \frac{(A^0 + A^z)}{\sqrt{2}} \xrightarrow{P} \frac{A^0 - A^z}{\sqrt{2}} \xrightarrow{T} \frac{A^0 + A^z}{\sqrt{2}} = A^+$$

$$\xi_T \xrightarrow{P} -\xi_T \xrightarrow{T} -\xi_T$$

under P and T

$$V(\infty, \xi^-; \xi_T) = p\exp\left[-ig\int_{\xi^-}^{\infty} d(-\lambda^-)\, A^+(-\lambda^-; -\xi_T)\right]$$

$\Downarrow$ change variable $-\lambda^- \to \lambda^-$

$$= p\exp\left[-ig\int_{-\xi^-}^{-\infty} d\lambda^-\, A^+(\lambda^-; -\xi_T)\right]$$

$$= V(-\infty, -\xi^-; -\xi_T)$$

$$\langle PS|\, \bar{\Psi}(0) \frac{\gamma^+}{2} \psi(\xi)\, |PS\rangle \xrightarrow{PT} \langle P-S|\, \bar{\Psi}(-\xi) \frac{\gamma^+}{2} \psi(0)\, |P-S\rangle$$

Now $\langle PS|\bar{\Psi}(0)\, V^{+}(\infty, 0; 0_T)\, \frac{\gamma^{+}}{2}\, V(\infty, \xi^{-}; \xi_T)\, \psi(\xi)\,|PS\rangle^{DIS}$

$\xrightarrow{PT}$

$$\langle P-S|\bar{\Psi}(-\xi)\, V(-\infty, 0; 0_T)\, \frac{\gamma^{+}}{2}\, V^{+}(-\infty, -\xi^{-}; -\xi_T)\, \psi(0)\,|P-S\rangle$$

$\Downarrow$ translational invariance

$$V(x, y) \longrightarrow V(x+a, y+a)$$

e.g. $\quad e^{iP\cdot a}\, V(x,y)\, e^{-iP\cdot a} = V(x+a, y+a)$

shift $\quad -\xi \xrightarrow{+\xi} 0$

$0 \xrightarrow{+\xi} \xi$

$-\infty \xrightarrow{+\xi} -\infty$

$$= \langle P-S|\bar{\Psi}(0)\, V(-\infty, \xi^{-}; \xi_T)\, \frac{\gamma^{+}}{2}\, V^{+}(-\infty, 0; 0_T)\, \psi(\xi)\,|P-S\rangle$$

$$= \langle P-S|\bar{\Psi}(0)\, V^{+}(-\infty, 0; 0_T)\, \frac{\gamma^{+}}{2}\, V(-\infty, \xi^{-}; \xi_T)\, \psi(\xi)\,|P, -S\rangle_{DY}$$

$$\Rightarrow \quad f_q^{DIS}(x, k_T, S_T) = f_q^{DY}(x, k_T, -S_T)$$

$$f_q^{DIS}(x, k_T, S_T) = f_q^{DIS}(x, k_T) - \frac{\epsilon^{\rho\sigma} k_{T\rho} S_{T\sigma}}{M} f_{1T}^{\perp DIS}(x, k_T)$$

$\Updownarrow$ equal to each other

$$f_q^{DY}(x, k_T, -S_T) = f_q^{DY}(x, k_T) + \frac{\epsilon^{\rho\sigma} k_{T\rho} S_{T\sigma}}{M} f_{1T}^{\perp DY}(x, k_T)$$

$$\Rightarrow \qquad f_q^{DIS}(x, k_T) = f_q^{DY}(x, k_T) \qquad \text{[unpolarized TMDPDF]}$$

$$f_{1T}^{\perp\, DIS}(x, k_T) = - f_{1T}^{\perp\, DY}(x, k_T) \qquad \text{[Sivers function]}$$