Nucleon spin, form factor, GPD

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Plan

• Proton spin	Lecture 1,2
• Form factors	Lecture 3
• Generalized parton distribution	Lecture 4

Notations

Metric $g^{\mu\nu} = (+1, -1, -1, -1)$ $\mu, \nu = 0, 1, 2, 3$ $i, j = 1, 2$ (transverse)

Light-cone coordinates $P^{\pm} = \frac{1}{\sqrt{2}}(P^0 \pm P^3)$ $g^{+-} = 1$ $P^+ = P_-$

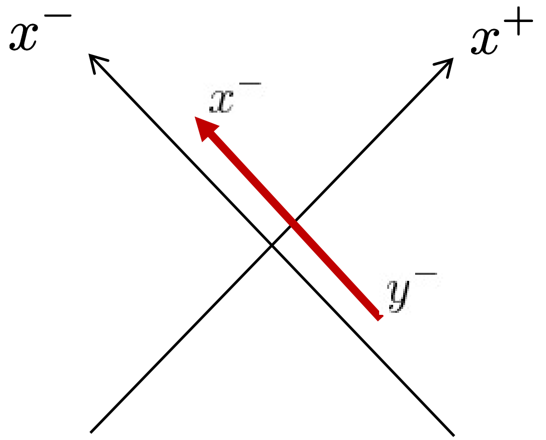
$$P \cdot x = P^+ x^- + P^- x^+ - P_\perp^i x_\perp^i$$

γ_5 , antisymmetric tensor (same as in Peskin) $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ $\epsilon^{0123} = +1$ $\epsilon^{12} = \epsilon_{12} = +1$

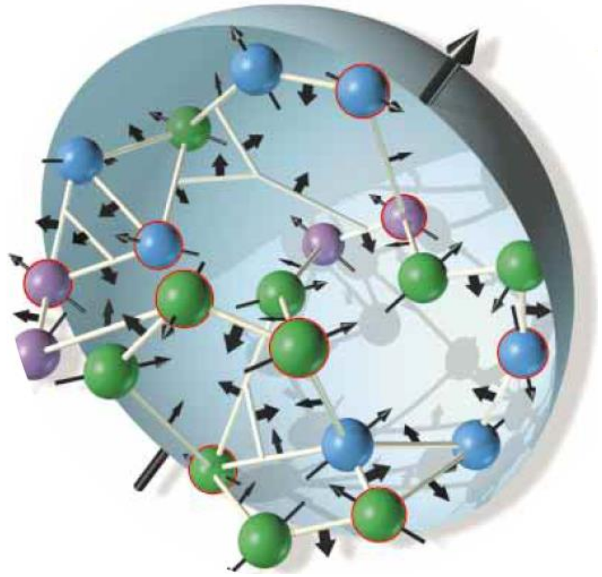
Coupling constant $D^\mu = \partial^\mu + igA^\mu$

Wilson line

$$W[x^-, y^-, x_\perp] = P\exp\left(-ig\int_{y^-}^{x^-} dz^- A^+(z^-, x_\perp)\right)$$



The proton spin problem



The proton has spin ½.

The proton is not an elementary particle.

$$\Rightarrow \quad \frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L^q + L^g$$

Quarks' helicity

Gluons' helicity

Orbital angular Momentum (OAM)

QCD angular momentum tensor

QCD Lagrangian → Lorentz invariant $x^\mu \to x^\mu + \omega^{\mu\nu} x_\nu$

→ Noether current $\partial_\mu M_{can}^{\mu\nu\lambda} = 0$

QCD angular momentum tensor

$$M_{can}^{\mu\nu\lambda} = x^\nu T_{can}^{\mu\lambda} - x^\lambda T_{can}^{\mu\nu} - \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\bar{\psi}\gamma_\rho\gamma_5\psi + F^{\mu\lambda}A^\nu - F^{\mu\nu}A^\lambda$$

quark helicity gluon helicity

canonical energy momentum tensor

$$T_{can}^{\mu\nu} = \bar{\psi} i\gamma^\mu \overleftrightarrow{\partial}^\nu \psi - F^{\mu\alpha}\partial^\nu A^\alpha - g^{\mu\nu}\mathcal{L}$$

→ Quark OAM → Gluon OAM

Exercise: Derive the canonical angular momentum tensor $M^{\mu\nu\lambda}_{can}$

Hint: Under an infinitesimal Lorentz transformation

$$\delta\psi = -\omega^{\mu\nu}\left(\frac{1}{2}(x_\nu\partial_\mu - x_\mu\partial_\nu)\psi - \frac{1}{8}[\gamma_\mu,\gamma_\nu]\psi\right)$$

$$\delta A^\alpha = -w^{\mu\nu}\left(x_\nu\partial_\mu A^\alpha - \frac{1}{2}(\delta^\alpha_\mu g_{\nu\beta} - g_{\mu\beta}\delta^\alpha_\nu)A^\beta\right)$$

$$\delta\mathcal{L} = -w^{\mu\nu}x_\nu\partial_\mu\mathcal{L}$$

Problems

$T^{\mu\nu}_{can}$ is not symmetric, not gauge invariant

$T^{\mu\nu}_{can}$ is conserved wrt the first index $\partial_\mu T^{\mu\nu}_{can} = 0$ but not the second $\partial_\nu T^{\mu\nu}_{can} \neq 0$

Jaffe-Manohar decomposition

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L^q_{can} + L^g_{can}$$

$\mu\nu\lambda = +12$ component of the canonical angular momentum tensor $M^{\mu\nu\lambda}_{can}$

Operators NOT gauge invariant except the quark helicity $\Delta\Sigma \sim \bar{\psi}\gamma^+\gamma_5\psi$

$$\Delta G \sim \epsilon^{ij} F^{+i} A^j \qquad L^q_{can} \sim \bar{\psi} x \times i\partial\psi \qquad L^g_{can} \sim F^{+\alpha} x \times \partial A_\alpha$$

To be understood in the light-cone gauge $A^+ = 0$

Naïve replacement $\partial^\mu \to D^\mu$ does not make this gauge invariant.

Quark helicity: definition $$2\Delta\Sigma S^\mu = \sum_f \langle PS|\bar{\psi}_f \gamma^\mu \gamma_5 \psi_f|PS\rangle$$ ← proton single-particle state,

spin 4-vector $$2S^\mu = \bar{u}(PS)\gamma^\mu\gamma_5 u(PS)$$ ← proton Dirac spinor

Exercise: Show that

1 $$P^\mu S_\mu = 0$$

2 $$S^\mu = \left(\vec{p}\cdot\vec{s},\ m\vec{s} + \frac{\vec{p}\cdot\vec{s}}{p^0+m}\vec{p}\right)$$

3 $$S^2 = -M^2$$

4 $$u(PS)\bar{u}(PS) = \frac{\not{P}+M}{2}\left(1+\frac{\gamma_5\not{S}}{M}\right)$$

$$u(p) = \frac{\not{p}+m}{\sqrt{2(p^0+m)}}\begin{pmatrix}\chi\\ \chi\end{pmatrix} = \begin{pmatrix}\sqrt{p\cdot\sigma}\chi\\ \sqrt{p\cdot\bar{\sigma}}\chi\end{pmatrix}$$

$$\vec{s} = \chi^\dagger\vec{\sigma}\chi \text{ and } |\vec{s}| = 1$$

Hint: use the Fierz identity and

$$2(p^\mu s^\nu - p^\nu s^\mu) = -im\bar{u}(ps)\sigma^{\mu\nu}\gamma_5 u(ps)$$

In the quark model,

$$\left|P,+\frac{1}{2}\right\rangle = \frac{1}{3\sqrt{2}}\left\{|uud\rangle(2|++-\rangle - |+-+\rangle - |-++\rangle) + \cdots\right\}$$

$$\frac{1}{2} = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \quad \Rightarrow \quad \Delta\Sigma = 1$$

With relativistic effects,

$j = \frac{1}{2}$ solution of the Dirac equation in a spherical potential

$$\psi = \frac{1}{\sqrt{4\pi}}\begin{pmatrix} f(r)\chi \\ -ig(r)\hat{r}\cdot\vec{\sigma}\chi \end{pmatrix}$$

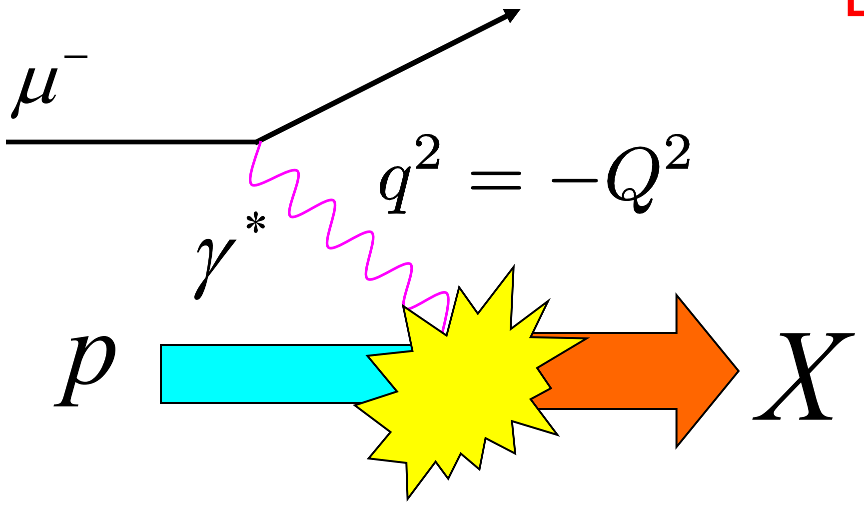
Upper component $l = 0$

Lower component $l = 1$

$$\Delta\Sigma = \int d^3r\psi^\dagger 2\Sigma^z\psi = 1 - \frac{4}{3}\int drr^2 g(r) \approx 0.7$$

The rest ~30% is quark OAM

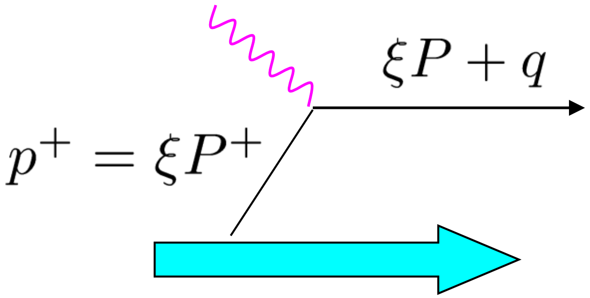
Deep inelastic scattering



Bjorken variable

$$
\begin{aligned}
x &= \frac{Q^2}{2P\cdot q} = \frac{Q^2}{(P+q)^2 + Q^2 - m_p^2} \\
&= \frac{Q^2}{Q^2 + m_X^2 - m_p^2} \\
&\sim \frac{Q^2}{s} \quad (x \ll 1)
\end{aligned}
$$

Physical meaning of x : momentum fraction carried by the struck parton



$$(\xi P + q)^2 = \xi^2 m_p^2 + 2\xi P\cdot q - Q^2 = 0$$

$$\xi \approx \frac{Q^2}{2Pq} = x$$

DIS structure functions

Unpolarized

$$\mathrm{Im}\frac{i}{2\pi}\int d^4y e^{iqy}\langle PS|T\{J^\mu(y)J^\nu(0)\}|PS\rangle\Big|_{sym}$$
$$= \left(-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)F_1(x,Q^2) + \left(P^\mu - \frac{P\cdot q}{q^2}q^\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2}q^\nu\right)\frac{F_2(x,Q^2)}{P\cdot q}$$

Polarized

$$\mathrm{Im}\frac{1}{2\pi}\int d^4y e^{iqy}\langle PS|T\{J^\mu(y)J^\nu(0)\}|PS\rangle\Big|_{asym} = -\epsilon^{\mu\nu\alpha\beta}\frac{q_\alpha}{P\cdot q}\left[S_\beta g_1 + \left(S_\beta - \frac{S\cdot q}{P\cdot q}P_\beta\right)g_2\right]$$

Longitudinal polarization → $g_1(x)$ $\quad S^\mu \approx \delta^\mu_+ S^+$

Transverse polarization → $g_1(x) + g_2(x) = g_T(x)$ $\quad S^\mu = \delta^\mu_i S^i_\perp$

Exercise

Forward Compton amplitude

$$T^{\mu\nu} = \frac{i}{2\pi}\int d^4x e^{iq\cdot x}\langle PS|\mathrm{T}\{J^\mu(x)J^\nu(0)\}|PS\rangle = T_S^{\mu\nu} + iT_A^{\mu\nu}$$

Not to be confused with real/imaginary decomposition

symmetric antisymmetric

Hadronic tensor

$$W^{\mu\nu} = \frac{1}{2\pi}\int d^4x e^{iq\cdot x}\langle PS|[J^\mu(x), J^\nu(0)]|PS\rangle = W_S^{\mu\nu} + iW_A^{\mu\nu}$$

Show that

$$2\mathrm{Im}T_S^{\mu\nu} = W_S^{\mu\nu}$$

$$2\mathrm{Im}T_A^{\mu\nu} = W_A^{\mu\nu}$$

Light-cone dominance

Want to study the correlator $$\int d^4y e^{iqy} \langle P|T\{J^\mu(y)J^\nu(0)|P\rangle$$

in the Bjorken limit $Q^2 \to \infty, \quad P\cdot q \to \infty, \quad x = \frac{Q^2}{2P\cdot q} = const.$

Naively the integral is dominated by $|y^\mu| \sim \frac{1}{|q^\mu|} \to 0 \quad ?$

Proton rest frame (photon in the minus direction) $$x = \frac{(q^3 - q^0)(q^3 + q^0)}{2m_p q^0} \simeq \frac{q^3 + q^0}{m_p}$$

$$y^+ \sim \frac{1}{q^-} \sim \frac{m_p x}{Q^2} \to 0, \qquad y^- \sim \frac{1}{q^+} \sim \frac{1}{m_p x} \qquad y^2 \sim \frac{1}{Q^2} \to 0$$

finite !

Operator product expansion

$$\int d^4y e^{iqy} \bar{\psi}\gamma^\mu\psi(y)\bar{\psi}\gamma^\nu\psi(0) = \bar{\psi} i(i\partial_\alpha + q_\alpha)\gamma^\mu\gamma^\alpha\gamma^\nu \frac{-1}{Q^2}\sum_n \left(\frac{2iq\cdot\partial}{Q^2}\right)^n \psi(0) + \cdots$$

$$+(\mu \to \nu, q \to -q)$$

c.f., Peskin (18.125)

Pick up the antisymmetric part

$$\gamma^\mu\gamma^\alpha\gamma^\nu = g^{\mu\alpha}\gamma^\nu - g^{\mu\nu}\gamma^\alpha + g^{\alpha\nu}\gamma^\mu + i\epsilon^{\mu\alpha\nu\rho}\gamma_\rho\gamma_5$$

$$\int d^4y e^{iqy}\bar{\psi}\gamma^\mu\psi(y)\bar{\psi}\gamma^\nu\psi(0) = 2\epsilon^{\mu\nu\lambda\alpha} q_\alpha \sum_n^{even} \frac{2q_{\mu_1}\cdots 2q_{\mu_n}}{Q^{2(n+1)}} \bar{\psi}\gamma_\lambda\gamma_5 i\partial^{\mu_1}\cdots i\partial^{\mu_n}\psi(0)$$

When $Q^2 \to \infty$, naively, the most important operators are those with smallest dimensions (smallest n)

Twist expansion

However, in the proton matrix element, $i\partial^\mu \to P^\mu$, and $\frac{2P\cdot q}{Q^2} = \frac{1}{x}$
is not small in the Bjorken limit $Q^2 \to \infty$, $x = const.$

The most important operators are those with lowest twist

(twist) = (dimension) – (spin)

In practice, the number of plus Lorentz indices

$\bar{\psi}\gamma^+\psi$ twist-2 $\bar{\psi}\gamma_\perp\psi$ twist-3

Twist-2 polarized quark operators
(symmetrized in all Lorentz indices and trace subtracted)

$$\bar{\psi}\gamma_5\gamma^{(\lambda}iD^{\mu_1}iD^{\mu_2}\cdots iD^{\mu_n)}\psi \; -(\text{traces})$$

Totally symmetric in all indices → twist-2

$$\langle PS|\frac{1}{n+1}\bar{\psi}\Big(\gamma_\lambda\gamma_5 i\partial_{(\mu_1}\cdots i\partial_{\mu_n)}+\sum_{i=1}^{n}\gamma_{\mu_i}\gamma_5 i\partial_{(\mu_1}\cdots i\partial_\lambda\cdots i\partial_{\mu_n)}\Big)\psi|PS\rangle\frac{2q^{\mu_1}\cdots 2q^{\mu_n}}{Q^{2n}}$$

$$\equiv\frac{a_n}{n+1}\Big(S_\lambda P_{\mu_1}\cdots P_{\mu_n}+\sum_{i=1}^{n}S_{\mu_i}P_{\mu_1}\cdots P_\lambda\cdots P_{\mu_n}\Big)\frac{2q^{\mu_1}\cdots 2q^{\mu_n}}{Q^{2n}}$$

$$=\frac{a_n}{n+1}\frac{1}{x^n}\left(S_\lambda+n\frac{S\cdot q}{P\cdot q}P_\lambda\right)$$

$$=\frac{a_n}{x^n}S_\lambda-\frac{na_n}{n+1}\frac{1}{x^n}\left(S_\lambda-\frac{S\cdot q}{P\cdot q}P_\lambda\right)$$

Only even-n terms contribute after adding $\mu\to\nu, q\to -q$

feed into $g_1(x)$

feed into $g_2(x)$

Anti-symmetric in λ and $\mu_1, \mu_2, \ldots$ → One twist higher (twist-3)

$$\langle PS|\frac{1}{n+1}\sum_{i=1}^{n}\bar{\psi}\Big(\gamma_\lambda\gamma_5 i\partial_{(\mu_1}\cdots i\partial_{\mu_n)} - \gamma_{\mu_i}\gamma_5 i\partial_{(\mu_1}\cdots i\partial_\lambda\cdots i\partial_{\mu_n)}\Big)\psi|PS\rangle\frac{2q^{\mu_1}\cdots 2q^{\mu_n}}{Q^{2n}}$$

$$\equiv \frac{d_n}{n+1}\sum_{i=1}^{n}\Big(S_\lambda P_{\mu_1}\cdots P_{\mu_n} - S_{\mu_i}P_{\mu_1}\cdots P_\lambda\cdots P_{\mu_n}\Big)\frac{2q^{\mu_1}\cdots 2q^{\mu_n}}{Q^{2n}}$$

$$= \frac{nd_n}{n+1}\frac{1}{x^n}\left(S_\lambda - \frac{S\cdot q}{P\cdot q}P_\lambda\right)$$

↓

feed into $g_2(x)$

$$g_1(x) = \frac{1}{2\pi}\mathrm{Im}\sum_n^{even}\frac{a_n}{x^{n+1}}, \qquad g_2(x) = \frac{1}{2\pi}\mathrm{Im}\sum_n^{even}\frac{n(d_n - a_n)}{n+1}\frac{1}{x^{n+1}}$$

g_1 structure function

$$g_1(x) = \frac{1}{2\pi}\mathrm{Im}\sum_{n=0}^{even}\frac{a_n}{x^{n+1}} = \frac{1}{2\pi S^+}\mathrm{Im}\sum_{n=0}^{even}\frac{1}{(P^+)^n x^{n+1}}\langle PS|\bar{\psi}\gamma_5\gamma^+(iD^+)^n\psi|PS\rangle + \cdots$$

Convergent only when $|x| > 1$!

All terms are real!

$$= \frac{1}{2\pi S^+}\mathrm{Im}\sum_{n=0}^{even}\frac{1}{x^{n+1}}\int\frac{dk^+}{2\pi}\left(\frac{k^+}{P^+}\right)^n\int dx^- e^{ik^+x^-}\langle PS\,\bar{\psi}(0)\gamma^+\gamma_5 W[0,x^-]\psi(x^-)|PS\rangle$$

Wilson line

Exercise: show this

$$= \frac{P^+}{4\pi S^+}\mathrm{Im}\int\frac{dk^+}{2\pi}\left(\frac{1}{xP^+ + k^+} + \frac{1}{xP^+ - k^+}\right)\int dx^- e^{ik^+x^-}\langle PS|\,\bar{\psi}(0)\gamma^+\gamma_5 W[0,x^-]\psi(x^-)|PS\rangle$$

Analytic continuation from

$|x| > 1$ to $1 > x > 0$ $\quad x \to x - i\epsilon$

(cf. $s \to s + i\epsilon$)

$$= \frac{P^+}{8\pi S^+} \int dx^- e^{ixP^+x^-} \langle PS|\bar{\psi}(0)\gamma^+\gamma_5 W[0, x^-]\psi(x^-)|PS\rangle + (x \to -x)$$

$$= \frac{1}{2}(\Delta q(x) + \Delta \bar{q}(x))$$

Polarized quark and antiquark distributions

Note the sign difference

$q(-x) = -\bar{q}(x)$ unpolarized quark PDF

$\Delta q(-x) = \Delta \bar{q}(x)$ polarized quark PDF

Exercise: Show that for n even, $\int_0^1 dx x^n g_1(x) = \frac{a_n}{4}$

$g_2(x)$ structure function

Similarly,

$$g_2(x) = \frac{1}{2\pi}\mathrm{Im}\sum_n \frac{n(d_n - a_n)}{n+1}\frac{1}{x^{n+1}} \qquad \int_0^1 dx x^n g_2(x) = \frac{n(d_n - a_n)}{4(n+1)}$$

Exercise: Invert these relations and get

$$g_2(x) = \underline{-g_1(x) + \int_x^1 \frac{dz}{x} g_1(z)} + \underline{\bar{g}_2(x)}$$

Wandzura, Wilczek (1977)

Wandzura-Wilczek part
related to twist-2 PDF

`genuine twist-3' part
$q\bar{q}g$ correlation functions

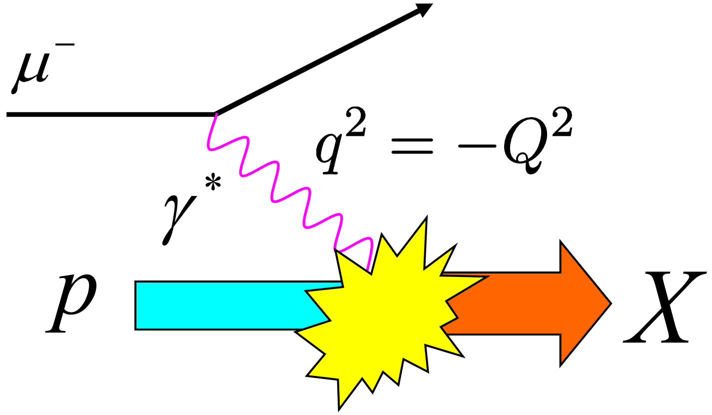
$$\int_0^1 dx x^2 \bar{g}_2(x) = \frac{d_2}{6} \qquad \langle PS|\bar{\psi}\gamma^+ g F^{+i}\psi|PS\rangle = 2d_2(P^+)^2\epsilon^{ij}S_j$$

Shuryak, Vainshtein (1982)

$\Delta\Sigma$ from polarized DIS

Longitudinal double spin asymmetry in polarized DIS

$$A_{LL} = \frac{\mu^{\uparrow}p^{\downarrow} - \mu^{\uparrow}p^{\uparrow}}{\mu^{\uparrow}p^{\uparrow} + \mu^{\uparrow}p^{\downarrow}} \sim \left(1 + \frac{\sigma_L}{\sigma_T}\right)\frac{2xg_1}{F_2}$$



$$\int_0^1 dx g_1(x, Q^2) = \frac{1}{2}\sum_f e_f^2 \int_0^1 dx(\Delta q_f(x, Q^2) + \Delta\bar{q}_f(x, Q^2)) + \cdots$$

Flavor SU(3) decomposition

$$\sum_f e_f^2 = \begin{pmatrix} \frac{4}{9} & & \\ & \frac{1}{9} & \\ & & \frac{1}{9} \end{pmatrix} = \frac{2}{9} + \frac{1}{6}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} + \frac{1}{18}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

$$\Delta\Sigma$$

$$\int_0^1 dx g_1(x) = \frac{1}{9}(\Delta u + \Delta d + \Delta s) + \frac{1}{12}(\Delta u - \Delta d) + \frac{1}{36}(\Delta u + \Delta d - 2\Delta s) + \mathcal{O}(\alpha_s)$$

nucleon isovector axial charge

$$\langle p|\bar{q}\gamma^\mu\gamma_5 t^3 q|p\rangle \sim g_A^{(3)}$$

→ from neutron beta decay

$$n(udd) \to p(uud) + e^- + \bar{\nu}_e$$

$$\langle p|\bar{u}\gamma^\mu\gamma_5 d|n\rangle \sim g_A^{(3)}$$

octet axial charge $\langle p|\bar{q}\gamma^\mu\gamma_5 t^8 q|p\rangle \sim g_A^{(8)}$

→ from hyperon semileptonic decay

$$\Xi^-(dss) \to \Lambda(uds) + e^- + \bar{\nu}_e$$

$$\langle \Lambda|\bar{u}\gamma^\mu\gamma_5 s|\Xi^-\rangle \sim 3F - D = g_A^{(8)}$$

Axial vector current baryon form factors

$$\langle B_b|A^\mu_a|B_c\rangle = \bar{u}_b(p')\left((F(q^2)T^a_{bc} + D(q^2)D^a_{bc})\gamma^\mu\gamma_5 + \cdots\right)u_c(p)$$

$$A^\mu_a = \bar{q}\gamma^\mu\gamma_5 t_a q$$

$$T^a_{bc} = -if_{abc},\ D^a_{bc} = d_{abc}$$

$$B^a t^a = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & -2\frac{\Lambda}{\sqrt{6}} \end{pmatrix}$$

$$\langle p|A^\mu_8|p\rangle = \left(\frac{-iF}{2}(if_{845} - if_{854}) + \frac{D}{2}(d_{844} + d_{855})\right)\bar{u}_p\gamma^\mu\gamma_5 u_p = \frac{1}{2\sqrt{3}}(3F - D)\bar{u}_p\gamma^\mu\gamma_5 u_p$$

$$\langle\Lambda|A^\mu_4 + iA^\mu_5|\Xi^-\rangle = \left(\sqrt{\frac{3}{2}}F + \frac{D}{\sqrt{2}}(d_{484} + d_{585})\right)\bar{u}_\Lambda\gamma^\mu\gamma_5 u_{\Xi^-} = \frac{1}{\sqrt{6}}(3F - D)\bar{u}_\Lambda\gamma^\mu\gamma_5 u_{\Xi^-}$$

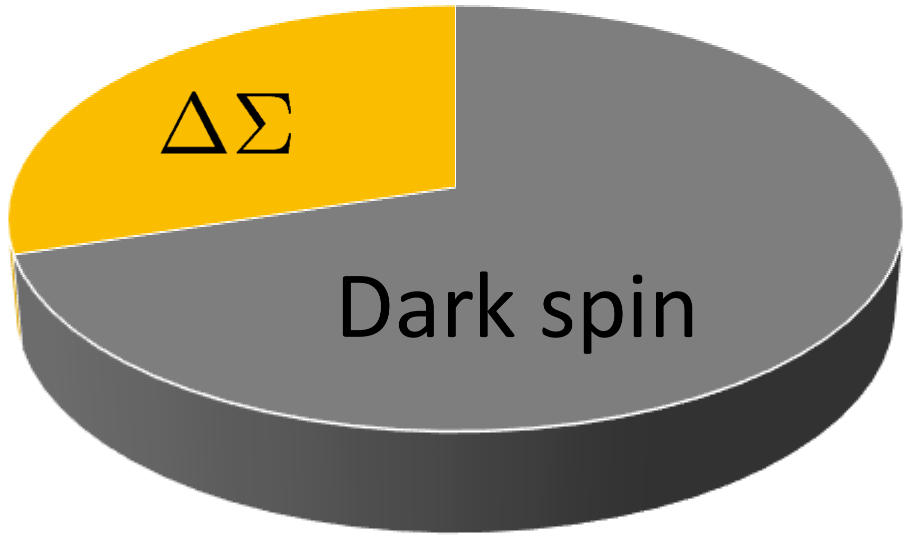
`Spin crisis'

In 1987, EMC (European Muon Collaboration) announced a very small value of the quark helicity contribution

$$\Delta\Sigma = 0.12 \pm 0.09 \pm 0.14 \ !?$$

Recent value from NLO QCD global analysis

$$\Delta\Sigma = 0.25 \sim 0.3$$



Gluon polarization $\Delta G = \int_0^1 dx \Delta G(x)$

Polarized gluon distribution

$$\Delta G(x) = \frac{i}{xS^+} \int \frac{dy^-}{2\pi} e^{ixP^+y^-} \langle PS|F^{+\alpha}(0)\tilde{F}^+_{\alpha}(y^-)|PS\rangle$$

$$\epsilon^\mu_{R/L} = \frac{-1}{\sqrt{2}}(0, \pm 1, i, 0)$$

$$iF^{+i}\tilde{F}^+_i = (F^{+R})^\dagger F^{+R} - (F^{+L})^\dagger F^{+L}$$

Non-local, even after taking a moment.

$$\int_0^1 dx \Delta G(x) \quad = \quad -\frac{1}{2S^+} \int dy^- \theta(y^-) \langle PS|F^{+\alpha}(0)\tilde{F}^+_{\alpha}(y^-)|PS\rangle$$

Depends on the prescription of the pole $1/x$.
The value of ΔG independent of the prescription.

In the light-cone gauge $A^+ = 0$, it reduces to the local operator in the Jaffe-Manohar decomposition.

Polarized DGLAP evolution

$$\frac{d}{d\ln Q^2}\begin{pmatrix}\Delta\Sigma(x)\\ \Delta G(x)\end{pmatrix} = \frac{\alpha_s(Q^2)}{2\pi}\int_x^1 \frac{dz}{z}\begin{pmatrix}\Delta P_{qq}(z) & \Delta P_{qg}(z)\\ \Delta P_{gq}(z) & \Delta P_{gg}(z)\end{pmatrix}\begin{pmatrix}\Delta\Sigma(x/z)\\ \Delta G(x/z)\end{pmatrix}$$

$$\Delta P_{qq}(z) = C_F\left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right),$$

$$\Delta P_{qg}(z) = n_f(2z-1),$$

$$\Delta P_{gq}(z) = C_F(2-z),$$

$$\Delta P_{gg}(z) = 6\left(\frac{1}{(1-z)_+} - 2z + 1\right) + \frac{\beta_0}{2}\delta(z-1)$$

Integrate over x $\quad \frac{d}{d\ln Q^2}\left(\frac{1}{2}\Delta\Sigma(Q^2) + \Delta G(Q^2)\right) \neq 0$ Helicity is not conserved!

RG evolution of $\Delta\Sigma$ and ΔG

In practice, the scale dependence of $\Delta\Sigma$ very weak $\int_0^1 dz \Delta P_{qq}(z) = 0$

Evolution starts at 2-loop in perturbation theory

$$\frac{d}{d\ln Q^2}\Delta\Sigma = -12 C_F T_F n_f \left(\frac{\alpha_s}{4\pi}\right)^2 \Delta\Sigma$$

Kodaira (1980)

On the other hand, ΔG has a strong scale dependence

$$\frac{d}{d\ln Q^2}\Delta G(Q^2) \approx \frac{\beta_0}{2}\frac{\alpha_s(Q^2)}{2\pi}\Delta G(Q^2) \qquad \frac{d\alpha_s(Q^2)}{d\ln Q^2} = -\frac{\beta_0}{4\pi}\alpha_s^2$$

$$\alpha_s(Q^2)\Delta G(Q^2) = const.$$

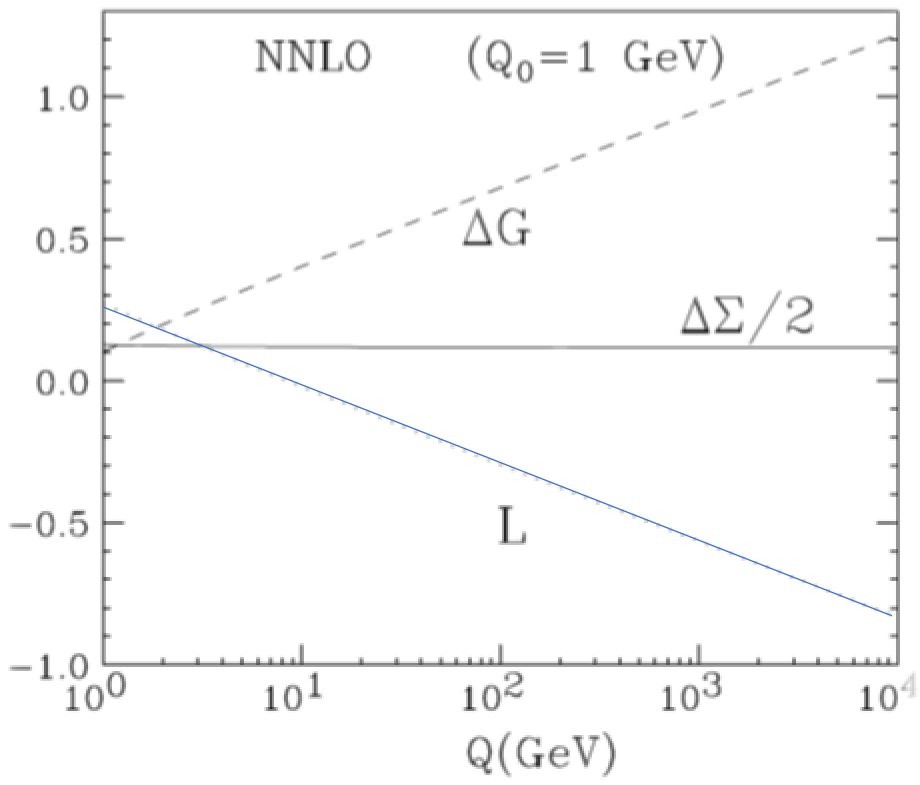
(at one loop)

Accident? No!

Deep connection to QCD chiral anomaly

$$\frac{\partial}{\partial \ln \mu^2}\left(\Delta\Sigma(\mu^2) + n_f \frac{\alpha_s(\mu^2)}{2\pi}\Delta G(\mu^2)\right) = 0.$$

de Florian, Vogelsang 1902.04636



Only the sum of helicity and OAM is conserved.

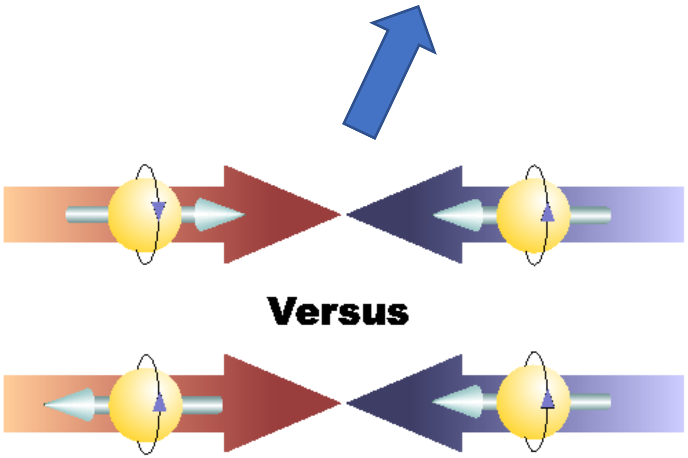
$$\frac{d}{d\ln Q^2}\left(\frac{1}{2}\Delta\Sigma(Q^2) + \Delta G(Q^2) + L_q(Q^2) + L_g(Q^2)\right) = 0$$

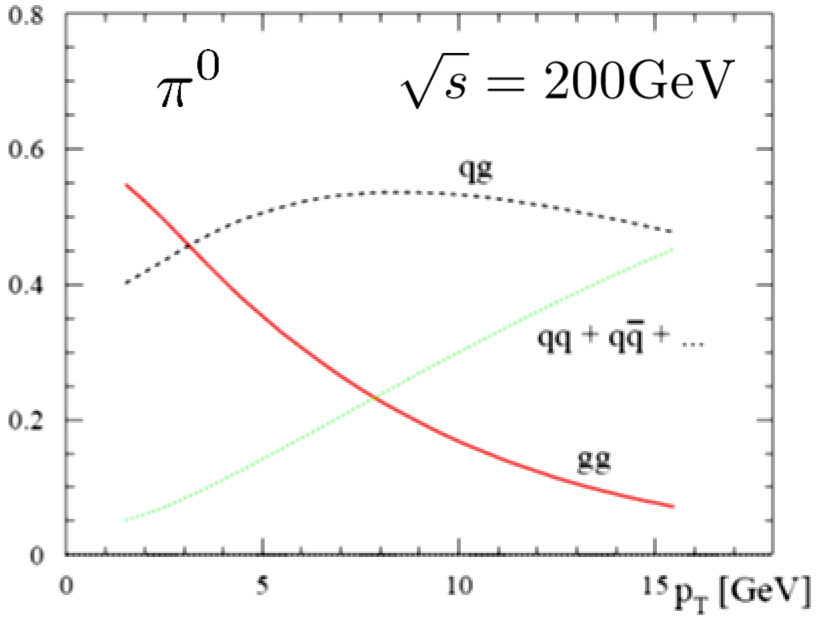
Determination of ΔG at RHIC

Double spin asymmetry of pions and photons in polarized pp.

$$A_{LL} = \frac{d\sigma^{++} - d\sigma^{+-}}{d\sigma^{++} + d\sigma^{+-}}$$

$$\propto \sum_{a,b} \Delta f_a \otimes \Delta f_b(x) \otimes \Delta\sigma_{ab}$$





Direct Photons Point to Positive Gluon Polarization

Results from 'golden measurement' at RHIC's PHENIX experiment show the spins of gluons align with the spin of the proton they're in

June 21, 2023

analysis of data from the PHENIX detector at the Relativistic Heavy Ion r (RHIC) gives fresh insight into how gluons contribute to proton spin.

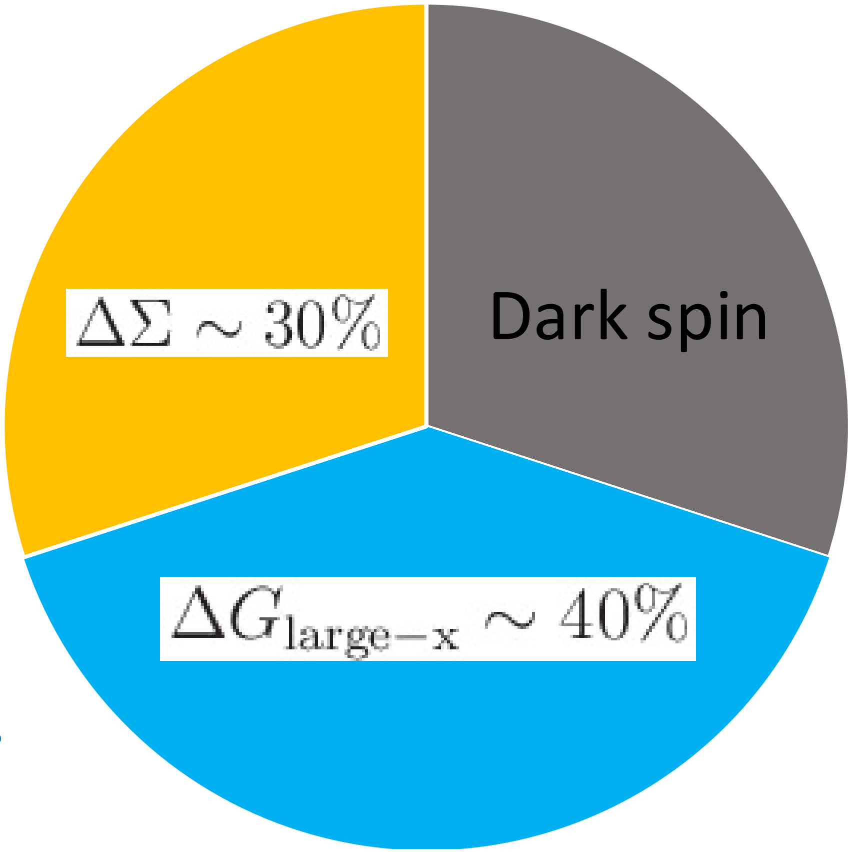
Evidence of nonzero gluon helicity $\Delta G = \int_0^1 dx \Delta G(x)$

A major achievement of the RHIC spin program!

$$\int_{0.05}^{1} dx \Delta G(x, Q^2 = 10\text{GeV}^2) = 0.20^{+.06}_{-.07}$$ DSSV

$$\int_{0.05}^{0.2} dx \Delta G(x, Q^2 = 10\text{GeV}^2) = 0.17 \pm 0.06$$ NNPDF

$$\int_{0.05}^{1} dx \Delta G(x, Q^2 = 10\text{GeV}^2) = 0.23 \pm 0.03$$ JAM

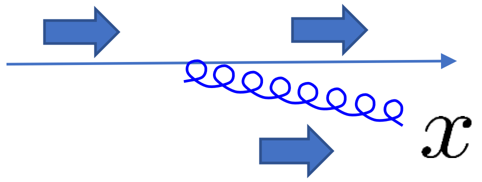


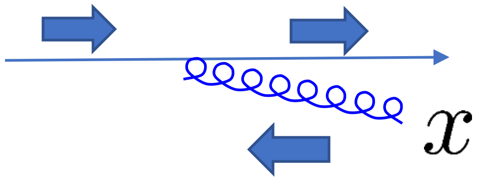
NNLO global analysis became available last year. Borsa et al. 2407.11635

Huge uncertainty from the small-x region --> EIC
Renewed interest in the small-x resummation of helicity PDFs

Helicity PDF at small-x

Consider q→ qg splitting





unpolarized splitting function

$$P_{gq}(x) = C_F \left(\frac{1}{x} + \frac{(1-x)^2}{x} \right)$$

$1/x$ enhancement (soft divergence)

$$G(x) \sim \frac{1}{x^{1+c}}$$

polarized splitting function

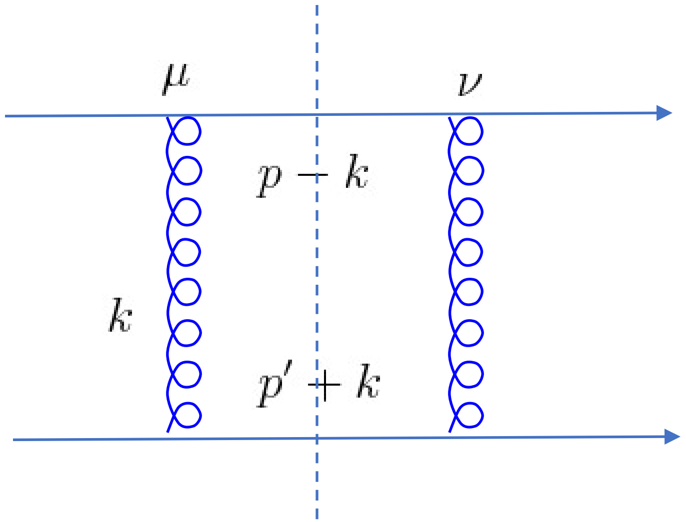
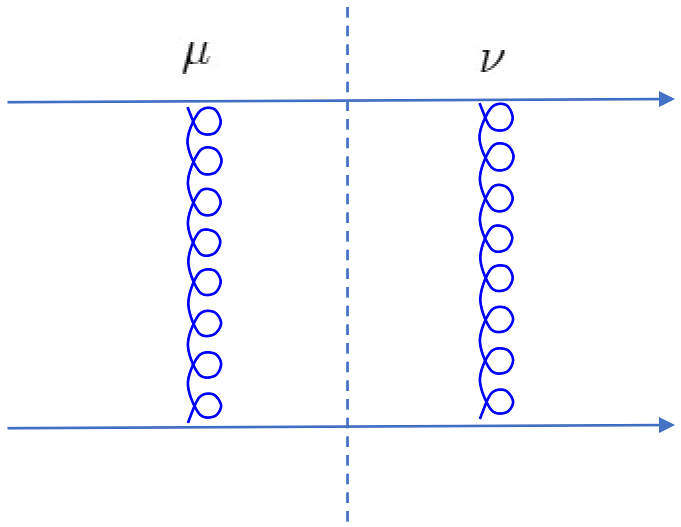
$$\Delta P_{gq}(x) = C_F \left(\frac{1}{x} - \frac{(1-x)^2}{x} \right) = C_F(2-x)$$

No $1/x$ enhancement,

$$\Delta G(x) \sim \frac{1}{x^{\alpha}}$$

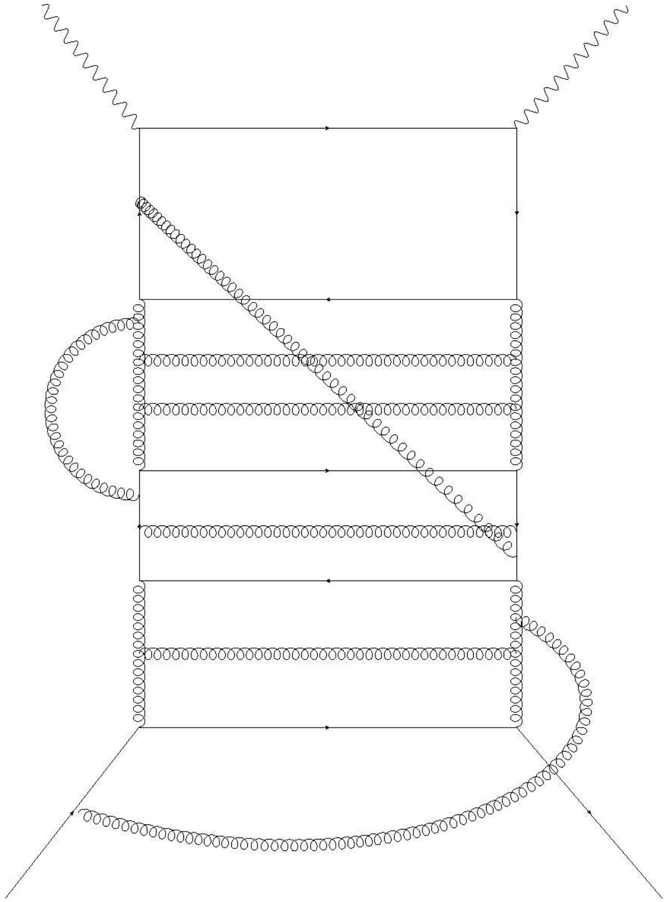
Spin effects are always suppressed by $x \sim (\text{energy})^{-1}$

But the value of α still matters!

Unpolarized	Polarized
	
$\mathrm{Tr}\not{p}\gamma^\mu(\not{p}-\not{k})\gamma^\nu \approx 8p^\mu p^\nu$	$\mathrm{Tr}\gamma_5 \not{S}\gamma^\mu(\not{p}-\not{k})\gamma^\nu \approx -4i\epsilon^{-\mu i\nu}S^+ k_i$
$g^4\frac{(p\cdot p')^2}{(k^2)^2} \sim \alpha_s^2 \frac{s^2}{k_\perp^4}$	$g^4\frac{p\cdot p' k_\perp^2}{(k^2)^2} \sim \alpha_s^2 \frac{s}{k_\perp^2}$
Neglect k in the numerator → Eikonal approximation	Either μ or ν is transverse (sub-eikonal) $d^2k_\perp$ integral logarithmic
Single logs (BFKL) $(\alpha_s \ln 1/x)^n$	Double logs $(\alpha_s \ln^2 1/x)^n$

Double logarithmic approximation

All-order resummation of small-x double logarithms $(\alpha_s \ln^2 1/x)^n$ for helicity distributions



Unlike BFKL, we need to include quark ladders

Unlike BFKL, we need to include non-ladder, `Bremsstrahlung' gluons

Resummation very hard, but can be done!

Kirschner, Lipatov (1983)

Bartels, Ermolaev, Ryskin (1996)
Kovchegov, Pitonyak, Sievert (2016~)

Regge intercept at small-x, revisited

Bartels, Ermolaev, Ryskin (1996)

$$\Delta\gamma_{GG}^{BER}(\omega) = \frac{4\,\bar{\alpha}_s}{\omega} + \frac{8\,\bar{\alpha}_s^2}{\omega^3} + \frac{56\,\bar{\alpha}_s^3}{\omega^5} + \frac{504\,\bar{\alpha}_s^4}{\omega^7} + \ldots$$

Borden, Kovchegov (2023)

$$\Delta\gamma_{GG}^{us}(\omega) = \frac{4\,\bar{\alpha}_s}{\omega} + \frac{8\,\bar{\alpha}_s^2}{\omega^3} + \frac{56\,\bar{\alpha}_s^3}{\omega^5} + \frac{496\,\bar{\alpha}_s^4}{\omega^7} + \ldots$$

Discrepancy at 4-loops!

$$\Delta q(x), \Delta G(x) \sim \frac{1}{x^\alpha}$$

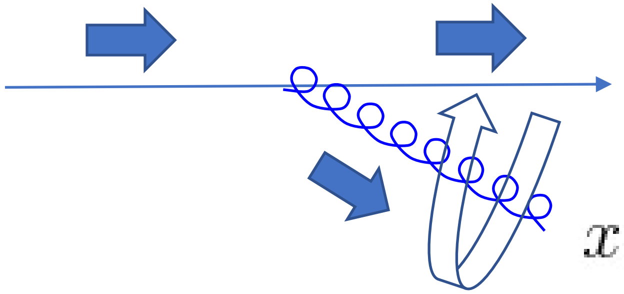
$$\alpha_{BER} \approx 3.664 \sqrt{\frac{\alpha_s N_c}{2\pi}} \qquad \alpha_{BK} \approx 3.661 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

OAM at small-x

Suppose an energetic quark emits a very soft gluon.

Nothing happens to the quark.

Due to angular momentum conservation, gluon helicity and OAM must cancel.



$x \ll 1$

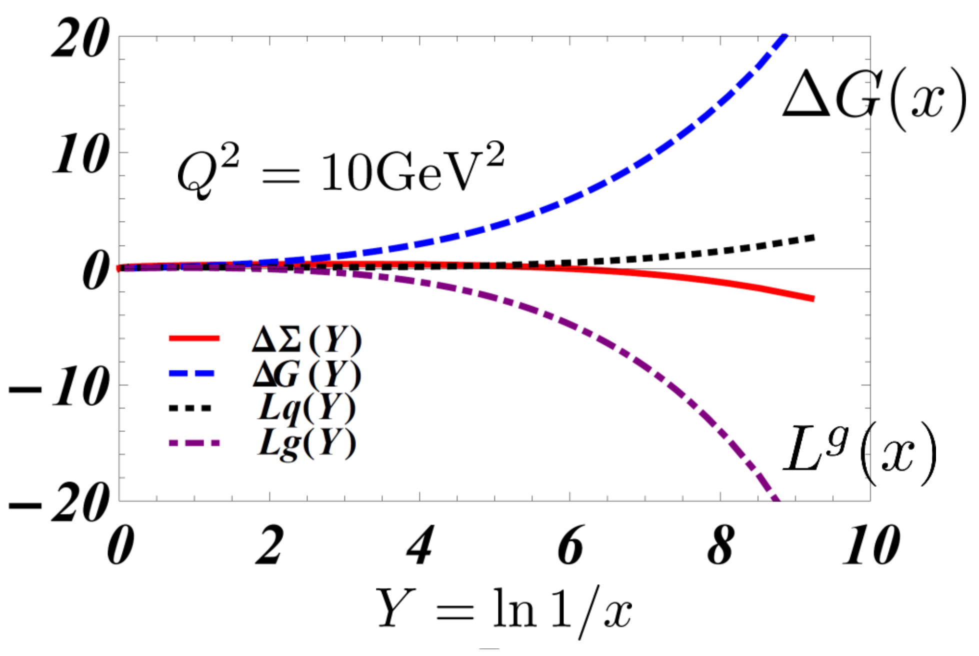
$$\frac{d}{d\ln Q^2}L_g(x) = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}(-2C_F + \cdots)\Delta q(x/z)$$

$$\frac{d}{d\ln Q^2}\Delta G(x) = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}(+2C_F + \cdots)\Delta q(x/z)$$

Helicity-OAM cancellation at small-x

If $\Delta G(x) \sim \frac{1}{x^\alpha}$, then $L_g(x) \approx -\frac{2}{1+\alpha}\Delta G(x)$ Boussarie, YH, Yuan (2019)

YH, Yang (2018)



There might be a sizable contribution to ΔG from the small-x region.

But there will be even larger L_g from the same x-region with an opposite sign.

Helicity is only half of the story.
Can EIC seriously address OAM?

Gauge invariant completion of JM decomposition

YH (2011) see also Chen et al. 0806.3166

$$\langle PS|\epsilon^{ij}F^{i+}A^j_{phys}|PS\rangle = 2S^+\Delta G$$

$$\lim_{\Delta\to 0}\langle P'S|\bar{\psi}\gamma^+ i\overleftrightarrow{D}^i_{pure}\psi|PS\rangle = iS^+\epsilon^{ij}\Delta_{\perp j}L^q_{can}$$

$$\lim_{\Delta\to 0}\langle P'S|F^{+\alpha}\overleftrightarrow{D}^i_{pure}A^{phys}_\alpha|PS\rangle = -i\epsilon^{ij}\Delta_{\perp j}S^+L^g_{can}$$

where

$$A^\mu_{phys} = -\int_{x^-}^{\infty} dz^- W[x^-,z^-]F^{+\mu}(z^-,x_\perp)$$

$$D^\mu_{pure} = D^\mu - igA^\mu_{phys} \ (= \partial^\mu \text{ in the light cone gauge})$$

Twist structure of OAM

YH, Yoshida (2012)

$$L^q_{can}(x) = x\int_x^{\epsilon(x)} \frac{dx'}{x'}(H_q(x') + E_q(x')) - x\int_x^{\epsilon(x)} \frac{dx'}{x'^2}\tilde{H}_q(x')$$

Wandzura-Wilczek part

$$-x\int_x^{\epsilon(x)} dx_1 \int_{-1}^{1} dx_2 \Phi_F(x_1,x_2)\mathcal{P}\frac{3x_1 - x_2}{x_1^2(x_1-x_2)^2}$$
$$-x\int_x^{\epsilon(x)} dx_1 \int_{-1}^{1} dx_2 \tilde{\Phi}_F(x_1,x_2)\mathcal{P}\frac{1}{x_1^2(x_1-x_2)} \, .$$

genuine twist-3

$$\Phi_F \sim \langle P'|\bar{\psi}\gamma^+ F^{+i}\psi|P\rangle$$

$$M_F \sim \langle P'|F^{+\mu}F^{+i}F^+_\mu|P\rangle$$

$$L^g_{can}(x) = \frac{x}{2}\int_x^{\epsilon(x)} \frac{dx'}{x'^2}(H_g(x') + E_g(x')) - x\int_x^{\epsilon(x)} \frac{dx'}{x'^2}\Delta G(x')$$
$$+2x\int_x^{\epsilon(x)} \frac{dx'}{x'^3}\int dX \Phi_F(X,x') + 2x\int_x^{\epsilon(x)} dx_1 \int_{-1}^{1} dx_2 \tilde{M}_F(x_1,x_2)\mathcal{P}\frac{1}{x_1^3(x_1-x_2)}$$
$$+2x\int_x^{\epsilon(x)} dx_1 \int_{-1}^{1} dx_2 M_F(x_1,x_2)\mathcal{P}\frac{2x_1 - x_2}{x_1^3(x_1-x_2)^2}$$

Wigner distribution in quantum mechanics

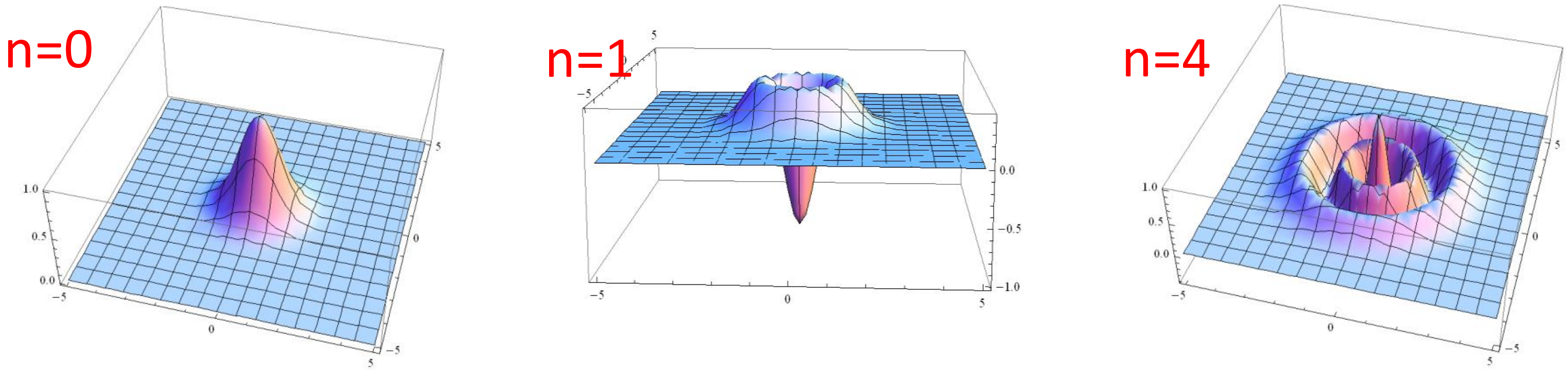
Phase space distribution in QM

$$f_W(q,p) = \int dx e^{-ipx/\hbar} \langle \psi | q - x/2 \rangle \langle q + x/2 | \psi \rangle$$

Reduces to q and p distributions upon integration

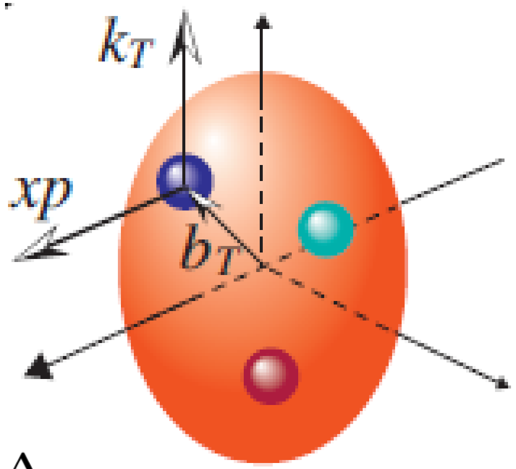
$$\int \frac{dq}{2\pi\hbar} f_W(q,p) = |\langle \psi | p \rangle|^2 , \qquad \int \frac{dp}{2\pi\hbar} f_W(q,p) = |\langle \psi | q \rangle|^2 .$$

Not positive definite, no probabilistic interpretation



n-th excited state of 1D harmonic oscillator

QCD Wigner distribution



Phase space distribution of partons in QCD—the `mother distribution'

Belitsky, Ji, Yuan (2004)

$$W(x,\vec{k}_\perp,\vec{b}_\perp)$$
$$= \int \frac{d^2\Delta_\perp}{(2\pi)^2}\frac{dz^- d^2 z_\perp}{16\pi^3} e^{ixP^+z^- - i\vec{k}_\perp\cdot\vec{z}_\perp}\langle P-\frac{\Delta}{2}|\bar{q}(b-z/2)\gamma^+ q(b+z/2)|P+\frac{\Delta}{2}\rangle$$

$\int d\vec{b}_\perp$ ↓ $\quad$ ↓ $\int d\vec{k}_\perp$

TMD $f(x,\vec{k}_\perp)$ $\quad$ GPD $f(x,\vec{b}_\perp)$

$\int d\vec{k}_\perp$ ↘ $\quad$ ↙ $\int d\vec{b}_\perp$ $\quad$ ↘ $\int dx$

$f(x)$ PDF $\quad$ $F(\vec{b}_\perp)$ Form factor

$\int dx$ ↘ $\quad$ ↙ $\int d\vec{b}_\perp$

Q charge

OAM from the Wigner distribution

Lorce, Pasquini (2011);
YH (2011);

Define $$L^q = \int dx \int d^2b_\perp d^2k_\perp (\vec{b}_\perp \times \vec{k}_\perp)_z W^q(x, \vec{b}_\perp, \vec{k}_\perp)$$

Go to momentum space $b_\perp \to \Delta_\perp$ and look for the component

$$W^{q,g} = i\frac{S^+}{P^+}\epsilon^{ij}k_\perp^i \Delta_\perp^j f^{q,g}(x, k_\perp) + \cdots$$

Then $$L^{q,g} = \int dx \int d^2k_\perp k_\perp^2 f^{q,g}(x, k_\perp)$$

Canonical OAM from the light-cone staple Wilson line

Make the Wigner distribution gauge invariant by attaching a staple-shaped Wilson line

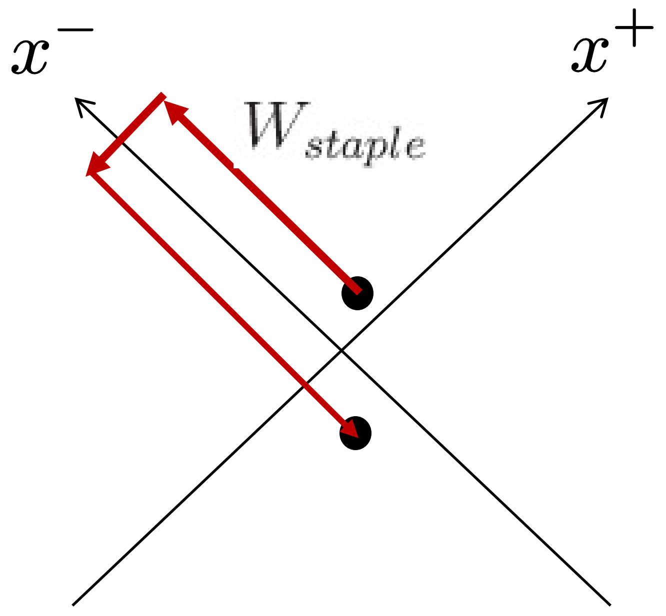
$$W^q \sim \int dz^- d^2 z_\perp e^{ixP^+ z^- - ik_\perp \cdot z_\perp} \langle \bar{\psi}(b - \frac{z}{2})\gamma^+ W_{staple} \psi(b + \frac{z}{2})\rangle$$

The resulting OAM is the gauge invariant canonical OAM

YH (2011)

$$\int d^2 k_\perp (b_\perp \times k_\perp) W^q(b_\perp, k_\perp) = \langle \bar{\psi} b_\perp \times iD_\perp^{pure} \psi \rangle$$

Proof: replace $k_\perp^i \to -i\frac{\partial}{\partial z_\perp^i}$



Wilson line derivative

$$\frac{\partial}{\partial z_\perp^i} W[\infty, z^-, z_\perp]\psi(z^-, z_\perp) = W\partial_{\perp i}\psi(z) - ig\int_{z^-}^{\infty} dx^- W[\infty, x^-]\underbrace{\frac{\partial A^+(x^-, z_\perp)}{\partial z_\perp^i}}_{D^+A_i - F^+{}_i} W[x^-, z^-]\psi(z)$$

Use the trick

$$\int_{z^-}^{\infty} dx^- W[\infty, x^-]D^+A_i(x^-, z_\perp)W[x^-, z^-] = \int_{z^-}^{\infty} dx^- \frac{d}{dx^-}\left(W[\infty, x^-]A_i(x^-)W[x^-, z^-]\right)$$

$$= A_i(\infty)W[\infty, x^-] - W[\infty, x^-]A_i(z^-)$$

Therefore,

$$\frac{\partial}{\partial z_\perp^i} W[\infty, z^-]\psi(z) = W[\infty, z^-]\left(D_{\perp i}\psi + ig\int_{z^-}^{\infty} dx^- W[z^-, x^-]F^+{}_i(x^-)W[x^-, z^-]\psi\right)$$

$$= W[\infty, z^-]D_{\perp i}^{pure}\psi(z)$$

Improved (Belinfante) energy momentum tensor

Return to the canonical angular momentum tensor and write

$$M_{can}^{\mu\nu\lambda} = x^\nu T_{can}^{\mu\lambda} - x^\lambda T_{can}^{\mu\nu} + H^{\mu\nu\lambda}$$

Define $$\tilde{T}^{\mu\nu} = T_{can}^{\mu\nu} + \partial_\rho G^{\rho\mu\nu}$$ ← One can add a total derivative.

where $$G^{\rho\mu\nu} = \frac{1}{2}(H^{\rho\mu\nu} - H^{\mu\rho\nu} - H^{\nu\rho\mu})$$

Exercise: Show that $\tilde{T}^{\mu\nu}$ is symmetric and conserved. Hint: use $\partial_\mu M_{can}^{\mu\nu\lambda} = 0$

Explicitly,

$$G^{\rho\mu\nu} = -\frac{1}{2}\epsilon^{\rho\mu\nu\sigma}\bar{\psi}\gamma_\sigma\gamma_5\psi + 2F^{\mu\rho}A^\nu$$

Exercise: Show that in QCD,

$$\tilde{T}^{\mu\nu} = \bar{\psi} i\gamma^{(\mu} \overleftrightarrow{D}^{\nu)}\psi - F^{\mu\rho}F^{\nu}{}_{\rho} - g^{\mu\nu}\mathcal{L} = \tilde{T}^{\mu\nu}_q + \tilde{T}^{\mu\nu}_g$$

$$A^{(\mu}B^{\nu)} \equiv \frac{A^\mu B^\nu + A^\nu B^\mu}{2} \qquad \overleftrightarrow{D}^\mu = \frac{D^\mu - \overleftarrow{D}^\mu}{2} \qquad \overleftarrow{D}^\mu = \overleftarrow{\partial}^\mu - igA^\mu$$

Hint: A useful identity

From the Dirac equation $(\not{D} + iM)\psi = \bar{\psi}(\overleftarrow{\not{D}} - iM) = 0$

$$\begin{aligned} 0 &= \bar{\psi}\gamma^\mu\gamma^\nu(\not{D} + iM)\psi - \bar{\psi}(\overleftarrow{\not{D}} - iM)\gamma^\nu\gamma^\mu\psi \\ &= \bar{\psi}(g^{\mu\nu}\gamma^\rho + g^{\nu\rho}\gamma^\mu - g^{\mu\rho}\gamma^\nu + i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma_5)D_\rho\psi \\ &\quad -\bar{\psi}\overleftarrow{D}_\rho(g^{\rho\nu}\gamma^\mu + g^{\nu\mu}\gamma^\rho - g^{\rho\mu}\gamma^\nu + i\epsilon^{\rho\nu\mu\sigma}\gamma_\sigma\gamma_5)\psi + 2iMg^{\mu\nu}\bar{\psi}\psi \\ &= 2\bar{\psi}(\gamma^\mu\overleftrightarrow{D}^\nu - \gamma^\nu\overleftrightarrow{D}^\mu)\psi + i\epsilon^{\rho\mu\nu\sigma}\partial_\rho(\bar{\psi}\gamma_\sigma\gamma_5\psi) \end{aligned}$$

Derivation from general relativity

If you are only interested in the symmetric form, there is a much quicker derivation.
Write down the action in curved space

$$S = \int d^4x \mathcal{L}[\psi, A] \to \int d^4x \sqrt{-g}\mathcal{L}[g^{\mu\nu}, \psi, A]$$

In my convention, $g^{\mu\nu} = (+1, -1, -1, -1)$ in the flat limit.

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int \sqrt{-g}\mathcal{L} \qquad T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \int \sqrt{-g}\mathcal{L}$$

beware the sign difference

Don't do this:

$$\bar{\psi}\gamma^\mu \partial_\mu \psi \to \bar{\psi} g^{\mu\nu} \gamma_\mu \partial_\nu \psi \qquad \frac{\delta}{\delta g^{\mu\nu}} \bar{\psi} g^{\mu\nu} \gamma_\mu \partial_\nu \psi = \bar{\psi}\gamma_\mu \partial_\nu \psi$$

Why is this incorrect?

Ji decomposition (operator form)

Improved angular momentum tensor

$$\tilde{M}^{\mu\nu\lambda} = x^\nu \tilde{T}^{\mu\lambda} - x^\lambda \tilde{T}^{\mu\nu}$$

$$= \underbrace{x^\nu \tilde{T}_q^{\mu\lambda} - x^\lambda \tilde{T}_q^{\mu\nu}}_{\text{total quark angular momentum}} + \underbrace{x^\nu \tilde{T}_g^{\mu\lambda} - x^\lambda \tilde{T}_g^{\mu\nu}}_{\text{total gluon angular momentum}}$$

Further decomposition in the quark part

$$\bar{\psi} i \gamma^{(\mu} \overleftrightarrow{D}^{\nu)} \psi = \underbrace{\bar{\psi} i \gamma^\mu \overleftrightarrow{D}^\nu \psi}_{\text{Quark orbital angular momentum}} - \frac{1}{4} \underbrace{\epsilon^{\mu\nu\rho\sigma} \partial_\rho (\bar{\psi} \gamma_\sigma \gamma_5 \psi)}_{\text{Quark spin}}$$

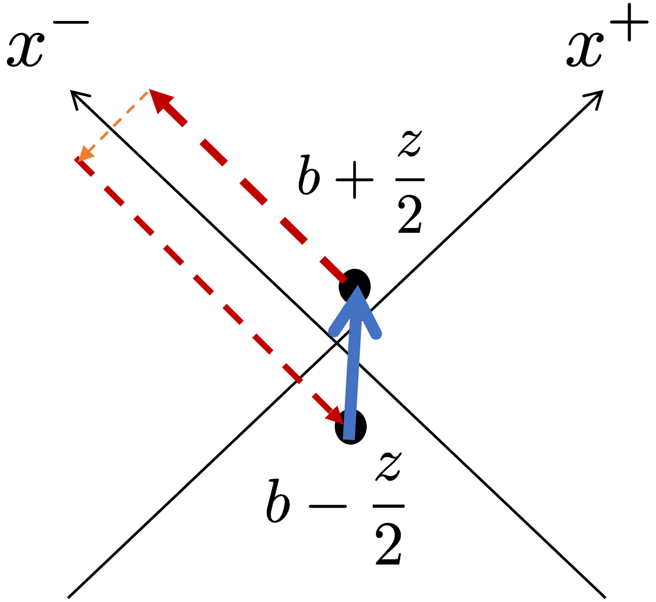
Quark spin $\rightarrow -\frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} \bar{\psi} \gamma_\sigma \gamma_5 \psi$

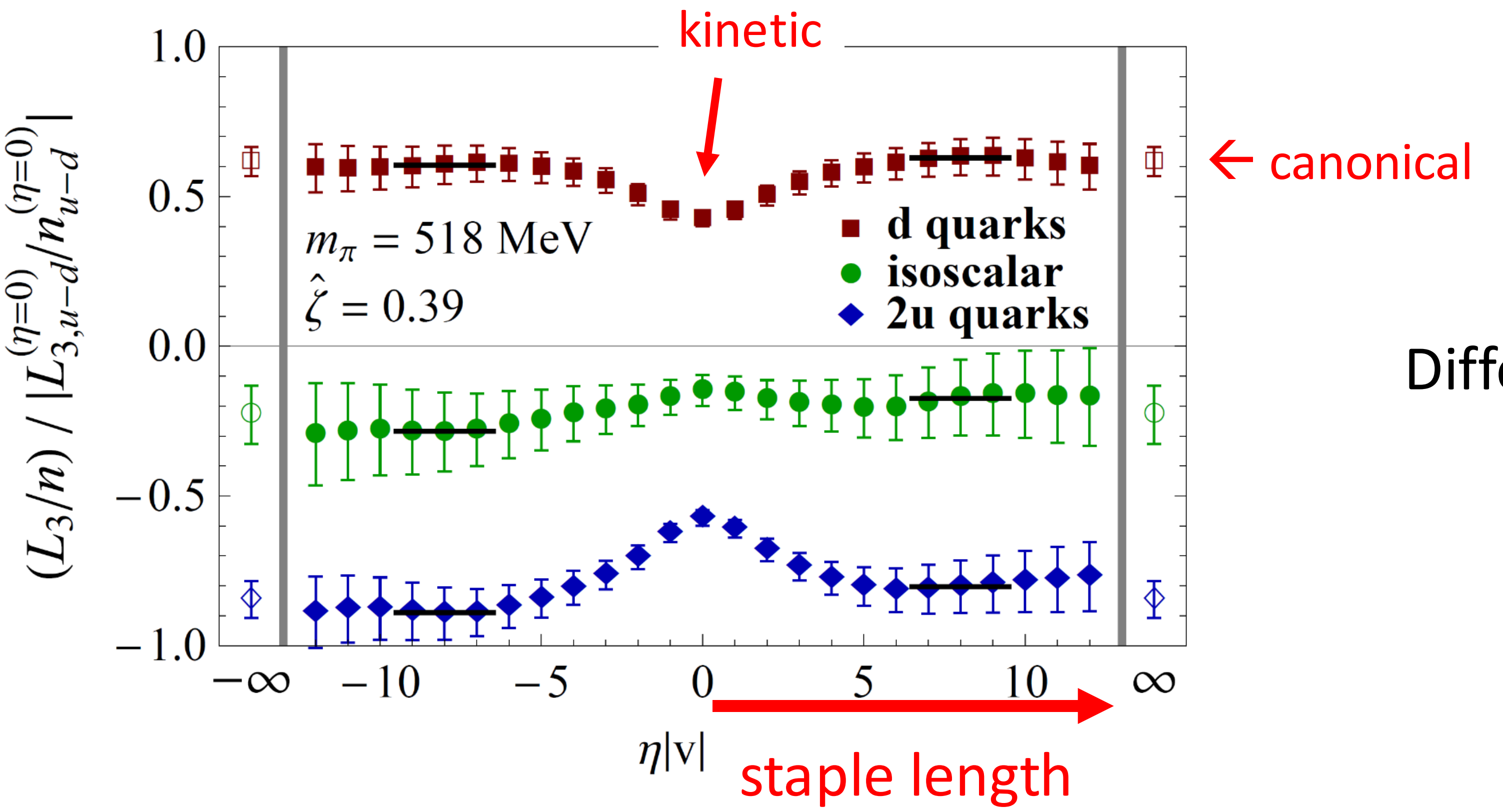
$$J^3 = M^{012} = L_q^3 + S_q^3 + J_g^3$$

Kinetic (Ji's) OAM from the straight Wilson line

Ji, Xiong, Yuan (2012)

$$\int d^2k_\perp (b_\perp \times k_\perp) W_{straight}(b_\perp, k_\perp) = \langle \bar{\psi} b_\perp \times iD_\perp \psi \rangle$$



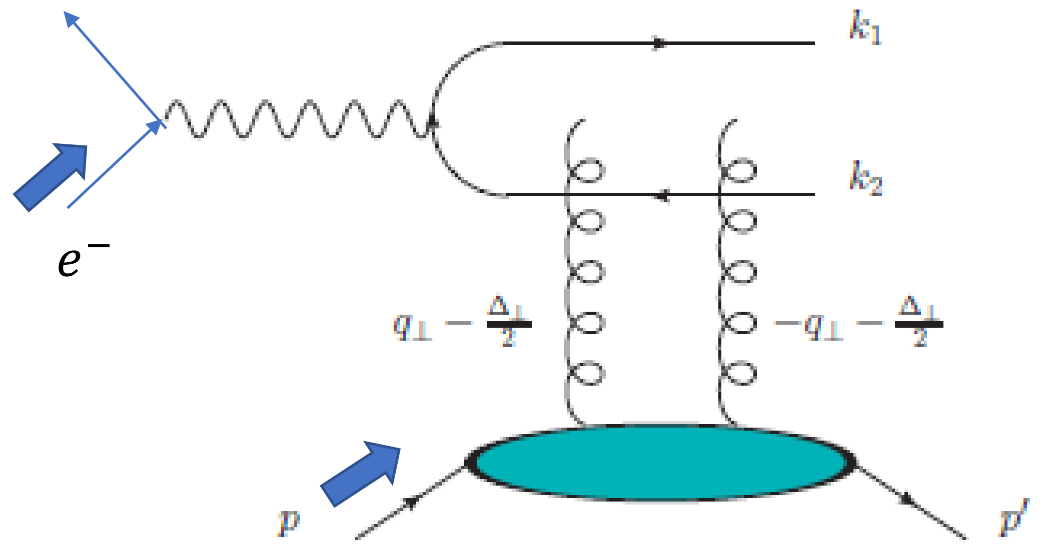


Difference seen in lattice QCD

Engelhardt (2017)

Gluon OAM from longitudinal double spin asymmetry in diffractive dijets

Bhattacharya, Boussarie, YH, (2022, 2024)
Kovchegov, Manley (2025,...)



$$L^z \sim b_\perp \times k_\perp$$

conjugate to $\Delta_\perp$
proton recoil momentum

correlated with jet
transverse momentum