Lecture 4:
Generalized Parton Distribution

Parton distribution function

$$q(x) = \int \frac{d\xi^-}{4\pi} e^{ixP^+\xi^-} \langle P|\bar{\psi}(0)\gamma^+\psi(\xi^-)|P\rangle, \qquad q(-x) = -\bar{q}(x)$$

$$G(x) = \frac{1}{xP^+} \int \frac{d\xi^-}{2\pi} e^{ixP^+\xi^-} \langle P|F_a^{+i}(0)F_a^{+i}(\xi^-)|P\rangle, \qquad G(-x) = -G(x)$$

$$\Delta q(x) = \frac{P^+}{S^+} \int \frac{d\xi^-}{4\pi} e^{ixP^+\xi^-} \langle P|\bar{\psi}(0)\gamma^+\gamma_5\psi(\xi^-)|P\rangle, \qquad \Delta q(-x) = \Delta\bar{q}(x)$$

$$\Delta G(x) = \frac{i}{xS^+} \int \frac{d\xi^-}{2\pi} e^{ixP^+\xi^-} \langle P|F_a^{+i}(0)\tilde{F}_a^{+i}(\xi^-)|P\rangle \qquad \Delta G(-x) = \Delta G(x)$$

Two-point correlation function along the light-cone (Wilson line implicit)

Function only of x (and the RG scale)

Number density interpretation in the light-cone gauge

DGLAP equation

$$\frac{d}{d\ln\mu^2} f_i(x,\mu) = \int_x^1 \frac{dx'}{x'} P_{ij}\left(\frac{x}{x'}\right) f_j(x',\mu)$$

Splitting function

Generalized parton distribution

Diehl, hep-ph/0307382
Belitsky, Radyushkin, hep-ph/0504030

forward → nonforward

$$\int d\xi^- e^{ixP^+\xi^-}\langle P|\bar{\psi}(0)\gamma^+\psi(\xi^-)|P\rangle \quad \Rightarrow \quad \int d\xi^- e^{ix\bar{P}^+\xi^-}\langle P'|\bar{\psi}(-\xi^-/2)\gamma^+\psi(\xi^-/2)|P\rangle$$

$$f(x,\mu) \to f(x,\eta,t,\mu)$$

$$\bar{P}^\mu = \frac{P^\mu + P'^\mu}{2} \qquad \Delta^\mu = P'^\mu - P^\mu$$

momentum transfer $t = \Delta^2 = (P' - P)^2$

$$\xi^- = \xi\cdot n \qquad n^\mu = \delta^\mu_-$$

skewness $\eta = \dfrac{-\Delta\cdot n}{2\bar{P}\cdot n} = \dfrac{P^+ - P'^+}{P^+ + P'^+}$

Quark GPDs

$$\Delta^\mu = P'^\mu - P^\mu$$

Unpolarized

$$\bar{P}^+ \int \frac{dz^-}{2\pi} e^{ix\bar{P}^+ z^-} \langle P'|\bar{q}(-z/2)\gamma^+ q(z/2)|P\rangle = \bar{u}(P')\left[\gamma^+ H_q(x,\eta,t) + \frac{i\sigma^{+\nu}\Delta_\nu}{2m_N} E_q(x,\eta,t)\right] u(P)$$

reduce to PDFs in the forward limit

no PDF counterpart

Polarized

$$\bar{P}^+ \int \frac{dz^-}{2\pi} e^{ix\bar{P}^+ z^-} \langle P'|\bar{q}(-z/2)\gamma^+\gamma_5 q(z/2)|P\rangle = \bar{u}(P')\left[\gamma^+\gamma_5 \tilde{H}_q(x,\eta,t) + \frac{\gamma_5\Delta^+}{2m_N} \tilde{E}_q(x,\eta,t)\right] u(P)$$

From PT symmetry, $H_q(x,\eta,t) = H_q(x,-\eta,t)$

From Hermiticity, $H_q^*(x,\eta,t) = H_q(x,-\eta,t)$

Gluon GPDs

$$\delta_{ij}\int\frac{dz^-}{2\pi\bar{P}^+}e^{ix\bar{P}^+z^-}\langle P'|F_a^{+i}(-z/2)F_a^{+j}(z/2)|P\rangle=\frac{1}{2\bar{P}^+}\bar{u}(P')\left(H_g\gamma^++E_g\frac{i\sigma^{+\nu}\Delta_\nu}{2m_N}\right)u(P)$$

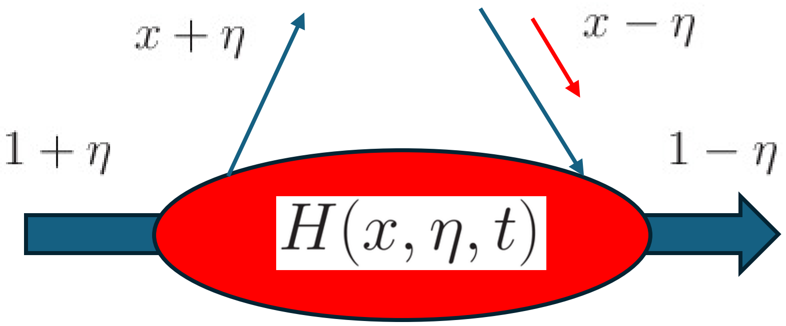
$$i\int\frac{dz^-}{2\pi\bar{P}^+}e^{ix\bar{P}^+z^-}\langle P'|\tilde{F}_a^{+\mu}(-z/2)F_{a\mu}^{+}(z/2)|P\rangle \;=\; \frac{1}{2\bar{P}^+}\bar{u}(P')\left(\tilde{H}_g\gamma^+\gamma_5+\tilde{E}_g\frac{\gamma_5\Delta^+}{2m_N}\right)u(P)$$

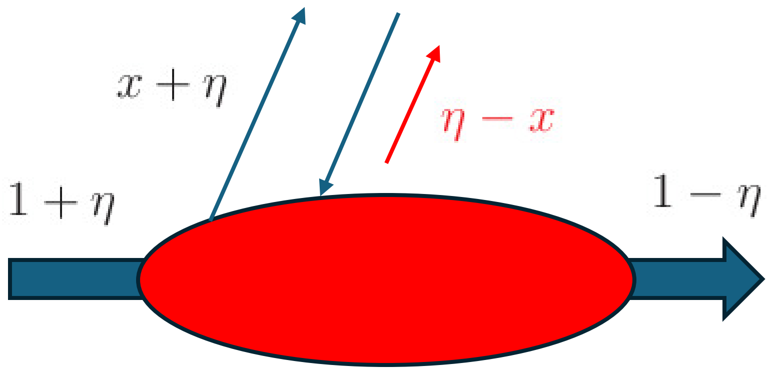
Note that $\quad F^{+i}\tilde{F}^{+i}=-F^{+\mu}\tilde{F}^{+}_{\mu}=\tilde{F}^{+\mu}F^{+}_{\mu}=-\epsilon_{ij}F^{+i}F^{+j}$

In the forward limit,

$$H_g(x)=xG(x)\qquad\tilde{H}_g(x)=x\Delta G(x)$$

DGLAP vs ERBL





$\eta < x < 1 \quad -1 < x < -\eta$

$-\eta < x < \eta$

DGLAP region

Efremov-Radyushkin-Brodsky-Lepage region

meson-like intermediate state

Different functional behavior, different physical interpretation

GPD and form factors

First moment of quark GPDs → contribution to elemag form factors

$$\int_{-1}^{1} dx H_q(x,\eta,t) = F_1^q(t) \qquad \int_{-1}^{1} dx E_q(x,\eta,t) = F_2^q(t)$$

$F_2^q(0)$ anomalous magnetic moment

Independent of η (why?)

Similarly, moments of polarized GPDs $\tilde{H}_q, \tilde{E}_q$ related to axial form factors

$$\langle P'|\bar{\psi}\gamma^\mu\gamma_5\psi|P\rangle = \bar{u}(P')\left[\gamma^\mu\gamma_5 g_A(t) + \frac{\Delta^\mu\gamma_5}{2m} g_P(t)\right] u(P)$$

$$g_A(t) = \sum_q \int_{-1}^{1} dx \tilde{H}_q(x,\xi,t) \qquad g_P(t) = \sum_q \int_{-1}^{1} dx \tilde{E}_q(x,\xi,t)$$

Polynomiality

Take the n-th moment of GPDs

$$\int dx x^n H_q(x,\eta,t) \quad = \quad h_0^n(t) + h_2^n(t)\eta^2 + \cdots + h_{n+1}^n(t)\eta^{n+1}$$

Polynomial in skewness!

$$H_q(x,\eta,t) = H_q(x,-\eta,t)$$ → only even powers

Reason:

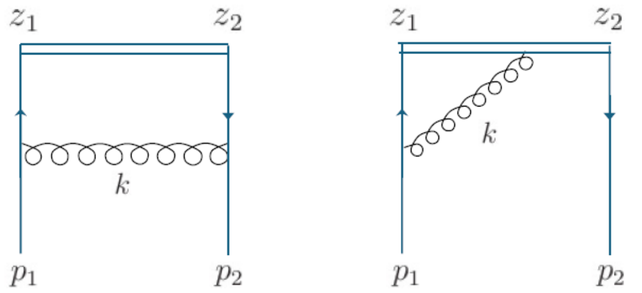
$$\int dx x^n H_q(x,\eta) \sim \langle P'|\bar{\psi}\gamma^+(D^+)^n\psi|P\rangle$$

$$\sim (\bar{P}^+)^{n+1} + (\bar{P}^+)^{n-1}(\Delta^+)^2 + \cdots + (\Delta^+)^{n+1} \qquad \Delta^+ \propto \eta$$

This is highly nontrivial to satisfy in GPD model building.

YH, Schoenleber 2508.01529

Example: one-loop calculation



Sudakov double log

$$
\begin{aligned}
H_q(x,\xi,t) = {}& \delta(1-x) + \frac{\alpha_s C_F}{2\pi}\theta(x-\xi)\Bigg[\frac{\delta(1-x)}{2}\left\{-\ln^2\frac{\mu_F^2}{-t} - 3\ln\frac{\mu_F^2}{-t} + \frac{\pi^2}{6} - \ln^2\frac{1+\xi}{1-\xi}\right\} \\
&+ \ln\frac{\mu_{\mathrm{UV}}^2}{-t}\left\{\frac{1+x^2-2\xi^2}{1-\xi^2}\left[\frac{1}{1-x}\right]_+^{\xi} + \delta(1-x)\left(\frac{3}{2} - \ln\frac{1+\xi}{1-\xi}\right)\right\} \\
&+ \frac{(1+x^2-2\xi^2)\ln(1-\xi^2)-(1-x)^2}{1-\xi^2}\left[\frac{1}{1-x}\right]_+^{\xi} - 2\frac{1+x^2-2\xi^2}{1-\xi^2}\left[\frac{\ln(1-x)}{1-x}\right]_+^{\xi}\Bigg] \\
&+ \frac{\alpha_s C_F}{2\pi}\theta(|\xi|-x)\Bigg\{\ln\frac{\mu_{\mathrm{UV}}^2}{-t}\frac{x+\xi}{1+\xi}\left(\frac{1}{2\xi}+\frac{1}{1-x}\right) - \frac{x+\xi}{2\xi(1+\xi)} \\
&+ \frac{1}{2\xi(1-x)(1-\xi^2)}\Bigg((1-x)(x+\xi)(1+\xi)\ln\left(\frac{4\xi^2}{\xi^2-x^2}\right) + 2\xi x^2\ln\frac{2\xi(1+\xi)}{(x+\xi)(1-x)} \\
&+ 2\xi^3\ln\left(\frac{\xi+x}{\xi-x}\right) + 2\xi\ln\left(\frac{(1+\xi)(\xi-x)}{2\xi(1-x)}\right) + 4\xi^3\ln\left(\frac{1-x}{1+\xi}\right)\Bigg)\Bigg\} + \mathcal{O}(\alpha_s^2).
\end{aligned}
$$

DGLAP region

ERBL region

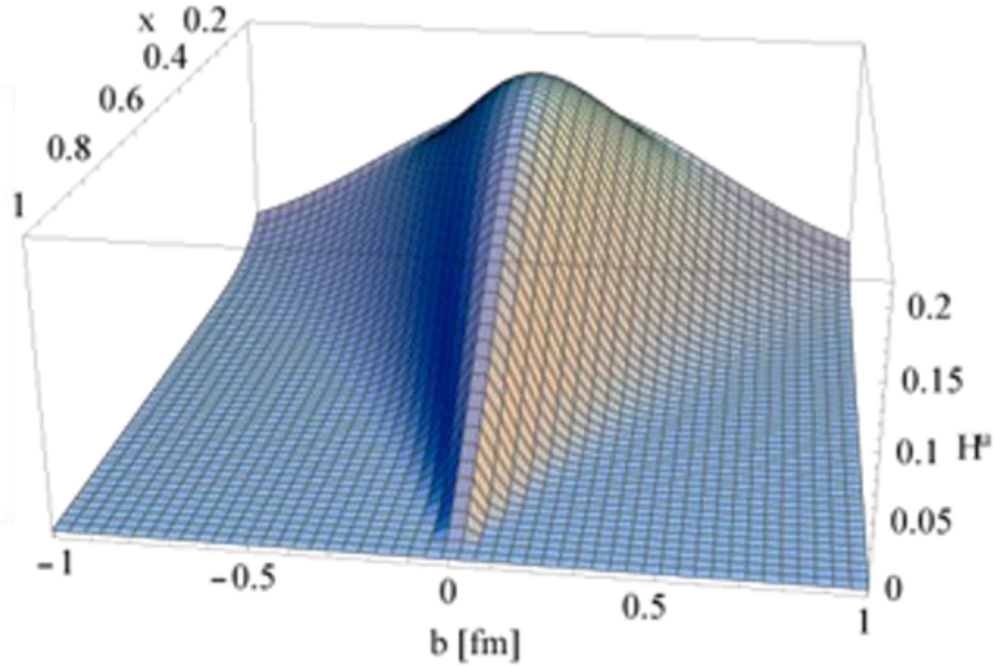
This function satisfies polynomiality, after a highly nontrivial cancellation between the DGLAP and ERBL regions!

GPD and 3D tomography

Set $\eta = 0$ and 2D Fourier transform $\Delta_\perp \leftrightarrow b_\perp$

Distribution of quarks in impact parameter $\vec{b}_\perp$ space

$$H_q(x, t = -\vec{\Delta}_\perp^2) \to H_q(x, \vec{b}_\perp)$$



2D spatial density invariant under Lorentz boost.

Large (infinite) longitudinal momentum suppresses recoil effects

More robust meaning as `coordinates' than the 3D vector $\vec{r}$
conjugate to $\vec{\Delta}$ in the Breit frame

Transverse spin deformation

For a transversely polarized nucleon,

$$\bar{u}\sigma^{+i}u \approx -\frac{2P^+}{m}\epsilon^{ij}S_j$$

Exercise: derive this

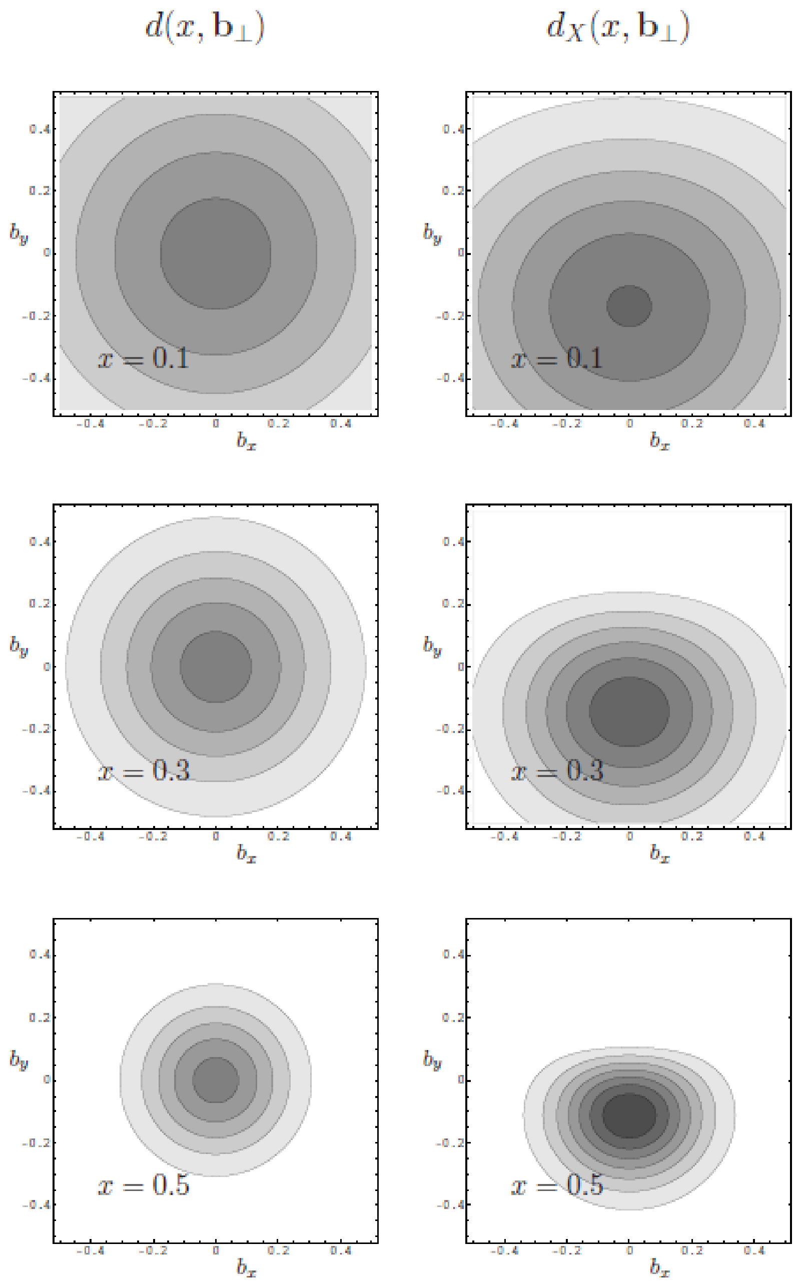
Get the linear combination

$$H_q(x,t) - \frac{i\epsilon^{ij}\Delta_i S_j}{2m^2}E_q(x,t)$$

$$\Rightarrow \quad H_q(x,b_\perp) - \frac{\epsilon^{ij}S_j}{2m^2}\frac{\partial}{\partial b_i}E_q(x,b_\perp)$$

deformation in xy plane

Burkardt (2002)



Connection to the gravitational form factors

Multiply by x and integrate over x .

$$\int_{-1}^{1} dx\,x (P^+)^2 \int \frac{dy^-}{2\pi} e^{ixP^+y^-} \langle \bar{\psi}(0)\gamma^+\psi(y^-)\rangle = \langle P'|\bar{\psi}\gamma^+ iD^+\psi|P\rangle = \langle P'|T_q^{++}|P\rangle$$

From Gordon identity,

$$\begin{aligned}\langle P'|T_q^{++}|P\rangle &= \bar{P}^+\bar{u}(P')\left[A_q(t)\gamma^+ + B_q(t)\frac{i\sigma^{+\lambda}\Delta_\lambda}{2m_N} + D_q(t)\frac{(\Delta^+)^2}{4m_N\bar{P}^+}\right]u(P) \\ &= \bar{P}^+\bar{u}(P')\left[\left(A_q(t)+\eta^2 D_q(t)\right)\gamma^+ + \left(B_q(t)-\eta^2 D_q(t)\right)\frac{i\sigma^{+\lambda}\Delta_\lambda}{2m_N}\right]u(P)\end{aligned}$$

$$\int_{-1}^{1} dx\,x H_q(x,\eta,t) = A_q(t) + \eta^2 D_q(t) \qquad \int_{-1}^{1} dx\,x E_q(x,\eta,t) = B_q(t) - \eta^2 D_q(t)$$

Ji sum rule (GPD version)

$$A_q(0) = \int dx x H_q(x,0,0) \qquad B_q(0) = \int dx x E_q(x,0,0)$$

$$J_q = \frac{1}{2}(A_q(0) + B_q(0)) = \frac{1}{2}\int_{-1}^{1} dx x (H_q(x,0,0) + E_q(x,0,0))$$

$$J_g = \frac{1}{2}(A_g(0) + B_g(0)) = \frac{1}{2}\int_{0}^{1} dx (H_g(x) + E_g(x))$$

GPD evolution

Significantly more complicated than DGLAP due to skewness dependence.

$$\frac{d}{d\ln\mu^2}\mathcal{O}(x,\eta,\mu^2) = \frac{C_F\alpha_s}{2\pi}\int dy K(x,y,\eta)\mathcal{O}(y;\eta)$$

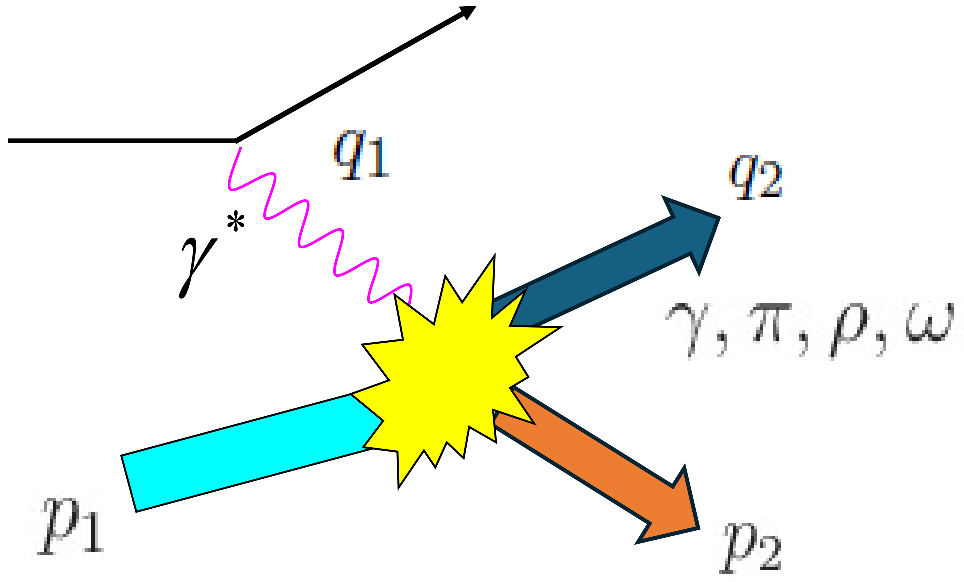
$$K(x,y,\eta)$$
$$= -\frac{x+\eta}{y+\eta}\left[\frac{2\eta-x+y}{2\eta(x-y)}(\theta(x+\eta)-\theta(x-y))\right]_+ - \frac{x-\eta}{y-\eta}\left[\frac{2\eta+x-y}{2\eta(x-y)}(\theta(x-\eta)-\theta(x-y))\right]_+ + \frac{3}{2}\delta(x-y)$$

Separate evolution in the DGLAP region $x > \eta$ and the ERBL region $-\eta < x < \eta$

State-of-the-art: 3-loop evolution equation Braun, Manashov, Moch, Stohmaier (2018)

Deeply Virtual Compton Scattering (DVCS)
Deeply Virtual Meson Production (DVMP)

GPD can be probed in exclusive processes
Small cross section, need high luminosity → JLab/EIC



$$q = \frac{q_1 + q_2}{2} \qquad P = \frac{p_1 + p_2}{2} \qquad Q^2 = -q^2$$

$$\Delta = p_2 - p_1 = q_1 - q_2$$

Bjorken variable $x_B = \dfrac{-q_1^2}{2p_1 \cdot q_1}$ generalized Bjorken variable $\xi = \dfrac{Q^2}{2P \cdot q}$

Skewness $\eta = \dfrac{-\Delta \cdot q}{2P \cdot q}$

Exercise: Show that when $q_2^2 = 0$ (DVCS) and $Q^2 \gg |\Delta^2|$ $\eta \approx \xi \approx \dfrac{x_B}{2 - x_B}$

Virtual Compton amplitude: derivation

$$T^{\mu\nu}(\xi,\eta,t) = \frac{i}{2\pi}\int d^4y e^{iq\cdot y}\langle p_2|\mathrm{T}\{\bar{\psi}\gamma^\mu\psi(y/2)\bar{\psi}\gamma^\nu\psi(-y/2)\}|p_1\rangle$$

Coordinate space propagator

$$\psi(y/2)\bar{\psi}(-y/2) = \frac{i\not{y}}{2\pi^2(y^2-i\epsilon)^2}$$

Dirac matrix identity

$$e^{-iq\cdot y}\gamma_\mu\not{y}\gamma_\nu + e^{iq\cdot y}\gamma_\nu\not{y}\gamma_\mu$$

$$= (e^{-iq\cdot y}+e^{iq\cdot y})(g_{\mu\rho}g_{\nu\tau}+g_{\nu\rho}g_{\mu\tau}-g_{\mu\nu}g_{\rho\tau})y^\rho\gamma^\tau + (e^{-iq\cdot y}-e^{iq\cdot y})i\epsilon_{\mu\rho\nu\lambda}y^\rho\gamma^\lambda\gamma_5$$

Connect to GPD in the leading twist approximation

$$\langle p_2|\bar{\psi}(-y/2)\gamma^\tau\psi(y/2)|p_1\rangle \approx \int dx e^{-ixP\cdot y}\int \frac{P^+ dy'^-}{2\pi} e^{ixP^+y'^-}\langle p_2|\bar{\psi}(-y'^-/2)\gamma^\tau\psi(y'^-/2)|p_1\rangle$$

Keep only $\tau = +$ (twist-2)

Do the d^4y integration

$$\int d^4y \frac{y^\rho}{2\pi^2 y^4}(e^{-iq\cdot y}+e^{iq\cdot y})e^{-ixP\cdot y} = \frac{-q^\rho - xP^\rho}{(-q-xP)^2} + \frac{q^\rho - xP^\rho}{(q-xP)^2}$$

Work in the $\gamma - P$ collinear frame and introduce a conjugate light-like vector

$$n^\mu = \frac{\delta^\mu_-}{P^+} \qquad P\cdot n = 1$$

Eliminate q in favor of n

$$q^\rho = \frac{Q^2}{2\xi}n^\rho - \xi P^\rho$$

(continued from the previous slide)

$$= \frac{\xi P^\rho - \frac{Q^2}{2\xi} n^\rho - x P^\rho}{-Q^2 + 2xq \cdot P + i\epsilon} + \frac{-\xi P^\rho + \frac{Q^2}{2\xi} n^\rho - x P^\rho}{-Q^2 - 2xq \cdot P + i\epsilon} = -\frac{n^\rho}{2} \left(\frac{1}{x - \xi + i\epsilon} + \frac{1}{x + \xi - i\epsilon} \right)$$

Similarly, in the antisymmetric part,

$$\frac{\xi P^\rho - \frac{Q^2}{2\xi} n^\rho - x P^\rho}{-Q^2 + 2xq \cdot P + i\epsilon} - \frac{-\xi P^\rho + \frac{Q^2}{2\xi} n^\rho - x P^\rho}{-Q^2 - 2xq \cdot P + i\epsilon} = \frac{1}{2P \cdot q} \left[-2P^\rho - \frac{Q^2}{2\xi} n^\rho \left(\frac{1}{x - \xi + i\epsilon} - \frac{1}{x + \xi - i\epsilon} \right) \right]$$

Remember, this is multiplied by $\epsilon_{\mu\rho\nu+} \langle \bar{\psi} \gamma^+ \gamma_5 \psi \rangle$

$\rho = -$, drop the $-2P^\rho$ term

Final result (leading order factorization formula)

unpolarized GPD

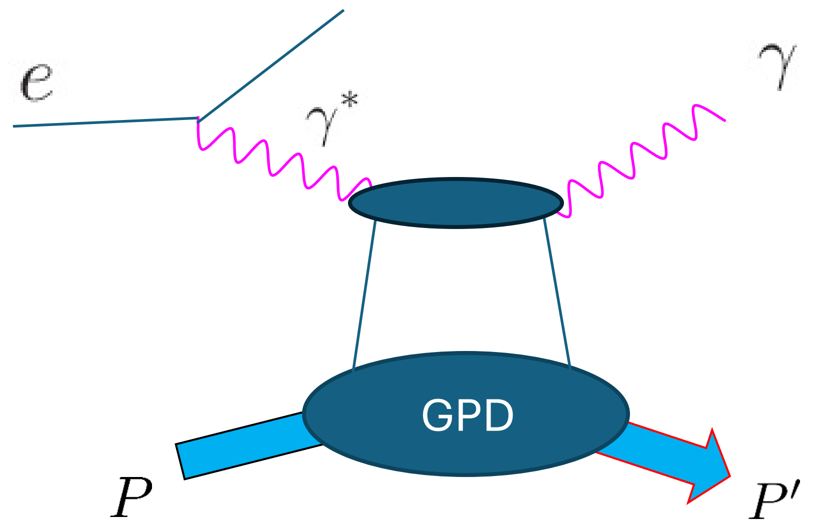
$$T^{\mu\nu}(\xi,\eta,t) = -(g^{\mu\rho}g^{\nu-} + g^{\mu-}g^{\nu\rho} - g^{\mu\nu}g^{\rho-})n_\rho \int \frac{dx}{4\pi}\left(\frac{1}{x+\xi-i\epsilon} + \frac{1}{x-\xi+i\epsilon}\right)\bar{u}(p_2)\left[H\gamma^+ + E\frac{i\sigma^{+\sigma}\Delta_\sigma}{2m}\right]u(p_1)$$
$$-i\epsilon^{\mu\nu\rho-}n^\rho \int \frac{dx}{4\pi}\left(\frac{1}{x+\xi-i\epsilon} - \frac{1}{x-\xi+i\epsilon}\right)\bar{u}(p_2)\left[\tilde{H}\gamma^+\gamma_5 + \tilde{E}\frac{\gamma_5\Delta^+}{2m}\right]u(p_1)$$

polarized GPD

It is the amplitude (not cross section) that factorizes.

Both the unpolarized and polarized GPDs contribute even in unpolarized Compton scattering.

State-of-the-art: next-to-next-to-leading order (NNLO) coefficient functions Braun, Ji, Schoenleber (2022)



GPD challenges

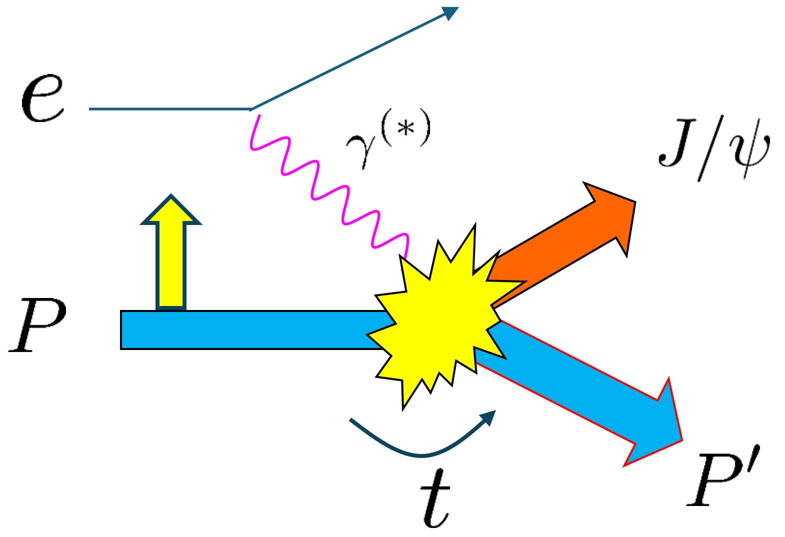
- More variables → more difficult to extract from experiments
- Many GPDs. `Polarized' GPDs contribute to unpolarized processes
- Severe inverse problem. How can one reconstruct $H(x,\xi,t)$ if one only knows

$$\int_{-1}^{1} dx \frac{1}{x-\xi+i\epsilon} H(x,\xi,t)$$

In contrast, PDF is directly related to the structure functions

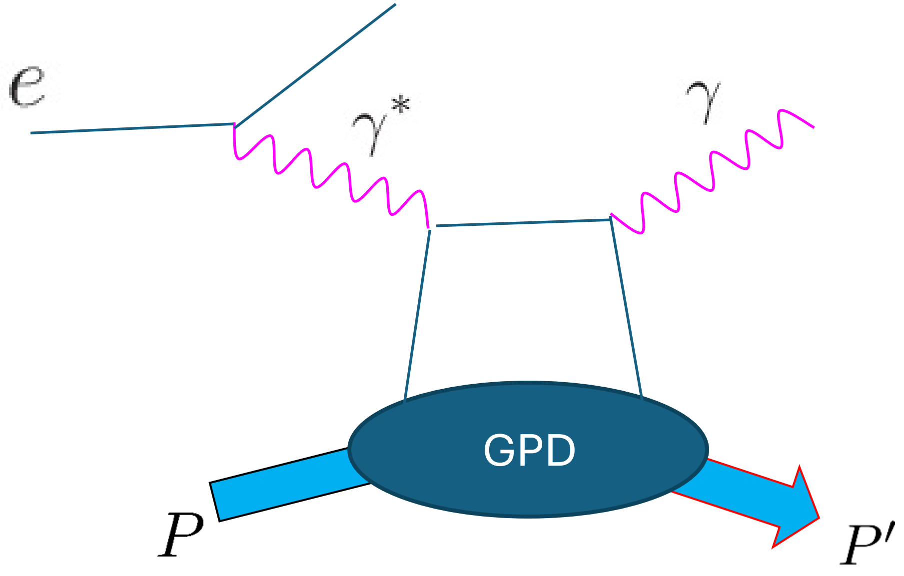
$$g_1(x) = \frac{1}{2}\sum_f e_f^2(\Delta q_f(x) + \Delta\bar{q}_f(x)) + \cdots$$

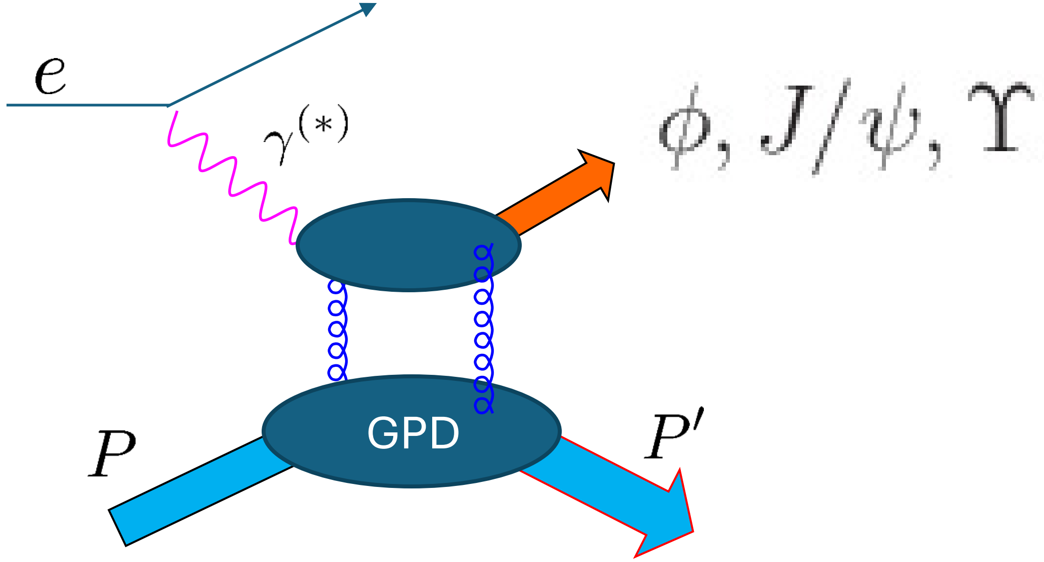
- Evolution equation complicated.
- Difficult to access gluon GPDs
- Yet, many recent progress in theory and lattice!



Gluon GPD E from exclusive single spin asymmetry

Can we access GFFs through GPD measurements?





Processes like DVCS and heavy meson production involve two photons/gluons
→ Can mimic a spin-2 graviton exchange!?

1+1=2

Polyakov-Weiss D-term

Remember polynomiality relation

$$\int dx x^n H_q(x,\eta,t) \quad = \quad h_0^n(t) + h_2^n(t)\eta^2 + \cdots + h_{n+1}^n(t)\eta^{n+1}$$

Only even powers
→ highest power n = odd

$$H_q(x,\eta,t) = H_q^{DD}(x,\eta,t) + \theta(\eta - |x|)D_q(x/\eta,t)$$

Radyushkin's
`double distribution'

Support in the ERBL region
Solely responsible for the highest power η^{n+1}

$$\int_{-1}^{1} dz z^n D_q(z,t) = h_{n+1}^n(t), \qquad D_q(z,t) = -D_q(-z,t)$$

In particular, when $n=1$, connection to GFF $\int_{-1}^{1} dz z D_q(z,t) = h_2^1(t) = D_q(t)$

Dispersion relation

Compton form factor has both real and imaginary parts

$$\mathcal{H}_q(\eta,t) = \int_{-1}^{1} dx \left(\frac{1}{\eta - x - i\epsilon} - \frac{1}{\eta + x - i\epsilon} \right) H_q(x,\eta,t)$$

$$\mathrm{Im}\mathcal{H}_q(\eta,t) = \pi(H_q(\eta,\eta,t) - H_q(-\eta,\eta,t))$$

One can show that Teryaev, arXiv:0510031, Anikin, Teryaev (2007)

$$\begin{aligned}\mathcal{H}_q(\eta,t) &= \int_{-1}^{1} dx \frac{1}{\eta - x - i\epsilon}(H_q(x,\eta,t) - H_q(-x,\eta,t)) \\ &= \int_{-1}^{1} dx \frac{1}{\eta - x - i\epsilon}\underbrace{(H_q(x,x,t) - H_q(-x,x,t))}_{\text{Proportional to the imaginary part}} + 2\int_{-1}^{1} dz \frac{D_q(z,t)}{1-z}\end{aligned}$$

Proof:

$$\int_{-1}^{1} dx \frac{1}{\eta - x - i\epsilon}\Big(H_q(x,\eta,t) - H_q(-x,\eta,t) - (H_q(x,x,t) - H(-x,x,t))\Big)$$

$$= \int_{-1}^{1} dx \sum_{n=0}^{\infty} \frac{1}{(n+1)!}(x-\eta)^n \frac{d^{n+1}}{d\eta^{n+1}}(H_q(x,\eta,t) - H(-x,\eta,t))$$

$$= \sum_{n=0}^{\infty}(1-(-1)^n)h^n_{n+1}(t) = 2\sum_{n=1}^{odd} h^n_{n+1}(t) = 2\int_{-1}^{1} dz \frac{D_q(z,t)}{1-z}$$

Because of polynomiality, only the highest power $(x-\eta)^n = x^n + \cdots$ contributes

Taking the real part, one finds the dispersion relation

$$\mathrm{Re}\mathcal{H}_q(\eta,t) = \frac{1}{\pi}\int_{-1}^{1} dx \mathrm{P}\frac{\mathrm{Im}\mathcal{H}_q(x,t)}{\eta - x} + 2\int_{-1}^{1} dz \frac{D_q(z,t)}{1-z}$$

subtraction constant

By separately measuring the real and imaginary part of the Compton form factor, one can extract the integral from the experimental data

$$\int_{-1}^{1} dz \frac{D_q(z,t)}{1-z}$$

BUT, this is not the D-term

$$\int_{-1}^{1} dz z D_q(z,t) = D_q(t)$$

What's going on? Taylor expand

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

Twist-2, spin-2
Energy momentum tensor

Twist-2, spin-4

Two-photon state couples to infinitely many twist-2 operators with arbitrary spin.

$$1 + 1 \neq 2$$

How can one extract only the spin-2 component?