Small x physics

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Outline

Lecture 1:

Introduction to Deep Inelastic Scattering
Parton model
Collinear framework: factorization
DGLAP evolution equations
Parton distribution functions from DGLAP
Nuclear structure functions
Nuclear PDFs
EIC prospects for inclusive DIS and nPDFs

Lecture 2:

Intro: Regge theory and Pomeron
Outline of BFKL construction:
- Effective Lipatov vertex
- Gluon reggeization: trajectory

BFKL equation
Eigenvalue. Collinear structure
Properties of the solution:
- Diffusion
- Increase with energy

Small x anomalous dimension

Lecture 3:

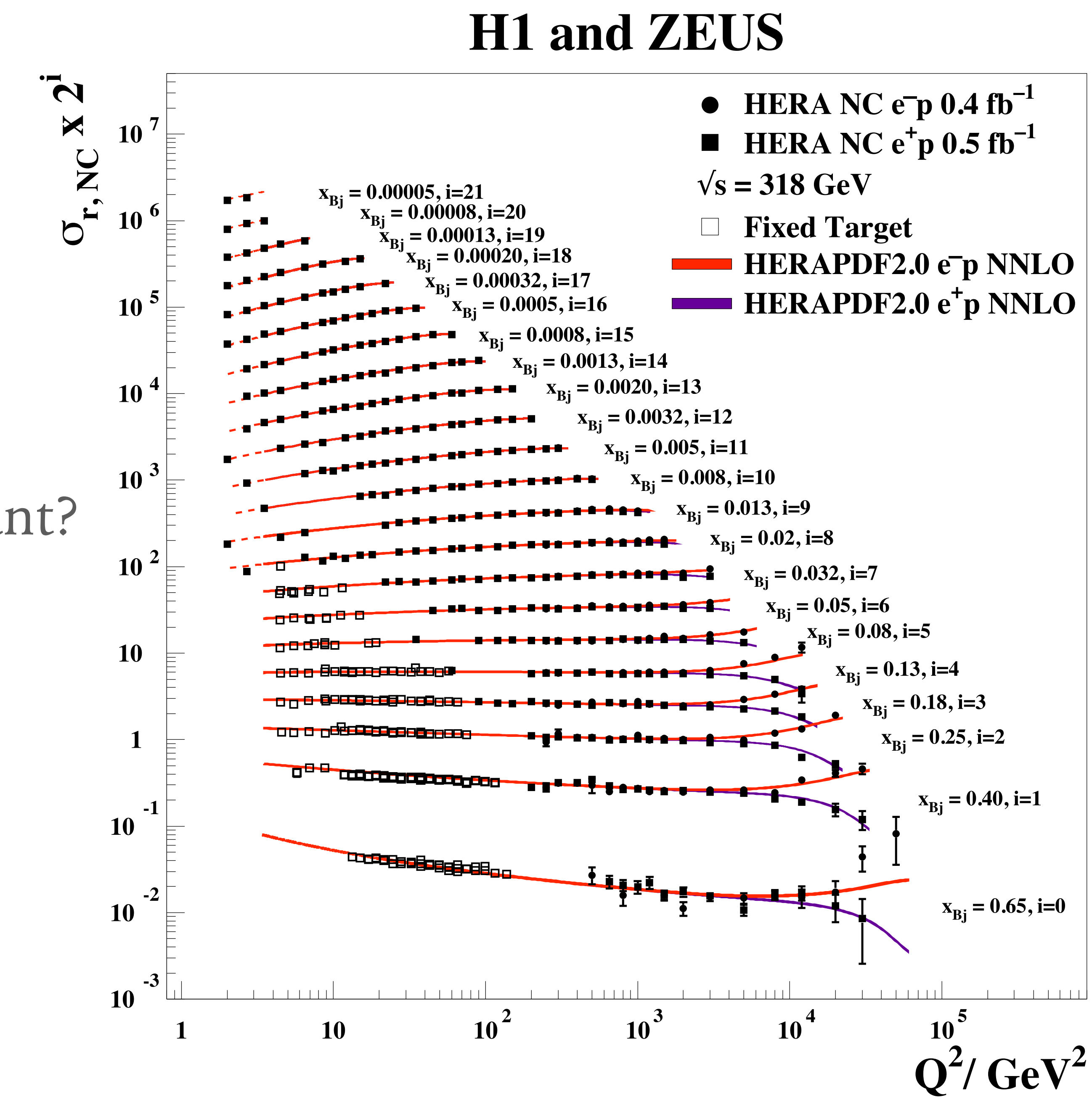
BFKL at NLO: large correction
Collinear limit of NLO BFKL
Resummation:
- Kinematical constraint, shifts of poles
- DGLAP anomalous dimension

Resummed result in Mellin space
Resummed result in momentum space
Improved small x splitting function
Phenomenology examples

Lecture 4:

Dipole model: GBW example
BFKL revisited: Mueller dipole evolution
Multiple rescattering: BK evolution
Properties of solution to BK equation
Saturation scale
Impact parameter dependence
Phenomenology examples: structure functions, diffraction, angular decorelations

Why small x ?



Collinear framework works very well

Why small x expansion should be important?

Motivation:

- Small x behavior of the collinear splitting function
- Regge theory: soft Pomeron

Collinear approximation: DGLAP evolution

DGLAP evolution: system of equations for parton densities, quarks and gluons

$$\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} q_i(x,\mu^2) \\ g(x,\mu^2) \end{pmatrix} = \sum_j \int_x^1 \frac{dz}{z} \begin{pmatrix} P_{q_i q_j}(\alpha_s, z) & P_{q_i g}(\alpha_s, z) \\ P_{g q_j}(\alpha_s, z) & P_{gg}(\alpha_s, z) \end{pmatrix} \begin{pmatrix} q_j(\frac{x}{z},\mu^2) \\ g(\frac{x}{z},\mu^2) \end{pmatrix}$$

q_j : quark density, g : gluon density

Splitting functions

calculated perturbatively

	LO	NLO	NNLO	N³LO

$$P_{ab}(\alpha_s, z) \equiv P_{b\to a}(\alpha_s, z) = \frac{\alpha_s}{2\pi} P_{ab}^{(0)}(z) + \left(\frac{\alpha_s}{2\pi}\right)^2 P_{ab}^{(1)}(z) + \left(\frac{\alpha_s}{2\pi}\right)^3 P_{ab}^{(2)}(z) + \left(\frac{\alpha_s}{2\pi}\right)^4 P_{ab}^{(3)}(z) + \dots$$

P_{ab}: describe splitting of parton b into parton a

How good is this expansion ?

Are higher orders necessarily smaller than lower orders for all values of z ?

Splitting function at N^3LO

Four-loop splitting functions in QCD
– The quark-quark case –

G. Falcioni[a], F. Herzog[a], S. Moch[b] and A. Vogt[c]

Four-loop splitting functions in QCD
– The gluon-to-quark case –

G. Falcioni[a], F. Herzog[a], S. Moch[b] and A. Vogt[c]

Four-loop splitting functions in QCD
– The quark-to-gluon case –

G. Falcioni[a,b], F. Herzog[c], S. Moch[d], A. Pelloni[e] and A. Vogt[f]

Four-loop splitting functions in QCD
– The gluon-gluon case –

G. Falcioni[a,b], F. Herzog[c], S. Moch[d], A. Pelloni[e] and A. Vogt[f]

$$P_{ab}(\alpha_s, z) = \frac{\alpha_s}{2\pi} P_{ab}^{(0)}(z) + \left(\frac{\alpha_s}{2\pi}\right)^2 P_{ab}^{(1)}(z) + \left(\frac{\alpha_s}{2\pi}\right)^3 P_{ab}^{(2)}(z) + \left(\frac{\alpha_s}{2\pi}\right)^4 P_{ab}^{(3)}(z) + \ldots$$

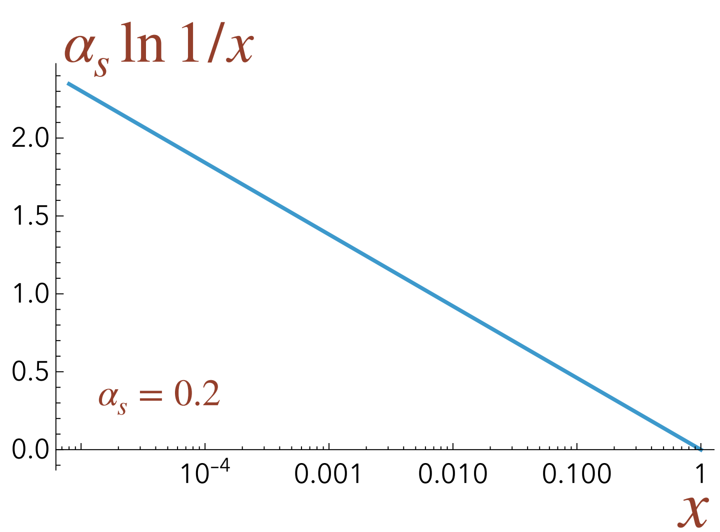
Abstract

We have computed the even-N moments $N \leq 20$ of the gluon-gluon splitting function P_{gg} at the fourth order of perturbative QCD via the renormalization of off-shell operator matrix elements. Our results, derived analytically for a general compact simple gauge group, agree with all results obtained for this function so far, in particular with the lowest five moments obtained via structure functions in deep-inelastic scattering. Using our new moments and all available endpoint constraints, we construct improved approximations for the four-loop $P_{gg}(x)$ that should be sufficient for a wide range of collider-physics applications. The N^3LO contributions to the scale derivative of the gluon distribution, resulting from these and the corresponding quark-to-gluon splitting functions, amount to 1% or less at $x \gtrsim 10^{-4}$ at a standard reference scale with $\alpha_s = 0.2$.

Gluon-gluon splitting function at N^{3}LO

Small x behavior of the collinear splitting function: shows presence of large terms (with alternating signs)

Small x: large logarithms of $\alpha_s \ln 1/x$



$\alpha_s \ln 1/x > 1$

for $x < 10^{-3} \div 10^{-2}$

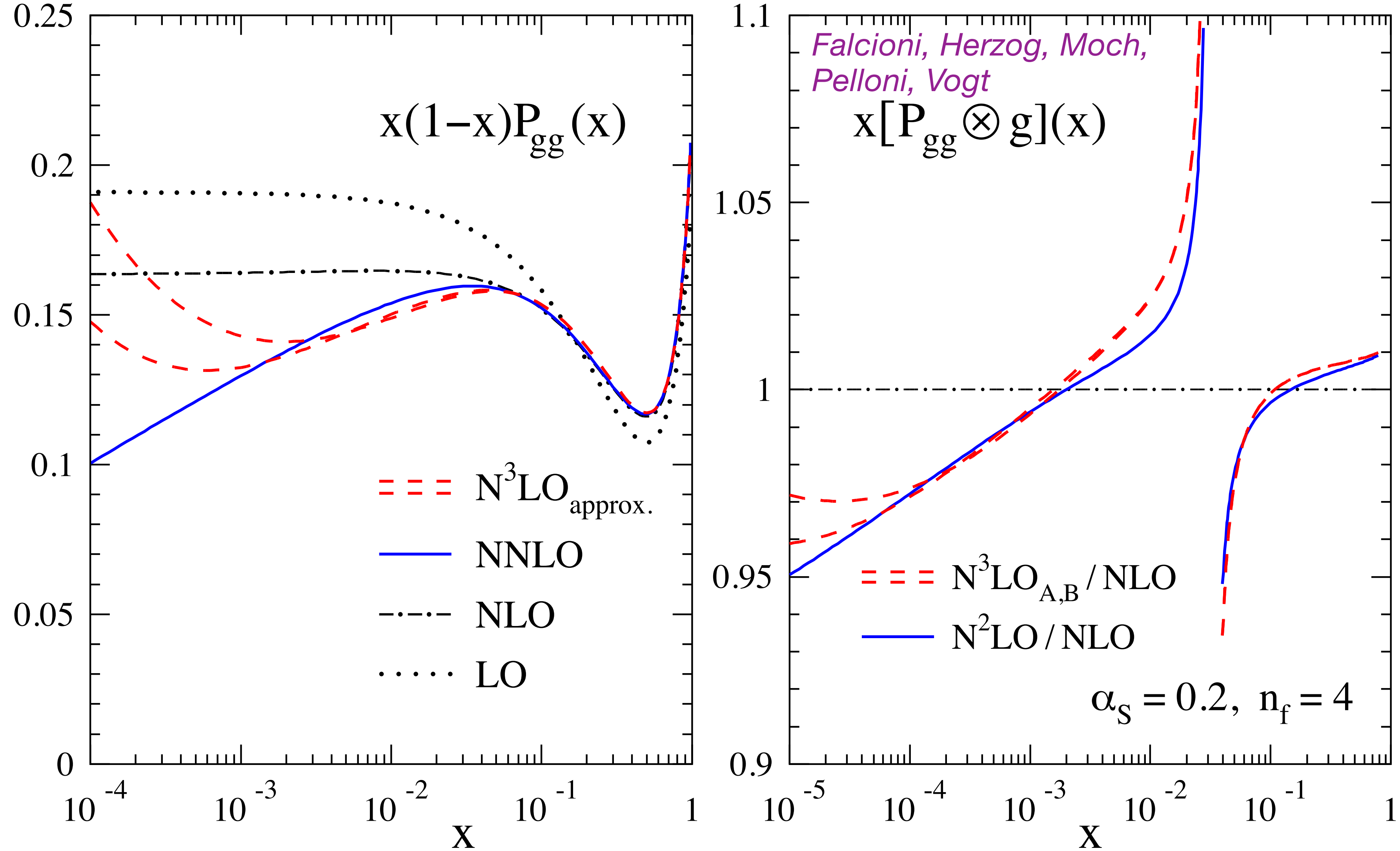


Figure 2: Left: the perturbative expansion of the splitting functions P_{gg} to N^{3}LO for $n_f = 4$ and $\alpha_s = 0.2$, using eq. (15) for the four-loop contribution. Right: the resulting N^{2}LO and N^{3}LO convolutions with the reference gluon distribution in eq. (18) normalized to the NLO result which changes sign at about $x = 0.3$.

$$xq_s(x,\mu_0^2) = 0.6\,x^{-0.3}(1-x)^{3.5}\,(1+5.0\,x^{0.8})$$
$$xg(x,\mu_0^2) = 1.6\,x^{-0.3}(1-x)^{4.5}\,(1-0.6\,x^{0.3})$$

For very small x need to consider resummation of $(\alpha_s \ln 1/x)^n$

S-matrix and Regge limit

Pre QCD… analysis of properties of S matrix

S matrix

$$S_{ab} = \langle b_{\rm out} | a_{\rm in} \rangle$$

Properties of S matrix:

- Lorentz invariance
- crossing
- unitarity
- analyticity

Scattering amplitude

$$S_{ab} = \delta_{ab} + i(2\pi)^4 \delta^{(4)}(\sum_a p_a - \sum_b p_b)\mathcal{A}_{ab}$$

Unitarity of S matrix

$$SS^\dagger = S^\dagger S = 1\!\!\mathrm{I}$$

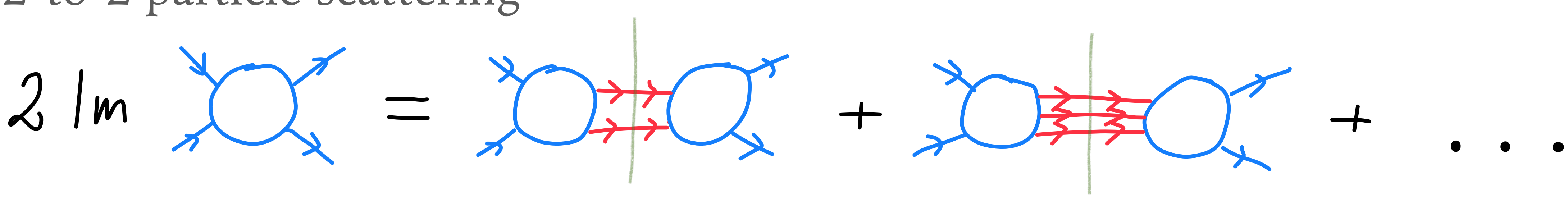
phase space n particles

$$d\Pi_n = \Pi_{j=1}^{n} \frac{d^3\mathbf{p}'_j}{(2\pi)^3 2E'_j}(2\pi)^4\delta^{(4)}(\sum_a p_a - \sum_{j=1}^{n} p'_j)$$

leads to the relation

$$2{\rm Im}\mathcal{A}_{ab} = \sum_n \int d\Pi_n \mathcal{A}_{an}\mathcal{A}^\dagger_{nb}$$

for 2-to-2 particle scattering



Optical theorem

Important special case of unitarity equations: **optical theorem**

If initial and final states are identical: forward elastic scattering amplitude

$$\mathcal{A}_{aa}(s,t=0)$$

Unitarity relations give

$$2\mathrm{Im}\mathcal{A}_{aa}(s,t=0) = \sum_n \int d\Pi_n |\mathcal{A}_{a\to n}|^2$$

Righthand side is the total cross section, times the flux factor

$$\sigma_{\rm tot} = \frac{1}{F}\sum_n \int d\Pi_n |\mathcal{A}_{a\to n}|^2$$

At high energy $F = 2s$

Therefore $$\sigma_{\rm tot} = \frac{1}{s}\,\mathrm{Im}\,\mathcal{A}_{\rm el}(s,t=0)$$

Total cross section for $1+2\to$ anything is given by the **imaginary** part of the **amplitude for elastic scattering** $1+2\to 1+2$ **in the forward direction (t=0)**

Regge limit

Regge limit: $s \to \infty \qquad t = \text{const}$

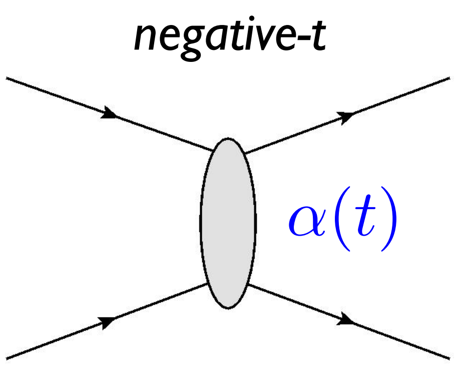
General principles

- Lorentz invariance
- crossing
- unitarity
- analyticity

General idea: analysis of analytic structure of the amplitudes.
Singularities in partial wave amplitudes in complex angular momentum.
These singularities will contribute to the scattering amplitude terms of the form

$$\mathcal{A}(s,t) \sim \tilde{\beta}(t) s^{\alpha(t)}$$

interpretation →



Amplitude dominated by exchange of the **Regge trajectory** $\alpha(t) = \alpha(0) + \alpha' t$

From optical theorem $\sigma_{\rm tot} = s^{-1} \text{Im} \mathcal{A}(s,0) \sim s^{\alpha(0)-1}$

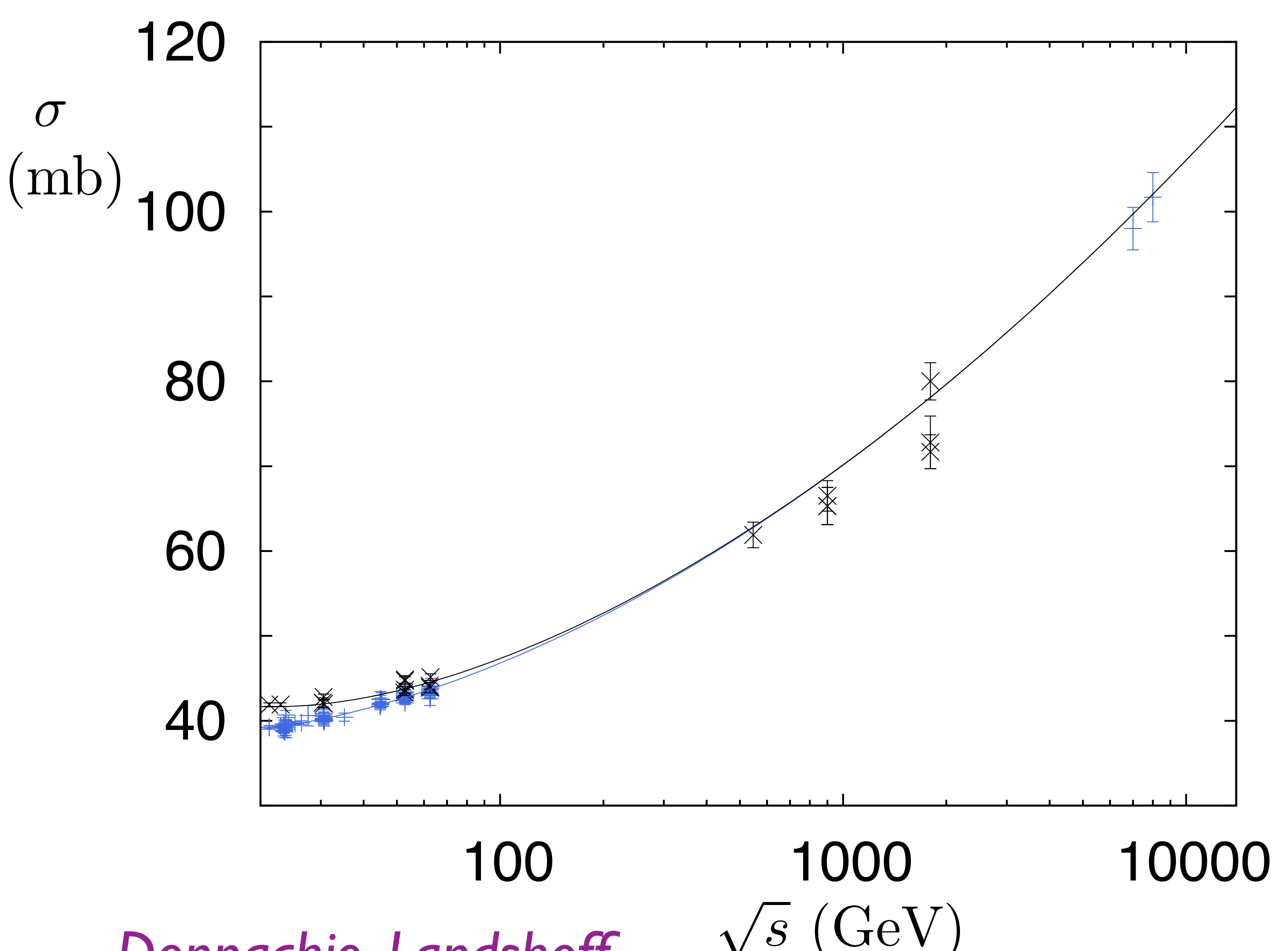
Intercept $\alpha(0)$ of Regge trajectory determines the behavior of the cross section

Pomeron

Pomeron:

Okun, Pomeranchuk; Foldy, Peierls

- Reggeon with intercept greater than unity.
- Corresponds to the exchange of the vacuum quantum numbers.
- Dominates the cross section at asymptotically high energies



Donnachie, Landshoff

Soft Pomeron

$$\alpha_P(t) = 1.11 + 0.165\mathrm{GeV}^{-2} t$$

(2013 parameters of fit to data including LHC)

$$\sigma_{\rm tot} \sim s^{\alpha_P(0)-1}$$

However, such soft pomeron power behavior is potentially in conflict with Froissart bound which stems from unitarity requirements:

$$\sigma^{\rm tot}(s) \leq C \log^2(s/s_0)$$

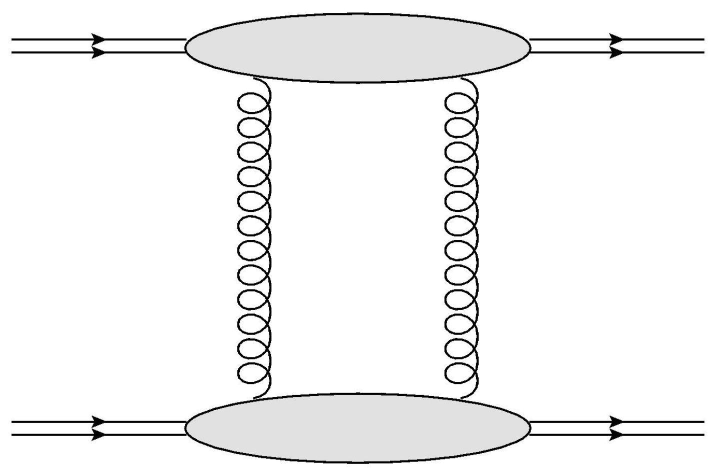
Note: the exact value of the constant C is of crucial importance here.

What is Pomeron in QCD ?

Simplest model for **Pomeron**:

2 gluons, color singlet

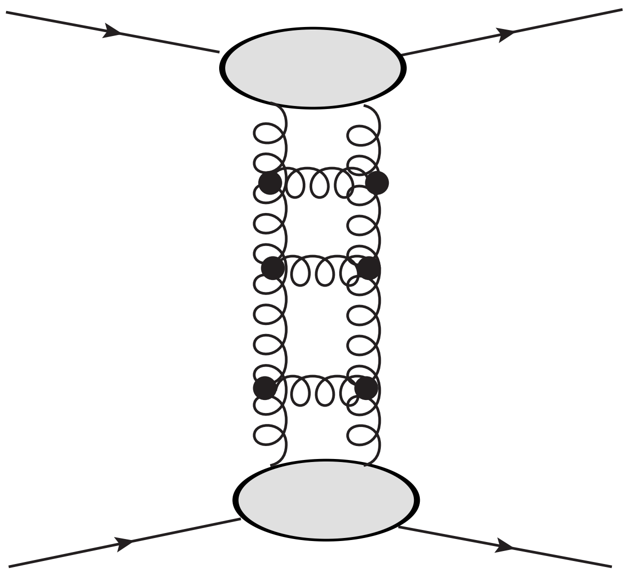
Low-Nussinov pomeron



Intercept would be : $\alpha(0) - 1 = 0$

Balitsky-Fadin-Kuraev-Lipatov

include logarithmically enhanced corrections



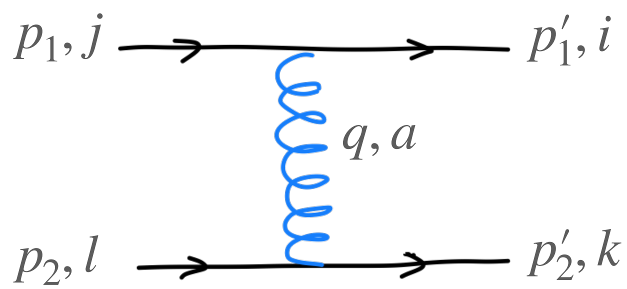
BFKL Pomeron: gluon ladder in the multi-Regge kinematics

Systematically sum leading logarithms of energy

Reggeization of the gluon

High-energy limit : one gluon exchange

Take quark-quark scattering (assuming quarks are different flavors)
One gluon exchange diagram (color octet exchange so does not contribute to the Pomeron, but important for gluon reggeization)



$$s = 2p_1 \cdot p_2 ,\ t = q^2$$

Result:

$$|\mathcal{A}^0_{qq' \to qq'}|^2 = g_s^4 \frac{N_c^2 - 1}{4N_c} 2 \frac{s^2 + u^2}{t^2} = g_s^4 \frac{4}{9} \frac{s^2 + u^2}{t^2}$$

High energy limit: $s \gg |t| \longrightarrow s \simeq -u$

$$|\mathcal{A}^0_{qq' \to qq'}|^2 = \frac{8}{9} g_s^4 \frac{s^2}{t^2}$$

For gluon gluon scattering:

$$|\mathcal{A}^0_{gg \to gg}|^2 = g_s^4 \frac{9}{2} \left(3 - \frac{tu}{s^2} - \frac{su}{t^2} - \frac{ts}{u^2} \right)$$

High energy limit: $s \gg |t| \longrightarrow s \simeq -u$

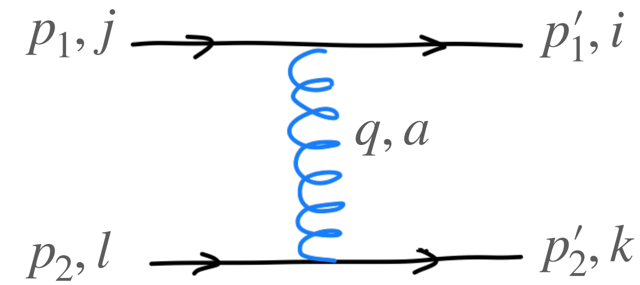
$$|\mathcal{A}^0_{gg \to gg}|^2 = g_s^4 \frac{9}{2} \frac{s^2}{t^2}$$

At large s, at this order the cross section for qq scattering is the same as for gg scattering, modulo different color factor

$$|\mathcal{A}^0_{gg \to gg}|^2 = \left(\frac{9}{4} \right)^2 |\mathcal{A}^0_{qq' \to qq'}|^2$$

$$\frac{C_A}{C_F} = \frac{9}{4}$$

Eikonal vertex



Another way of deriving it : introduce **eikonal vertex**

The upper line in the diagram is:

$$-ig_s\bar{u}(p_1+q)\gamma^\mu u(p_1)t^a_{ij}$$

Since $s \gg |t|$, all q components are small with respect to p_1, p_2 components and we have

$$-ig_s\bar{u}(p_1)\gamma^\mu u(p_1)t^a_{ij} = -2ig_s p_1^\mu t^a_{ij} \qquad \textit{eikonal vertex}$$

The same for lower line

$$-ig_s\bar{u}(p_2-q)\gamma^\mu u(p_2)t^a_{kl} \simeq -2ig_s p_2^\mu t^a_{kl}$$

Finally one gets an amplitude

$$\mathcal{A}^{(0)}_{qq'\to qq'} = g_s^2 2p_1^\mu \frac{g_{\mu\nu}}{q^2} 2p_2^\nu t^a_{ij}t^a_{kl} = g_s^2 \frac{4p_1\cdot p_2}{q^2} t^a_{ij}t^a_{kl} = 8\pi\alpha_s \frac{s}{t} t^a_{ij}t^a_{kl}$$

which leads to the same result as before

$$|\mathcal{A}^0_{qq'\to qq'}|^2 = \frac{8}{9} g_s^4 \frac{s^2}{t^2}$$

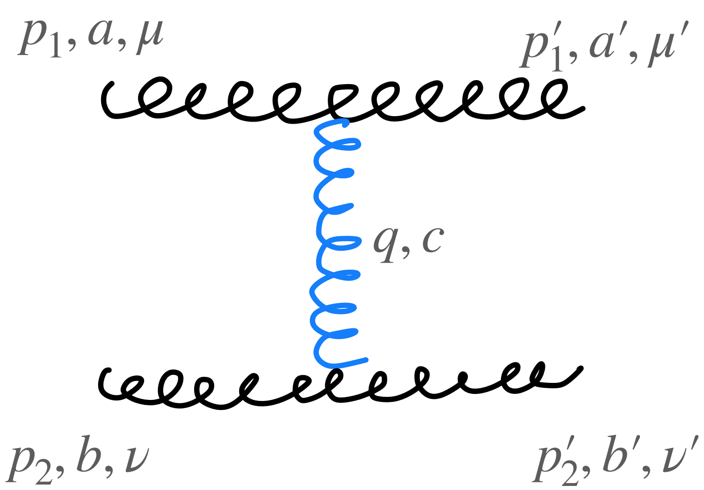
Eikonal approximation works for particles of any spin.

For example for gluon-scalar interaction we would have

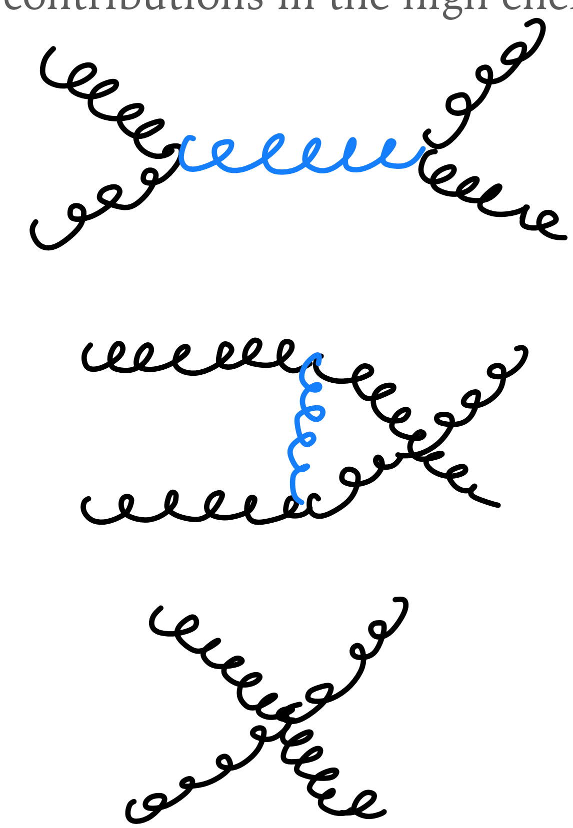
$$-ig_s(2p_1+q)^\mu t^a_{ij} \simeq -2ig_s p_1^\mu t^a_{ij}$$

$gg \to gg$ in high energy limit

t-channel diagram



at this order other diagrams give subleading contributions in the high energy limit(*)



Eikonal approximation can be done for 3-gluon vertex:

$$g_s f^{aa'c}[g_{\mu\mu'}(p_1+p_1')_\rho + g_{\rho\mu'}(-p_1'+q)_\mu - g_{\mu\rho}(p_1+q)_{\mu'}]$$

$$\simeq g_s f^{aa'c}[2g_{\mu\mu'}p_{1\rho} - g_{\rho\mu'}p_{1\mu} - g_{\mu\rho}p_{1\mu'}]$$

Using physical polarizations for external gluons: $p \cdot \varepsilon(p) = 0$

eikonal approximation for 3 -gluon vertex

$$\simeq 2g_s f^{aa'c} g_{\mu\mu'} p_{1\rho}$$

$$\mathcal{A}^{(0)}_{gg\to gg} = -ig_s^2 f^{aa'c} f^{bb'c} g_{\mu\mu'} \frac{4p_1\cdot p_2}{q^2} g_{\nu\nu'} \varepsilon^\mu_{\lambda_1}(p_1)\varepsilon^{\mu'}_{\lambda_1'}(p_1')\varepsilon^\nu_{\lambda_2}(p_2)\varepsilon^{\nu'}_{\lambda_2}(p_2')$$

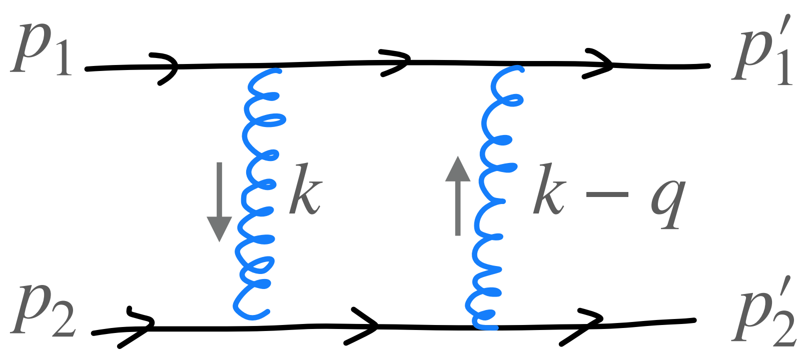
Squaring, summing over helicities and colors one gets in the high energy limit

$$|\mathcal{A}^{(0)}_{gg\to gg}|^2 = g_s^4 \frac{9}{2}\frac{s^2}{t^2}$$

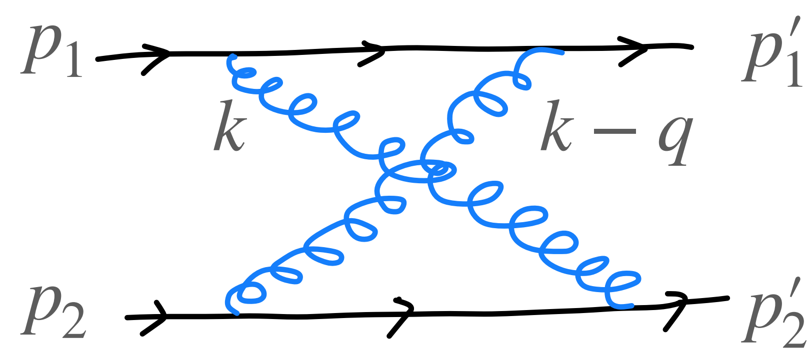
which is the same result as before

Two-gluon exchange

Consider qq scattering at one-loop. Dominant diagrams contributing to virtual corrections :

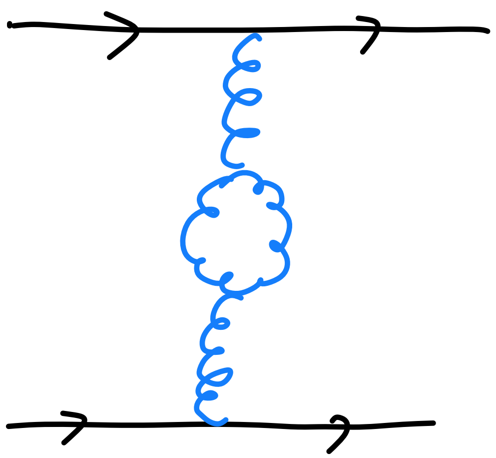


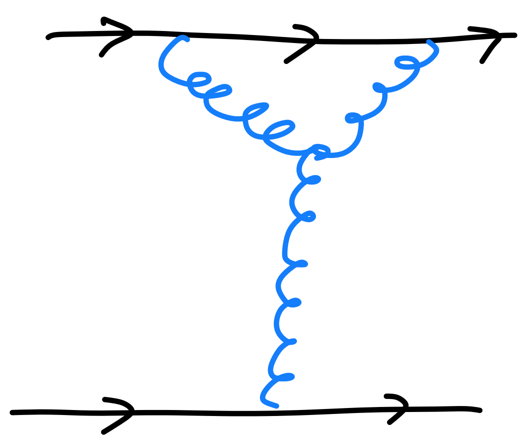
box



crossed

On the other hand : self energy insertions and vertex diagrams do not contribute(*) at this order



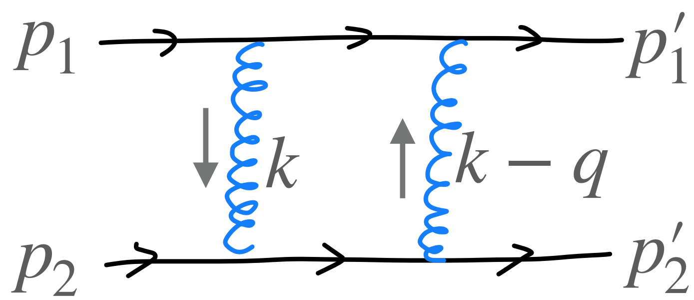


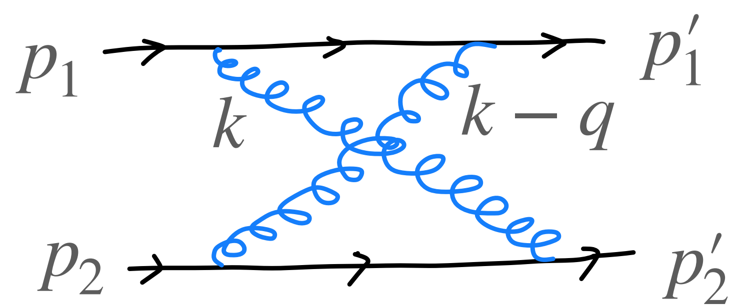
((*) that may though depend on gauge)

Strong coupling is fixed in the leading logarithmic calculation at high energy (->LL BFKL)
Strong coupling starts to run at next-to-leading logarithmic calculation (->NLL BFKL)

Two gluon exchange

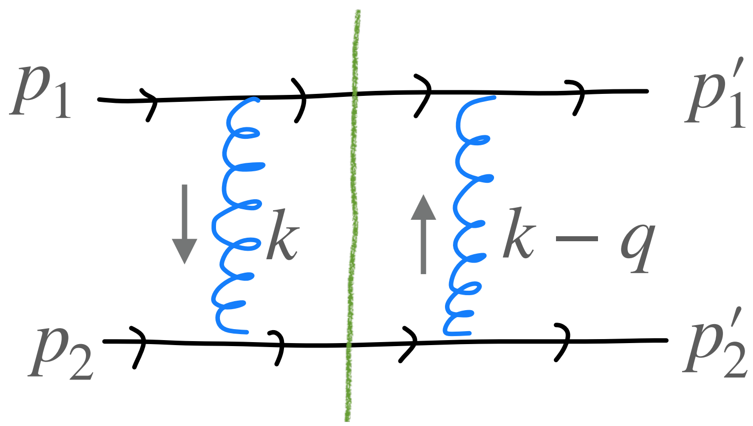
$\mathcal{A}_a$





$\mathcal{A}_b$

Can compute it for example using unitarity in the form of cutting rules



$$\mathrm{Im}\mathcal{A}_a = \frac{1}{2}\int d\Pi_2\, \mathcal{A}^{(0)}(s,k^2)\,\mathcal{A}^{(0)}(s,(q-k)^2)$$

where $\mathcal{A}^{(0)}$ is lowest order amplitude (single gluon exchange)

Phase-space integral $\displaystyle\int d\Pi_2 = \int \frac{d^4k}{(2\pi)^2}\delta((p_1-k)^2)\,\delta((p_2+k)^2)$

Sudakov decomposition

$$k = \alpha p_1 + \beta p_2 + k_\perp$$

$$p_1^\mu = (\sqrt{s}/2, \sqrt{s}/2, \mathbf{0})$$
$$p_2^\mu = (\sqrt{s}/2, -\sqrt{s}/2, \mathbf{0})$$

In the large energy limit $(s \gg |t|)$ delta functions give

$$\alpha \simeq \frac{\mathbf{k}^2}{s} \qquad |\beta| \simeq \frac{\mathbf{k}^2}{s} \qquad \alpha = |\beta| \ll 1$$

$$k^2 \simeq -\mathbf{k}^2 \qquad (k-q)^2 \simeq -(\mathbf{k}-\mathbf{q})^2$$

Virtualities of the exchanged gluons dominated by the transverse components

Two gluon exchange

Obtaining real part of the amplitude (dispersion relations)
and summing both contributions

$$\mathcal{A}_a + \mathcal{A}_b$$

The final result can be expressed as a projection onto **octet** and **singlet**

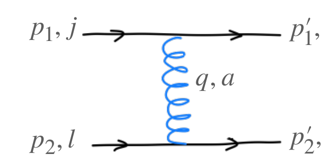
Amplitude decomposition into color representations R:

$$\mathcal{A}_{kl}^{ij}(s,t) = \sum_R P_{lk}^{ij}(R)\mathcal{A}_R(s,t)$$

projectors $P_{lk}^{ij}(R)$

singlet: $\mathcal{A}_{\underline{1}}^{(1)} \sim i\alpha_s^2 \frac{s}{t}\epsilon_g(t)$

octet: $\mathcal{A}_{\underline{8}}^{(1)} = \mathcal{A}_{\underline{8}}^{(0)}\,\epsilon_g(t)\,\ln\frac{s}{|t|}$ where $\mathcal{A}_{\underline{8}}^{(0)} = 8\pi\alpha_s \frac{s}{t} t_{ij}^a t_{kl}^a$



$$\epsilon_g(t) = \frac{N_c\alpha_s}{4\pi^2}\int d^2\mathbf{k}\frac{t}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2}$$

with $t = -\mathbf{q}^2$

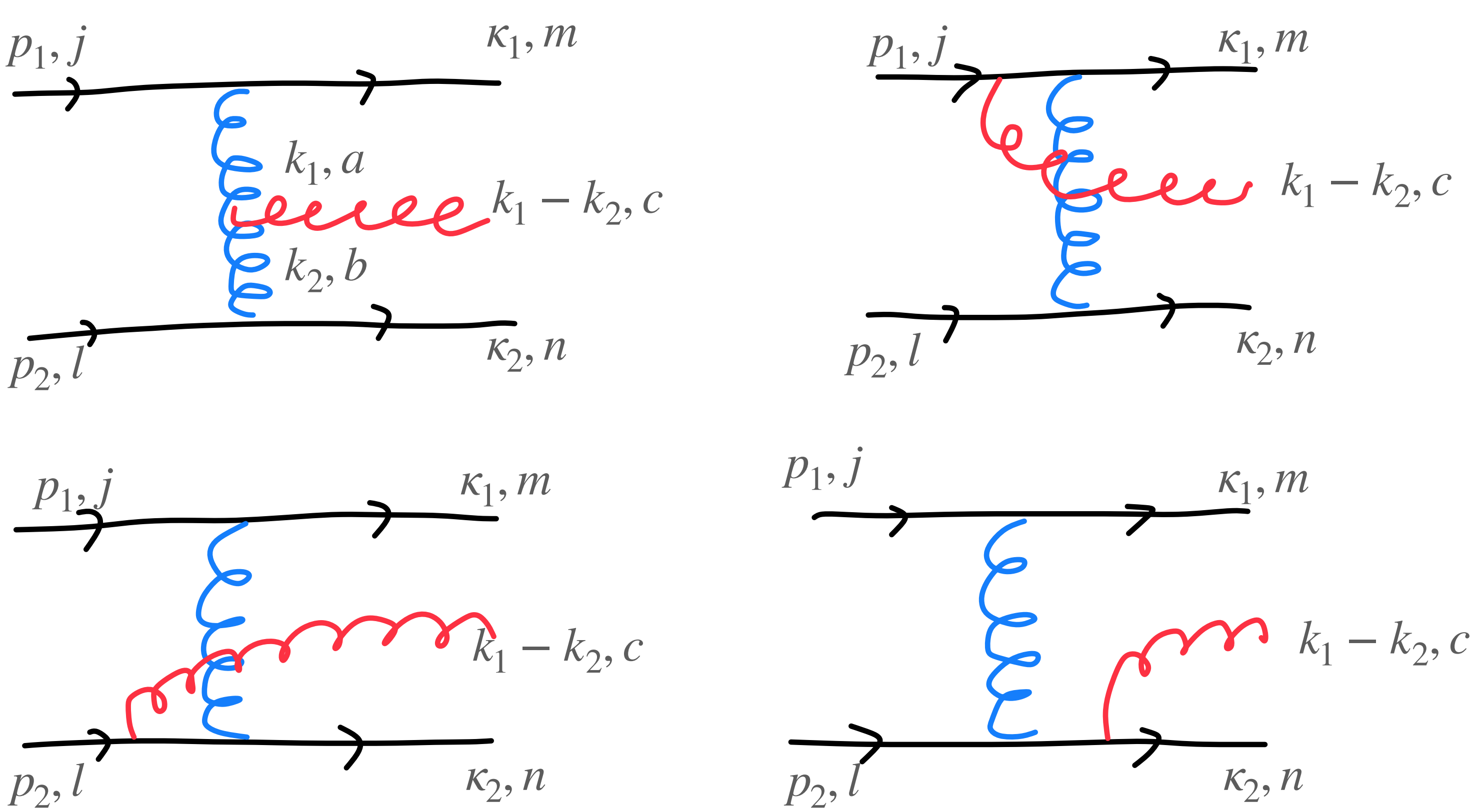
Octet amplitude at this order is proportional to lower amplitude times $\epsilon_g(t)\,\ln s/|t|$

It is infra-red divergent (massless quarks). In reality, they would be confined in the hadrons, which will provide off-shellness to the quarks, and thus IR regularization.

It is not necessary to introduce a cutoff since the final BFKL equation is free from IR divergencies.

Three parton production (real diagrams)

Next building block of the BFKL is the parton production in the high energy limit.
Consider following real gluon emission diagrams



Sudakov decomposition

$$k_1 = \alpha_1 p_1 + \beta_1 p_2 + k_{1\perp}$$

$$k_2 = \alpha_2 p_1 + \beta_2 p_2 + k_{2\perp}$$

Dominant contribution at high energy (when computing the two-loop graph) comes from region
strong ordering of longitudinal momenta

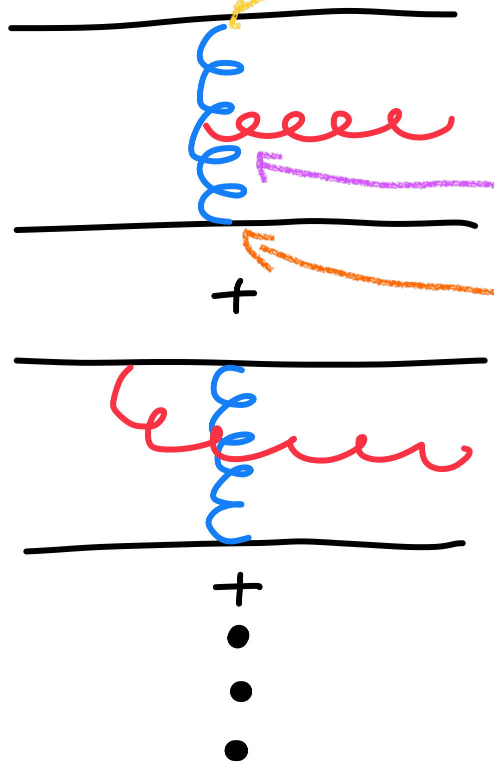
$$1 \gg \alpha_1 \gg \alpha_2$$

$$1 \gg |\beta_2| \gg |\beta_1|$$

virtualities of exchanged momenta dominated by transverse components

$$k_1^2 \simeq -\mathbf{k}_1^2 \qquad k_2^2 \simeq -\mathbf{k}_2^2$$

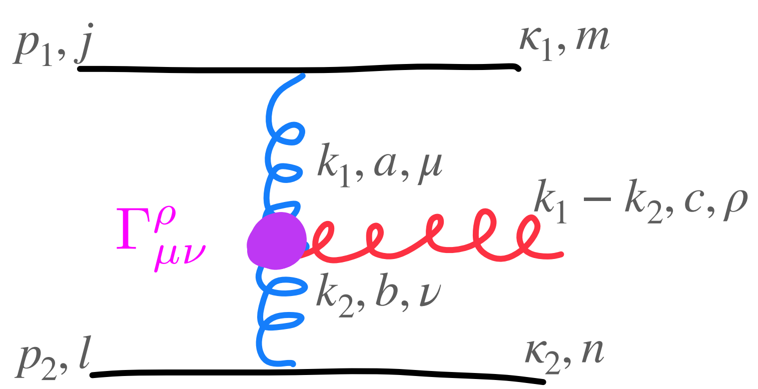
Lipatov vertex



$$(-2ig_s p_1^\mu)t^a_{mj}\left(\frac{-i}{k_1^2}\right)g_s f_{abc}\gamma^{\rho\mu\nu}\left(\frac{-i}{k_2^2}\right)(-2ig_s p_2^\nu)t^b_{nl}$$

$$(-2ig_s p_1^\mu)t^a_{j'j}\left(\frac{i}{(p_1-k_1-k_2)^2}\right)(-2ig_s)(p_1^\mu-k_1^\mu+k_2^\mu)t^b_{mj'}\left(\frac{-i}{k_2^2}\right)(-2ig_s p_2^\nu)t^b_{nl}$$

combining all 5 terms one obtains the result



$$A^\rho_{2\to 3} = -4ig_s^3\frac{p_1^\mu p_2^\nu}{\mathbf{k}_2^2\mathbf{k}_1^2}\,t^a_{mj}t^b_{nl}f_{abc}\,\Gamma^\rho_{\mu\nu}$$

Lipatov effective vertex

$$\Gamma^\rho_{\mu\nu} = \frac{2p_{2\mu}p_{1\nu}}{s}\left[\left(\alpha_1+\frac{2\mathbf{k}_1^2}{\beta_2 s}\right)p_1^\rho+\left(\beta_2+\frac{2\mathbf{k}_2^2}{\alpha_1 s}\right)p_2^\rho-(k^\rho_{1\perp}+k^\rho_{2\perp})\right]$$

Lipatov effective vertex

Lipatov effective vertex satisfies Ward identity (gauge invariant)

$$(k_{1\rho} - k_{2\rho})\Gamma^{\rho}_{\mu\nu}(k_1, k_2) = 0$$

For convenience one can introduce related quantity

$$C^{\rho}(k_1, k_2) = \left(\alpha_1 + \frac{2\mathbf{k}_1^2}{\beta_2 s}\right)p_1^{\rho} + \left(\beta_2 + \frac{2\mathbf{k}_2^2}{\alpha_1 s}\right)p_2^{\rho} - (k_{1\perp}^{\rho} + k_{2\perp}^{\rho})$$

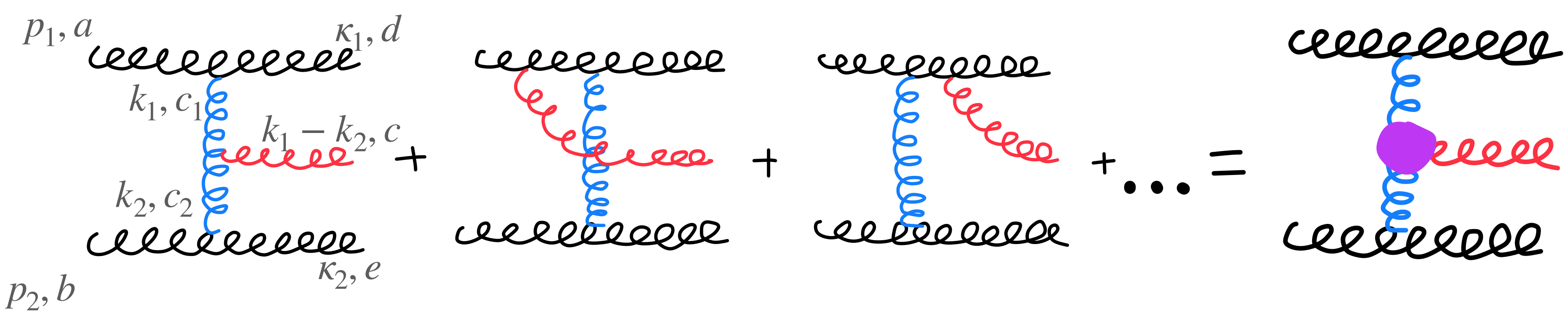
$$\Gamma^{\rho}_{\mu\nu} = \frac{2p_{2\mu}p_{1\nu}}{s}C^{\rho} \longrightarrow p_1^{\mu}p_2^{\nu}\Gamma^{\rho}_{\mu\nu} = \frac{2p_2\cdot p_1 p_1\cdot p_2}{s}C^{\rho} = \frac{s}{2}C^{\rho}$$

$$\mathcal{A}^{\rho}_{2q\to(2q+g)} = 2is\, g_s t^a_{mj}\left(\frac{i}{\mathbf{k}_1^2}\right) f_{abc} g_s C^{\rho}(k_1, k_2)\left(\frac{i}{\mathbf{k}_2^2}\right) g_s t^b_{nl}$$

Similar analysis can be performed with gluons in the initial state

Since eikonal vertex is independent of the spin of the particle

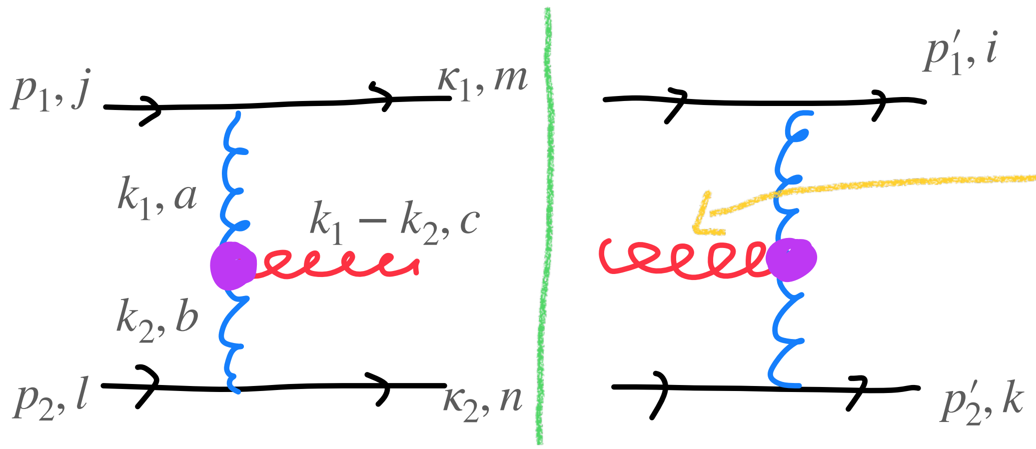
Form of amplitude the same(modulo color), the same effective vertex



$$\mathcal{A}^{\mu_a\mu_b\mu_d\rho\mu_e}_{abdce}(2g\to 3g) = 2is\, g_s f^{adc_1} g^{\mu_a\mu_d}\left(\frac{i}{\mathbf{k}_1^2}\right) \cdot f^{c_1cc_2} g_s C^{\rho}(k_1, k_2) \cdot \left(\frac{i}{\mathbf{k}_2^2}\right) g_s f^{bec_2} g^{\mu_b\mu_e}$$

Real emission gluon contribution to Im A

Real emission contribution to the imaginary part of qq scattering



sum over polarizations of the intermediate gluon

$$\mathrm{Im}\mathcal{A}_{\mathrm{real}}(s,t) = \frac{-g^{\rho\sigma}}{2}\int d\Pi_3\, \mathcal{A}^{\rho}_{2\to 3}(k_1,k_2)\, \mathcal{A}^{\sigma\dagger}_{2\to 3}(k_1-q,k_2-q)$$

Three body phase space

$$\int d\Pi_3 = \frac{1}{(2\pi)^5}\int d^4k_1 d^4k_2\, \delta((p_1-k_1)^2)\,\delta((k_1-k_2)^2)\,\delta((p_2+k_2)^2)$$

Sudakov decomposition

$k_1 = \alpha_1 p_1 + \beta_1 p_2 + k_{1\perp}$

$k_2 = \alpha_2 p_1 + \beta_2 p_2 + k_{2\perp}$

$$\int d\Pi_3 = \frac{s^2}{4(2\pi)^5}\int d\alpha_1 d\beta_1 d^2\mathbf{k}_1 \int d\alpha_2 d\beta_2 d^2\mathbf{k}_2\, \delta(-\beta_1(1-\alpha_1)s - \mathbf{k}_1^2)$$
$$\cdot\delta((\alpha_1-\alpha_2)(\beta_1-\beta_2)s - (\mathbf{k}_1-\mathbf{k}_2)^2)\,\delta(\alpha_2(1+\beta_2)s - \mathbf{k}_2^2)$$

High energy approximation: $1 \gg \alpha_1 \gg \alpha_2$ $\quad 1 \gg |\beta_2| \gg |\beta_1|$

$$\int d\Pi_3 = \frac{s^2}{4(2\pi)^5}\int d\alpha_1 d\beta_1 d^2\mathbf{k}_1 \int d\alpha_2 d\beta_2 d^2\mathbf{k}_2\, \delta(-\beta_1 s - \mathbf{k}_1^2)\delta(\alpha_1\beta_2 s - (\mathbf{k}_1-\mathbf{k}_2)^2)\,\delta(\alpha_2 s - \mathbf{k}_2^2)$$

$$= \frac{1}{4(2\pi)^5}\int_{\alpha_2}^{1}\frac{d\alpha_1}{\alpha_1} d^2\mathbf{k}_1 \int d\alpha_2 d^2\mathbf{k}_2\, \delta(\alpha_2 s - \mathbf{k}_2^2) = \frac{1}{4(2\pi)^5 s}\int d^2\mathbf{k}_1 \int d^2\mathbf{k}_2 \int_{\mathbf{k}_2^2/s}^{1}\frac{d\alpha_1}{\alpha_1}$$

logarithmic integration

Real emission gluon contribution to Im A

In leading logarithmic order in $\ln s$ one makes the approximation

$$\int d\Pi_3 = \frac{1}{4(2\pi)^5 s}\int_{\mathbf{k}^2/s}^{1} \frac{d\alpha_1}{\alpha_1} \int d^2\mathbf{k}_1 \int d^2\mathbf{k}_2$$

where $\mathbf{k}^2 \simeq \mathbf{k}_1^2 \simeq \mathbf{k}_2^2$ and assuming $\mathbf{k}^2 \simeq \mathbf{q}^2$

This is fine in leading logarithmic order since energy is high

$$\ln\frac{s}{s_0} = \ln\frac{s s_1}{s_0 s_1} = \ln\frac{s}{s_1} + \ln\frac{s_1}{s_0} \simeq \ln\frac{s}{s_1}$$

as long as $s \gg s_0, s_1, \quad s \to \infty$

Integrals over the longitudinal and transverse components **decouple**

Amplitudes

$$\mathcal{A}^{\rho}_{2\to 3}(k_1,k_2)\,\mathcal{A}^{\sigma\dagger}_{2\to 3}(k_1-q,k_2-q) = 4g_s^6 s^2 \mathcal{C}_{\rm real}\frac{C^{\rho}(k_1,k_2)C_{\rho}(-k_1+q,-k_2+q)}{\mathbf{k}_1^2\mathbf{k}_2^2(\mathbf{k}_2-\mathbf{q})^2(\mathbf{k}_1-\mathbf{q})^2}$$

Contraction of the Lipatov vertices

$$C^{\rho}(k_1,k_2)C_{\rho}(-k_1+q,-k_2+q) = -2\left[\mathbf{q}^2 - \frac{\mathbf{k}_1^2(\mathbf{k}_2-\mathbf{q})^2}{(\mathbf{k}_1-\mathbf{k}_2)^2} - \frac{\mathbf{k}_2^2(\mathbf{k}_1-\mathbf{q})^2}{(\mathbf{k}_1-\mathbf{k}_2)^2}\right]$$

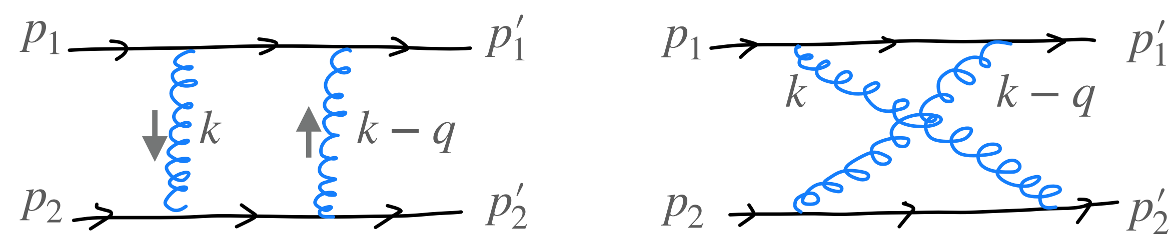
Contribution to the imaginary part of amplitude from real emission

$$\mathrm{Im}\mathcal{A}_{\rm real}(s,t) = \frac{2\alpha_s^3}{\pi^2}\mathcal{C}_{\rm real}\, s\left(\ln\frac{s}{|t|}\right)\cdot\int d^2\mathbf{k}_1\int d^2\mathbf{k}_1\left[\frac{\mathbf{q}^2}{\mathbf{k}_1^2\mathbf{k}_2^2(\mathbf{k}_1-\mathbf{q})^2(\mathbf{k}_2-\mathbf{q})^2} - \frac{1}{(\mathbf{k}_1-\mathbf{q})^2\mathbf{k}_2^2(\mathbf{k}_1-\mathbf{k}_2)^2} - \frac{1}{(\mathbf{k}_1-\mathbf{q})^2\mathbf{k}_1^2(\mathbf{k}_1-\mathbf{k}_2)^2}\right]$$

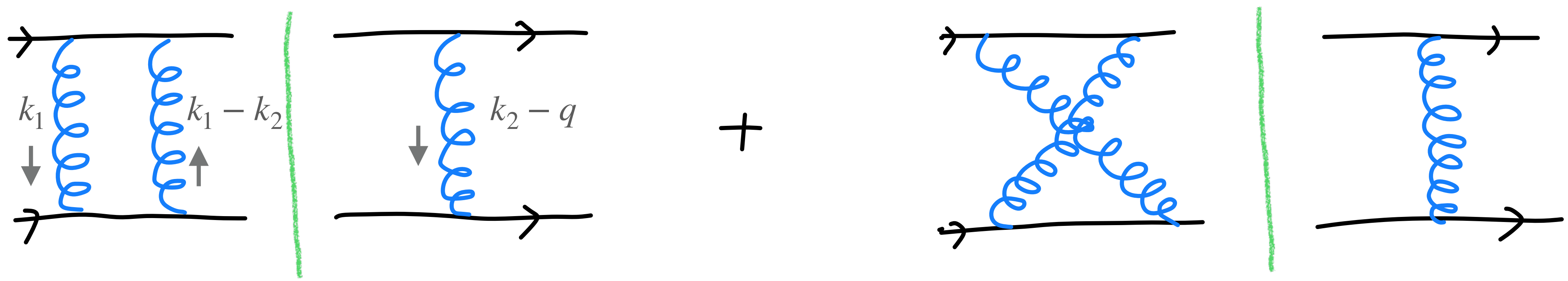
Logarithmic enhancement and factorization of longitudinal and transverse momenta

Virtual contribution at $\mathcal{O}(\alpha_s^3)$

We already computed diagrams at $\mathcal{O}(\alpha_s^2)$ of the kind



To accompany real emissions just computed, need virtual contributions $\mathcal{O}(\alpha_s^3)$ of the kind



other terms in which single gluon is to the left of the cut

$+ \ \ldots$

$$\mathrm{Im}\mathcal{A}^{(2)}_{\mathrm{virt}} = \frac{1}{2}\int d\Pi_2\, \mathcal{A}^{(1)}(s,k_2^2)\,\mathcal{A}^{(0)}(s,(q-k_2)^2) + \frac{1}{2}\int d\Pi_2\, \mathcal{A}^{(0)}(s,k_1^2)\,\mathcal{A}^{(1)}(s,(q-k_1)^2)$$

Recall: lower order (octet)

$$\mathcal{A}^{(0)}(s,(k_2-q)^2) = 8\pi\alpha_s \frac{s}{(k_2-q)^2} t^a_{mi} t^a_{nk}$$

$$\mathcal{A}^{(1)}(s,k_2^2) = 8\pi\alpha_s \frac{s}{k_2^2}\ln(s/k_2^2)\epsilon_g(k_2^2) t^b_{mj} t^b_{nl}$$

Putting everything together: imaginary part for virtual contributions at order $\mathcal{O}(\alpha_s^2)$

$$\mathrm{Im}\mathcal{A}_{\mathrm{virtual}}(s,t) = -\frac{2N_c\alpha_s^3}{\pi^2}\mathcal{C}_{\mathrm{virtual}}\, s\left(\ln\frac{s}{|t|}\right)\int d^2\mathbf{k}_1 \int d^2\mathbf{k}_1 \left[\frac{1}{(\mathbf{k}_1-\mathbf{q})^2\mathbf{k}_2^2(\mathbf{k}_1-\mathbf{k}_2)^2} + \frac{1}{(\mathbf{k}_2-\mathbf{q})^2\mathbf{k}_1^2(\mathbf{k}_1-\mathbf{k}_2)^2}\right]$$

Reggeized gluon

Steps to obtain the reggeized gluon:

Step 1: project onto color octet

Step 2: add real and virtual contributions:

$$\mathrm{Im}\mathcal{A}_{\underline{8}}(s,t) = \mathrm{Im}\mathcal{A}_{\underline{8},\mathrm{real}}(s,t) + \mathrm{Im}\mathcal{A}_{\underline{8},\mathrm{virtual}}(s,t)$$

Step 3: reconstruct the real part of the amplitude (dispersion relations)

Step 4: put everything together to obtain the $\mathcal{O}(\alpha_s^3)$ amplitude

$$\mathcal{A}_{\underline{8}}^{(2)}(s,t) = \mathcal{A}_{\underline{8}}^{(0)}\,\frac{1}{2}\,\epsilon_g^2(t)\,\ln^2\frac{s}{|t|}$$

Step 5: combine $\mathcal{O}(\alpha_s)$, $\mathcal{O}(\alpha_s^2)$, $\mathcal{O}(\alpha_s^3)$

$$\mathcal{A}_{\underline{8}}^{(2)}(s,t) = \mathcal{A}_{\underline{8}}^{(0)}\left[1 + \epsilon_g(t)\,\ln\frac{s}{|t|} + \frac{1}{2}\,\epsilon_g^2(t)\,\ln^2\frac{s}{|t|}\right]$$

Conjecture: three terms of the expansion exponentiate

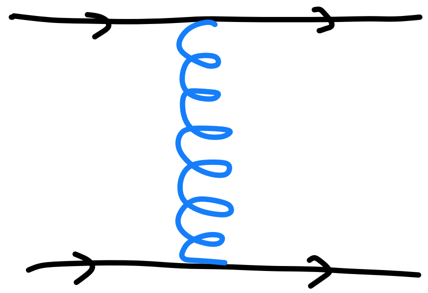
$$\mathcal{A}_{\underline{8}}(s,t) = \mathcal{A}_{\underline{8}}^{(0)}\left(\frac{s}{|t|}\right)^{\epsilon_g(t)} = 8\pi\alpha_s t^a_{ij} t^a_{kl}\,\frac{s}{t}\left(\frac{s}{|t|}\right)^{\epsilon_g(t)}$$

This conjecture is indeed true. One can show that it works to all orders in α_s in leading logarithmic approximation

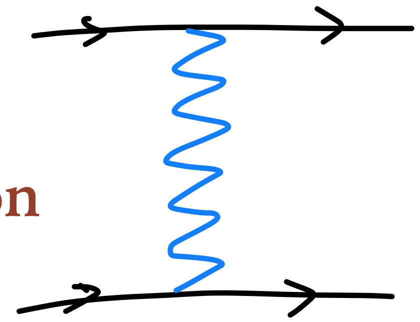
Reggeized gluon

The result for the **color octet** amplitude for qq elastic scattering in the leading logarithmic approximation at high energy suggests the following interpretation

lowest order: exchanged gluon in the t-channel



resummation in LL high energy limit: **reggeized gluon**



$$\mathcal{A}_{\underline{8}}(s,t) = \mathcal{A}^{(0)}_{\underline{8}} \left(\frac{s}{|t|}\right)^{\epsilon_g(t)} = 8\pi\alpha_s t^a_{ij} t^a_{kl} \frac{s}{t} \left(\frac{s}{|t|}\right)^{\epsilon_g(t)}$$

Ansatz: modify gluon propagators in the t-channel to account for virtual corrections

$$\frac{-i}{k^2} \longrightarrow \frac{-i}{k^2}\left(\frac{s}{-k^2}\right)^{\epsilon_g(k^2)}$$

More precisely

$$D_{\mu\nu}(s,k^2) = \frac{-ig_{\mu\nu}}{k^2}\left(\frac{s}{-k^2}\right)^{\epsilon_g(k^2)}$$

reggeized gluon
or
gluon reggeization

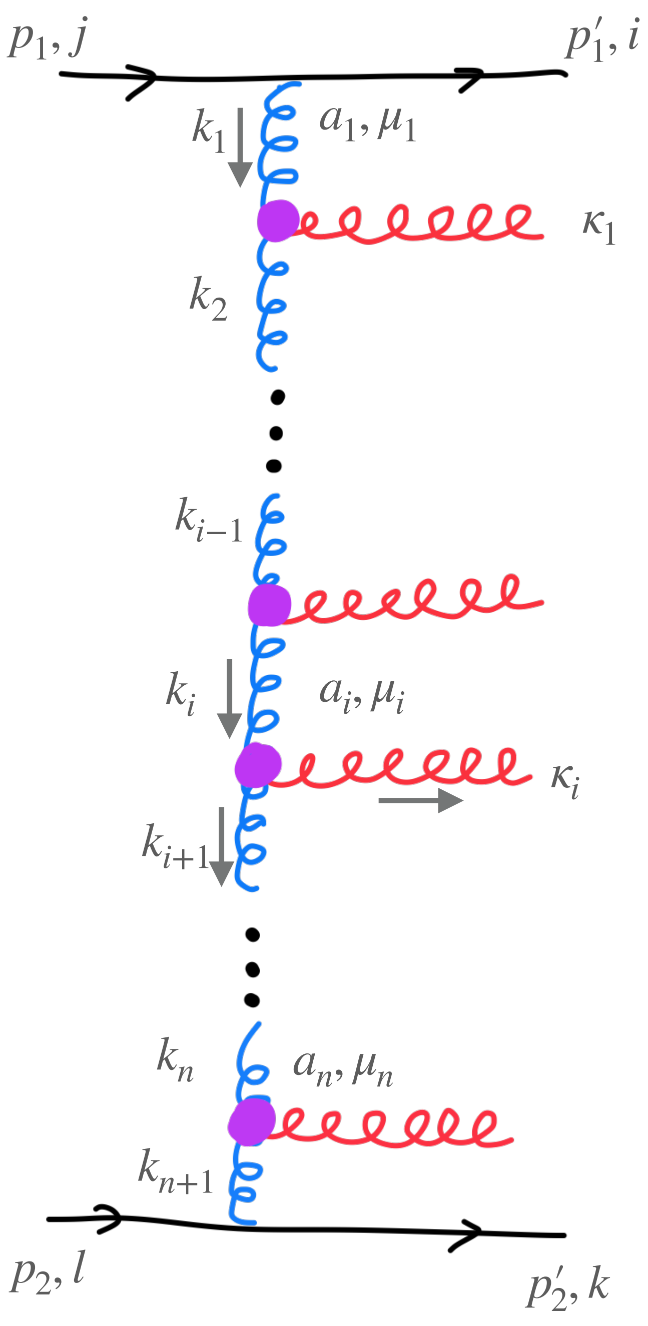
where $$\epsilon_g(t) = \frac{N_c\alpha_s}{4\pi^2}\int d^2\mathbf{k}\frac{t}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2}$$

$$\alpha_g(t) = 1 + \epsilon_g(t)$$

Building the BFKL : multigluon amplitude

qq scattering with emission of n gluons: 2(q)->2(q)+n(g) amplitude

Tree level amplitude



Sudakov decomposition $k_i = \alpha_i p_1 + \beta_i p_2 + k_{i\perp}$

multi-Regge kinematics

$$1 \gg \alpha_1 \gg \alpha_2 \gg \cdots \gg \alpha_{n+1}$$

$$1 \gg |\beta_{n+1}| \gg \cdots \gg |\beta_2| \gg |\beta_1|$$

Implies strong ordering in **rapidity** of outgoing particles

$$\kappa^\mu = (E_\kappa, \kappa_z, \boldsymbol{\kappa}) \qquad \kappa^\pm = E_\kappa \pm \kappa_z \qquad y = \frac{1}{2}\ln\frac{\kappa^+}{\kappa^-}$$

alternatively $\kappa^\mu = (\kappa^+, \kappa^-, \boldsymbol{\kappa})$

$$p_1 = (p_1^+, 0, \mathbf{0}), \quad p_2 = (0, p_2^-, \mathbf{0})$$

$$\kappa_i = k_i - k_{i+1} = ((\alpha_i - \alpha_{i+1})\sqrt{s}, (\beta_i - \beta_{i+1})\sqrt{s}, \mathbf{k}_i - \mathbf{k}_{i+1})$$

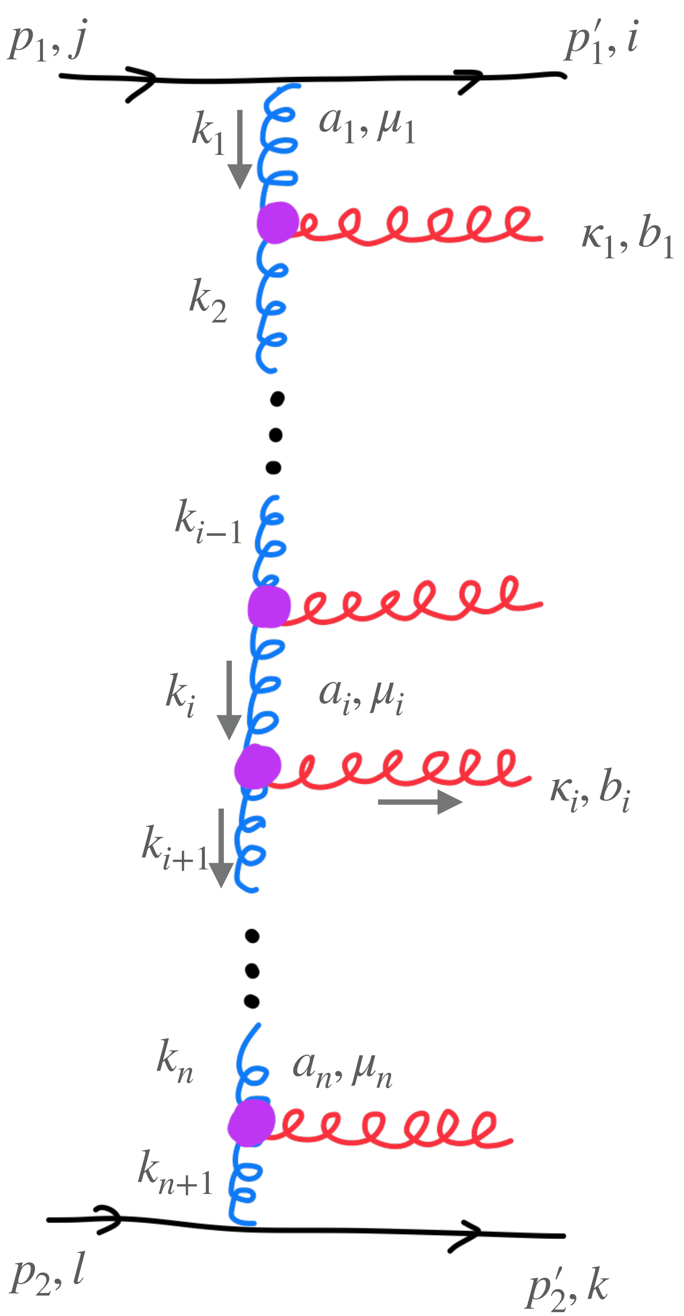
Rapidity of outgoing particles i and i+1

$$y_i = \frac{1}{2}\ln\frac{\kappa_i^+}{\kappa_i^-} = \frac{1}{2}\ln\frac{\alpha_i - \alpha_{i+1}}{\beta_i - \beta_{i+1}} \simeq \frac{1}{2}\ln\frac{\alpha_i}{|\beta_{i+1}|} \gg y_{i+1} \simeq \frac{1}{2}\ln\frac{\alpha_{i+1}}{|\beta_{i+2}|}$$

Multi-Regge kinematics: strong ordering in rapidity

$$y_0 \gg y_1 \gg y_2 \gg \cdots \gg y_{n-1} \gg y_n \gg y_{n+1}$$

Amplitude 2 to n particle in Regge limit

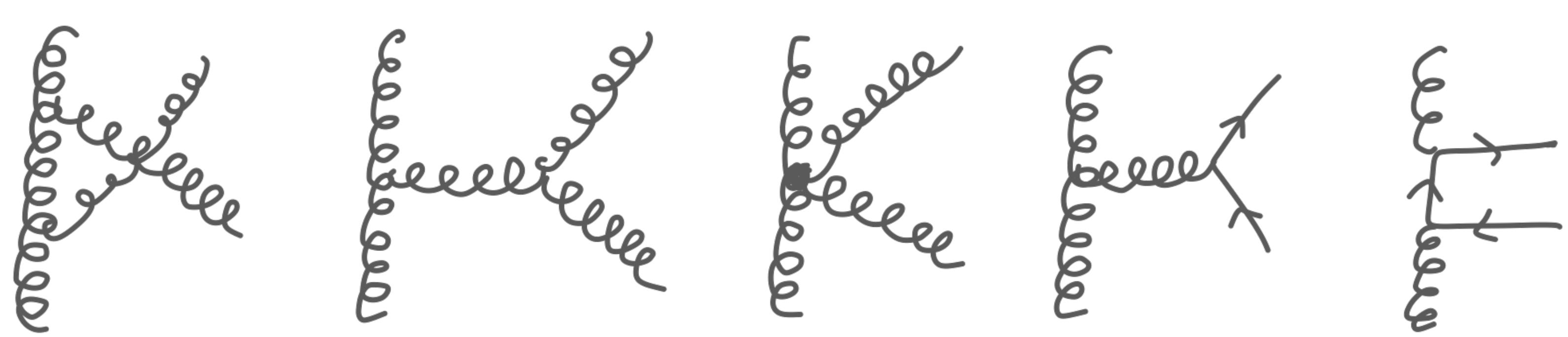


Tree level amplitude can be written as

$$\begin{aligned}
\mathcal{A}^{\rho_1\rho_2\ldots\rho_n}_{2\to n+2} &= 2isg_s t^{a_1}_{ij}\left(\frac{i}{\mathbf{k}_1^2}\right)\\
&\times g_s f_{a_1a_2b_1} C^{\rho_1}(k_1,k_2)\left(\frac{i}{\mathbf{k}_1^2}\right)\\
&\times g_s f_{a_2a_3b_2} C^{\rho_2}(k_2,k_3)\left(\frac{i}{\mathbf{k}_2^2}\right)\\
&\ldots\\
&\times g_s f_{a_na_{n+1}b_n} C^{\rho_n}(k_n,k_{n+1})\left(\frac{i}{\mathbf{k}_{n+1}^2}\right)\\
&\times g_s t^{a_{n+1}}_{kl}
\end{aligned}$$

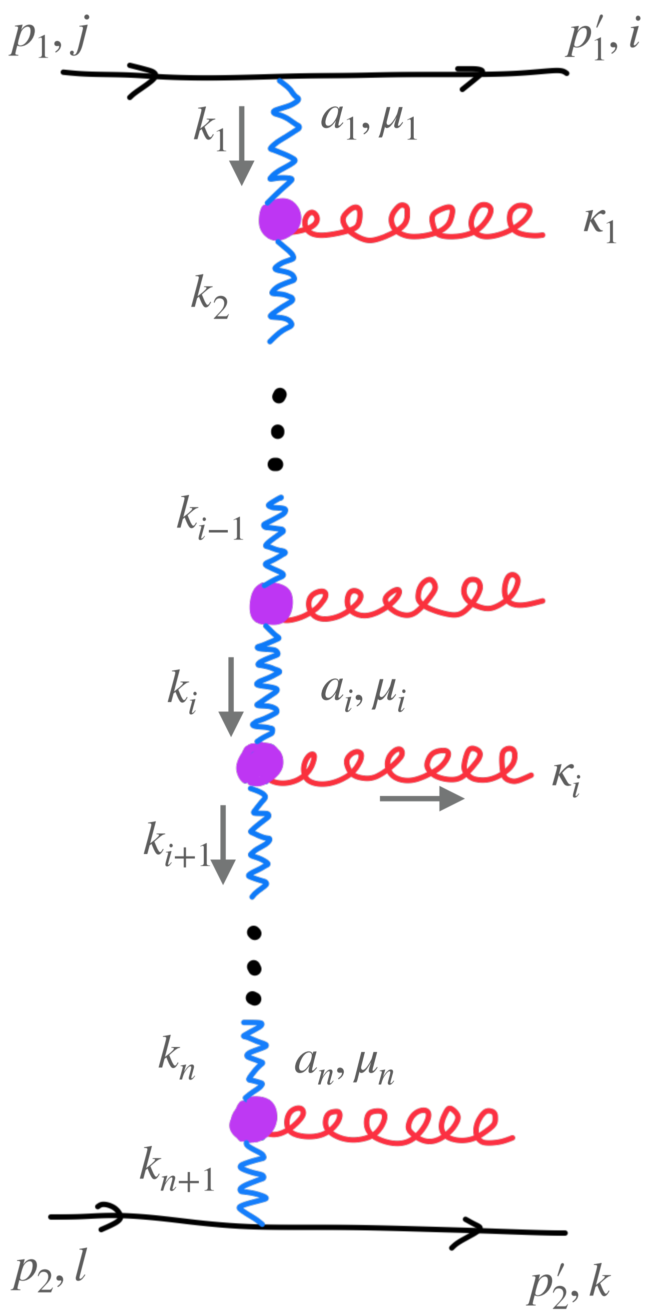
composition of effective Lipatov vertices and propagators for gluons in the t-channel

Example of diagrams which do not contribute at leading-logarithmic accuracy

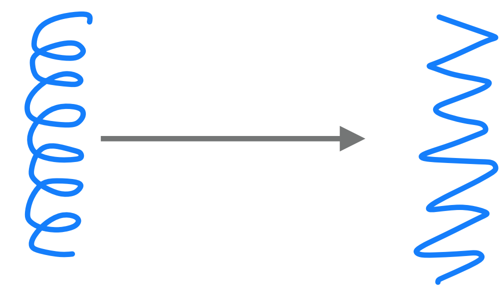


2 to 2+n amplitude with virtual corrections

But, need to incorporate virtual corrections. Include them through the reggeization of the gluon



replace the gluon propagator



with the reggeized gluon propagator

$$\mathcal{A}_{2\to n+2}^{\rho_1\rho_2\dots\rho_n} = 2isg_s t_{ij}^{a_1}\left(\frac{i}{\mathbf{k}_1^2}\right)\left(\frac{1}{\alpha_1}\right)^{\epsilon(k_1^2)}$$
$$\times\, g_s f_{a_1a_2b_1} C^{\rho_1}(k_1,k_2)\left(\frac{i}{\mathbf{k}_2^2}\right)\left(\frac{\alpha_1}{\alpha_2}\right)^{\epsilon(k_2^2)}$$
$$\times\, g_s f_{a_2a_3b_2} C^{\rho_2}(k_2,k_3)\left(\frac{i}{\mathbf{k}_3^2}\right)\left(\frac{\alpha_2}{\alpha_3}\right)^{\epsilon(k_3^2)}$$
$$\dots$$
$$\times\, g_s f_{a_na_{n+1}b_n} C^{\rho_n}(k_n,k_{n+1})\left(\frac{i}{\mathbf{k}_{n+1}^2}\right)\left(\frac{\alpha_n}{\alpha_{n+1}}\right)^{\epsilon(k_{n+1}^2)}$$
$$\times\, g_s t_{kl}^{a_{n+1}}$$

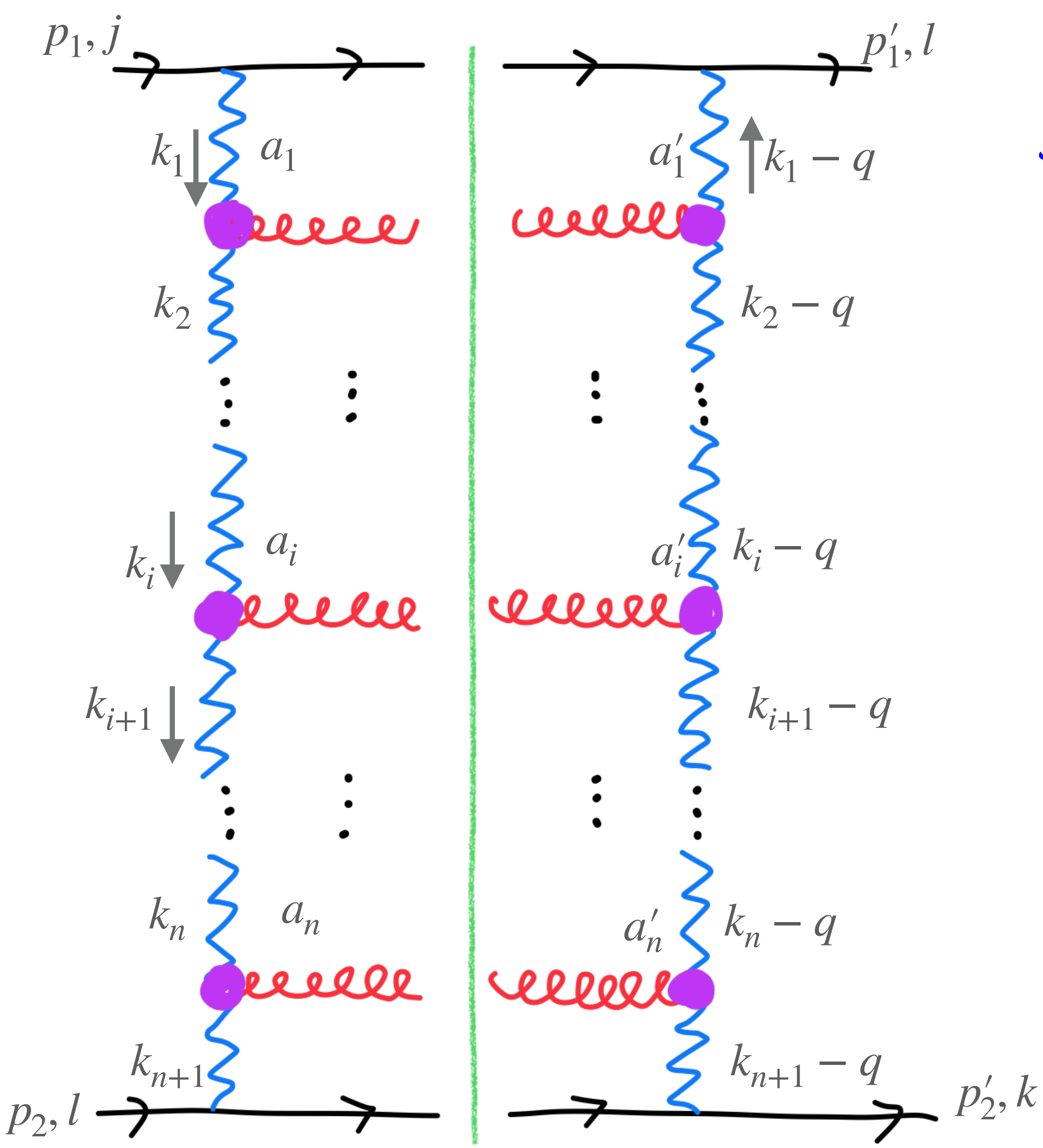
Lipatov; Bartels; Kuraev, Lipatov,Fadin

also using $$\frac{-i}{k_i^2}\left(\frac{s_i}{-k_i^2}\right)^{\epsilon_g(k_i^2)} \simeq \frac{-i}{k_i^2}\left(\frac{\alpha_{i-1}}{\alpha_i}\right)^{\epsilon_g(k_i^2)}$$

Towards BFKL equation

Having 2 t o2+n amplitude in high energy limit one can use unitarity relations to obtain the imaginary part of elastic qq scattering

$$\mathrm{Im}\mathcal{A}(s,t) = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \int d\Pi_{n+2}\, \mathcal{A}^{\rho_1\ldots\rho_n}_{(2\to n+2)}(k_1,k_2,\ldots,k_n)\mathcal{A}^{\dagger}_{(2\to n+2)\,\rho_1\ldots\rho_n}(k_1-q,k_2-q,\ldots,k_n-q)$$



$$\mathcal{A}^{\rho_1\rho_2\ldots\rho_n}_{2\to n+2} = 2isg_s^2 t^{a_1}_{ij} t^{a_{n+1}}_{kl} \left(\frac{i}{\mathbf{k}_1^2}\right)\left(\frac{1}{\alpha_1}\right)^{\epsilon(k_1^2)}$$

$$\times\, \Pi_{i=1}^{n}\left[g_s f_{a_i a_{i+1} b_i} C^{\rho_i}(k_i,k_{i+1})\left(\frac{i}{\mathbf{k}_{i+1}^2}\right)\left(\frac{\alpha_i}{\alpha_{i+1}}\right)^{\epsilon(k_{i+1}^2)}\right]$$

and similarly for complex conjugate amplitude
Contraction of Lipatov vertices

$$C^{\rho_i}(k_i,k_{i+1})C_{\rho_i}(-k_i+q,-k_{i+1}+q) =$$

$$-2\left[\mathbf{q}^2 - \frac{\mathbf{k}_i^2(\mathbf{k}_{i+1}-\mathbf{q})^2}{(\mathbf{k}_i-\mathbf{k}_{i+1}^2)} - \frac{\mathbf{k}_{i+1}^2(\mathbf{k}_i-\mathbf{q})^2}{(\mathbf{k}_i-\mathbf{k}_{i+1}^2)}\right]$$

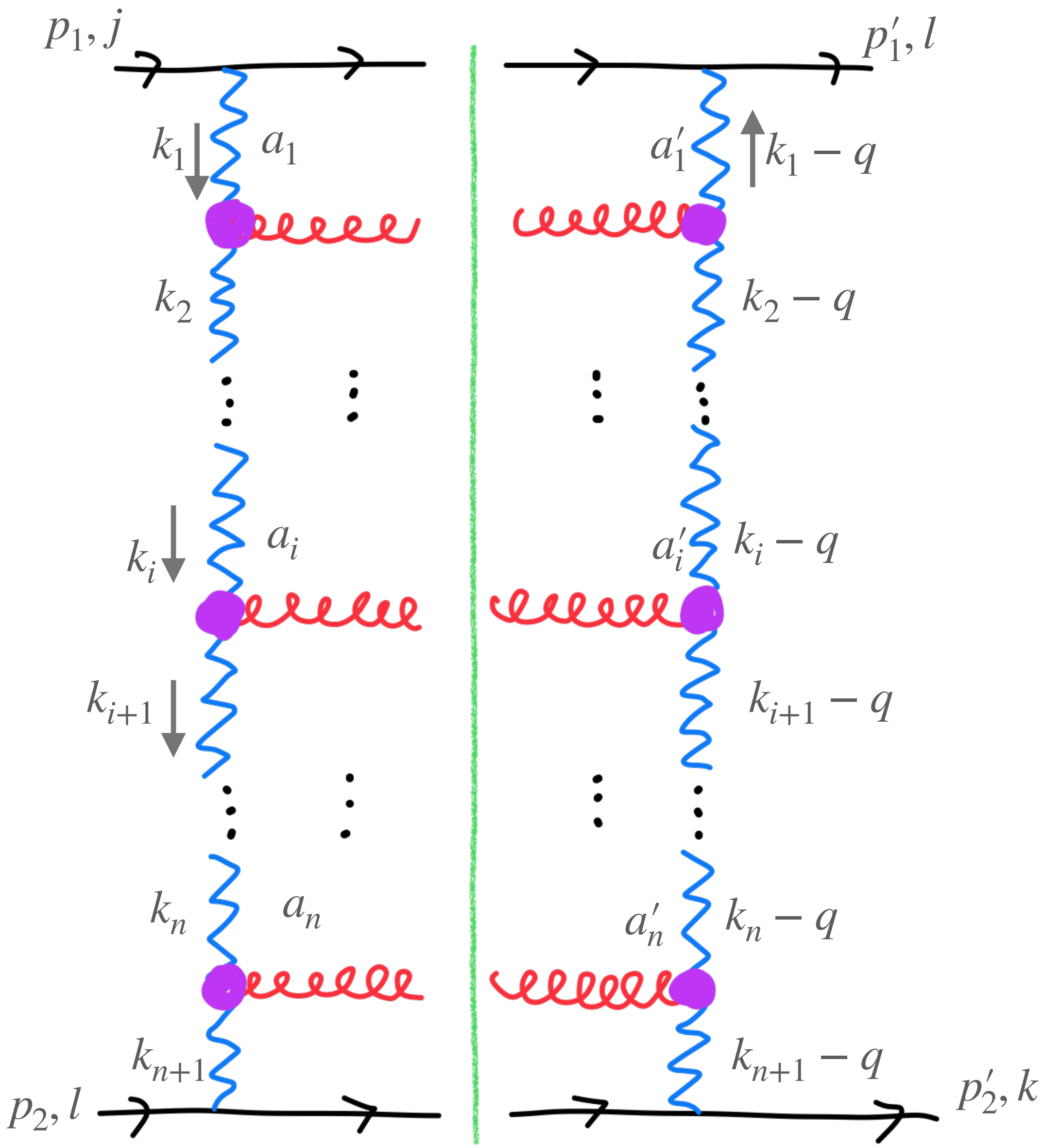
$$= -2\mathcal{K}_{\mathrm{real}}(\mathbf{k}_i,\mathbf{k}_{i+1})$$

q is momentum transfer through the ladder

Towards BFKL

Explicit form for the n+2 body phase space

$$\int d\Pi_{n+2} \equiv \int \prod_{i=0}^{n+1} \frac{d^4\kappa_i}{(2\pi)^3}\delta(\kappa_i^2)\,(2\pi)^4\delta^{(4)}(p_1+p_1-\sum_{i=0}^{n+1}\kappa_i)$$



changing integration over intermediate momenta and using Sudakov variables

$$\int d\Pi_{n+2} = \frac{s^{n+1}}{2^{n+1}(2\pi)^{3n+2}}\int\prod_{i=1}^{n+1} d\alpha_i d\beta_i d^2\mathbf{k}_i$$

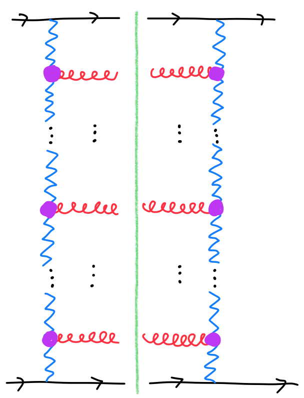
$$\times\,\delta(-\beta_1(1-\alpha_1)s-\mathbf{k}_1^2)\,\delta(\alpha_{n+1}(1+\beta_{n+1})s-\mathbf{k}_{n+1}^2)$$

$$\times\prod_{j=1}^{n}\delta((\alpha_j-\alpha_{j+1})(\beta_j-\beta_{j+1})s-(\mathbf{k}_j-\mathbf{k}_{j+1})^2)$$

Using multi-Regge kinematics, and integrating over β_i

$$\int d\Pi_{n+2} = \frac{1}{2^{n+1}(2\pi)^{3n+2}}\prod_{i=1}^{n+1}\int_{\alpha_{i+1}}^{1}\frac{d\alpha_i}{\alpha_i}\int_0^1 d\alpha_{n+1}\int\prod_{j=1}^{n+1}d^2\mathbf{k}_j\,\delta(\alpha_{n+1}s-\mathbf{k}^2)$$

Imaginary part of the elastic amplitude



$$\mathrm{Im}\mathcal{A}(s,t) = \frac{1}{2}\sum_{n=0}^{\infty} 4s^2 g_s^4 \mathcal{G}_R \int d\Pi_{n+2} \frac{1}{\mathbf{k}_1^2(\mathbf{k}_1-\mathbf{q})^2}\left(\frac{1}{\alpha_1}\right)^{\epsilon(k_1^2)+\epsilon((k_1-q)^2)}$$

$$\times \prod_{i=1}^{n}\left[\frac{g_s^2}{\mathbf{k}_{i+1}^2(\mathbf{k}_{i+1}-\mathbf{q})^2}(-2\eta_R)\mathcal{K}_{\mathrm{real}}(\mathbf{k}_i,\mathbf{k}_{i+1})\left(\frac{\alpha_i}{\alpha_{i+1}}\right)^{\epsilon(k_{i+1}^2)+\epsilon((k_{i+1}-q)^2)}\right]$$

Exchange of **two** reggeized gluons in the t-channel

Iteration of many emissions down the ladder given by the kernel $\mathcal{K}_{\mathrm{real}}$, with reggeized gluons in the t-channel

The above form is already projected onto the color representation R.
Factors $\mathcal{G}_R, \eta_R$ depend on the color representation: singlet, octet
For BFKL Pomeron one takes singlet

Recall that phase space

$$\int d\Pi_{n+2} = \frac{1}{2^{n+1}(2\pi)^{3n+2}}\prod_{i=1}^{n+1}\int_{\alpha_{i+1}}^{1}\frac{d\alpha_i}{\alpha_i}\int_0^1 d\alpha_{n+1}\int\prod_{j=1}^{n+1}d^2\mathbf{k}_j\,\delta(\alpha_{n+1}s-\mathbf{k}^2)$$

Contains nested integrals over α_i. Can disentangle using **Mellin** transform

Mellin transform recap

$$\mathcal{F}(\omega) = \int_1^\infty d\left(\frac{s}{s_0}\right)\left(\frac{s}{s_0}\right)^{-\omega-1} f(s) \qquad\qquad f(s) = \frac{1}{2\pi i}\int_C d\omega \left(\frac{s}{s_0}\right)^{\omega} \mathcal{F}(\omega)$$

Mellin transform allows to 'de-convolute' convolutions...

suppose we have convolutions over set of functions

$$f(s) = \prod_{i=1}^{n} \int_{\alpha_{i+1}}^{1} \frac{d\alpha_i}{\alpha_i}\, g_i(\frac{\alpha_{i-1}}{\alpha_i})\, s_0\delta(\alpha_n s - s_0) \qquad \text{with} \qquad \alpha_0 = 1, \;\; \alpha_{n+1} = 0$$

Take the mellin transform

$$\mathcal{F}(\omega) = \int_1^\infty d\left(\frac{s}{s_0}\right)\left(\frac{s}{s_0}\right)^{-\omega-1}$$

$$\times \prod_{i=1}^{n} \int_{\alpha_{i+1}}^{1} \frac{d\alpha_i}{\alpha_i}\, g_i(\alpha_{i-1}/\alpha_i)\, s_0\delta(\alpha_n s - s_0) = \prod_{i=1}^{n} \int_{\alpha_{i+1}}^{1} \frac{d\alpha_i}{\alpha_i}\, g_i(\alpha_{i-1}/\alpha_i)\, \alpha_n^{\omega}$$

Change variables $\rho_i = \frac{\alpha_i}{\alpha_{i-1}}$ so that $\alpha_n = \rho_1 \dots \rho_n$

Finally $\mathcal{F}(\omega) = \prod_{i=1}^{n} \int_0^1 d\rho_i\, \rho_i^{\omega-1}\, g_i(\frac{1}{\rho_i}) \;\; = \prod_{i=1}^{n} \mathcal{G}_i(\omega)$ Product of Mellin transforms of g_i

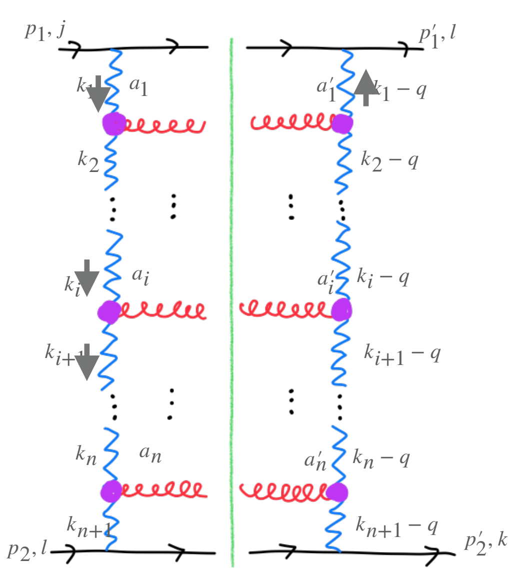
Result for Im A in Mellin space

Putting together result for imaginary part and taking Mellin transform

$$\mathrm{Im}\mathcal{A}(s,t)=\frac{1}{2}\sum_{n=0}^{\infty}4s^2g_s^4\mathcal{G}_R\int d\Pi_{n+2}\frac{1}{\mathbf{k}_1^2(\mathbf{k}_1-\mathbf{q})^2}\left(\frac{1}{\alpha_1}\right)^{\epsilon(k_1^2)+\epsilon((k_1-q)^2)}$$

$$\times\prod_{i=1}^{n}\left[\frac{g_s^2}{\mathbf{k}_{i+1}^2(\mathbf{k}_{i+1}-\mathbf{q})^2}(-2\eta_R)\mathcal{K}_{\mathrm{real}}(\mathbf{k}_i,\mathbf{k}_{i+1})\left(\frac{\alpha_i}{\alpha_{i+1}}\right)^{\epsilon(k_{i+1}^2)+\epsilon((k_{i+1}-q)^2)}\right]$$

$$\int d\Pi_{n+2}=\frac{1}{2^{n+1}(2\pi)^{3n+2}}\prod_{i=1}^{n+1}\int_{\alpha_{i+1}}^{1}\frac{d\alpha_i}{\alpha_i}\int_0^1 d\alpha_{n+1}\int\prod_{j=1}^{n+1}d^2\mathbf{k}_j\,\delta(\alpha_{n+1}s-\mathbf{k}^2)$$



$$f_R(\omega,t)=\int_1^{\infty}d\left(\frac{s}{|t|}\right)\left(\frac{s}{|t|}\right)^{-\omega-1}\frac{\mathrm{Im}\mathcal{A}_R(s,t)}{s}$$

$$f_R(\omega,t)=(4\pi\alpha_s)^2\mathcal{G}_R\sum_{n=0}^{\infty}\prod_{i=1}^{n+1}\frac{d^2\mathbf{k}_i}{(2\pi)^2}$$

Integrals over the transverse momenta

$$\times\frac{1}{\mathbf{k}_1^2(\mathbf{k}_1-\mathbf{q})^2}\frac{1}{\omega-\epsilon_g(k_1^2)-\epsilon_g((k_1-q)^2)}$$

$$\times(-2\alpha_s\eta_R)\mathcal{K}_{\mathrm{real}}(\mathbf{k}_1,\mathbf{k}_2)$$

Real radiative corrections

$$\times\frac{1}{\mathbf{k}_2^2(\mathbf{k}_2-\mathbf{q})^2}\frac{1}{\omega-\epsilon_g(k_2^2)-\epsilon_g((k_2-q)^2)}$$

Propagators

$$\dots$$

$$\times(-2\alpha_s\eta_R)\mathcal{K}_{\mathrm{real}}(\mathbf{k}_n,\mathbf{k}_{n+1})$$

$$\times\frac{1}{\mathbf{k}_{n+1}^2(\mathbf{k}_{n+1}-\mathbf{q})^2}\frac{1}{\omega-\epsilon_g(k_{n+1}^2)-\epsilon_g((k_{n+1}-q)^2)}$$

Mellin transforms of

$$\int_0^1 d\rho_i\rho_i^{\omega-1}\left(\frac{1}{\rho_i}\right)^{\epsilon(k_i^2)+\epsilon((k_i-q)^2)}$$

BFKL equation

Define the quantity unintegrated over the transverse momenta $\mathcal{F}_R(\omega, \mathbf{k}, \mathbf{q})$

$$f_R(\omega, t = -\mathbf{q}^2) = (4\pi\alpha_s)^2 \mathcal{G}_R \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{\mathcal{F}_R(\omega, \mathbf{k}, \mathbf{q})}{\mathbf{k}^2\,(\mathbf{k}-\mathbf{q})^2}$$

The solution can then be written in the form of this function which satisfies integral equation

$$[\omega - \epsilon_g(-\mathbf{k}^2) - \epsilon_g(-(\mathbf{k}-\mathbf{q})^2)]\,\mathcal{F}_R(\omega, \mathbf{k}, \mathbf{q}) \;=\; 1 - \frac{2\alpha_s \eta_R}{4\pi^2} \int d^2\mathbf{k}' \,\frac{\mathcal{K}_{\rm real}(\mathbf{k}, \mathbf{k}')}{\mathbf{k}'^2(\mathbf{k}'-\mathbf{q})^2}\, \mathcal{F}_R(\omega, \mathbf{k}', \mathbf{q})$$

Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation

$$\mathcal{K}_{\rm real}(\mathbf{k}, \mathbf{k}') \;=\; \left[\mathbf{q}^2 - \frac{\mathbf{k}^2(\mathbf{k}'-\mathbf{q})^2}{(\mathbf{k}-\mathbf{k}'^2)} - \frac{\mathbf{k}'^2(\mathbf{k}-\mathbf{q})^2}{(\mathbf{k}-\mathbf{k}'^2)}\right]$$

Real emission kernel: contraction of two Lipatov vertices

$$\epsilon_g(-\mathbf{k}^2), \epsilon_g(-(\mathbf{k}-\mathbf{q})^2)$$

terms responsible for virtual emissions

The octet case, has the solution which corresponds to the amplitude for the exchange of the reggeized gluon, i.e. $\sim \frac{s}{t}\left(\frac{s}{|t|}\right)^{\epsilon_g(t)}$

This justifies a posteriori the ansatz on the reggeization $\frac{-i}{k^2} \longrightarrow \frac{-i}{k^2}\left(\frac{s}{-k^2}\right)^{\epsilon_g(k^2)}$

BFKL Pomeron

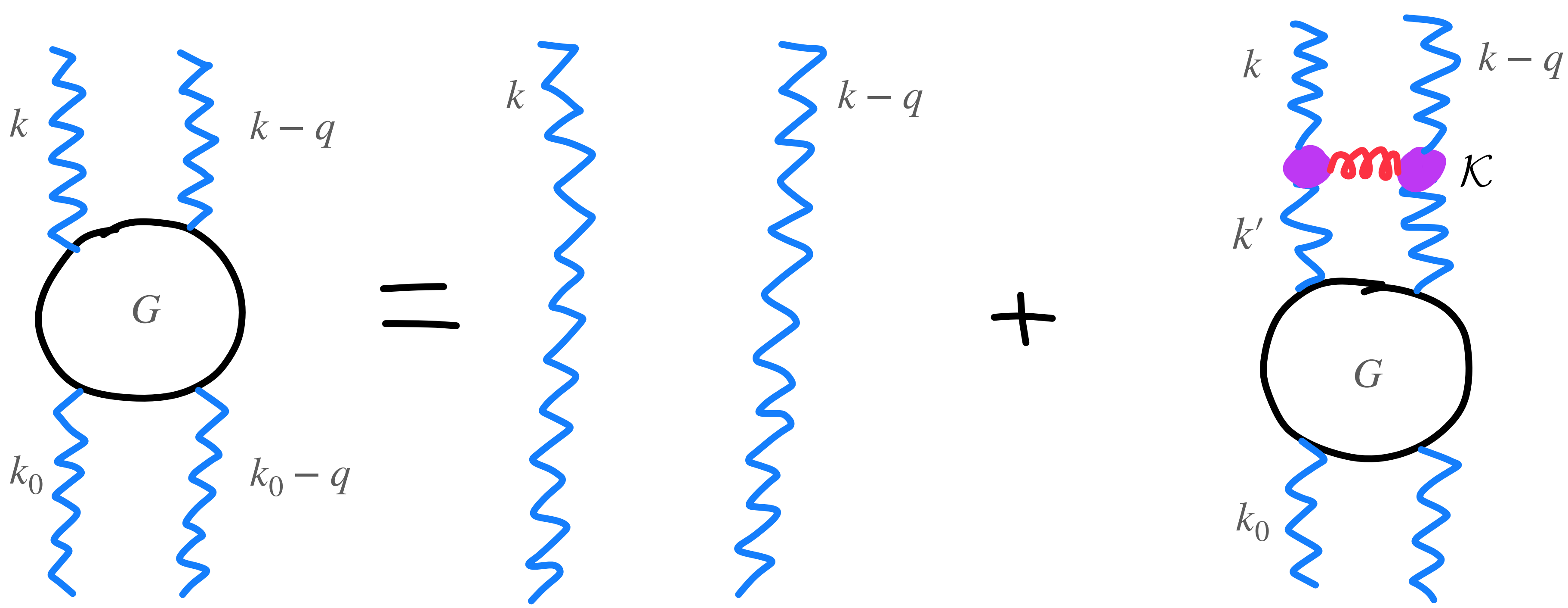
The BFKL equation for the **Pomeron** is obtained by taking the **singlet**

We can introduce another useful function

$$\mathcal{F}(\omega, \mathbf{k}, \mathbf{q}) = \int d^2\mathbf{k}_0 \, \frac{\mathbf{k}^2}{\mathbf{k}_0^2} \, G(\omega, \mathbf{k}, \mathbf{k}_0, \mathbf{q})$$

BFKL gluon Green's function

$$[\omega - \epsilon_g(-\mathbf{k}^2) - \epsilon_g(-(\mathbf{k}-\mathbf{q})^2)] \, G(\omega, \mathbf{k}, \mathbf{k}_0, \mathbf{q}) \; = \; \delta^{(2)}(\mathbf{k} - \mathbf{k}_0) - \frac{N_c \alpha_s}{2\pi^2} \int d^2\mathbf{k}' \, \frac{\mathcal{K}_{\rm real}(\mathbf{k}, \mathbf{k}')}{\mathbf{k}^2 (\mathbf{k}' - \mathbf{q})^2} G(\omega, \mathbf{k}', \mathbf{k}_0, \mathbf{q})$$



Infrared properties of BFKL

$$[\omega - \epsilon_g(-\mathbf{k}^2) - \epsilon_g(-(\mathbf{k}-\mathbf{q})^2)]\, G(\omega,\mathbf{k},\mathbf{k}_0,\mathbf{q}) \;=\; \delta^{(2)}(\mathbf{k}-\mathbf{k}_0) - \frac{N_c\alpha_s}{2\pi^2}\int d^2\mathbf{k}'\,\frac{\mathcal{K}_{\rm real}(\mathbf{k},\mathbf{k}')}{\mathbf{k}^2(\mathbf{k}'-\mathbf{q})^2}G(\omega,\mathbf{k}',\mathbf{k}_0,\mathbf{q})$$

Use explicit forms of trajectories and move them to r.h.s of equation (and perform shifts of integration variable to rearrange the terms)

$$\begin{aligned}\omega\, G(\omega,\mathbf{k},\mathbf{k}_0,\mathbf{q}) \;=\;& \delta^{(2)}(\mathbf{k}-\mathbf{k}_0)\\ &+\frac{\bar\alpha_s}{2\pi}\int d^2\mathbf{k}'\Bigg[\frac{-\mathbf{q}^2}{\mathbf{k}^2(\mathbf{k}'-\mathbf{q})^2}\,G(\omega,\mathbf{k}',\mathbf{k}_0,\mathbf{q})\\ &+\frac{1}{(\mathbf{k}'-\mathbf{k})^2}\Bigg(\underbrace{G(\omega,\mathbf{k}',\mathbf{k}_0,\mathbf{q})}_{real}-\mathbf{k}^2\underbrace{\frac{G(\omega,\mathbf{k},\mathbf{k}_0,\mathbf{q})}{\mathbf{k}'^2+(\mathbf{k}'-\mathbf{k})^2}}_{virt.}\Bigg)\\ &+\frac{1}{(\mathbf{k}'-\mathbf{k})^2}\Bigg(\underbrace{\frac{(\mathbf{k}-\mathbf{q})^2\mathbf{k}'^2G(\omega,\mathbf{k}',\mathbf{k}_0,\mathbf{q})}{(\mathbf{k}'-\mathbf{q})^2\mathbf{k}^2}}_{real}-(\mathbf{k}-\mathbf{q})^2\underbrace{\frac{G(\omega,\mathbf{k},\mathbf{k}_0,\mathbf{q})}{(\mathbf{k}'-\mathbf{q})^2+(\mathbf{k}'-\mathbf{k})^2}}_{virt.}\Bigg)\Bigg]\end{aligned}$$

IR finite, terms in(…) multiplying $\frac{1}{(\mathbf{k}'-\mathbf{k})^2}$ vanish when $(\mathbf{k}'-\mathbf{k})\to 0$

Cancellation between **real** and **virtual** terms.

Cancellation of IR divergencies justifies strong ordering in longitudinal momenta giving leading logarithms in energy

Solution to BFKL

Consider t=0. Relevant for the total cross section.

$$\omega\, G(\omega,\mathbf{k},\mathbf{k}_0,\mathbf{q}=0) \;=\; \delta^{(2)}(\mathbf{k}-\mathbf{k}_0) + \frac{\bar{\alpha}_s}{\pi}\int d^2\mathbf{k}'\,\frac{1}{(\mathbf{k}'-\mathbf{k})^2}\left(G(\omega,\mathbf{k}',\mathbf{k}_0,\mathbf{q}=0) - \mathbf{k}^2\frac{G(\omega,\mathbf{k},\mathbf{k}_0,\mathbf{q}=0)}{\mathbf{k}'^2+(\mathbf{k}'-\mathbf{k})^2}\right)$$

$$\omega\, G(\omega,\mathbf{k},\mathbf{k}_0) \;=\; \delta^{(2)}(\mathbf{k}-\mathbf{k}_0) + \mathcal{K}\otimes G$$

where kernel contains real and virtual

Linear equation in G. The solution can be constructed by finding eigenfunctions

$$\mathcal{K}\otimes\phi(\mathbf{k}) = \lambda\,\phi(\mathbf{k}) \qquad \phi_\nu^n(\mathbf{k}) = \frac{1}{\pi\sqrt{2}}(k^2)^{\gamma-1}e^{in\phi} \qquad \gamma = \frac{1}{2}+i\nu$$

so that they satisfy completeness $\delta^{(2)}(\mathbf{k}-\mathbf{k}_0) = \sum_n \frac{1}{2\pi i}\int_C d\gamma\,\phi_\nu^n(k)\phi_\nu^{*n}(k_0)$ +orthogonality

solution

$$G(\omega,\mathbf{k},\mathbf{k}_0) = \sum_n \frac{1}{2\pi i}\int_C d\gamma\frac{\phi_\nu^n(k)\phi_\nu^{*n}(k_0)}{\omega-\bar{\alpha}_s\chi(\gamma,n)}$$

where C is defined as: $\int_{1/2-i\infty}^{1/2+i\infty}$

$$\int d^2\mathbf{k}'\mathcal{K}(\mathbf{k},\mathbf{k}')\phi_\gamma^n(\mathbf{k}') = \bar{\alpha}_s\,\chi(\gamma,n)\phi_\gamma^n(\mathbf{k})$$

BFKL LLx eigenvalue

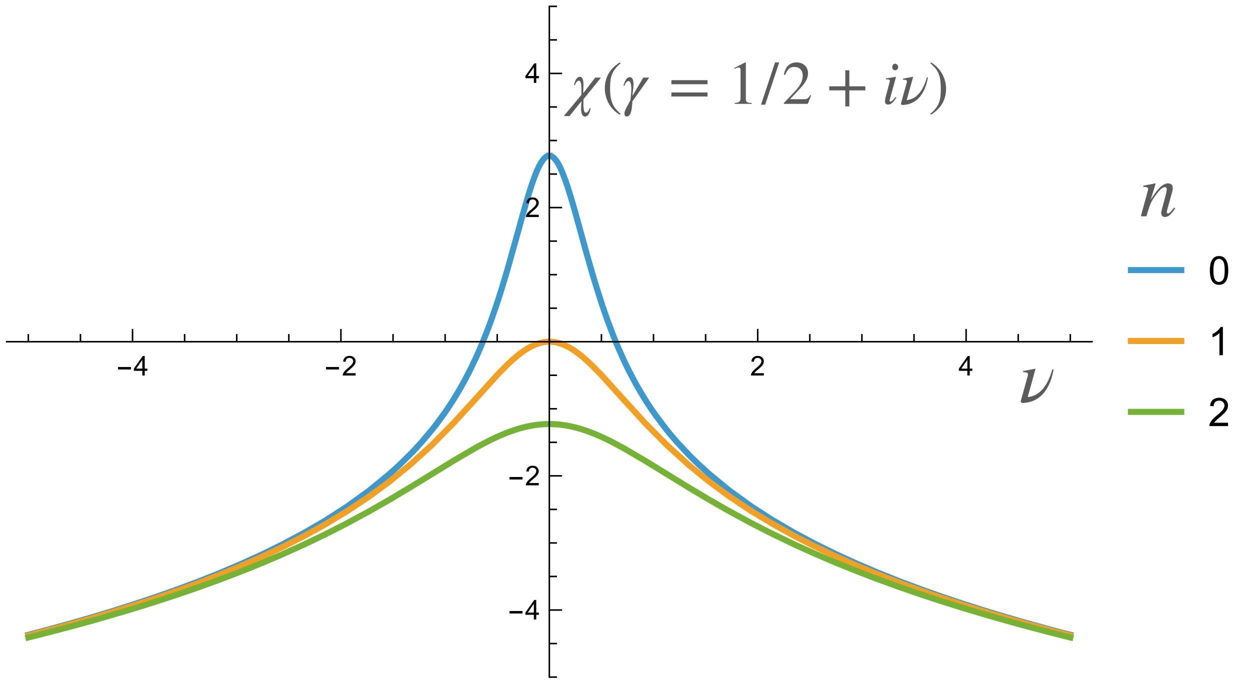
$$\chi(\gamma,n) = 2\psi(1) - \psi(\gamma+\frac{|n|}{2}) - \psi(1-\gamma+\frac{|n|}{2})$$

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) \qquad \psi(x) = \int_0^1 dt\,\frac{1-t^{x-1}}{1-t} - \gamma_E \qquad \psi(1) = -\gamma_E$$

Solution to BFKL

$$\chi(\gamma, n) = 2\psi(1) - \psi(\gamma + \frac{|n|}{2}) - \psi(1 - \gamma + \frac{|n|}{2})$$

Eigenvalue on the line $\gamma = 1/2 + i\nu$



Maximum for $\nu = 0$

Highest for n=0

$$\chi(\gamma = \frac{1}{2} + i\nu) \simeq \chi_0(\frac{1}{2}) - a^2\nu^2 + \ldots$$

$$\chi_0(\frac{1}{2}) = 4\ln 2 \simeq 2.77 \qquad a^2 = 14\zeta(3)$$

In this approximation the solution is
$$G(\omega, \mathbf{k}, \mathbf{k}_0) \simeq \frac{1}{\pi k k_0} \int_{-i\infty}^{+i\infty} \frac{d\nu}{2\pi} \left(\frac{k^2}{k_0^2}\right)^{i\nu} \frac{1}{\omega - \bar{\alpha}_s \chi_0(\frac{1}{2}) + \bar{\alpha}_s a^2 \nu^2}$$

Can invert to get the energy dependence s

$$\mathcal{G}(s, \mathbf{k}, \mathbf{k}_0) = \frac{1}{2\pi i} \int_{C_\omega} d\omega \left(\frac{s}{s_0}\right)^{\omega} G(\omega, \mathbf{k}, \mathbf{k}_0)$$

The energy dependence of this quantity is giving the energy dependence of the imaginary part of the forward elastic scattering amplitude and hence the cross section

Solution to BFKL

Performing the integral over ω gives

$$\mathcal{G}(s,\mathbf{k},\mathbf{k}_0) \simeq \frac{1}{2\pi^2 k k_0} \int_{-i\infty}^{+i\infty} d\nu \left(\frac{k^2}{k_0^2}\right)^{i\nu} \left(\frac{s}{s_0}\right)^{\omega_0 - \bar{\alpha}_s a^2 \nu^2}$$

$$\omega_0 = \bar{\alpha}_s \chi_0(\frac{1}{2}) = \bar{\alpha}_s 4 \ln 2$$

one can now perform the ν integral to obtain

$$\mathcal{G}(s,\mathbf{k},\mathbf{k}_0) \simeq \frac{1}{k k_0} \left(\frac{s}{s_0}\right)^{\omega_0} \frac{1}{\sqrt{\pi \ln s/s_0}} \frac{1}{2\pi a} \exp\left(-\frac{\ln^2(k^2/k_0^2)}{4\bar{\alpha}_s a^2 \ln(s/s_0)}\right)$$

The solutions shows growth with energy s as a power: s^{ω_0}

The intercept of the BFKL Pomeron,
or intercept of the hard Pomeron

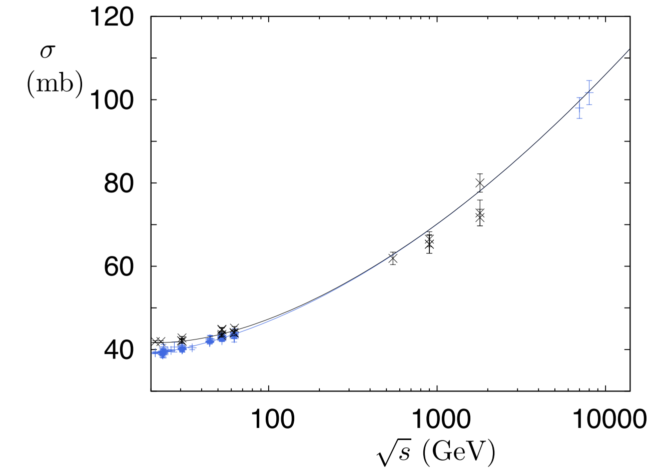
$$\alpha_{IP}(0) = 1 + \omega_0 > 1$$

Example $\bar{\alpha}_s = \frac{\alpha_s N_c}{\pi} \simeq 0.2$

gives $\alpha_{IP}^{\mathrm{BFKL}} - 1 \simeq 0.55$

very strong growth !

This value is larger than the intercept of the soft Pomeron (eg. Donnachie-Landshoff fits to pp cross section)



$$\alpha_{IP}^{\mathrm{soft}} - 1 \simeq 0.1$$

Numerical solution to BFKL

$$\omega\, G(\omega,\mathbf{k},\mathbf{k}_0) \;=\; \delta^{(2)}(\mathbf{k}-\mathbf{k}_0) + \mathcal{K}\otimes G$$

Can rewrite BFKL in ω space as a differential equation in $\ln s/s_0$ and integrate it.

$$\mathcal{G}(s,\mathbf{k},\mathbf{k}_0) = \frac{1}{2\pi i}\int_{C_\omega} d\omega \left(\frac{s}{s_0}\right)^{\omega} G(\omega,\mathbf{k},\mathbf{k}_0)$$

$$\frac{\partial}{\partial \ln s/s_0}\mathcal{G}(s,\mathbf{k},\mathbf{k}_0) = \frac{1}{2\pi i}\int_{C_\omega} d\omega\, \omega \left(\frac{s}{s_0}\right)^{\omega} G(\omega,\mathbf{k},\mathbf{k}_0)$$

$$\mathcal{G}(Y;k,k_0) = \mathcal{G}^{(0)}(k,k_0) + \int_0^Y dy \int dk'^2 \mathcal{K}(k,k')\,\mathcal{G}(y;k,k_0)$$

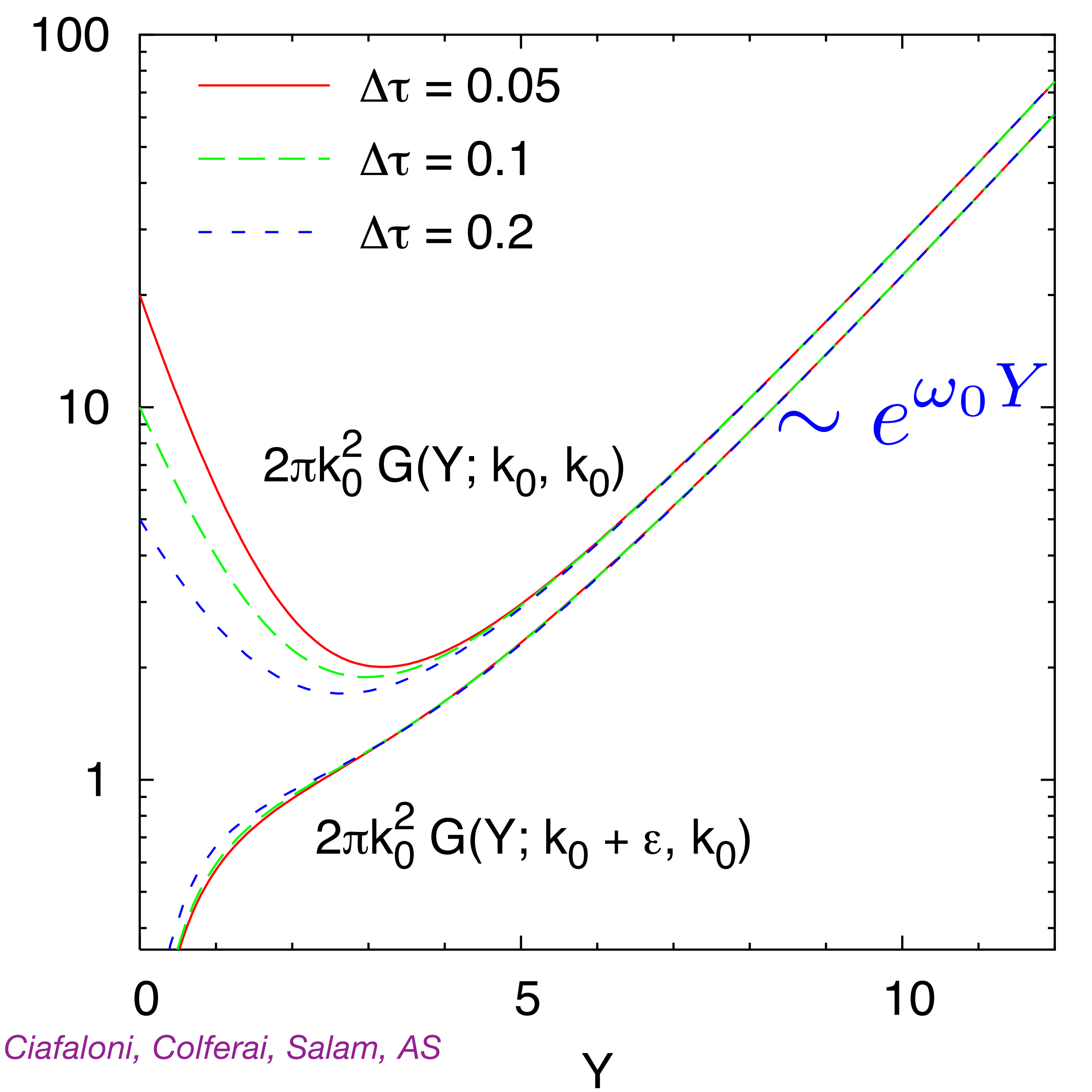
here integrated over the angle $k^2 \equiv \mathbf{k}^2$

with $\mathscr{G}^{(0)}$ being the initial condition and evolution variable

$$Y = \ln s/s_0$$

In principle $\mathscr{G}^{(0)} \sim \delta(k-k_0)$

In practice $\mathscr{G}^{(0)} \sim \dfrac{1}{\Delta\tau}\delta_{m0}$ where m is the point on the grid in $\ln k$, and $\Delta\tau$ is the spacing on the grid



Solution to LLx BFKL with $\bar{\alpha}_s = 0.2$

Exponential growth with rapidity Y

The discretization dependence is negligible when the gluon Green's function is integrated over with smooth impact factor

Off set for illustration by ε

Diffusion properties of BFKL

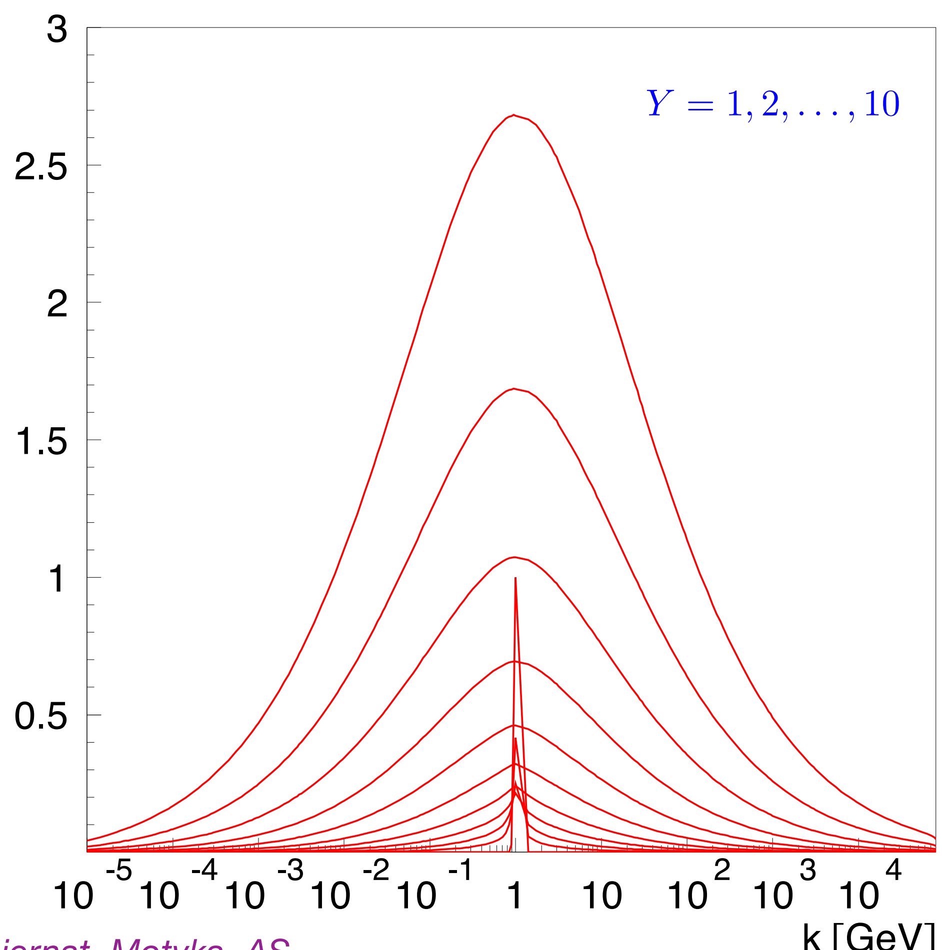
$$\mathcal{G}(s,\mathbf{k},\mathbf{k}_0) \simeq \frac{1}{kk_0}\left(\frac{s}{s_0}\right)^{\omega_0} \frac{1}{\sqrt{\pi \ln s/s_0}} \frac{1}{2\pi a} \exp\left(-\frac{\ln^2(k^2/k_0^2)}{4\bar{\alpha}_s a^2 \ln(s/s_0)}\right) \qquad a^2 \sim \chi''(1/2)$$

The **exponential** is responsible for the **diffusion** of the transverse momenta along the ladder

Gaussian distribution in $\ln(k^2/k_0^2)$ with a width proportional to $\alpha_s Y = \alpha_s \ln s/s_0$

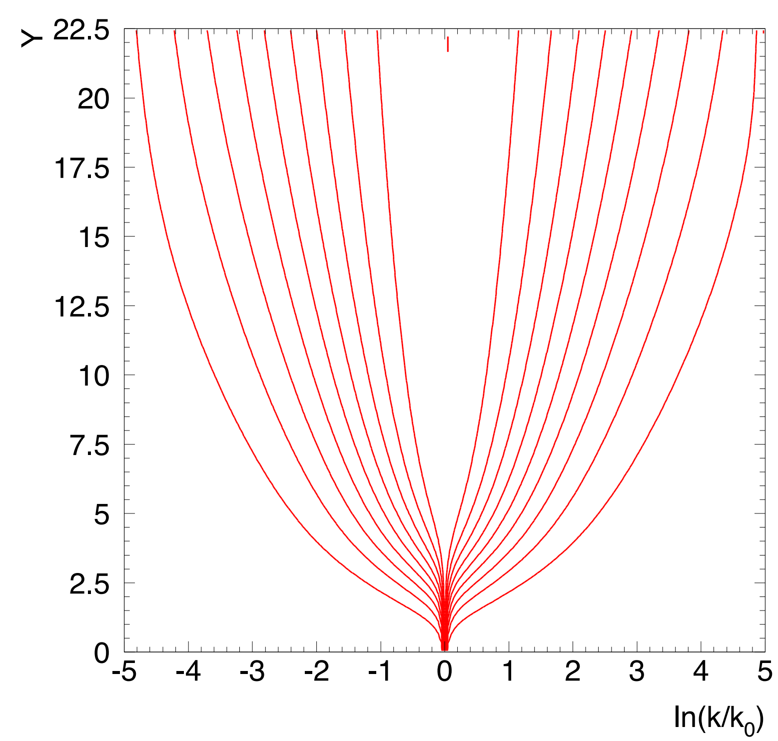
Numerical s**olution to LLx BFKL** with $\bar{\alpha}_s = 0.2$

$k\mathcal{G}(Y;k,k_0)$

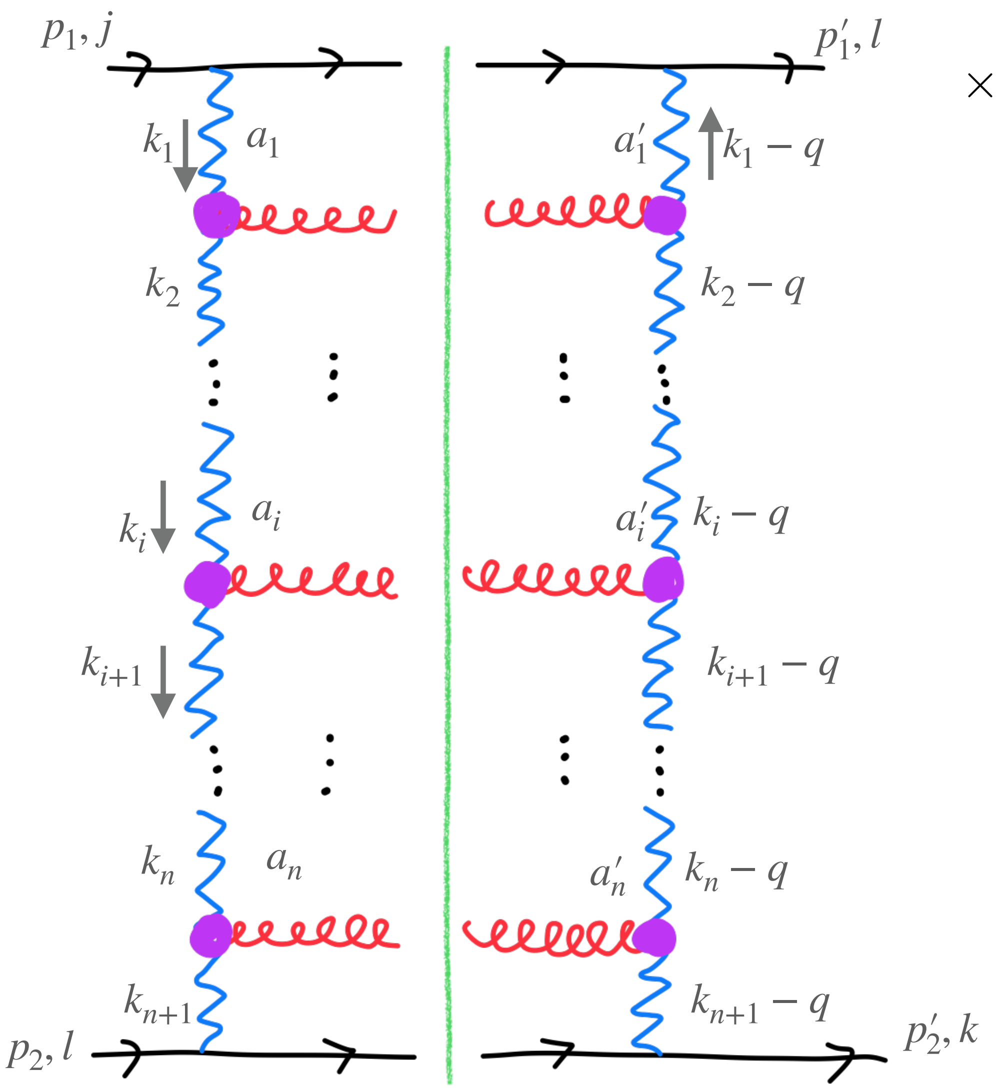


Golec-Biernat, Motyka, AS

contours of normalized solution $\frac{kG(Y,k,k_0)}{k_{\max}G(Y,k_{\max},k_0)}$



Diffusion properties of BFKL



$$\mathrm{Im}\mathcal{A}(s,t) = \frac{1}{2}\sum_{n=0}^{\infty} 4s^2 g_s^4 \mathcal{G}_R \int d\Pi_{n+2} \frac{1}{\mathbf{k}_1^2(\mathbf{k}_1-\mathbf{q})^2}\left(\frac{1}{\alpha_1}\right)^{\epsilon(k_1^2)+\epsilon((k_1-q)^2)}$$

$$\times \prod_{i=1}^{n}\left[\frac{g_s^2}{\mathbf{k}_{i+1}^2(\mathbf{k}_{i+1}-\mathbf{q})^2}(-2\eta_R)\mathcal{K}_{\mathrm{real}}(\mathbf{k}_i,\mathbf{k}_{i+1})\left(\frac{\alpha_i}{\alpha_{i+1}}\right)^{\epsilon(k_{i+1}^2)+\epsilon((k_{i+1}-q)^2)}\right]$$

$$\int d\Pi_{n+2} = \frac{1}{2^{n+1}(2\pi)^{3n+2}}\prod_{i=1}^{n+1}\int_{\alpha_{i+1}}^{1}\frac{d\alpha_i}{\alpha_i}\int_0^1 d\alpha_{n+1}\int\prod_{j=1}^{n+1} d^2\mathbf{k}_j\,\delta(\alpha_{n+1}s-\mathbf{k}^2)$$

Multi-Regge kinematics: strong ordering in rapidity of produced particles

$$y_0 \gg y_1 \gg y_2 \gg \cdots \gg y_{n-1} \gg y_n \gg y_{n+1}$$

No ordering of the transverse momenta of the ladder

Integrals over $\mathbf{k}_j$ are unrestricted

Eventually, sensitivity to **non-perturbative** region

Becomes particularly important when strong coupling runs

(though at NLLx order)

More on the diffusion in BFKL

BFKL solution has the following property

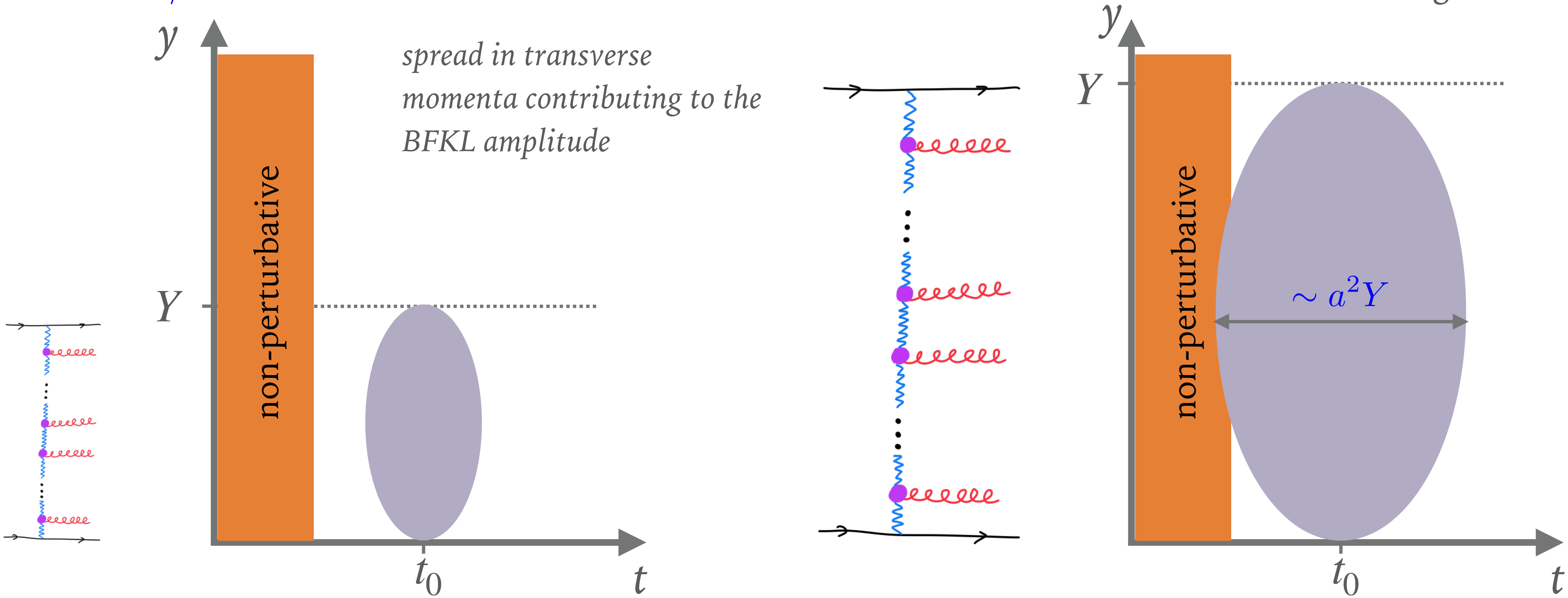
$$\mathcal{G}(Y,\mathbf{k},\mathbf{k}_0) = \int d^2\mathbf{k}'\,\mathcal{G}(y,\mathbf{k},\mathbf{k}')\,\mathcal{G}(Y-y,\mathbf{k}',\mathbf{k}_0)$$

$$Y = \ln\frac{s}{s_0} \qquad y = \ln\frac{s'}{s_0}$$

BFKL amplitude can be constructed from convolution of two amplitudes with arbitrary partition in energy (rapidity)

$t = \ln k^2/\Lambda^2$

Bartels' cigar



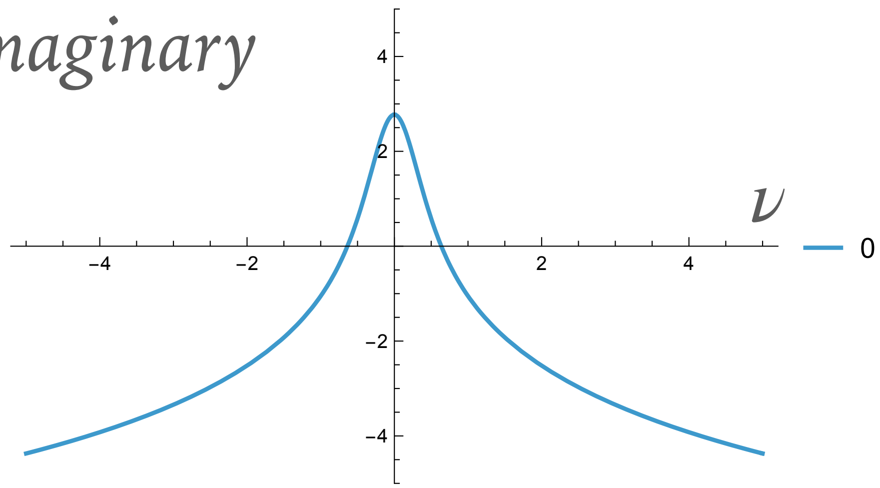
Even when (transverse) external scales are large, at sufficiently high energies transverse momenta along the ladder can diffuse into non-perturbative region

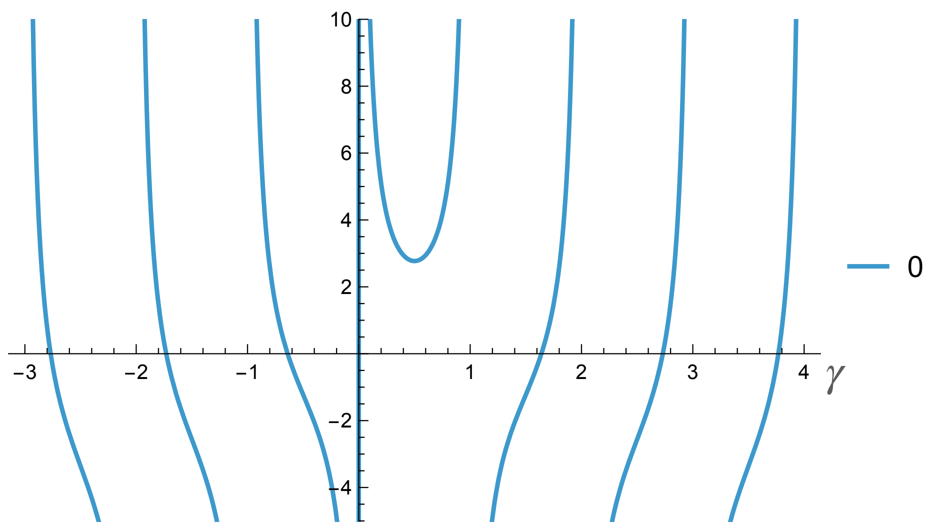
Collinear structure of BFKL kernel

BFKL eigenvalue at LLx $n = 0$

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$$

imaginary





given that

$$\psi(z) = -\gamma_E + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+z}\right)$$

real axis

$$\gamma = \frac{1}{2} + i\nu$$

$\chi(\gamma)$ has simple poles at

$\gamma = 0, \pm 1, \pm 2, \ldots$

Collinear structure of BFKL kernel

$$\omega\, G(\omega, \mathbf{k}, \mathbf{k}_0, \mathbf{q}=0) \;=\; \delta^{(2)}(\mathbf{k}-\mathbf{k}_0) + \frac{\bar{\alpha}_s}{\pi}\int d^2\mathbf{k}'\, \frac{1}{(\mathbf{k}'-\mathbf{k})^2}\left(G(\omega, \mathbf{k}', \mathbf{k}_0, \mathbf{q}=0) - \mathbf{k}^2 \frac{G(\omega, \mathbf{k}, \mathbf{k}_0, \mathbf{q}=0}{\mathbf{k}'^2 + (\mathbf{k}'-\mathbf{k})^2}\right)$$

Take real contribution and assume azimuthal angle independence

$$\frac{\bar{\alpha}_s}{\pi}\int d^2\mathbf{k}'\, \frac{1}{(\mathbf{k}'-\mathbf{k})^2}\,\phi(k') + \text{virt.} \;\longrightarrow\; \bar{\alpha}_s \int dk'^2\, \frac{1}{|k'^2-k^2|}\,\phi(k') + \text{virt.}$$

If we take $k \gg k'$ can drop virtual contribution, then expand the kernel and take $\phi(k') = (k'^2)^{\gamma-1}$

$$\frac{1}{(k^2)^{\gamma-1}}\int_0^{k^2-\epsilon_k} dk'^2\, \frac{1}{|k'^2-k^2|}\,(k'^2)^{\gamma-1} \;=\; \frac{1}{(k^2)^{\gamma-1}}\int_0^{k^2-\epsilon_k} dk'^2\, \frac{1}{k^2}\,\frac{1}{1-k'^2/k^2}\,(k'^2)^{\gamma-1}$$

the divergence at 1 is canceled by the virtual term anyway

$$= \int_0^{1-\epsilon} \frac{du}{u}\,\frac{1}{1-u}\,u^{\gamma} = \sum_{n=0}^{\infty}\int_0^{1-\epsilon} du\, u^{\gamma-1+n} = \sum_{n=0}^{\infty}\frac{1}{\gamma+n}$$

Poles at $0, -1, -2, \ldots$ In particular pole $\frac{1}{\gamma}$ comes from the first term in the expansion assuming strong ordering $k \gg k'$

Collinear pole $\frac{1}{\gamma}$: corresponds to strong ordering case $k \gg k'$

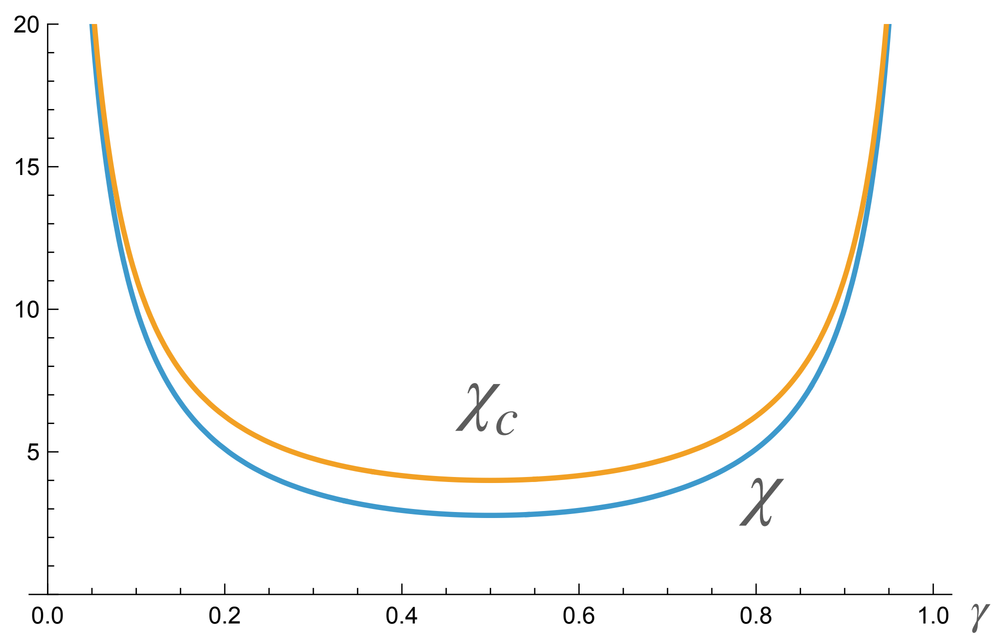
Similarly, one can expand for $k' \gg k$ and obtain poles at $1, 2, 3, \ldots$

Anti-collinear pole $\frac{1}{1-\gamma}$: corresponds to strong ordering case $k' \gg k$

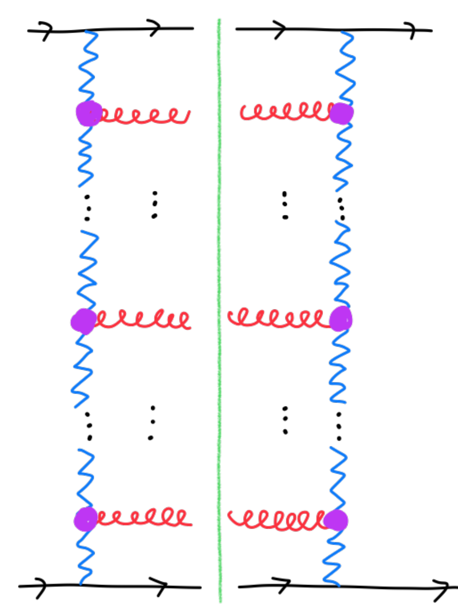
Collinear structure of BFKL

Leading collinear poles of BFKL:

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma) \longrightarrow \frac{1}{\gamma} + \frac{1}{1-\gamma} \equiv \chi_c$$



Such collinear model can be thought of as a simple model for BFKL



$$k_{1\perp}^2 \gg k_{2\perp}^2 \gg \cdots \gg k_{n+1,\perp}^2$$

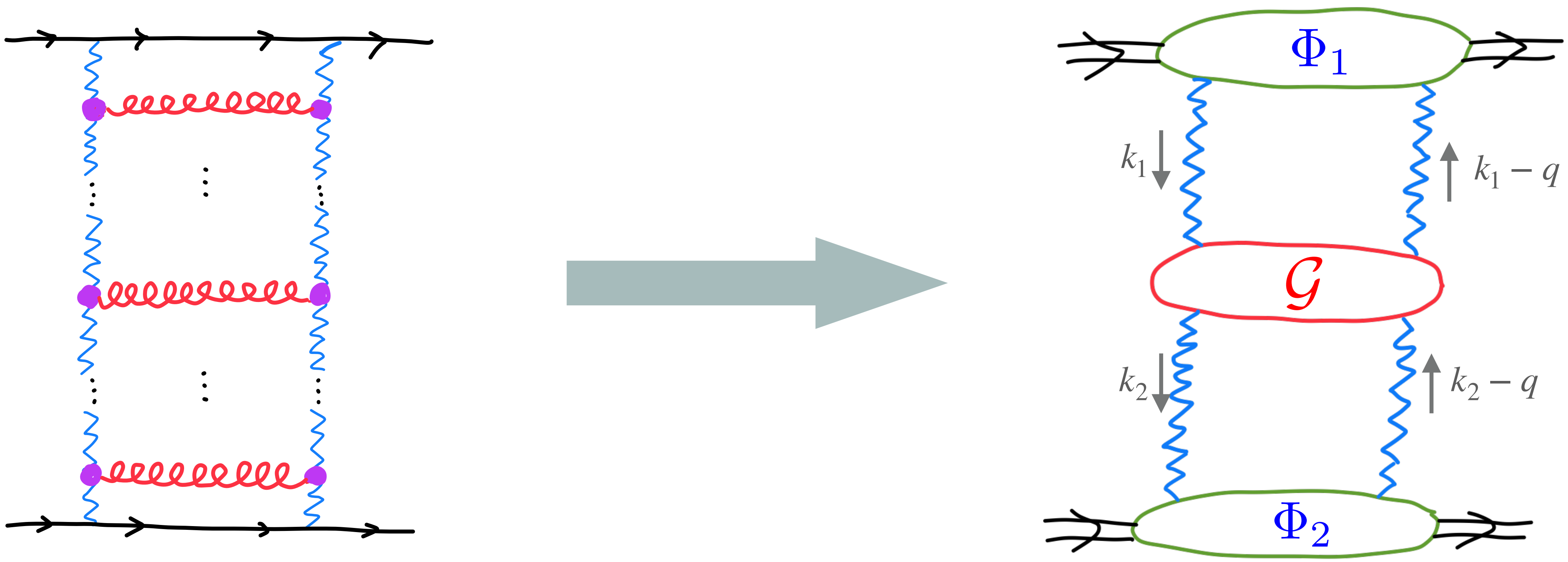
$$k_{1\perp}^2 \ll k_{2\perp}^2 \ll \cdots \ll k_{n+1,\perp}^2$$

Collinear model: combines two strong orderings in transverse momenta

BFKL in physical process

So far we considered parton-parton scattering. How to apply BFKL in the physical processes ?

quark



$$\mathcal{A}(s,t) = \int \frac{d^2\mathbf{k}_1}{(2\pi)^2}\frac{d^2\mathbf{k}_2}{(2\pi)^2}\frac{\Phi_1(\mathbf{k}_1,\mathbf{q})\,\Phi_2(\mathbf{k}_2,\mathbf{q})}{\mathbf{k}_2^2(\mathbf{k}_1-\mathbf{q})^2}\,\mathcal{G}(s,\mathbf{k}_1,\mathbf{k}_2,\mathbf{q})$$

Impact factors Φ_1, Φ_2 : process dependent

The energy dependence should come from the BFKL Pomeron

High energy factorization (k_T factorization): BFKL universal at high energy

BFKL in DIS

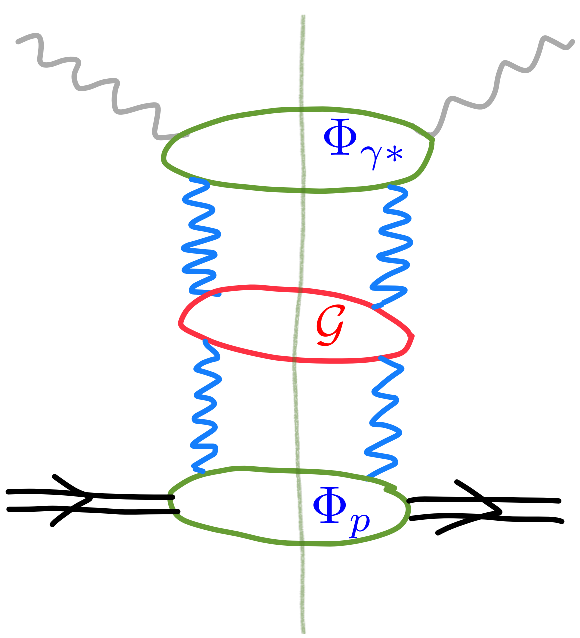
Inclusive DIS cross section for $lp \to lX$ (l charged lepton, $Q^2 \ll M_Z^2$, $s \gg M_p^2$)

$$\frac{d^2\sigma}{dx dQ^2} = \frac{2\pi\alpha_{\rm em}^2}{Q^4 x}\left[(1+(1-y)^2)F_2(x,Q^2) - y^2 F_L(x,Q^2)\right]$$

Rewrite in terms of the cross-sections for scattering transverse or longitudinal photons off the proton

$$F_2(x,Q^2) = \frac{Q^2}{4\pi^2\alpha_{\rm em}}\left[\sigma_T(x,Q^2)+\sigma_L(x,Q^2)\right] \qquad F_L(x,Q^2) = \frac{Q^2}{4\pi^2\alpha_{\rm em}}\,\sigma_L(x,Q^2)$$

$$\sigma_\lambda(x,Q^2) = \int \frac{d^2\mathbf{k}_1}{(2\pi)^2}\frac{d^2\mathbf{k}_2}{(2\pi)^2}\frac{\Phi^\lambda_{\gamma^*}(\mathbf{k}_1,Q)\,\Phi_p(\mathbf{k}_2)}{\mathbf{k}_1^2\mathbf{k}_2^2}\,\mathcal{G}(x,\mathbf{k}_1,\mathbf{k}_2)$$



$\Phi^\lambda_{\gamma^*}(\mathbf{k},Q)$ virtual photon-gluon impact factor polarization λ. Calculable in pQCD

$\Phi_p(\mathbf{k})$ proton impact factor, non-perturbative

BFKL in DIS: k_T factorization

Introduce unintegrated gluon density:

$$f(x,\mathbf{k}) = \int \frac{d^2\mathbf{k}_2}{(2\pi)^3} \frac{\Phi_p(\mathbf{k}_2)}{\mathbf{k}_2^2} \mathbf{k}^2 \, \mathcal{G}(x,\mathbf{k},\mathbf{k}_2)$$

kT factorization formula

Catani, Ciafaloni, Hautmann
Collins, Ellis

$$\sigma_\lambda(x,Q^2) = \int \frac{d^2\mathbf{k}_1}{(2\pi)\mathbf{k}_1^4} \Phi_\lambda^{\gamma^*}(\mathbf{k}_1,Q) \, f(x,\mathbf{k}_1)$$

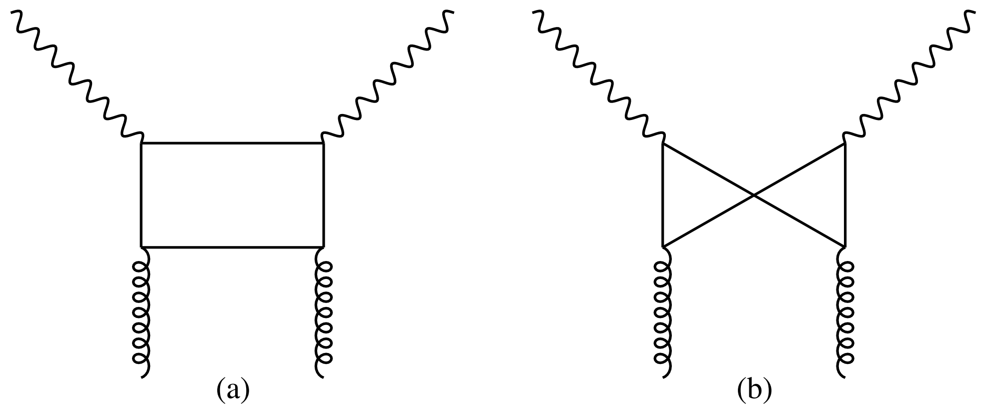
f(x,k) contains the non-perturbative information on the proton but satisfies the BFKL evolution equation.

For structure functions can perform angular integration and one obtains the form for BFKL for f
Initial condition needs to be parametrized

$$\frac{\partial}{\partial \ln 1/x} f(x,k^2) = \bar{\alpha}_S k^2 \int \frac{dk'^2}{k'^2} \left\{ \frac{f(x,k'^2)-f(x,k^2)}{|k'^2-k^2|} + \frac{f(x,k^2)}{[4k'^4+k^4]^{\frac{1}{2}}} \right\}$$

$$\Phi^\lambda_{\gamma^*}(\mathbf{k},Q)$$

virtual photon-gluon impact factor polarization λ. Calculable in pQCD



Diagrams contributing to the lowest order photon-gluon impact factor

Relation between collinear and small x evolutions

BFKL evolution equation for unintegrated gluon density (written as integral equation)

$$f(x,k^2) = f^{(0)} + \overline{\alpha}_S k^2 \int_x^1 \frac{dz}{z} \int \frac{dk'^2}{k'^2} \left\{ \frac{f(x/z,k'^2) - f(x/z,k^2)}{|k'^2 - k^2|} + \frac{f(x/z,k^2)}{[4k'^4 + k^4]^{\frac{1}{2}}} \right\}$$

DGLAP evolution equation for collinear gluon density

$$\frac{\partial g(x,Q^2)}{\partial \ln Q^2} = \int_x^1 \frac{dz}{z} P_{gg}(z)\, g(x/z,Q^2)$$

$$P_{gg}(z) = \frac{\alpha_S}{2\pi} P_{gg}^{(0)}(z) + \dots$$

Take the collinear limit of the BFKL, that is $k \gg k'$

$$f(x,k^2) = f^{(0)} + \overline{\alpha}_S k^2 \int_x^1 \frac{dz}{z} \int^{k^2} \frac{dk'^2}{k'^2} \frac{1}{k^2} f(x/z,k'^2) = f^{(0)} + \frac{\alpha_S N_c}{\pi} \int_x^1 \frac{dz}{z} \int^{k^2} \frac{dk'^2}{k'^2} f(\frac{x}{z},k'^2)$$

Take the small x limit of the DGLAP, that is

$$P_{gg}^{(0)}(z) = 2C_A \Big[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) + \delta(1-z) \frac{11C_A - 4n_f T_R}{6} \Big] \to \frac{2C_A}{z}$$

$$\frac{\partial xg(x,Q^2)}{\partial \ln Q^2} = \frac{N_c \alpha_S}{\pi} \int_x^1 \frac{dz}{z} \frac{x}{z} g(x/z,Q^2)$$

DLLA: double leading logarithmic approximation

$$\alpha_s^n \ln^n Q^2/Q_0^2 \ln^n 1/x$$

Equations are the same if we identify

$$xg(x,Q^2) = \int^{Q^2} \frac{dk^2}{k^2} f(x,k^2)$$

$$Q^2 \to \infty, \quad x \to 0$$

Relation between collinear and small x approaches

In order to inspect the relation between collinear and small x can use **Mellin** space

Recall factorization formula

$$\sigma_\lambda(x,Q^2) = \int \frac{d^2\mathbf{k}}{(2\pi)\mathbf{k}^4}\, \Phi_\lambda^{\gamma^*}(\mathbf{k},Q)\, f(x,\mathbf{k})$$

$$F_2(x,Q^2) = \frac{Q^2}{4\pi^2\alpha_{\rm em}}[\sigma_T(x,Q^2)+\sigma_L(x,Q^2)]$$

$$F_L(x,Q^2) = \frac{Q^2}{4\pi^2\alpha_{\rm em}}\sigma_L(x,Q^2)$$

To be precise, the k_T factorization formula derived is (written for structure functions)

Askew, Kwiecinski, Martin, Sutton

$$F_\lambda(x,Q^2) = \int \frac{dk^2}{k^2}\int_x^1 \frac{dz}{z}\hat{F}_\lambda^{\gamma^*}(z,k^2,Q^2)\, f(x/z,k^2)$$

where impact factor is

$$\int_x^1 \frac{dz}{z}\hat{F}_\lambda^{\gamma^*}(z,k^2,Q^2) \leftrightarrow \Phi^\lambda_{\gamma^*}$$

The difference between the two formulae is in the argument of unintegrated gluon density f

In the high energy limit the difference is subleading

But, the second result is more accurate, as it takes into account exact kinematics in the diagrams

Mellin transform:

$$\overline{f}(\omega,k^2) \equiv \int_0^1 \frac{dx}{x}\, x^\omega f(x,k^2),$$

$$\overline{F}_\lambda(\omega,Q^2) \equiv \int_0^1 \frac{dx}{x}\, x^\omega F_\lambda(x,Q^2)$$

$$\overline{\hat{F}}_\lambda(\omega,k^2,Q^2) \equiv \int_0^1 \frac{dz}{z}\, z^\omega \hat{F}_\lambda(z,k^2,Q^2)$$

Allows to recast the factorization formula

$$\overline{F}_\lambda(\omega,Q^2) = \int \frac{dk^2}{k^2}\overline{\hat{F}}_\lambda^{\gamma^*}(\omega,k^2,Q^2)\overline{f}(\omega,k^2)$$

Relation between collinear and small x approaches

Take again Mellin moments in $k \to \gamma$:

$$\tilde{\hat{F}}^{\gamma^*}_\lambda(\omega,\gamma) = \int d\tau\, \tau^{-\gamma-1}\, \overline{\hat{F}}^{\gamma^*}_\lambda(\omega,\tau)$$

where $\tau = Q^2/k^2$

(assuming fixed coupling)

$$\tilde{f}(\omega,\gamma) = \int dk^2\, (k^2)^{-\gamma-1}\, \overline{f}(\omega,k^2)$$

Then the factorization formula becomes

$$\overline{F}_\lambda(\omega,Q^2) = \int \frac{dk^2}{k^2} \overline{\hat{F}}^{\gamma^*}_\lambda(\omega,k^2,Q^2)\overline{f}(\omega,k^2) \longrightarrow \overline{F}_\lambda(\omega,Q^2) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\gamma\, \tilde{\hat{F}}^{\gamma^*}_\lambda(\omega,\gamma)\,\tilde{f}(\omega,\gamma)(Q^2)^\gamma$$

The unintegrated gluon density satisfies BFKL equation with the solution

$$\tilde{f}(\omega,\gamma) = \frac{\tilde{f}^0(\omega,\gamma)}{1-(\overline{\alpha}_S/\omega)\chi(\gamma)} \qquad \chi(\gamma) = 2\psi(1) - \psi(1-\gamma) - \psi(\gamma)$$

Large Q^2 behavior of structure function $\overline{F}_\lambda(\omega,Q^2)$ is controlled by the pole which lies to the left and is closest to the contour of integration

There are two sources of poles:

From the solution for $\tilde{f}(\omega,\gamma)$ when $1-(\overline{\alpha}_S/\omega)\chi(\gamma) = 0$

From $\tilde{\hat{F}}^{\gamma^*}_\lambda(\omega,\gamma)$

Relation between collinear and small x approaches

$$\overline{F}_\lambda(\omega,Q^2)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}d\gamma\,\hat{\tilde{F}}_\lambda^{\gamma^*}(\omega,\gamma)\,\tilde{f}(\omega,\gamma)(Q^2)^\gamma$$

Let's inspect the poles from $\tilde{f}$. The BFKL kernel eigenvalue can be shown to have an expansion

$$\chi(\gamma)=2\psi(1)-\psi(1-\gamma)-\psi(\gamma)=\frac{1}{\gamma}\left[1+\sum_{n=1}^{\infty}2\zeta(2n+1)\gamma^{2n+1}\right]=\frac{1}{\gamma}\left[1+\sum_{n=1}^{\infty}c_n\gamma^{2n+1}\right]$$

$$\frac{1}{1-(\overline{\alpha}_S/\omega)\chi(\gamma)}=\frac{\gamma R}{\gamma-\bar{\gamma}}$$ where $$R=\left(1-\frac{\overline{\alpha}_S}{\omega}\left.\frac{d(\gamma\chi)}{d\gamma}\right|_{\overline{\gamma}}\right)^{-1}$$

and solution has expansion $$\overline{\gamma}=\sum_{k=1}^{\infty}a_k\left(\frac{\bar{\alpha}_s}{\omega}\right)^k$$

Plugging expansions into the equation $1-(\overline{\alpha}_S/\omega)\chi(\gamma)=0$ ⟶

$$\sum_{k=1}^{\infty}a_k\left(\frac{\overline{\alpha}_s}{\omega}\right)^k-\frac{\overline{\alpha}_s}{\omega}-\frac{\overline{\alpha}_s}{\omega}\sum_{n=1}^{\infty}c_n\left(\sum_{k=1}^{\infty}a_k\left(\frac{\overline{\alpha}_s}{\omega}\right)^k\right)^{2n+1}=0$$

solution:

$$a_1=\frac{\bar{\alpha}_s}{\omega}\qquad a_2=a_3=0$$

$$a_4=c_1a_1^3=2\zeta(3)\qquad a_5=3c_1a_1^2a_2=0$$

$$a_6=3c_1(a_1a_2^2+a_1^2a_3)+c_2a_1^5=c_2a_1^5=2\zeta(5)$$

BFKL anomalous dimension

$$\overline{\gamma}=\frac{\overline{\alpha}_S}{\omega}+2\zeta(3)\left(\frac{\overline{\alpha}_S}{\omega}\right)^4+2\zeta(5)\left(\frac{\overline{\alpha}_S}{\omega}\right)^6+\dots$$

Jaroszewicz

Small x anomalous dimension

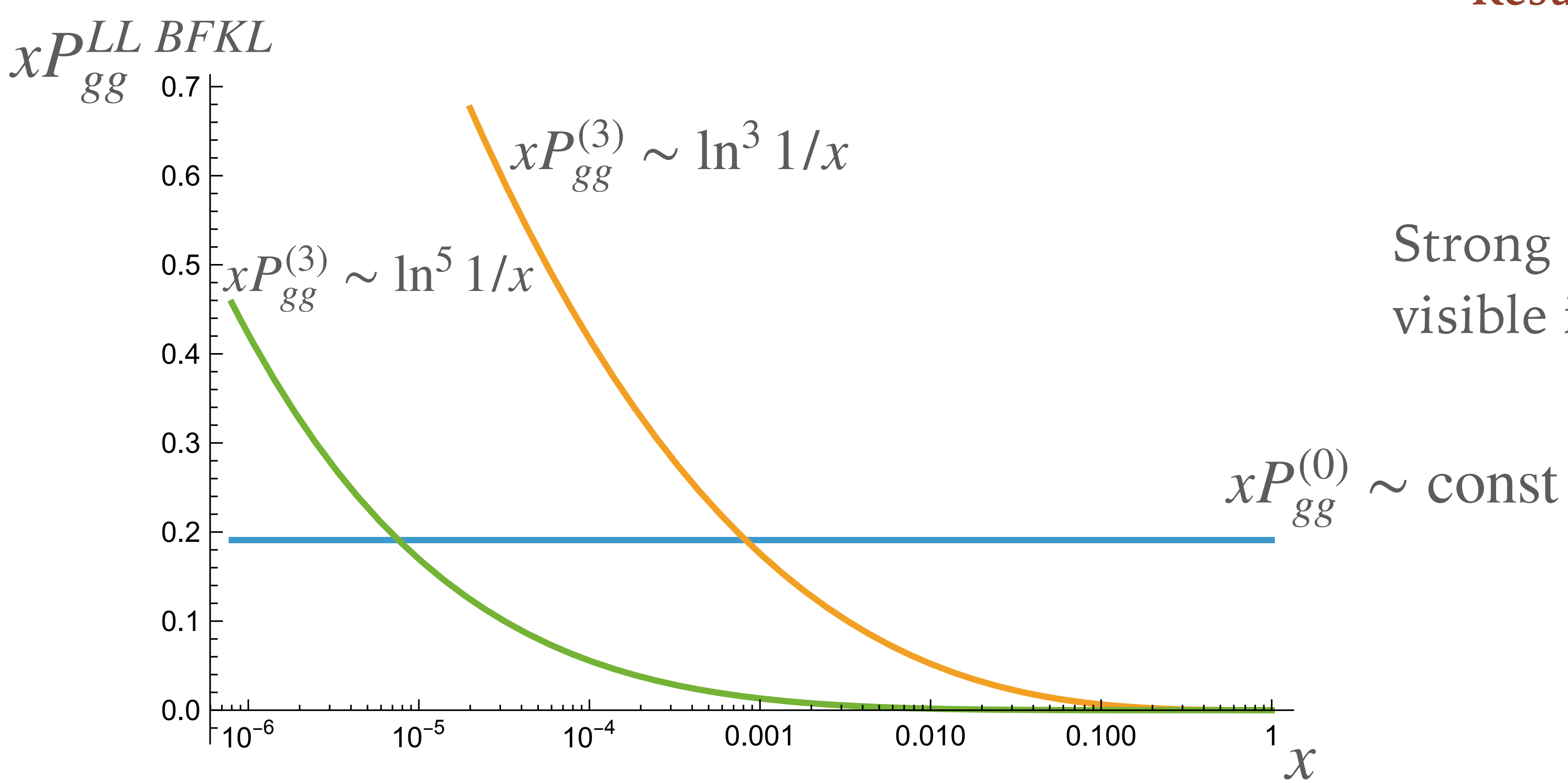
BFKL anomalous dimension $$\overline{\gamma} = \frac{\overline{\alpha}_S}{\omega} + 2\zeta(3)\left(\frac{\overline{\alpha}_S}{\omega}\right)^4 + 2\zeta(5)\left(\frac{\overline{\alpha}_S}{\omega}\right)^6 + \ldots$$

First term agrees with LO DGLAP, the next term only appear at N^{3}LO DGLAP

Inverting to get collinear splitting function (using the convention from previous lecture)

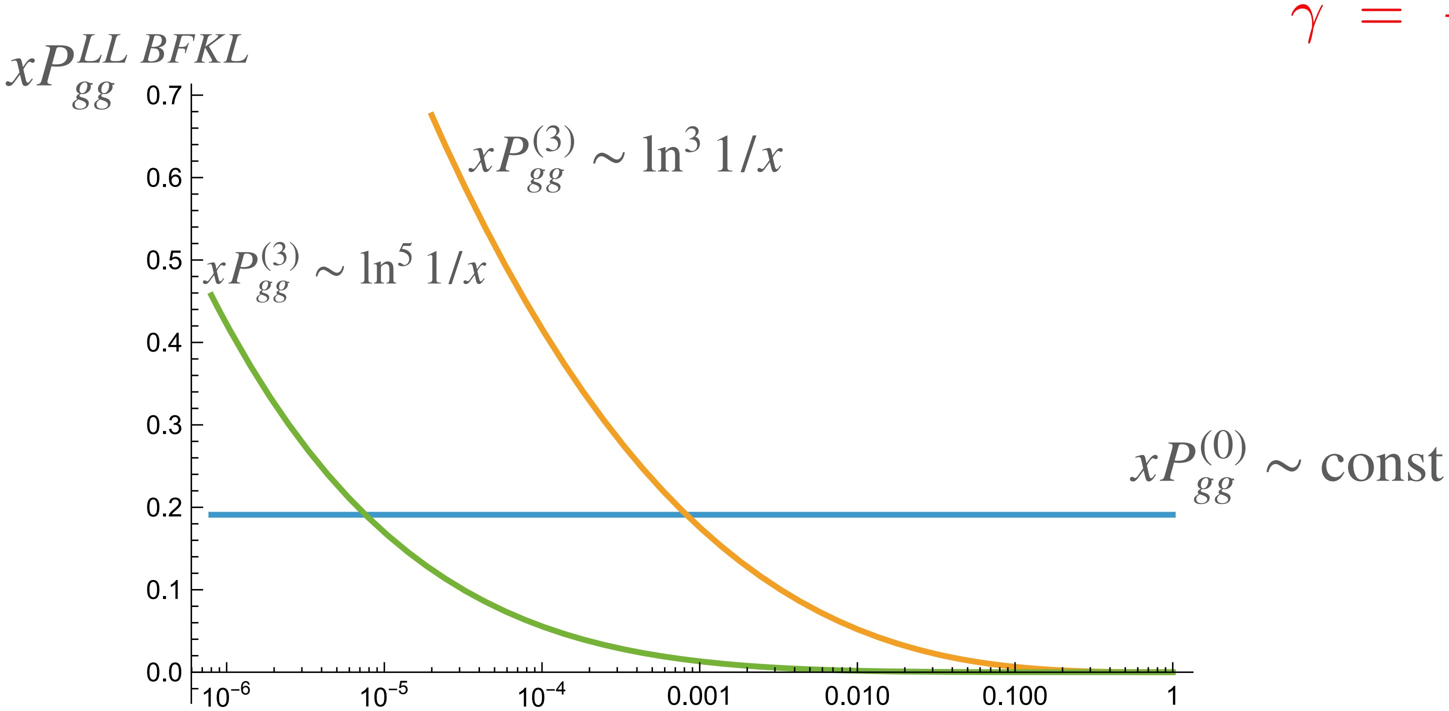
$$\frac{1}{2\pi i}\int_C d\omega x^{-\omega-1}\overline{\gamma}(\omega) = \frac{1}{2\pi i}\int_C d\omega x^{-\omega-1}\sum_{k=1}^{\infty} a_k\left(\frac{\bar{\alpha}_s}{\omega}\right)^k = \sum_{k=1}^{\infty}\frac{a_k}{(k-1)!}\frac{\bar{\alpha}_s}{x}(\bar{\alpha}_s \ln 1/x)^{k-1}$$

Resummation of large logarithms

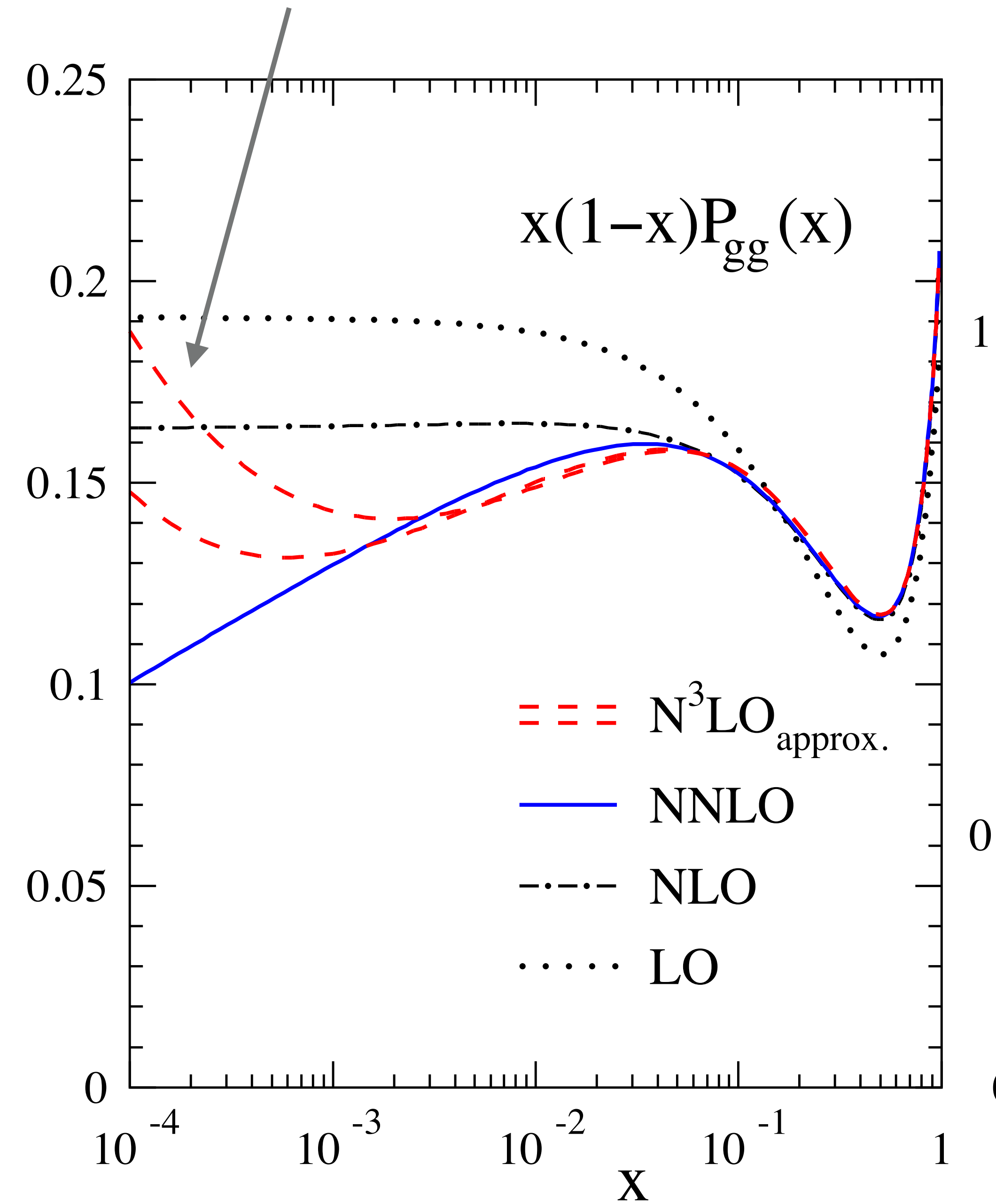


Strong growth towards small x visible in anomalous dimension

Small x anomalous dimension



$$\overline{\gamma} = \frac{\overline{\alpha}_S}{\omega} + 2\zeta(3)\left(\frac{\overline{\alpha}_S}{\omega}\right)^4 + 2\zeta(5)\left(\frac{\overline{\alpha}_S}{\omega}\right)^6 + \dots$$



Small x anomalous dimension from BFKL appear in higher order anomalous dimension in DGLAP

Relation BFKL to collinear

Inserting pole $$\frac{1}{1-(\overline{\alpha}_S/\omega)\chi(\gamma)} = \frac{\gamma R}{\gamma-\bar{\gamma}}$$ $$\overline{\gamma} = \frac{\overline{\alpha}_S}{\omega} + 2\zeta(3)\left(\frac{\overline{\alpha}_S}{\omega}\right)^4 + 2\zeta(5)\left(\frac{\overline{\alpha}_S}{\omega}\right)^6 + \ldots$$

into $$\overline{F}_\lambda(\omega,Q^2) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\gamma\, \hat{\tilde{F}}_\lambda^{\gamma^*}(\omega,\gamma)\, \tilde{f}(\omega,\gamma)(Q^2)^\gamma$$

Gives $$\overline{F}_\lambda(\omega,Q^2) = \hat{\tilde{F}}_\lambda^{\gamma^*}(\omega,\overline{\gamma})\ \overline{\gamma}R\left(\frac{\overline{\alpha}_S}{\omega}\right)\ \tilde{f}^0(\omega,\overline{\gamma})(Q^2)^{\overline{\gamma}}$$

this analysis is for fixed coupling constant

equivalent to collinear factorization

$$\overline{F}_\lambda(\omega,Q^2) = C_\lambda(\omega,\overline{\gamma})\, g(\omega,Q^2)$$

where the moment of the coefficient function

$$C_\lambda(\omega,\overline{\gamma}) = \hat{\tilde{F}}_\lambda^{\gamma^*}(\omega,\overline{\gamma})\, R\left(\frac{\overline{\alpha}_S}{\omega}\right)$$

the moment of the integrated gluon density

$$g(\omega,Q^2) = (Q_0^2)^{\overline{\gamma}}\,\overline{\gamma}\,\tilde{f}^0(\omega,\overline{\gamma})\left(\frac{Q^2}{Q_0^2}\right)^{\overline{\gamma}}$$

Recall DGLAP evolution

$$Q^2\frac{d}{dQ^2}g(x,Q^2) = \int_x^1 \frac{dz}{z}P(\alpha_s,z)g\left(\frac{x}{z},Q^2\right)$$

Recast in moment space

$$Q^2\frac{d}{dQ^2}g(\omega,Q^2) = \gamma_{gg}(\omega)g(\omega,Q^2)$$

With solution

$$g(\omega,Q^2) = g(\omega,Q_0^2)\left(\frac{Q^2}{Q_0^2}\right)^{\gamma_{gg}(\omega)}$$

with fixed strong coupling constant

Relation BFKL to collinear

What about contribution from other poles, e.g. $\gamma = 0$?

$$\overline{F}_\lambda(\omega, Q^2) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} d\gamma\, \hat{\tilde{F}}_\lambda^{\gamma^*}(\omega,\gamma)\, \tilde{f}(\omega,\gamma)(Q^2)^\gamma$$

Strictly speaking analysis presented is valid for F_L. One can show that

$$\hat{\tilde{F}}_L^{\gamma^*}(\omega,\gamma) \sim \frac{1}{\gamma}$$

But this is canceled by numerator :

$$\frac{1}{1-(\overline{\alpha}_S/\omega)\chi(\gamma)} = \frac{\gamma R}{\gamma - \bar{\gamma}}$$

So the analysis holds (in leading twist)

For F_2 one can show that:

$$\hat{\tilde{F}}_2^{\gamma^*}(\omega,\gamma) \sim \frac{1}{\gamma^2}$$

Therefore contour integration encloses simple pole

$$\hat{\tilde{F}}_\lambda^{\gamma^*}(\omega,\gamma)\, \tilde{f}(\omega,\gamma) \sim \frac{1}{\gamma}$$

Such pole will result in the term which is independent of Q^2

To extract the effects of the pole at $\gamma = \overline{\gamma}$ one can consider

$$\frac{\partial \overline{F}_2(\omega, Q^2)}{\partial \ln Q^2}$$

Therefore

$$\frac{\partial \overline{F}_2(\omega, Q^2)}{\partial \ln Q^2} = \hat{\tilde{F}}_2(\omega,\overline{\gamma}) R \overline{\gamma}^2 \tilde{f}^{(0)}(\omega,\overline{\gamma})(Q^2)^{\overline{\gamma}}$$

Relation BFKL to collinear framework

Obtained formula

$$Q^2 \frac{\partial \overline{F}_2(\omega, Q^2)}{\partial Q^2} = \hat{\hat{F}}_2(\omega, \overline{\gamma}) R \overline{\gamma}^2 \tilde{f}^{(0)}(\omega, \overline{\gamma}) (Q^2)^{\overline{\gamma}}$$

is the small x resummed version of the collinear one and it can be recast as

$$Q^2 \frac{\partial \overline{F}_2(\omega, Q^2)}{\partial Q^2} = \sum_q 2e_q^2 \, P_{qg}(\omega, \overline{\gamma}) \, g(\omega, Q^2)$$

Therefore the term $P_{qg}(\omega, \overline{\gamma})$ also contains small x resummation

$$\bar{\alpha}_s \sum_{k=1}^{\infty} b_k \left(\frac{\bar{\alpha}_s}{\omega} \right)^k$$

Comments:

- Numerically, small x resummation in P_{qg} turns out to be much more important in the HERA regime that the small x resummation in P_{gg}
- Shown analysis is for leading twist. There are non-leading twist contributions by going away from strict $Q^2 \to \infty$
- The analysis of the relation of BFKL and collinear framework in the case of running coupling can also be performed, though it is much more complicated

Kwiecinski, Martin

BFKL solution at non-zero t

In general BFKL solution depends on t, momentum transfer
The solution can be obtained in coordinate space and utilizing symmetry properties of the kernel

Lipatov

$$\begin{aligned}\omega\, G(\omega,\mathbf{k},\mathbf{k}_0,\mathbf{q}) &= \delta^{(2)}(\mathbf{k}-\mathbf{k}_0)\\ &+\frac{\bar{\alpha}_s}{2\pi}\int d^2\mathbf{k}'\left[\frac{-\mathbf{q}^2}{\mathbf{k}^2(\mathbf{k}'-\mathbf{q})^2}G(\omega,\mathbf{k}',\mathbf{k}_0,\mathbf{q})\right.\\ &+\frac{1}{(\mathbf{k}'-\mathbf{k})^2}\left(G(\omega,\mathbf{k}',\mathbf{k}_0,\mathbf{q})-\mathbf{k}^2\frac{G(\omega,\mathbf{k},\mathbf{k}_0,\mathbf{q})}{\mathbf{k}'^2+(\mathbf{k}'-\mathbf{k})^2}\right)\\ &\left.+\frac{1}{(\mathbf{k}'-\mathbf{k})^2}\left(\frac{(\mathbf{k}-\mathbf{q})^2\mathbf{k}'^2G(\omega,\mathbf{k}',\mathbf{k}_0,\mathbf{q})}{(\mathbf{k}'-\mathbf{q})^2\mathbf{k}^2}-(\mathbf{k}-\mathbf{q})^2\frac{G(\omega,\mathbf{k},\mathbf{k}_0,\mathbf{q})}{(\mathbf{k}'-\mathbf{q})^2+(\mathbf{k}'-\mathbf{k})^2}\right)\right]\end{aligned}$$

Introduce transverse coordinates conjugate to transverse momenta:

$$\mathbf{r}_1,\mathbf{r}'_1,\mathbf{r}_2,\mathbf{r}'_2 \longrightarrow \mathbf{k},\mathbf{k}',\mathbf{q}-\mathbf{k},\mathbf{q}-\mathbf{k}'$$

BFKL Green's function in coordinate space is defined as:

$$F(\omega,\mathbf{r}_1,\mathbf{r}'_1,\mathbf{r}_2,\mathbf{r}'_2)=\int d^2\mathbf{k}d^2\mathbf{k}'d^2\mathbf{q}\,\exp[i(\mathbf{k}\cdot\mathbf{r}_1+(\mathbf{q}-\mathbf{k})\cdot\mathbf{r}_2-\mathbf{k}'\cdot\mathbf{r}'_1-(\mathbf{q}-\mathbf{k}')\cdot\mathbf{r}'_2)]\\ \times\frac{G(\omega,\mathbf{k},\mathbf{k}',\mathbf{q})}{\mathbf{k}'^2(\mathbf{k}-\mathbf{q})^2}$$

BFKL solution at non-zero t

BFKL equation in coordinate space reads then

$$\omega\nabla_1^2\nabla_2^2 F(\omega,\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}'_1,\mathbf{r}'_2) = (2\pi)^4\delta^2(\mathbf{r}_1-\mathbf{r}'_1)\delta^2(\mathbf{r}_2-\mathbf{r}'_2)$$
$$+\frac{\bar{\alpha}_s}{2\pi}\left\{(2\pi)^2\delta^2(\mathbf{r}_{12})(\nabla_1+\nabla_2)^2\,F(\omega,\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}'_1,\mathbf{r}'_2)\right.$$
$$+\nabla_1^2\int\frac{d^2\mathbf{r}_0}{\mathbf{r}_{10}^2}\left[\nabla_2^2\,F(\omega,\mathbf{r}_0,\mathbf{r}_2,\mathbf{r}'_1,\mathbf{r}'_2)-\frac{\mathbf{r}_{12}^2}{\mathbf{r}_{10}^2+\mathbf{r}_{20}^2}\,\nabla_2^2F(\omega,\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}'_1,\mathbf{r}'_2)\right]$$
$$+\nabla_2^2\int\frac{d^2\mathbf{r}_0}{\mathbf{r}_{20}^2}\left[\nabla_1^2\,F(\omega,\mathbf{r}_1,\mathbf{r}_0,\mathbf{r}'_1,\mathbf{r}'_2)-\frac{\mathbf{r}_{12}^2}{\mathbf{r}_{10}^2+\mathbf{r}_{20}^2}\,\nabla_1^2F(\omega,\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}'_1,\mathbf{r}'_2)\right]$$

$\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$

∇_i^2 2-d laplacian

The solution can be constructed by using the complete set of eigenfunctions which have the following form

$$\phi_{n,\nu}(\rho_{10},\rho_{20}) = \left(\frac{\rho_{12}}{\rho_{10}\rho_{20}}\right)^h\left(\frac{\rho_{12}^*}{\rho_{10}^*\rho_{20}^*}\right)^{\bar{h}}$$

$$\mathcal{K}\otimes\phi_{n,\nu} = \omega(n,\nu)\,\phi_{n,\nu}$$

conformal weights

$$h = \frac{1+n}{2} - i\nu$$
$$\bar{h} = \frac{1-n}{2} - i\nu$$

and complex variables have been introduced to represent the 2-D coordinates

$$\rho'_k = x'_k + i\,y'_k\,,\quad k=1,2$$
$$\rho_k = x_k + i\,y_k\,,\quad k=1,2$$

BFKL solution at non-zero t

The eigenfunctions satisfy the completeness relation

$$\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\nu \int d^2\mathbf{r}_0\, 16(\nu^2+\frac{n^2}{4})\, \frac{\phi_{n,\nu}(\mathbf{r}_{10},\mathbf{r}_{20})\phi^*_{n,\nu}(\mathbf{r}_{1'0},\mathbf{r}_{2'0})}{\mathbf{r}_{12}^2\mathbf{r}_{1'2'}^2} \;=\; (2\pi)^4\,\delta^2(\mathbf{r}_{11'})\delta^2(\mathbf{r}_{22'})$$

And the solution is expressed as

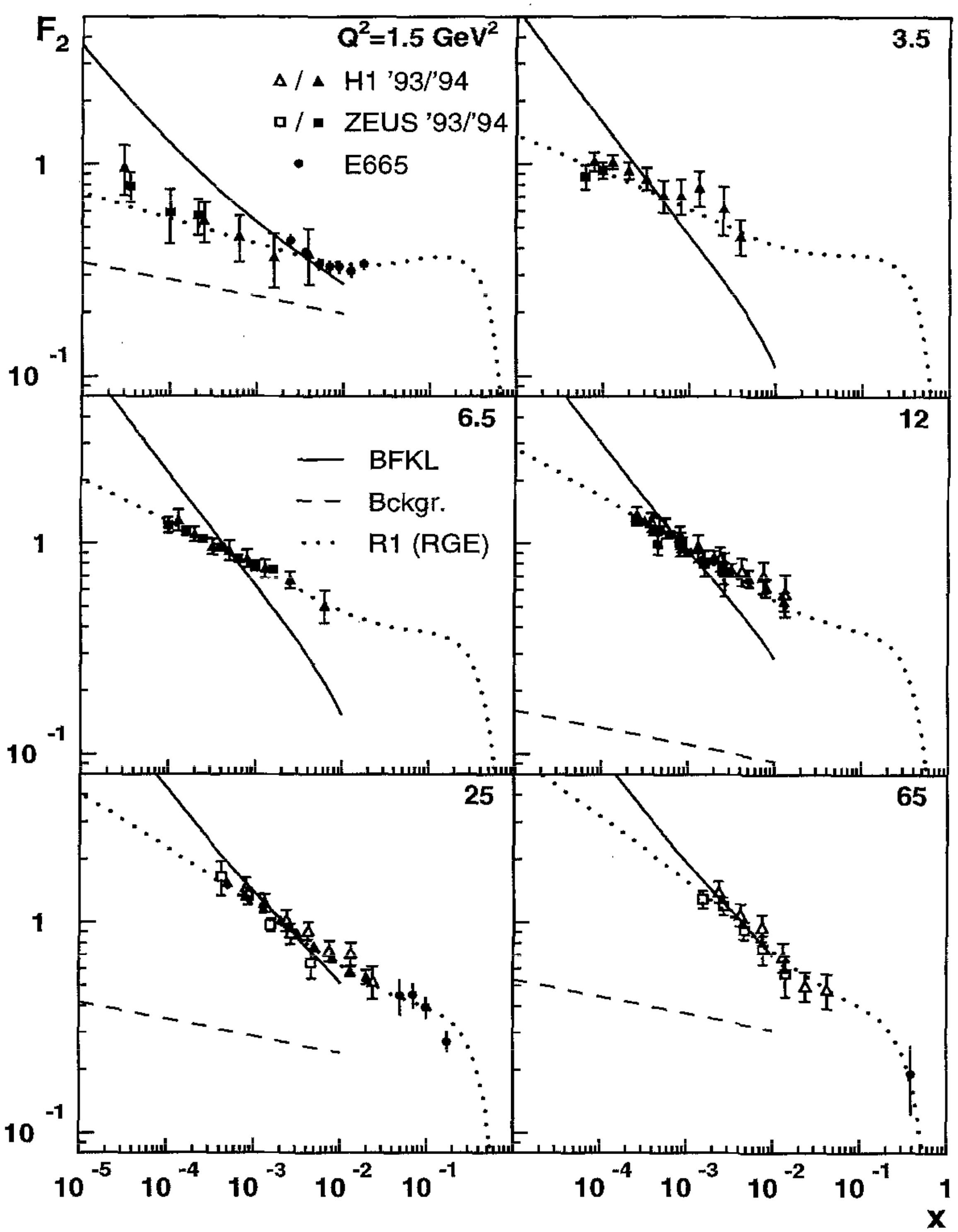
$$F(\omega,\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}'_1,\mathbf{r}'_2) = \sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\nu \int d^2\mathbf{r}_0\, \frac{(\nu^2+\frac{n^2}{4})}{[\nu^2+((n-1)/2)^2][\nu^2+((n+1)/2)^2]} \times \frac{\phi_{n,\nu}(\mathbf{r}_{10},\mathbf{r}_{20})\phi^*_{n,\nu}(\mathbf{r}_{1'0},\mathbf{r}_{2'0})}{\omega-\bar{\alpha}_s\omega(n,\nu)}$$

With the same eigenvalue as before

$$\chi(\gamma=\frac{1}{2}+i\nu,n)\equiv\omega(\nu,n)=2\psi(1)-\psi(\frac{1}{2}+i\nu+\frac{|n|}{2})-\psi(\frac{1}{2}-i\nu+\frac{|n|}{2})$$

Example: BFKL at LL and HERA data

Bojak, Ernst



see also: Ball, Forte

- Rise with energy (or decreasing Bjorken x) too steep for the phenomenology
- Cannot describe HERA data with LL BFKL

HERA: $F_2 \sim x^{-(0.2\sim0.3)}$

BFKL: $F_2 \sim x^{-0.5}$

- Need **higher order**, next-to-leading logarithmic, **NLL** terms.
- Powerlike growth eventually violates unitarity bounds for amplitudes. Will need to cinsider also other class of corrections: **saturation**.

Example: application of LL BFKL to $\gamma^*\gamma^*$ scattering

CERN-EP-2001-064
31 August 2001

Measurement of the Hadronic Cross-Section for the Scattering of Two Virtual Photons at LEP

The OPAL Collaboration

Abstract

The interaction of virtual photons is investigated using the reaction $e^+e^- \to e^+e^-$ hadrons based on data taken by the OPAL experiment at e^+e^- centre-of-mass energies $\sqrt{s_{ee}} = 189-209$ GeV, for $W > 5$ GeV and at an average Q^2 of 17.9 GeV^2. The measured cross-sections are compared to predictions of the Quark Parton Model (QPM), to the Leading Order QCD Monte Carlo model PHOJET to the NLO prediction for the reaction $e^+e^- \to e^+e^- q\bar{q}$, and to BFKL calculations. PHOJET, NLO $e^+e^- \to e^+e^- q\bar{q}$, and QPM describe the data reasonably well, whereas the cross-section predicted by a Leading Order BFKL calculation is too large.

CERN-EP/2001-075
October 31, 2001

Double-Tag Events in Two-Photon Collisions at LEP

The L3 Collaboration

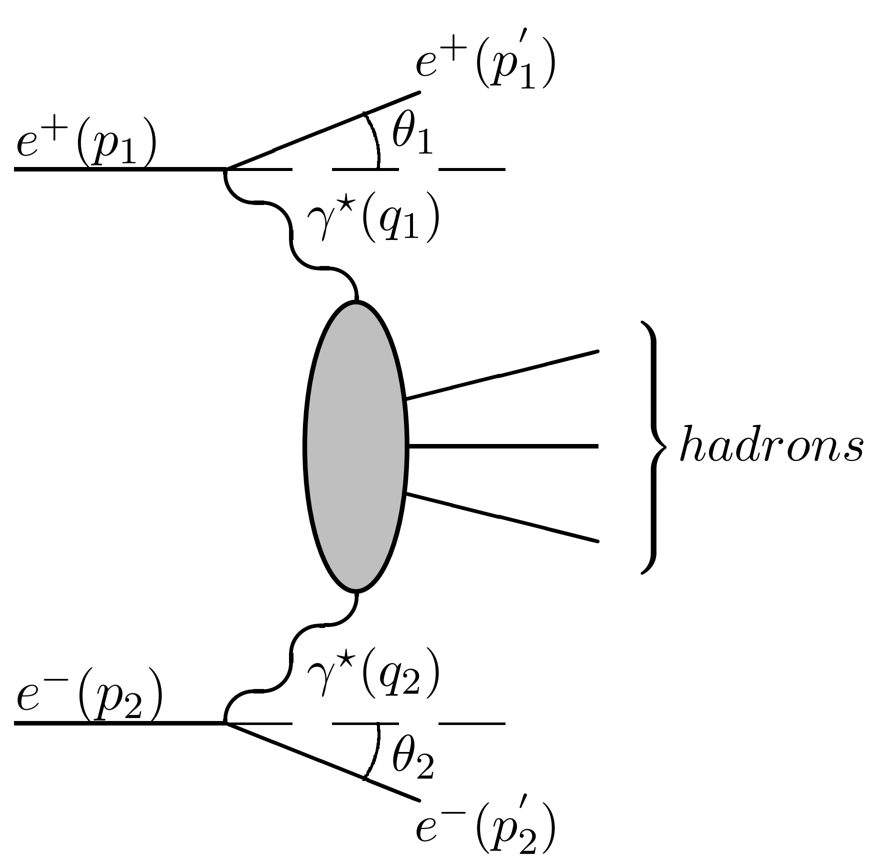
Abstract

Double-tag events in two-photon collisions are studied using the L3 detector at LEP centre-of-mass energies from $\sqrt{s} = 189$ GeV to 209 GeV. The cross sections of the $e^+e^- \to e^+e^-$*hadrons* and $\gamma^*\gamma^* \to$ *hadrons* processes are measured as a function of the product of the photon virtualities, $Q^2 = \sqrt{Q_1^2 Q_2^2}$, of the two-photon mass, $W_{\gamma\gamma}$, and of the variable $Y = \ln(W_{\gamma\gamma}^2/Q^2)$. The average photon virtuality is $\langle Q_1^2 \rangle = \langle Q_2^2 \rangle = 16$ GeV^2. The results are in agreement with next-to-leading order calculations for the process $\gamma^*\gamma^* \to q\bar{q}$ in the interval $2 \leq Y \leq 5$. An excess is observed in the interval $5 < Y \leq 7$, corresponding to $W_{\gamma\gamma}$ greater than 40 GeV. This may be interpreted as a contribution of resolved photon QCD processes or the onset of BFKL phenomena.

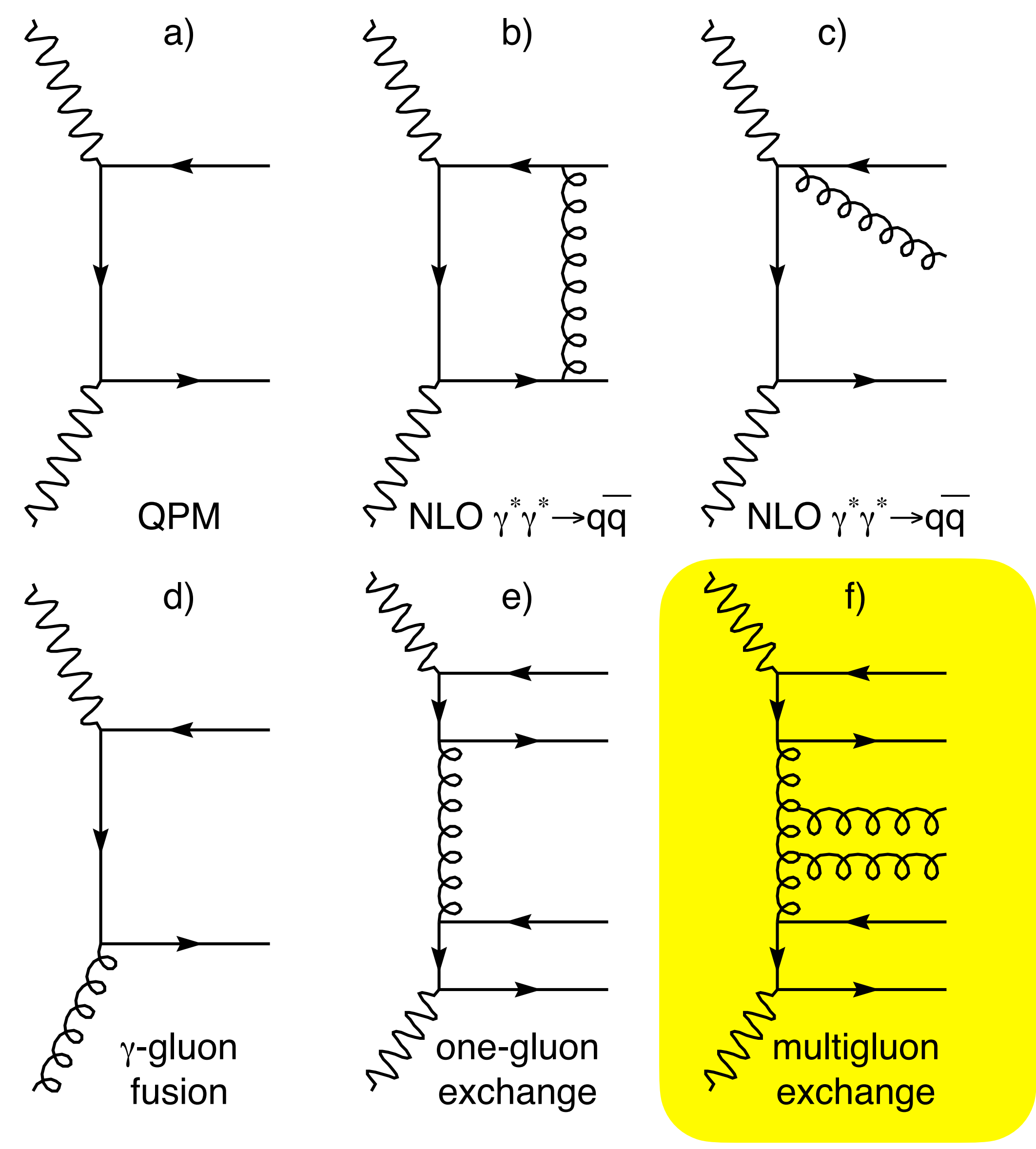
Example: application of LL BFKL to γ*γ* scattering

Process: $e^+e^- \longrightarrow e^+e^- + \text{hadrons}$

doubly-tagged events

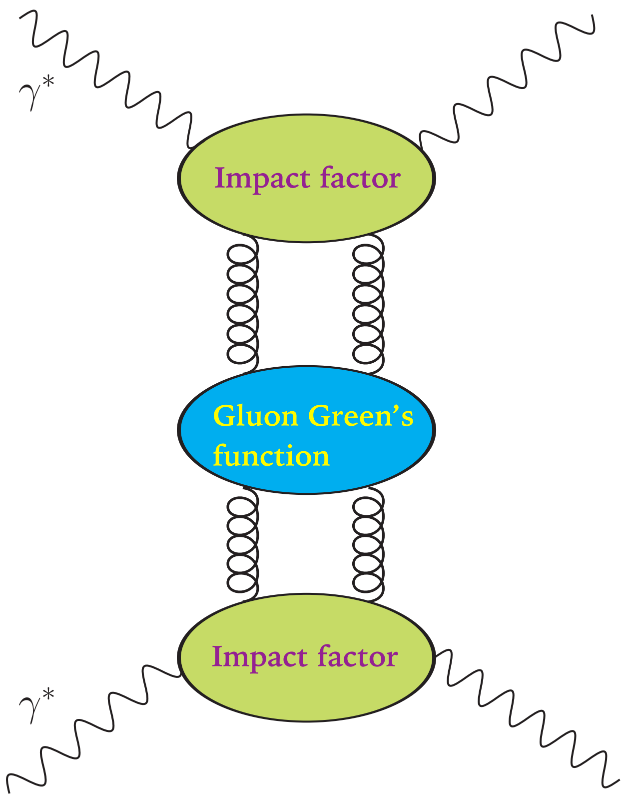


Sample diagrams which contribute



Dominant at large energy (rapidity)

Example: application of LL BFKL to γ*γ* scattering



High-energy factorization

Two virtual photons provide large perturbative scales: Q_1^2, Q_2^2

rapidity $Y = \ln \frac{W_{\gamma\gamma}^2}{\sqrt{Q_1^2 Q_2^2}}$

$s = W_{\gamma\gamma}^2$

$$\sigma^{(jk)}(s, Q_1, Q_2) = \frac{1}{2\pi Q_1 Q_2} \int \frac{d\omega}{2\pi i} \left(\frac{s}{s_0}\right)^{\omega} \int \frac{d\gamma}{2\pi i} \left(\frac{Q_1^2}{Q_2^2}\right)^{\gamma - \frac{1}{2}} \phi^{(j)}(\gamma)\, G(\omega,\gamma)\, \phi^{(k)}(1-\gamma)$$

$Q_1^2 = -q_1^2, Q_2^2 = -q_2^2$ are negative photon virtualities

$s = (q_1 + q_2)^2$ for the $\gamma^*\gamma^*$ process

j, k photon polarizations

$\phi^{(j,k)}$ impact factors: known up to NLO

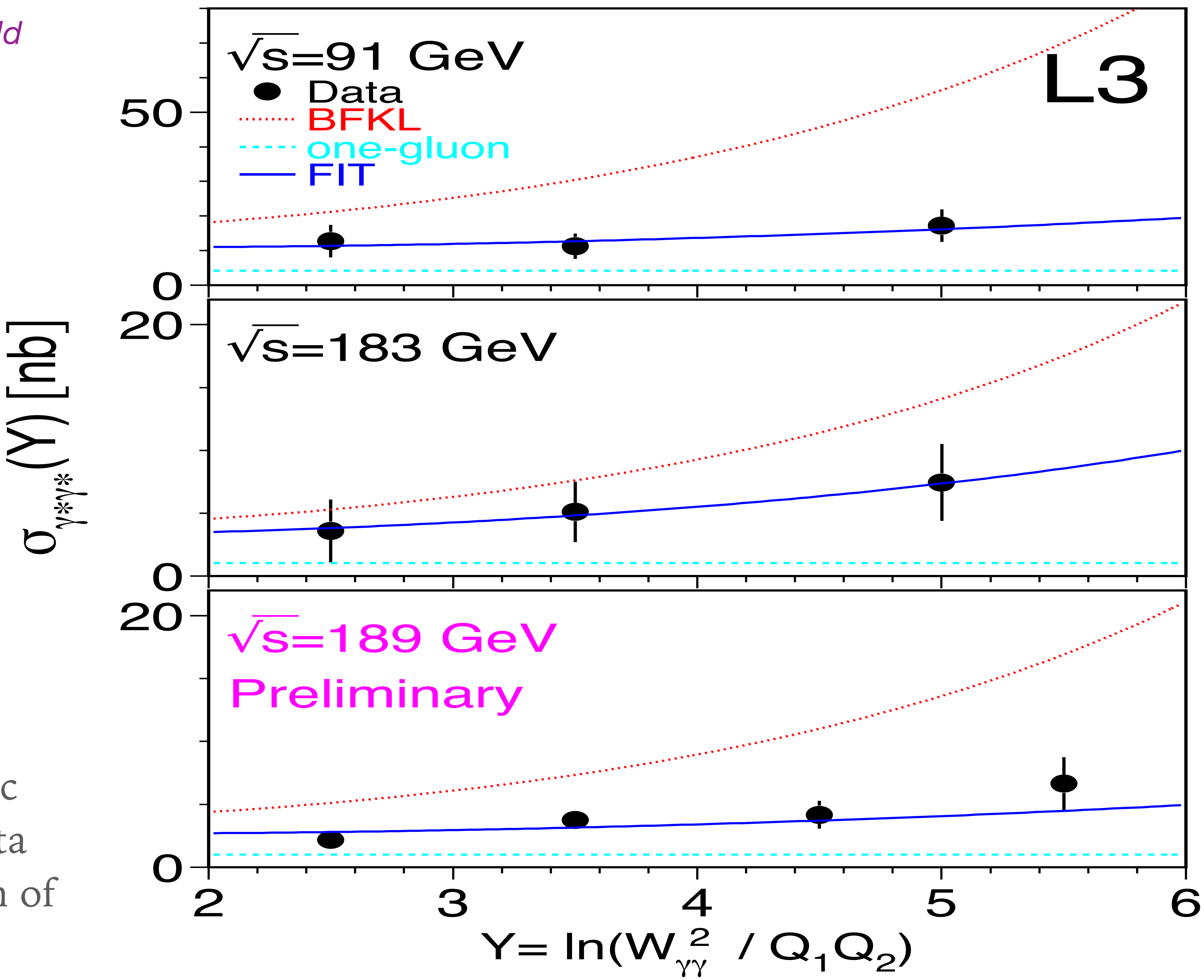
Balitsky, Chirilli

$G(\omega, \gamma)$ BFKL gluon Green's function

$$G(\omega,\gamma) = \frac{1}{\omega - \bar{\alpha}_s \chi(\gamma)}$$

Example: application of LL BFKL to γ*γ* scattering

Donnachie,Soldner-Rembold



BFKL at leading logarithmic order overestimates the data and gives too steep growth of the cross section