Kennedy-Tasaki transformation and noninvertible symmetry in lattice models beyond one dimension

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Outline

- ❑ Introduction: Kramer-Wannier transformation & Kennedy-Tasaki transformation in (1+1)d
- ❑ (2+1)d and higher-dimensional generalization of KW transformation (with circuits + projections) to generalized Ising models on hypercubes
- ❑ (2+1)d and higher-dimensional generalization of KT transformation from *subsystem* symmetry-protected topological (SSPT) phases to *subsystem* spontaneous symmetry-breaking (SSSB) phases (with circuits + KW): $Z_2 \times Z_2$ case and Z_2 case
- ❑ (2+1)d and higher-dimensional generalization of non-invertible symmetry protected SSPT phases with corner gapless modes
- ❑ Summary

Kramer-Wannier transformation

- Kramer-Wannier (1941) in classical Ising models: duality in partition functions (from one lattice to its dual lattice)

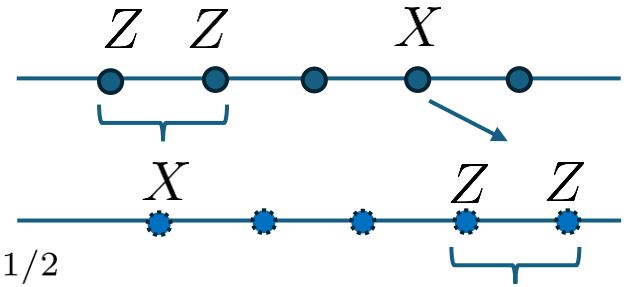
- In quantum spin chains (transverse-field Ising)

$$\mathbf{H}_{\mathrm{TFI}} = -\sum_{i=1}^{L} Z_i Z_{i+1} - \lambda \sum_{i=1}^{L} X_i$$

[Kogut, Rev. Mod. Phys. 1979] to dual lattice:

$$Z_n Z_{n+1} \to X_{n+1/2}, \quad \prod_{m<n} X_m \to Z_{n+1/2}$$

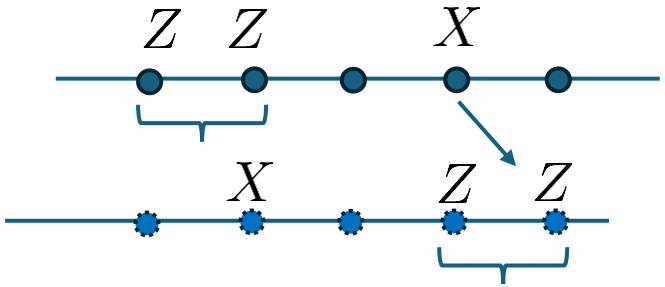
the latter implies $X_{n-1} \to Z_{n-1/2} Z_{n+1/2}$



- Nowadays, we consider mapping to the same lattice (Hilbert space):

$$Z_n Z_{n+1} \to X_{n+1} (\text{instead of } n \text{ on r.h.s.})$$

$$X_n \to Z_n Z_{n+1}$$

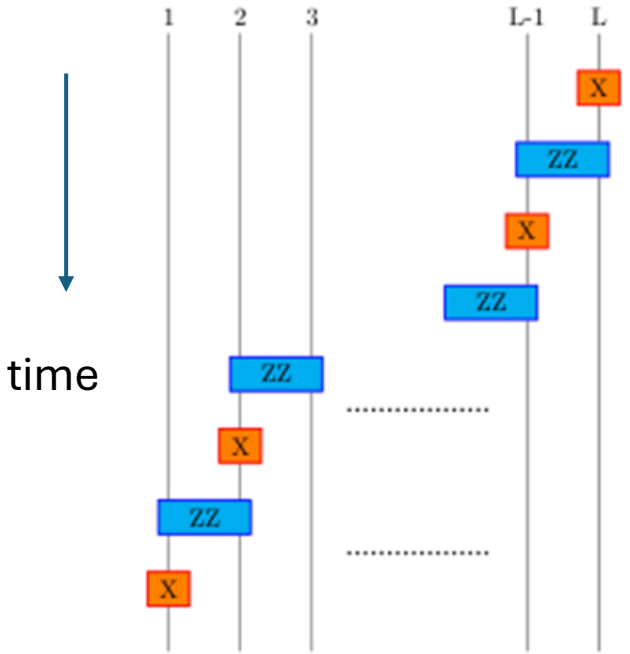


➔ How to realize this as a quantum operation?
(cannot be done by a unitary circuit only)

Achieving KW transformation: $Z_j Z_{j+1} \to X_{j+1},\ X_j \to Z_j Z_{j+1}$ (p.b.c.)

❑ Consider a unitary circuit: $\tilde{\mathbf{D}} \equiv \left(\prod_{j=1}^{L-1} e^{i\frac{\pi}{4} X_j} e^{i\frac{\pi}{4} Z_j Z_{j+1}} \right) e^{i\frac{\pi}{4} X_L}$

[Ho & Hsieh 2019, Seiberg & Shao 2024, Chen et al. 2024]



➢ Can verify that

$$\tilde{\mathbf{D}} X_j = Z_j Z_{j+1} \tilde{\mathbf{D}} \text{ for } j \neq L$$

$$\tilde{\mathbf{D}} X_L = Z_L Z_1 \eta \tilde{\mathbf{D}}, \qquad \eta \equiv \prod_{j=1}^{L} X_j$$

is the symmetry action that commutes $\tilde{D}$

$$\tilde{\mathbf{D}} Z_j = Y_1 \prod_{k=2}^{j} X_k \tilde{\mathbf{D}} \quad \text{thus} \quad \tilde{\mathbf{D}} Z_j Z_{j+1} = X_{j+1} \tilde{\mathbf{D}}$$

➢ Absorb η by including projection to yield the resultant KW transformation: [Seiberg & Shao 2024]

$$\mathbf{D} \equiv \tilde{\mathbf{D}} \mathbf{P} = \tilde{\mathbf{D}}(1+\eta)/2$$

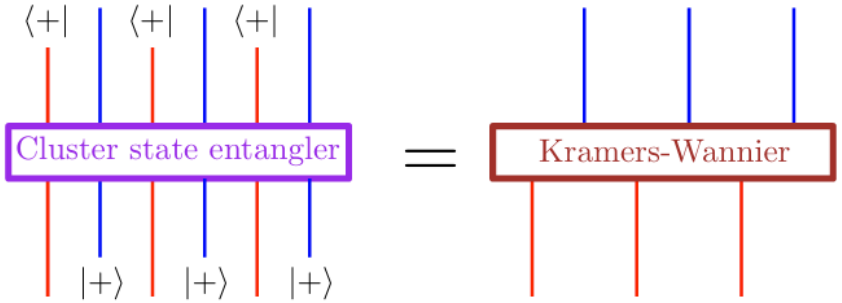
X $\equiv e^{i(\pi/4)X}$ ZZ $\equiv e^{i(\pi/4) Z_j Z_{j+1}}$

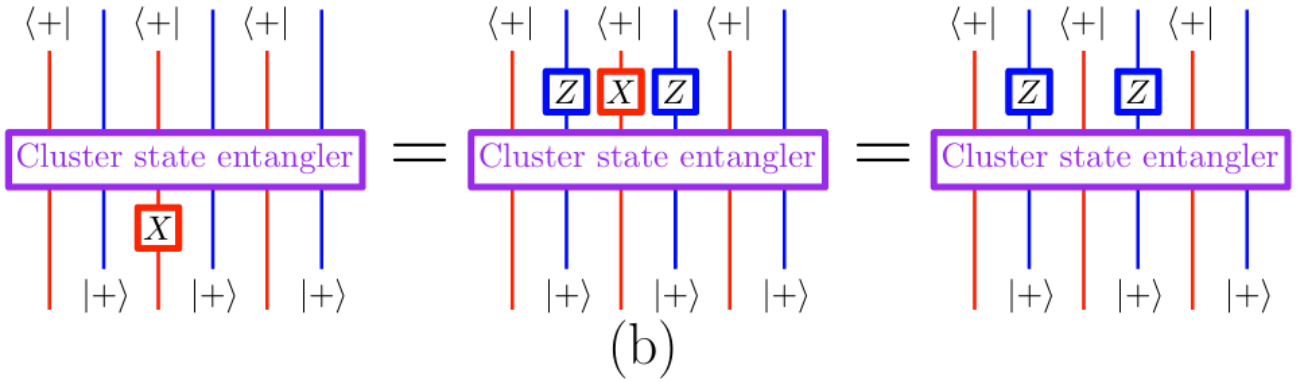
$$\mathbf{D} X_L = Z_L Z_1 \mathbf{D}$$ (thus, non-invertible)

Alternative implementation of KW via measurement

- Comment: KW transformation is a gauging process, which can be implemented by entangling, measurement, and post-selection (or correction)

[Tantivasadakarn et al. 2004]





Relation between the cluster state entangler and Kramers-Wannier duality in arbitrary dimensions

$$T \equiv \prod_{j} CZ_{j,j+1} = \prod_{\langle e,o \rangle} CZ_{e,o}$$

Proof of this equality at the level of operators, where X on the red sites is interchanged with ZZ on the blue sites

KW is a non-invertible symmetry @ λ=1 for TFI model

$$Z_j Z_{j+1} \to X_{j+1},\ X_j \to Z_j Z_{j+1}\ \text{(p.b.c)}$$

$$\mathbf{H}_{\text{TFI}} = -\sum_{i=1}^{L} Z_i Z_{i+1} - \lambda \sum_{i=1}^{L} X_i \quad \Longrightarrow -\sum_{i=1}^{L} X_i - \lambda \sum_{i=1}^{L} Z_i Z_{i+1}$$

$$\mathbf{D}\,\mathbf{H}_{\text{TFI}}(\lambda = 1) = \mathbf{H}_{\text{TFI}}(\lambda = 1)\,\mathbf{D}$$

- Open boundary condition can also be considered and the KW is unitary [Nussinov, Ortiz & Cobanera 2012]
- Can include (1) twist in the periodic chain (2) even or odd symmetry section → unitary in the extended Hilbert space [Li, Oshikawa & Zheng 2023, Seiberg & Shao 2023]
- KW symmetry will impose nontrivial GS degeneracy for 1st order transition [Seiberg, Seifnashri & Shao 2024]
- Discussed in terms of topological defect lines and their fusions [Aasen, Mong &Fendley 2016, ..]

Generalized symmetry

- Non-invertible symmetry first discussed in topological defect lines in RCFT
  [Fröhlich, Fuchs, Runkel & Schweigert, 2007 and 2010]

- Non-group composition rule ➔ fusion category
  [Chang, Lin, Shao, Wang & Yin, 2019]

- Non-invertible symmetry is part of the bigger picture of generalized symmetry: invertible and non-invertible 0- and higher-form symmetries
  ➔ fusion d-category $\mathscr{C}$

  [Gaiotto, Kapustin, Seiberg & Willett 2015, Kapustin & Thorgren 2017, Thorngren & Wang 2019, Kong et al. 2020, Freed, Moore & Teleman 2022, ...]

Kennedy-Tasaki transformation

[Kennedy & Tasaki '92]

- Spin chain with S=1:

$$H_1 = \sum_i [S_i^s S_{i+1}^x + S_i^y S_{i+1}^y + \lambda S_i^z S_{i+1}^z + D(S_i^z)^2]$$

$$U = \prod_{j<k} \exp(i\pi S_j^z S_k^x)$$

[Oshikawa '92]

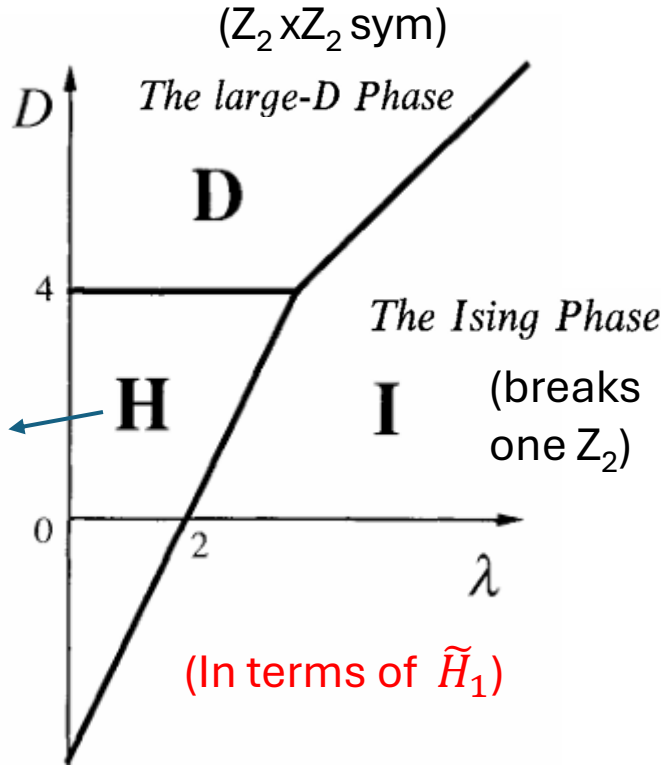
$$\tilde{H}_1 = \sum_i h_i + (1-\lambda) S_i^z S_{i+1}^z + D(S_i^z)^2$$

$$h_i = -S_i^x S_{i+1}^x + S_i^y \exp\{i\pi(S_i^z + S_{i+1}^x)\} S_{i+1}^y - S_i^z S_{i+1}^z$$

Haldane phase

(full breaking of $Z_2 \times Z_2$)

($Z_2 \times Z_2$ sym)



- Modern picture: KT transforms the $Z_2 \times Z_2$ SPT phase to a (fully) spontaneous symmetry breaking phase

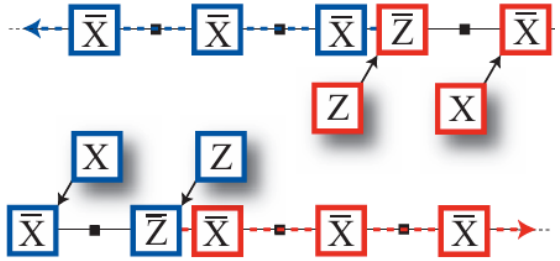
Kennedy-Tasaki transformation on the cluster chain

- Spin chain with S=1/2 (λ=0 nontrivial SPT phase, a.k.a. cluster phase):

$$\mathbf{H}_{\text{cluster}} = -\sum_{i=1}^{2M} Z_{i-1}X_iZ_{i+1} - \lambda\sum_{i=1}^{2M} X_i \qquad \text{(p.b.c.)}$$

symmetric under: $\eta^{\text{o}} = \prod_{i\in\text{odd}} X_i, \quad \eta^{\text{e}} = \prod_{i\in\text{even}} X_i$

- Doherty and Bartlett ('09) found a mapping to transform this Hamiltonian (with o.b.c.) to two copies of transverse-field Ising models



➔ KT transformation for qubits

- The transformation is non-invertible

$$\text{KT}\, X_i = X_i\, \text{KT}$$

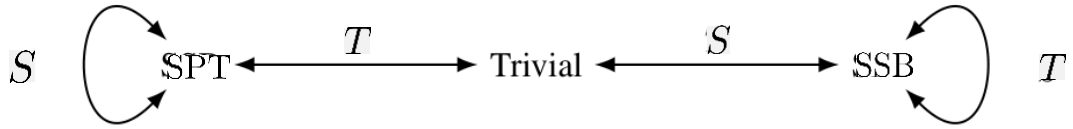
$$\text{KT}\, Z_{i-1}X_iZ_{i+1} = Z_{i-1}Z_{i+1}\, \text{KT}$$

See also [Li, Oshikawa & Zheng 2023]

How to “implement” KT transformation?

- [Field theoretical formation] STS or TST
  (S: gauging, T: stacking an SPT phase)

See [Li, Oshikawa & Zheng 2023]



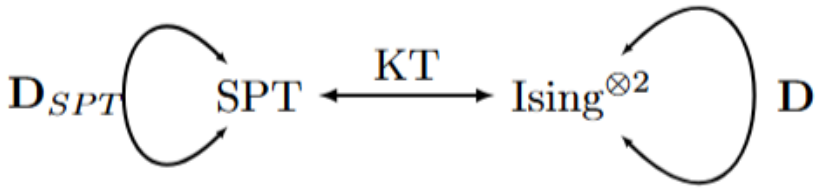
- Our construction via KW on two sublattices (even/odd): $\mathrm{KT} \equiv \mathbf{D}_{odd}^{\dagger}\mathbf{D}_{even}^{\dagger}T\mathbf{D}_{even}\mathbf{D}_{odd}$

$$T \equiv \prod_{j} CZ_{j,j+1} = \prod_{\langle e,o \rangle} CZ_{e,o} \qquad T\left(Z_{j-1}X_jZ_{j+1}\right)T^{\dagger} = X_j$$

Later, we will also define a “KW” transformation on cluster such that $Z_{j-1}Z_{j+1} \leftrightarrow X_j$

➔ Late, we will also ask “What are inequivalent SPT phases protected by non-invertible symmetry?” (in one- and higher-dimensions)

An emerged picture



Kramer-Wannier duality

Kennedy-Tasaki transformation

Symmetry-Protected Topological Phases (trivial and non-trivial)

Spontaneous symmetry-breaking (ordered)

Symmetry-preserved (disordered)

"Kramer-Wannier duality"

trivial

nontrivial

See also [Li, Oshikawa & Zheng 2023]

Non-invertible symmetry and phases

- ❑ These transformations cannot be implemented unitarily, i.e., not by a unitary circuit alone, projection is needed ➔ non-invertible

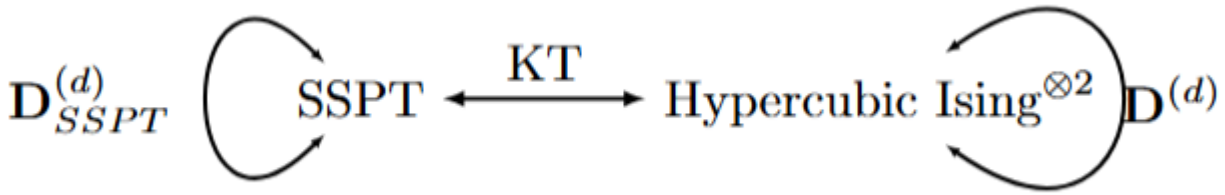
- ❑ SPT phases are dual to symmetry-breaking phases?
  How do we connect them operationally?

- ❑ Can non-invertible symmetry impose further constraints on the phases and give finer-grained classification on phases of matter?

See e.g. [Seifnashri & Shao 2024] for (1+1)d case

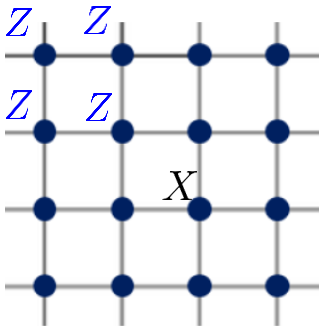
- ➢ Much was understood in (1+1)d; we are interested in exploring these questions in two and higher spatial dimensions

Our results



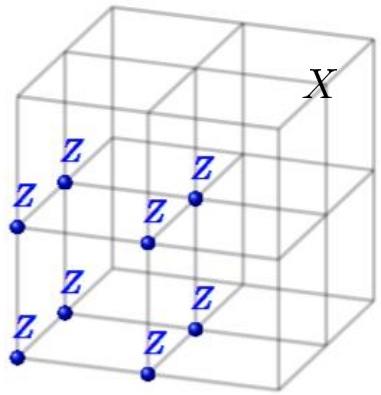
- (2+1)d and higher dimensional generalization of KW transformation (with circuits + projections) to generalized Ising models on hypercubes
- (2+1)d and higher dimensional generalization of KT transformation from *subsystem* symmetry-protected topological (SSPT) phases to *subsystem* spontaneous symmetry-breaking (SSSB) phases (with circuits + KW)
  - ✓ $Z_2 x Z_2$ case (two sublattices: red and blue)
  - ✓ Z_2 case
- (2+1)d and higher dimensional generalization of non-invertible symmetry protected SSPT phases with corner gapless modes
  - ✓ Analyze what symmetry combinations are anomaly-free (various methods including symmetry defects)

Higher dimensional KW transformation



Plaquette Ising

$$\boldsymbol{H}_{\mathrm{TFPI}} = -\sum_{i,j} (Z_{i,j} Z_{i+1,j} \times Z_{i+1,j+1} Z_{i,j+1} + \lambda X_{i,j})$$



Cubic (hypercubic) Ising

[Parayil Mana, Li, Sukeno & Wei, Phys.Rev.B 109, 245129 (2024)]

$$\boldsymbol{H}_{\mathrm{TFHCI}} = -\sum_{c\in\Delta_c} \prod_{v\in\partial c} Z_v - \lambda \sum_{v\in\Delta_v} X_v$$

- These models have subsystem symmetry (e.g.)

direction

$$\eta_j^x = \prod_{i=1}^{L} X_{i,j}$$

$$\eta_i^y = \prod_{j=1}^{L} X_{i,j}$$

- - - -

- KW can be implemented along lines in all directions (but with *Hadamard* transformation, i.e. basis change in between)

$$\mathbf{D}^{(d)} \equiv \mathbf{P}\tilde{\mathbf{D}}_d \mathbf{H}^{\otimes(d)} \tilde{\mathbf{D}}_{d-1} \mathbf{H}^{\otimes(d)} \cdots \tilde{\mathbf{D}}_1 \mathbf{P}$$

$\boldsymbol{H}^{\otimes(d)}$: Hadamard gates on all sites

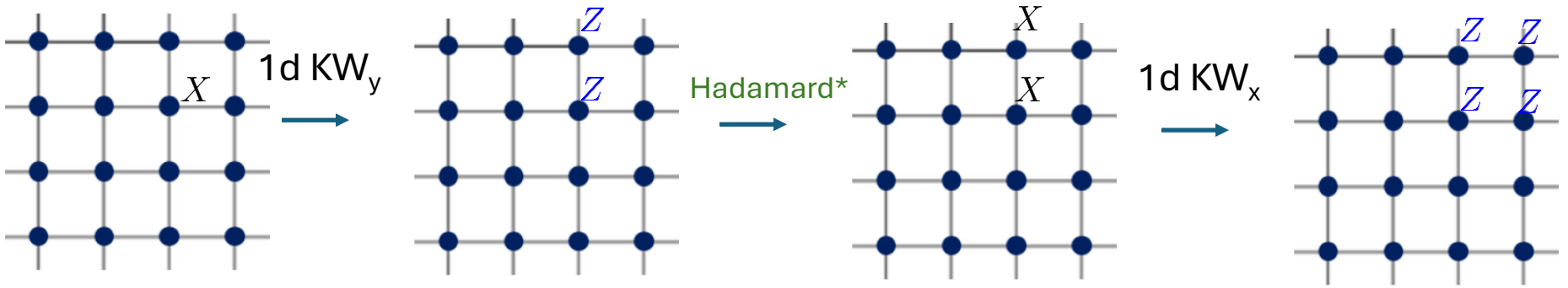
$$\mathbf{P} \equiv \prod_{j\perp x} \frac{(1+\eta_j^x)}{2} \prod_{i\perp y} \frac{(1+\eta_i^y)}{2} \prod_{k\perp z} \frac{(1+\eta_k^z)}{2} \cdots$$

(projection onto subsystem symmetric subspace)

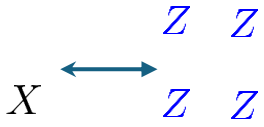
Higher dimensional KW: e.g. 2d

[Parayil Mana, Li, Sukeno & Wei (2024)]

- It turns out generalization beyond 1d is simple: using 1d KW and Hadamard transformation* (Z ←→X)



- KW transformation:



$$\boldsymbol{H}_{\mathrm{TFPI}} = -\sum_{i,j}\left(Z_{i,j}Z_{i+1,j} \times Z_{i+1,j+1}Z_{i,j+1} + \lambda X_{i,j}\right)$$

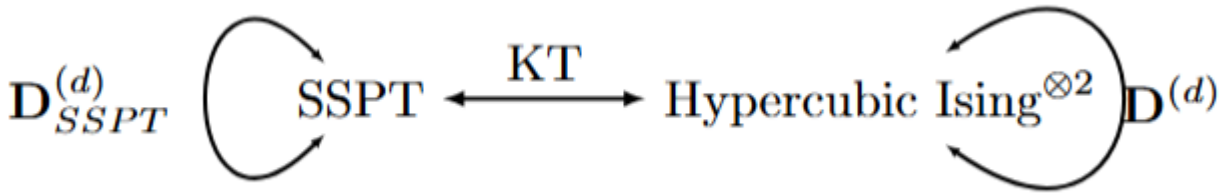
becomes a symmetry at λ=1

- This easily generalizes to higher dimensions

$$\boldsymbol{H}_{\mathrm{TFHCI}} = -\sum_{c\in\Delta_c}\prod_{v\in\partial c} Z_v - \lambda\sum_{v\in\Delta_v} X_v$$

*Hadamard switches between X and Z

Our results



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- ❑ (2+1)d and higher dimensional generalization of KT transformation from *subsystem* symmetry-protected topological (SSPT) phases to *subsystem* symmetry-breaking phases (with circuits + KW)
  - ✓ $Z_2 x Z_2$ case (two sublattices: red and blue)
  - ✓ Z_2 case

- ❑ (2+1)d and higher dimensional generalization of non-invertible symmetry protected SSPT phases with corner gapless modes
  - ✓ Analyze what symmetry combinations are anomaly-free (various methods including symmetry defects)

Subsystem symmetry protected phases

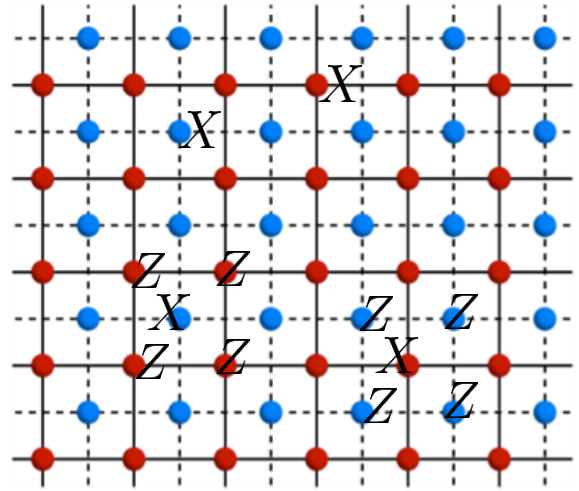
- 1d $Z_2 \times Z_2$ SPT is generalized to higher-dimensional $Z_2 \times Z_2$ subsystem SPT (SSPT)

[You, Devakul, Burnell & Sondhi 2018]

- In (2+1)d SSPT phases with abelian symmetry G are classified by

[Devakul, Williamson & You, 2018]

$$\mathcal{C}[G] \equiv H^2(G^2, U(1)) / \left(H^2(G, U(1))\right)^3$$



2d with two sublattices (red & blue)

- A prominent example in 2d is the cluster model, which would be trivial if one considers $Z_2 \times Z_2$ global (0-form) symmetry, instead of subsystem symmetry

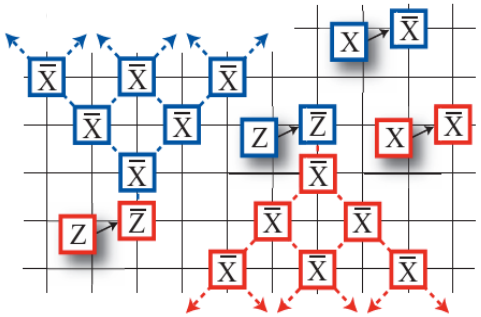
$$\mathcal{C}[Z_2 \times Z_2] = Z_2 \times Z_2 \times Z_2$$

$$\begin{aligned} \boldsymbol{H}_{\text{2dcluster}}(\lambda) \quad = \quad & -\sum_{v_r} X_{v^r} \prod_{v^b \in \partial p^b} Z_{v^b} - \lambda \sum_{v^r} X_{v^r} \\ & -\sum_{v^b} X_{v^b} \prod_{v^r \in \partial p^r} Z_{v^r} - \lambda \sum_{v^b} X_{v^b} \end{aligned}$$

λ=0: nontrivial; λ→∞: trivial

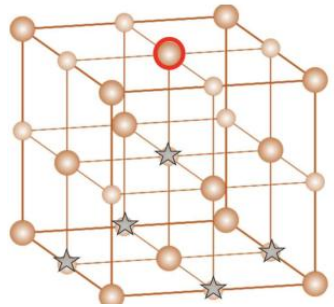
Higher dimensional KT transformation

- Doherty and Bartlett (2009) were the first to construct such KT for the cluster model [motivation from quantum computation]



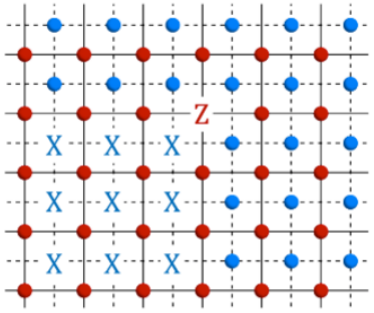
- The light-cone pattern generalized by You et al. to 3d (2018) in their study of 3d subsystem SPT

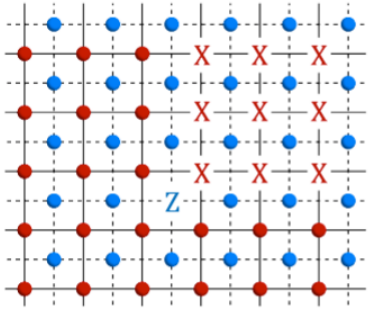
$$\sigma_x \to \sigma_x, \tau_x \to \tau_x;$$
$$\sigma_z \to \sigma_z \prod_{i \in P_a} \tau_x^i, \tau_z \to \tau_z \prod_{i \in P_b} \sigma_x^i$$



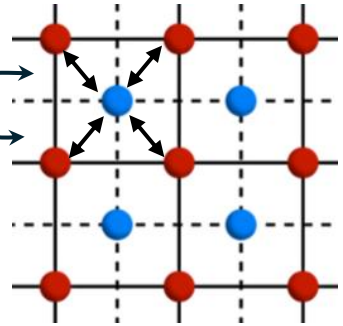
[Parayil Mana et al. (2024)]

- We can derive these patterns using $\mathrm{KT}^{(d)} \equiv \boldsymbol{D}_b^{(d)\dagger} \boldsymbol{D}_r^{(d)\dagger} T^{(d)} \boldsymbol{D}_r^{(d)} \boldsymbol{D}_b^{(d)}$



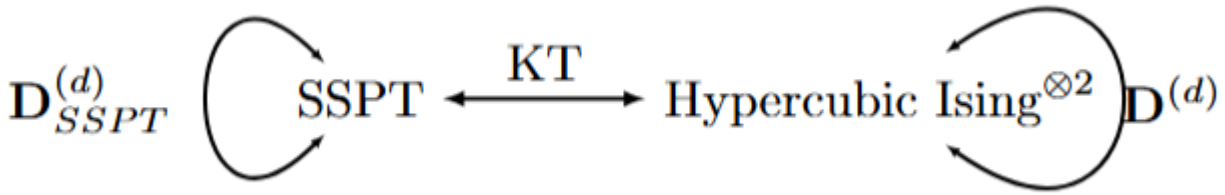


$T^{(d)}$ is the collection of CZ gates between sites on different sublattices



(lattice rotated by 45 compared to Doherty & Bartlett and You et al.)

Our results



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Z_2 subsystem symmetry protected phases

- In (2+1)d SSPT phases with abelian symmetry G are classified by

$$\mathcal{C}[G] \equiv H^2(G^2, U(1)) / \left(H^2(G, U(1))\right)^3$$ [Devakul, Williamson & You, 2018]

$$\mathcal{C}[Z_2] = Z_2$$ ➔ can we construct the nontrivial one ? [Parayil Mana, Li, Sukeno & Wei (2024)]

- Yes (lines of X symmetry: horizontal, vertical & diagonal)

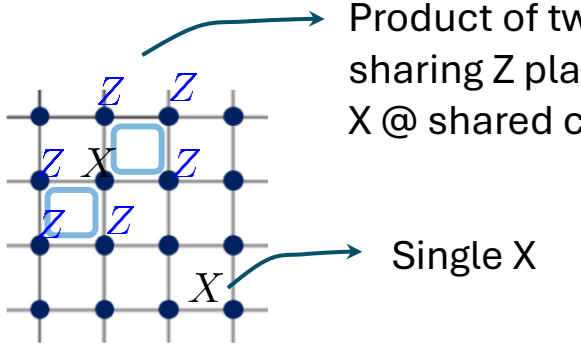
$$H_{\mathrm{SSPT}}(\lambda) = -\sum_{i,j} \left[\begin{array}{ccc} & Z & Z \\ Z & X & Z \\ Z & Z & \end{array} + \lambda X \right]$$

λ=0: nontrivial; λ→∞: trivial

KT

$$H_{\mathrm{DPIM}}(\lambda) = -\sum_{i,j} \left[\begin{array}{ccc} & Z & Z \\ Z & & Z \\ Z & Z & \end{array} + \lambda X \right]$$

KW



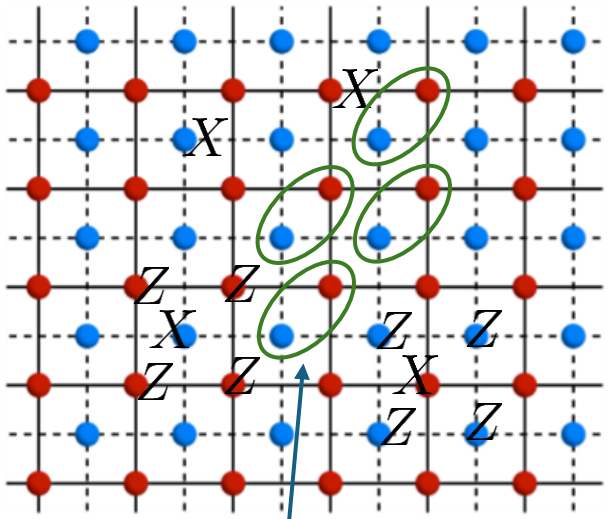
From $Z_2 \times Z_2$ to Z_2

[Parayil Mana, Li, Sukeno & Wei (2024)]

❑ Adding a coupling term (with $g \to \infty$)

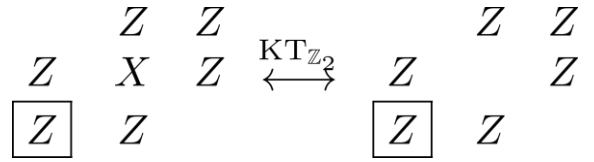
$-g \sum Z_b$ Z_r to

$$\boldsymbol{H}_{\text{2dcluster}} = -\sum_{v_r} X_{v^r} \prod_{v^b \in \partial p^b} Z_{v^b} - \lambda \sum_{v^r} X_{v^r}$$
$$-\sum_{v^b} X_{v^b} \prod_{v^r \in \partial p^r} Z_{v^r} - \lambda \sum_{v^b} X_{v^b}$$



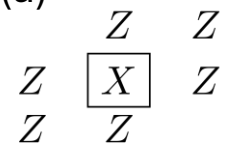
❑ KT transformation from Z_2 SSPT to Z_2 SSSB:

➔ $\mathrm{KT}_{\mathbb{Z}_2} \equiv \mathbf{D}_{\mathrm{DPIM}}^{\dagger} T_{\mathrm{SSPT}} \mathbf{D}_{\mathrm{DPIM}}$ gives

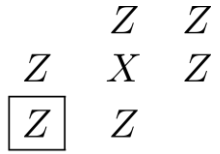


where:

(a)



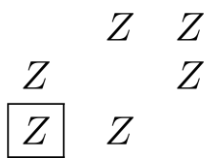
$\mathbf{D}_{\mathrm{DPIM}}$



(KW)

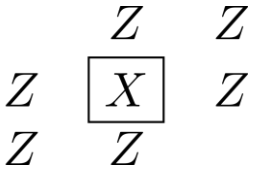
(b)

$\mathbf{D}_{\mathrm{DPIM}}$



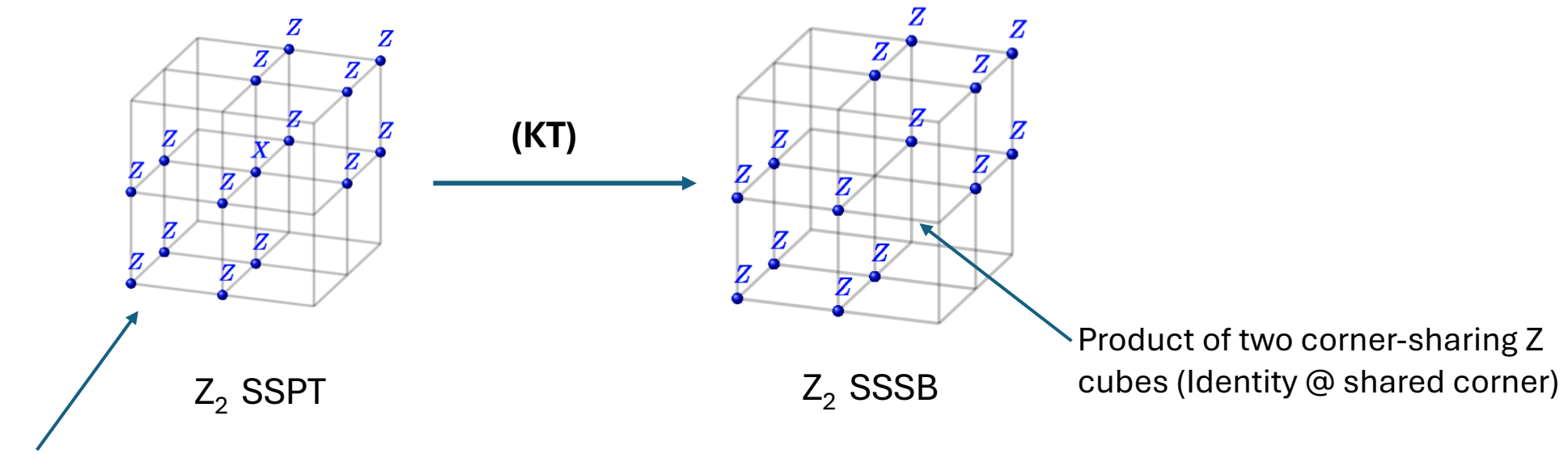
(c)

T_{SSPT} (entangler)

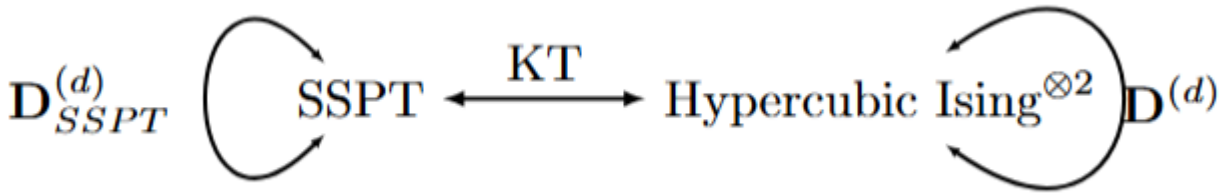


(3+1)d KT transformation on Z_2 SSPT model

[Parayil Mana, Li, Sukeno & Wei (2024)]



Our results



- ❑ (2+1)d and higher dimensional generalization of KW transformation (with circuits + projections) to generalized Ising models on hypercubes

- ❑ (2+1)d and higher dimensional generalization of KT transformation from *subsystem* symmetry-protected topological (SSPT) phases to *subsystem* symmetry-breaking phases (with circuits + KW)

  - ✓ $Z_2 \times Z_2$ case (two sublattices: red and blue)
  - ✓ Z_2 case

- ❑ (2+1)d and higher dimensional generalization of non-invertible symmetry protected SSPT phases with corner gapless modes

  - ✓ Analyze what symmetry combinations are anomaly-free (various methods including symmetry defects)

Non-invertible SPT phases

- 1d cluster state is a nontrivial $Z_2 \times Z_2$ SPT state
- Seifnashri and Shao (2024) showed that by imposing non-invertible (KW) symmetry, there are 3 inequivalent non-invertible SPT phases (with the cluster phase being one)
  - Key technique is to use Kennedy-Tasaki transformation and consider different symmetry breaking patterns (on this side, non-invertible symmetry becomes CZ entangler)

$$\mathrm{H}_{\text{1D-cluster}} = -\sum_{i=1}^{2L} Z_{i-1} X_i Z_{i+1} \xrightarrow{\text{KT}} \mathrm{H}_{\text{Ising}^2} = -\sum_{i:\text{even}} \hat{Z}_i \hat{Z}_{i+1} - \sum_{i:\text{odd}} \hat{Z}_i \hat{Z}_{i+1}$$

Symmetries:

$$\mathbf{D} = \mathbf{T}^{-1}\mathbf{D}_e\mathbf{D}_o$$ (which swaps Z Z and X)

$$\eta_e = \prod_{j:\text{even}} X_j\,, \quad \eta_o = \prod_{j:\text{odd}} X_j$$

KT →

$$\hat{\mathrm{V}} = \prod_{i=1}^{2L} CZ_{i,i+1}, \quad \hat{\eta}_e = \prod_{j:\text{even}} \hat{X}_j\,, \quad \hat{\eta}_o = \prod_{j:\text{odd}} \hat{X}_j$$

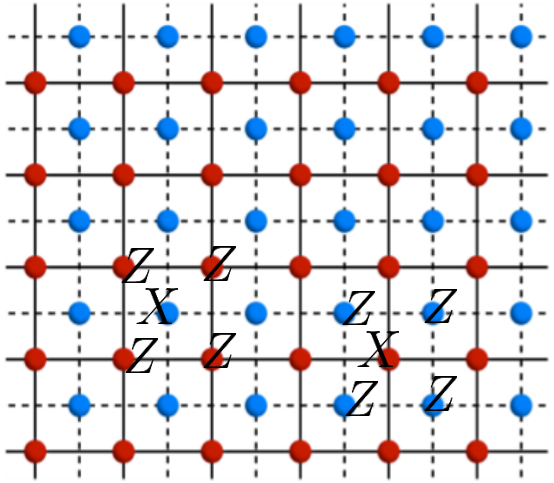
➔As $\hat{\eta}_e$ and $\hat{\eta}_o$ are spontaneously broken and $\hat{V}\hat{\eta}_e\hat{\eta}_o$ is anomalous, there remain three choices:

$\hat{V}$, $\hat{V}\hat{\eta}_e$ and $\hat{V}\hat{\eta}_o$ → distinct phases

Higher-dim non-invertible symmetry protected SSPT phases?

❑ Begin with (2+1)d cluster-state Hamiltonian:

$$\boldsymbol{H}_{\text{2dcluster}} = -\sum_{v_r} X_{v^r} \prod_{v^b \in \partial p^b} Z_{v^b} - \sum_{v^b} X_{v^b} \prod_{v^r \in \partial p^r} Z_{v^r}$$



➢ Subsystem symmetry:

$$\eta^x_{r,j} = \prod_i X_{i,j}\,, \quad \eta^y_{r,i} = \prod_j X_{i,j}\,,$$

$$\eta^x_{b,j} = \prod_i X_{i+\frac{1}{2},j+\frac{1}{2}}\,, \quad \eta^y_{b,i} = \prod_j X_{i+\frac{1}{2},j+\frac{1}{2}}$$

➢ Non-invertible symmetry:

$$\mathbf{D}^{(2)} = \mathbf{T}^{-1}_{\frac{1}{2},\frac{1}{2}} \mathbf{D}^{(2)}_r \mathbf{D}^{(2)}_b \quad \text{which maps:} \quad X_{v_r} \leftrightarrow \begin{matrix} Z_{v_b} & Z_{v_b} \\ Z_{v_b} & Z_{v_b} \end{matrix}\,, \qquad X_{v_b} \leftrightarrow \begin{matrix} Z_{v_r} & Z_{v_r} \\ Z_{v_r} & Z_{v_r} \end{matrix}$$

➢ Under KT: $\boldsymbol{D}^{(2)}$ becomes cluster entangler $\hat{V}^{(2)}$; Hamiltonian becomes

$$\mathrm{H}^{(2)}_{\text{SSSB}} = -\sum_{v_r} \begin{matrix} \hat{Z}_{v_r} & \hat{Z}_{v_r} \\ \hat{Z}_{v_r} & \hat{Z}_{v_r} \end{matrix} - \sum_{v_b} \begin{matrix} \hat{Z}_{v_b} & \hat{Z}_{v_b} \\ \hat{Z}_{v_b} & \hat{Z}_{v_b} \end{matrix}$$

Possible remaining symmetry

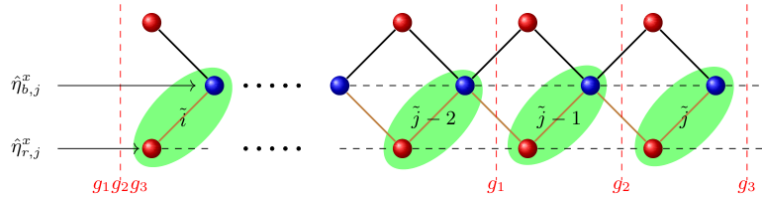
[Parayil Mana, Li, Sukeno & Wei, arXiv:2505.181199]

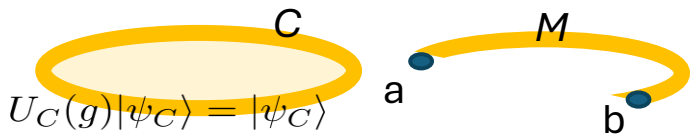
- All subsystem symmetries are spontaneously broken:

$$\eta^x_{r,j} = \prod_i X_{i,j}\,, \quad \eta^y_{r,i} = \prod_j X_{i,j}\,,$$
$$\eta^x_{b,j} = \prod_i X_{i+\frac{1}{2},j+\frac{1}{2}}\,, \quad \eta^y_{b,i} = \prod_j X_{i+\frac{1}{2},j+\frac{1}{2}}$$

- But many combinations are possible (much more than 1d case), as long as they are not anomalous

→ can be analyzed using various methods: (1) defect Hamiltonians, (2) symmetry defect fusion (generalized to 2d) [Seifnashri 2024], (3) Else-Nayak-type symmetry truncation argument



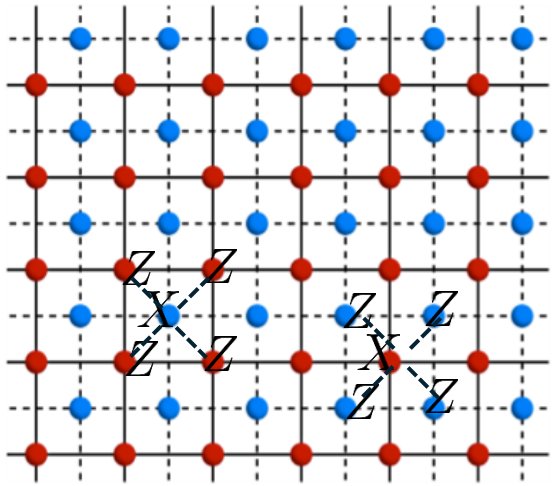


- We will consider analogous combinations considered in 1d: $\hat{V}^{(2)}$, $\hat{V}^{(2)}\hat{\eta}_r$ and $\hat{V}^{(2)}\hat{\eta}_b$

$$\hat{\eta}_r = \prod_{(i,j)\in \text{red}} X_{i,j} \qquad \hat{\eta}_b = \prod_{(i,j)\in \text{blue}} X_{i,j}$$

Only $\hat{V}^{(2)}$ is preserved (SSSB side)

- SSB side: two copies of plaquette Ising model (spontaneous subsystem symmetry breaking)



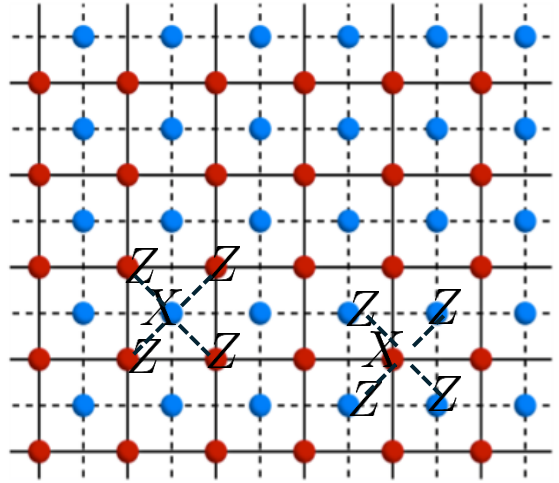
- SSPT: The 2d cluster model is the example

$$\boldsymbol{H}_{\text{2dcluster}} = -\sum_{v_r} X_{v^r} \prod_{v^b \in \partial p^b} Z_{v^b} - \sum_{v^b} X_{v^b} \prod_{v^r \in \partial p^r} Z_{v^r}$$

(a graph state, with graph being the square lattice)

Only $\hat{V}^{(2)}$ unbroken, all line symmetries broken

$$\boldsymbol{H}_{\text{2dcluster}} = -\sum_{v_r} X_{v^r} \prod_{v^b \in \partial p^b} Z_{v^b} - \sum_{v^b} X_{v^b} \prod_{v^r \in \partial p^r} Z_{v^r}$$



- How do we know that only $\hat{V}^{(2)}$ is unbroken?
  - We can construct order parameters to check if all line symmetries are broken

On the symmetry breaking side, the order parameters for this phase are $\{\hat{Z}_{i,1}, \hat{Z}_{i+\frac{1}{2}\,\frac{3}{2}}, \hat{Z}_{1,j}, \hat{Z}_{\frac{3}{2},j+\frac{1}{2}}\}_{i=1,\ldots,L_x;j=2,\ldots,L_y}$.
There are in total $2(L_x + L_y - 1)$ independent order parameters. The order parameters commute with $\hat{\mathrm{V}}^{(2)}$ and therefore $\hat{\mathrm{V}}^{(2)}$ is unbroken.

A useful Lemma to check symmetry & breaking

Suppose $\{\hat{O}_i\}$ are a set of order parameters with non-zero eigenvalues restricted to the ground space for a symmetry breaking Pauli Hamiltonian H, with symmetry group G, defined on a lattice with broken symmetry generators $\{\hat{g}_i\}$ for $i \in S$ where S is a finite indexing set. Consider $\hat{\mathrm{V}}$ such that $[\mathrm{H}, \hat{\mathrm{V}}] = 0$ and $\hat{\mathrm{V}}^2 = \mathrm{I}$ satisfying

- $[\hat{O}_i, \mathrm{H}] = 0 \ \forall i \in S$,
- $[\hat{O}_i, \hat{\mathrm{V}}] = 0 \ \forall i \in S$,
- $[\hat{O}_i, \hat{O}_j] = 0 \ \forall i, j \in S$,
- $\exists$ linearly independent broken symmetry generators $\{\tilde{g}_i\}$ such that $[\tilde{g}_i, \hat{O}_j] = 0$ when $i \neq j$ and $\{\tilde{g}_i, \hat{O}_i\} = 0$,
- H has $2^{|S|}$ ground states.

Then $\hat{\mathrm{V}}$ is an unbroken symmetry.

[Parayil Mana, Li, Sukeno & Wei, arXiv:2505.181199]

➔ Key: for each broken symmetry, we can find an order parameter that anti-commutes with it

Only diagonal $\hat{V}^{(2)}$ $\hat{\eta}_r$ is preserved (SSSB side)

$$\hat{\mathrm{H}}_{\mathrm{blue}} = \sum_{v_b} \begin{matrix} & \hat{Z}_{v_b} & \\ \hat{Z}_{v_b} & \boxed{\hat{Z}_{v_b}} & \hat{Z}_{v_b} \end{matrix} - \sum_{v_r} \begin{matrix} & \hat{Y}_{v_r} & \\ \hat{Y}_{v_r} & \boxed{\hat{Y}_{v_r}} & \hat{Y}_{v_r} \end{matrix} - \sum_{v_r} \begin{matrix} \hat{Z}_{v_b} & & \hat{Z}_{v_b} \\ & \boxed{\hat{Y}_{v_r}} & \\ \hat{Z}_{v_b} & & \hat{Z}_{v_b} \end{matrix}$$

- Construction of Hamiltonian: ensured only the diagonal part of the symmetry is preserved

The order parameters for this phase are of the form $\hat{Z}_{v_b}$ for v_b of the form $(i+\frac{1}{2}, \frac{3}{2})$ and $(\frac{3}{2}, j+\frac{1}{2})$ and $\hat{Y}_{v_r}\begin{pmatrix} & \hat{Z}_{v_b} & \hat{Z}_{v_b} \\ 1- & & \\ & \hat{Z}_{v_b} & \hat{Z}_{v_b} \end{pmatrix}$ for v_r of the form $(i,1)$ and $(1,j)$ for $i = 1, ..., L_x$ and $j = 1, ..., L_y$. Then the ground state configuration is specified by the values of $\hat{Z}_{i+\frac{1}{2},\frac{3}{2}} = \pm 1$, $\hat{Z}_{\frac{3}{2},j+\frac{1}{2}} = \pm 1$, $\hat{Y}_{i,1} = \pm 1$ and $\hat{Y}_{1,j} = \pm 1$ for $i = 1, ..., L_x$ and $j = 1, ..., L_y$.

Only diagonal $D^{(2)}$ η_r is preserved (SSPT side)

- By applying KT to $\hat{H}_{\text{blue}}$, we obtain

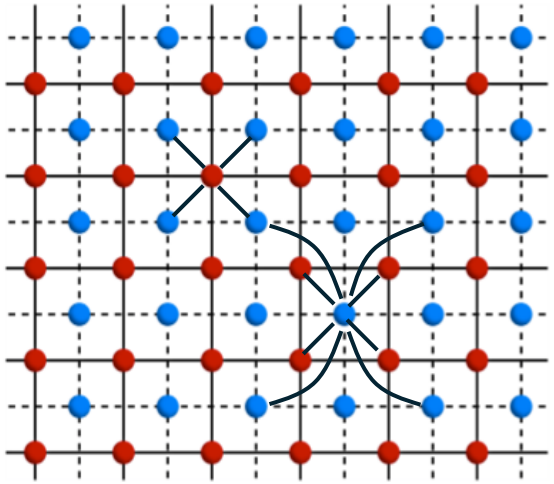
$$\mathrm{H}_{\text{blue}} = \sum_{v_r} Z_{v_b} Z_{v_b} \boxed{X_{v_r}} Z_{v_b} Z_{v_b} - \sum_{v_b} Y_{v_r} Y_{v_r} \boxed{X_{v_b}} Y_{v_r} Y_{v_r} - \sum_{v_b} Z_{v_b} Z_{v_b} Z_{v_r} Z_{v_r} \boxed{X_{v_b}} Z_{v_r} Z_{v_r} Z_{v_b} Z_{v_b}$$

As far as the $Z_2 \times Z_2$ subsystem symmetries are concerned, this Hamiltonian belongs to the cluster phase; but it is in a different phase when $\boldsymbol{D}^{(2)}$ is also included in the set of symmetries.

➔Ground state is:

$$|\text{blue}\rangle = \prod_{v_b} CZ_{v_b, v_b+(1,1)} CZ_{v_b, v_b+(-1,1)} \prod_{v_r} \prod_{v_b \in \partial(p_b = v_r)} CZ_{v_r, v_b} |+\rangle^{\otimes \Delta_{v_b}} |-\rangle^{\otimes \Delta_{v_r}}$$

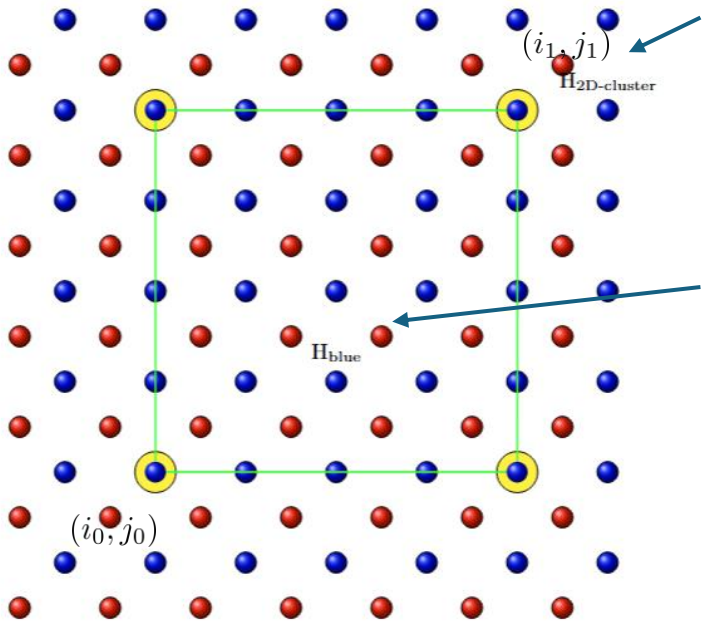
(essentially a graph state)



Only diagonal $\hat{V}^{(2)}$ $\hat{\eta}_b$ is preserved (SSSB)

On the symmetry breaking side, we get a Hamiltonian $\hat{\mathrm{H}}_{\mathrm{red}}$ obtained by $v_b \leftrightarrow v_r$. After applying the KT transformation, we get a Hamiltonian $\mathrm{H}_{\mathrm{red}}$ and its unique ground state $|\mathrm{red}\rangle$ obtained again by $v_b \leftrightarrow v_r$ and $p_b \leftrightarrow p_r$.

Gapless corner modes between H_{blue} and H_{cluster}



$$\mathrm{H}_{\text{2D-cluster}} = -\sum_{v_r} \begin{matrix} & Z_{v_b} & \\ Z_{v_b} & \boxed{X_{v_r}} & Z_{v_b} \\ & Z_{v_b} & \end{matrix} - \sum_{v_b} \begin{matrix} & Z_{v_r} & \\ Z_{v_r} & \boxed{X_{v_b}} & Z_{v_r} \\ & Z_{v_r} & \end{matrix}$$

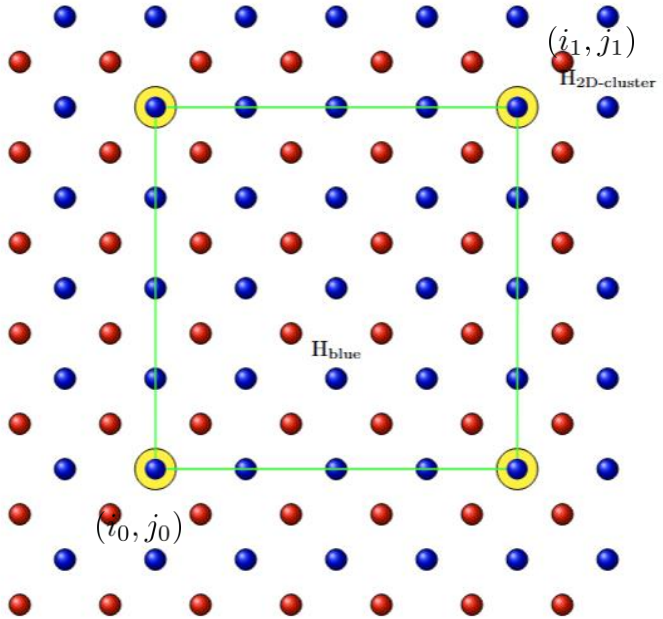
Gapless corner modes between H_{blue} and H_{cluster}

- We could add terms to the Hamiltonian along the interface (except at the corners) that respect subsystem and non-invertible symmetries and commute with each other

$$\mathrm{H}_{\text{interface}} = -\sum_{\substack{v_b=(i+\frac{1}{2},j_1+\frac{1}{2}) \\ i_0<i<i_1}} \begin{matrix} Z & & Z \\ & X_{v_b} & \\ Y & & Y \end{matrix} \begin{pmatrix} & Z & & Z^2_{v_b} & \\ 1+ & & X & & X \\ & Z & & Z^2 & \end{pmatrix} \begin{matrix} Z \\ \\ Z \end{matrix} - \sum_{\substack{v_b=(i+\frac{1}{2},j_0+\frac{1}{2}) \\ i_0<i<i_1}} \begin{matrix} Y & & Y \\ & X_{v_b} & \\ Z & & Z \end{matrix} \begin{pmatrix} & Z & & Z^2 & \\ 1+ & & X & & X \\ & Z & & Z^2_{v_b} & \end{pmatrix} \begin{matrix} Z \\ \\ Z \end{matrix}$$

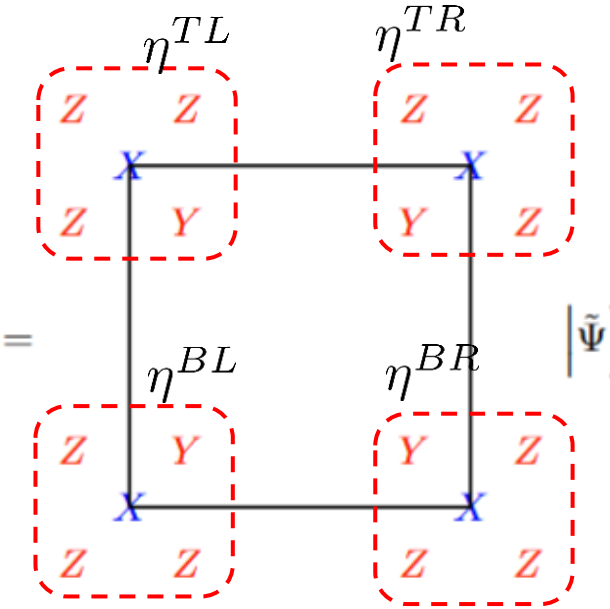
$$- \sum_{\substack{v_b=(i_1+\frac{1}{2},j+\frac{1}{2}) \\ j_0<j<j_1}} \begin{matrix} Y & & Z \\ & X_{v_b} & \\ Y & & Z \end{matrix} \begin{pmatrix} & Z & & Z & \\ & & X & & \\ 1+ & Z^2 & & Z^2_{v_b} & \\ & & X & & \\ & Z & & Z & \end{pmatrix} - \sum_{\substack{v_b=(i_0+\frac{1}{2},j+\frac{1}{2}) \\ j_0<j<j_1}} \begin{matrix} Z & & Y \\ & X_{v_b} & \\ Z & & Y \end{matrix} \begin{pmatrix} & Z & & Z & \\ & & X & & \\ 1+ & Z^2_{v_b} & & Z^2 & \\ & & X & & \\ & Z & & Z & \end{pmatrix}$$

Ground state $\tilde{\Psi}$ properties with interface terms: η's action



❑ Global symmetry η acting on ground state:

$$\eta_r \left|\tilde{\Psi}\right\rangle = \left|\tilde{\Psi}\right\rangle , \quad \eta_b \left|\tilde{\Psi}\right\rangle = \left|\tilde{\Psi}\right\rangle \equiv \begin{matrix} \eta_b^{TL} & \eta_b^{TR} \\ \eta_b^{BL} & \eta_b^{BR} \end{matrix} \left|\tilde{\Psi}\right\rangle$$



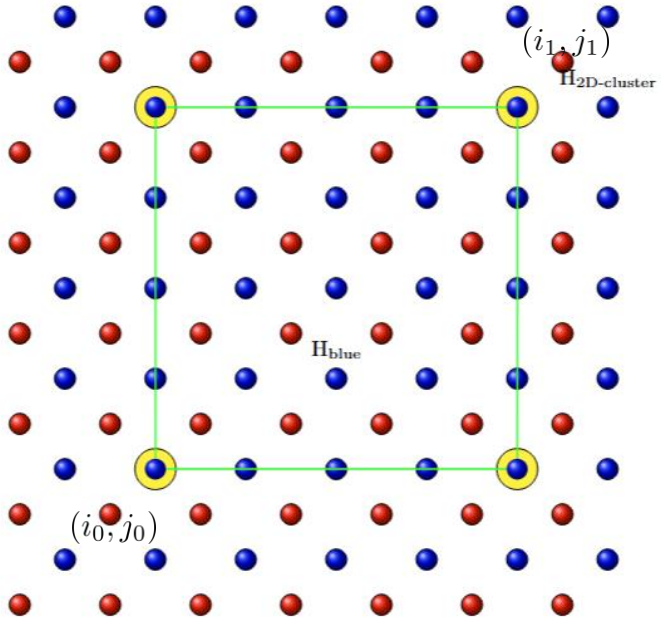
→ Commutation of localized η:

$$\{\eta^{TL}, Z^{TL}\} = 0 , \quad \{\eta^{TR}, Z^{TR}\} = 0$$
$$\{\eta^{BL}, Z^{BL}\} = 0 , \quad \{\eta^{BR}, Z^{BR}\} = 0$$

❑ Subsystem symmetries acting on GS:

$$\eta^x_{r,j}|\tilde{\Psi}\rangle = |\tilde{\Psi}\rangle \quad \forall j, \qquad \eta^y_{r,i}|\tilde{\Psi}\rangle = |\tilde{\Psi}\rangle \quad \forall i, \qquad \eta^x_{b,j}|\tilde{\Psi}\rangle = |\tilde{\Psi}\rangle \quad \forall j \neq j_0, j_1 , \qquad \eta^y_{b,i}|\tilde{\Psi}\rangle = |\tilde{\Psi}\rangle \quad \forall i \neq i_0, i_1 ,$$
$$\eta^x_{b,j_0}|\tilde{\Psi}\rangle = \eta_b^{BL}\eta_b^{BR}|\tilde{\Psi}\rangle , \quad \eta^x_{b,j_1}|\tilde{\Psi}\rangle = \eta_b^{TL}\eta_b^{TR}|\tilde{\Psi}\rangle , \quad \eta^y_{b,i_0}|\tilde{\Psi}\rangle = \eta_b^{BL}\eta_b^{TL}|\tilde{\Psi}\rangle , \quad \eta^x_{b,i_1}|\tilde{\Psi}\rangle = \eta_b^{BR}\eta_b^{TR}|\tilde{\Psi}\rangle$$

Ground state $\tilde{\Psi}$ properties with interface terms: $D^{(2)}$ action



❑ Commutation btwn D and η

$$\mathbf{D}^{(2)}\eta_b^{TL} = -\eta_b^{TL}\mathbf{D}^{(2)}\,, \quad \mathbf{D}^{(2)}\eta_b^{TR} = -\eta_b^{TR}\mathbf{D}^{(2)}\,,$$
$$\mathbf{D}^{(2)}\eta_b^{BL} = -\eta_b^{BL}\mathbf{D}^{(2)}\,, \quad \mathbf{D}^{(2)}\eta_b^{BR} = -\eta_b^{BR}\mathbf{D}^{(2)}\,.$$

$$\mathbf{D}^{(2)} \begin{matrix} Z^{TL} & Z^{TR} \\ Z^{BL} & Z^{BR} \end{matrix} |\tilde{\Psi}\rangle = \begin{matrix} Z^{TL} & Z^{TR} \\ Z^{BL} & Z^{BR} \end{matrix} \mathbf{D}^{(2)} |\tilde{\Psi}\rangle$$

❑ D's action on GS ➔ localized actions

$$\mathbf{D}^{(2)}|\tilde{\Psi}\rangle \sim \big(Z^{TR}Z^{TL}Z^{BR}Z^{BL} + \mathrm{D}^{TL}\mathrm{D}^{TR}Z^{BL}Z^{BR} + \mathrm{D}^{TL}\mathrm{D}^{BR}Z^{TR}Z^{BL} + \mathrm{D}^{BR}\mathrm{D}^{TR}Z^{BL}Z^{TL}$$
$$+\mathrm{D}^{TL}\mathrm{D}^{BL}Z^{TR}Z^{BR} + \mathrm{D}^{BL}\mathrm{D}^{TR}Z^{BR}Z^{TL} + \mathrm{D}^{BR}\mathrm{D}^{BL}Z^{TL}Z^{TR} + \mathrm{D}^{TL}\mathrm{D}^{TR}\mathrm{D}^{BL}\mathrm{D}^{BR}\big)|\tilde{\Psi}\rangle$$

$$\mathrm{D}^{TL} = Z^{TL}\eta_b^{TL}\,, \quad \mathrm{D}^{TR} = Z^{TR}\eta_b^{TR}\,, \quad \mathrm{D}^{BL} = Z^{BL}\eta_b^{BL}\,, \quad \mathrm{D}^{BR} = Z^{BR}\eta_b^{BR}$$

❑ Projective algebra:

➢ guarantee gapless corner modes

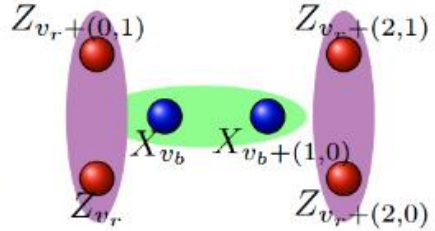
$$\{\mathrm{D}^{TL}, Z^{TL}\} = 0\,, \quad \{\mathrm{D}^{TL}, \eta_b^{TL}\} = 0\,, \quad \{Z^{TL}, \eta_b^{TL}\} = 0\,,$$
$$\{\mathrm{D}^{TR}, Z^{TR}\} = 0\,, \quad \{\mathrm{D}^{TR}, \eta_b^{TR}\} = 0\,, \quad \{Z^{TR}, \eta_b^{TR}\} = 0\,,$$
$$\{\mathrm{D}^{BL}, Z^{BL}\} = 0\,, \quad \{\mathrm{D}^{BL}, \eta_b^{BL}\} = 0\,, \quad \{Z^{BL}, \eta_b^{BL}\} = 0\,,$$
$$\{\mathrm{D}^{BR}, Z^{BR}\} = 0\,, \quad \{\mathrm{D}^{BR}, \eta_b^{BR}\} = 0\,, \quad \{Z^{BR}, \eta_b^{BR}\} = 0\,.$$

[See also results by Y. Furukawa, arXiv:2505.11419]

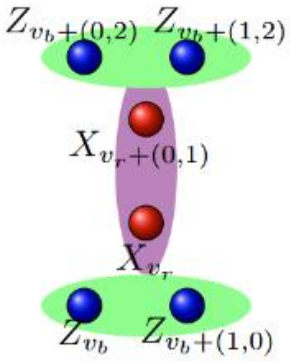
$\hat{V}^{(2)}\hat{\eta}_r\hat{\eta}_b$ is not anomalous in 2d and higher

- Unlike in 1d, we show that $\hat{V}^{(2)}\hat{\eta}_r\hat{\eta}_b$ is anomaly-free by explicitly constructing a gapped Hamiltonian with a unique GS that respects the symmetry

$$\mathrm{H}_{XX-ZZ} = -\sum_{\substack{v_b=(i+\frac{1}{2},j+\frac{1}{2}) \\ i+j=0 \bmod 2}} X_{v_b} X_{v_b+(1,0)} - \sum_{\substack{v_b=(i+\frac{1}{2},j+\frac{1}{2}) \\ i+j=0 \bmod 2}} Z_{v_b} Z_{v_b+(1,0)} - \sum_{\substack{v_b=(i+\frac{1}{2},j+\frac{1}{2}) \\ i+j=0 \bmod 2}}$$

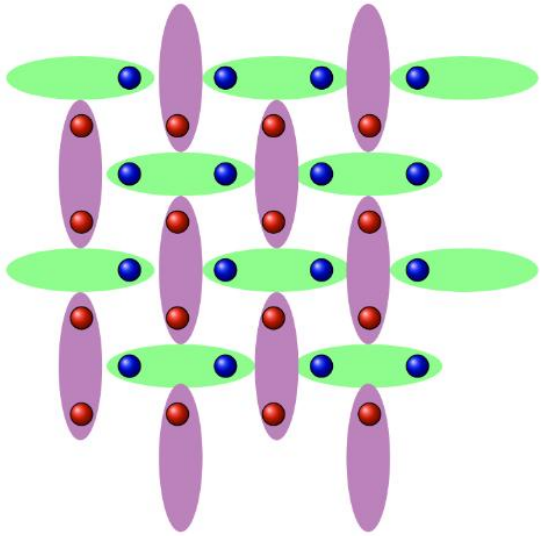


$$- \sum_{\substack{v_r=(i,j) \\ i+j=0 \bmod 2}} X_{v_r+(0,1)} X_{v_r} - \sum_{\substack{v_r=(i,j) \\ i+j=0 \bmod 2}} Z_{v_r+(0,1)} Z_{v_r} - \sum_{\substack{v_r=(i,j) \\ i+j=0 \bmod 2}}$$



,

➔Ground state is a product of (00+11)



- However, we have not been able to construct the associated non-invertible SSPT phases ☹

Summary

- ❑ Gave (2+1)d and higher-dimensional generalization of KW transformation (with circuits + projections) to generalized Ising models on hypercubes

- ❑ Gave (2+1)d and higher-dimensional generalization of KT transformation from *subsystem* symmetry-protected topological (SSPT) phases to *subsystem* spontaneous symmetry-breaking (SSSB) phases (with circuits + KW)

  - ✓ $Z_2 \times Z_2$ case (two sublattices: red and blue)
  - ✓ Z_2 case

- ❑ Gave (2+1)d and higher dimensional generalization of non-invertible symmetry protected SSPT phases with corner gapless modes

  - Also another (3+1)d model with gapless hinge modes (not discussed)

- ❑ Still searching SSPT models with $D^{(2)}\ \eta_r\ \eta_b$ non-invertible symmetry