# Topological defects in lattice models and affine Temperley–Lieb algebra


Jesper L. Jacobsen [1,2,3]

[1] 'Ecole Normale Supérieure, Paris (Laboratoire de Physique)

[2] Sorbonne Université, Paris (Faculté des Sciences et Ingénierie)

[3] Commisariat à l'Energie Atomique, Saclay (Institut de Physique Théorique)


New advancements on defects and their applications,
Yukawa Institute of Theoretical Physics, Kyoto, 15 July 2025

Collaborators: Jonathan Belletête, Azat Gainutdinov, Hubert Saleur, Thiago Tavares

# Topological defects

## In Conformal Field Theory (CFT)

- Non-contractible curve separating two CFT
- Matching condition between two sides
- Consider two identical CFT with continuity of stress tensor
- Toplological defect:
  Curve can be deformed (without crossing field insertions)

# Topological defects

## In Conformal Field Theory (CFT)

- Non-contractible curve separating two CFT
- Matching condition between two sides
- Consider two identical CFT with continuity of stress tensor
- Toplological defect:
  Curve can be deformed (without crossing field insertions)

## Lattice structures

- Lattice version of Virasoro algebra [Koo-Saleur]
- Fusion of primaries in a lattice construction
- Modular invariance on the lattice
- Topological defects on the lattice [Kadanoff-Ceva]

## A different perspective

- Topological defects often defined in minimal models (Ising etc).
- More innate topological control if working outright with models defined in terms of non-local objects.
- Braided tensor categories.
- Gives access to non-unitary, non-rational CFT (in 2D).

## A different perspective

- Topological defects often defined in minimal models (Ising etc).
- More innate topological control if working outright with models defined in terms of non-local objects.
- Braided tensor categories.
- Gives access to non-unitary, non-rational CFT (in 2D).

## Examples

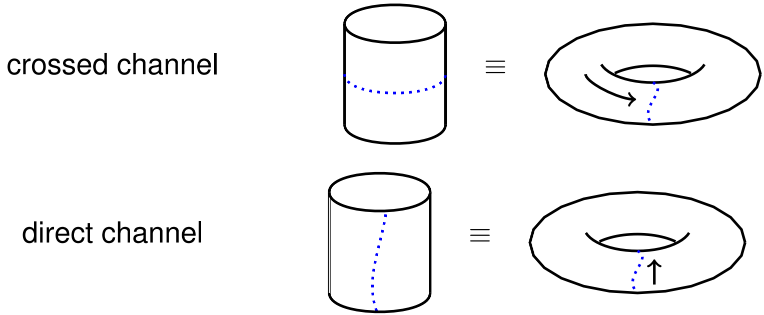
- $Q$-state Potts model $\leftrightarrow$ Temperley-Lieb algebra $\leftrightarrow$ $A_1^{(1)}$ spin chain
- O($n$) loop model $\leftrightarrow$ Dilute TL algebra $\leftrightarrow$ $A_2^{(2)}$ spin chain
- $A_2$ web model $\leftrightarrow$ Kuperberg spider $\leftrightarrow$ $G_2^{(1)}$ spin chain
- Global symmetries ($S_Q$, O($n$), $W_3$) beyond Virasoro.
- Quantum groups ($U_q(sl_2)$, $U_q(sl_3)$) by Schur-Weyl duality.

# Non-unitary (or even logarithmic) CFT

- Define lattice topological defects from an algebraic approach
- Loop models and (affine) Temperley-Lieb algebra
- Radial quantisation

# Non-unitary (or even logarithmic) CFT

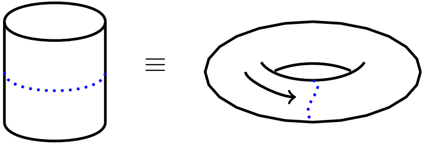
- Define lattice topological defects from an algebraic approach
- Loop models and (affine) Temperley-Lieb algebra
- Radial quantisation



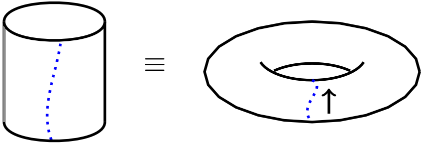
# Non-unitary (or even logarithmic) CFT

- Define lattice topological defects from an algebraic approach
- Loop models and (affine) Temperley-Lieb algebra
- Radial quantisation

crossed channel



direct channel



- In crossed channel, topological defect $X$ such that $[L_n, X] = 0 = [\bar{L}_n, X]$

# Affine Temperley-Lieb algebra

- $n$ lattice sites
- Complex parameter $\mathfrak{q}$, corresponding to loop weight $\mathfrak{q} + \mathfrak{q}^{-1}$
- Definition in terms of generators and relations

# Affine Temperley-Lieb algebra

- $n$ lattice sites
- Complex parameter $\mathfrak{q}$, corresponding to loop weight $\mathfrak{q} + \mathfrak{q}^{-1}$
- Definition in terms of generators and relations

## First (usual) definition

- Arc generators $e_1, \ldots, e_n$ with index defined modulo $n$
- Shift generators $u, u^{-1}$
- Relations

$$
\begin{aligned}
e_i e_i &= (\mathfrak{q} + \mathfrak{q}^{-1}) e_i , \\
e_i e_{i\pm 1} e_i &= e_i , \\
e_i e_j &= e_j e_i \ \text{ if } |i-j| \geq 2 , \\
u e_i &= e_{i+1} u , \\
u^2 e_{n-1} &= e_1 \ldots e_{n-1} ,
\end{aligned}
$$

## Second (alternative) definition

- Arc generators $e_1, \ldots, e_{n-1}$
- Blob generators $b, b^{-1}$
- Hoop operator (central element!) $Y \equiv -\mathfrak{q} b - \mathfrak{q}^{-1} b^{-1}$
- Relations

$$
\begin{aligned}
e_i e_i &= (\mathfrak{q} + \mathfrak{q}^{-1}) e_i, \\
e_i e_{i\pm 1} e_i &= e_i, \\
e_i e_j &= e_j e_i \qquad \text{if } |i-j| \geq 2, \\
e_i b &= b e_i \qquad \text{if } i \geq 2, \\
e_1 b e_1 &= (\underbrace{\mathfrak{q} b + \mathfrak{q}^{-1} b^{-1}}_{\equiv -Y}) e_1 = e_1 (\mathfrak{q} b + \mathfrak{q}^{-1} b^{-1}),
\end{aligned}
$$

- Looks similar to blob algebra [Martin-Saleur]
- But here $b$ is invertible, and $Y$ is an operator
- In blob algebra, $b^2 = b$ and $Y$ is replaced by a complex number (quotient)

- Looks similar to blob algebra [Martin-Saleur]
- But here $b$ is invertible, and $Y$ is an operator
- In blob algebra, $b^2 = b$ and $Y$ is replaced by a complex number (quotient)

## Relation between formulations: Braid translation

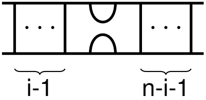
- Take

$$\begin{aligned} e_i &= e_i, \qquad 1 \leq i \leq n-1, \\ b &= (-\mathfrak{q})^{-3/2} g_1^{-1} \dots g_{n-1}^{-1} u^{-1}, \\ b^{-1} &= (-\mathfrak{q})^{3/2} u\, g_{n-1} \dots g_1, \end{aligned}$$

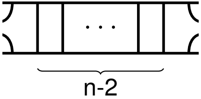
- With braid generators $g_i^{\pm 1} = (-\mathfrak{q})^{\pm 1/2} 1 + (-\mathfrak{q})^{\mp 1/2} e_i$
- They satisfy the braid relations $g_i g_{i\pm1} g_i = g_{i\pm1} g_i g_{i\pm1}$

# Easier to work with diagrams

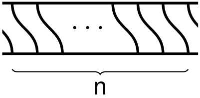
- Generators

$e_i =$ 

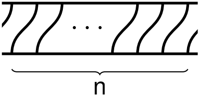
,

$e_n =$ 

,

$u =$ 

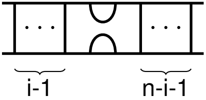
,

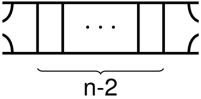
$u^{-1} =$ 

.

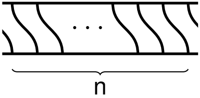
# Easier to work with diagrams

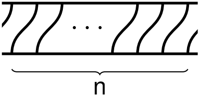
- Generators

$e_i =$ 

, $e_n =$ 

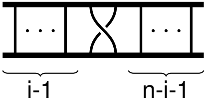
,

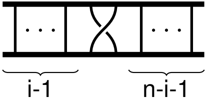
$u =$ 

, $u^{-1} =$ 

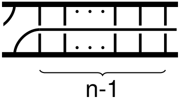
.

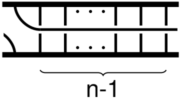
- Braids

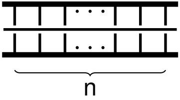
$$g_i = (-\mathfrak{q})^{\frac{1}{2}} \mathbf{1} + (-\mathfrak{q})^{-\frac{1}{2}} e_i =$$



$$g_i^{-1} = (-\mathfrak{q})^{-\frac{1}{2}} \mathbf{1} + (-\mathfrak{q})^{\frac{1}{2}} e_i =$$


# The central element $Y = -(\mathfrak{q}b + \mathfrak{q}^{-1}b^{-1})$

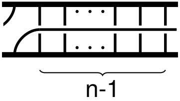
$Y = (-\mathfrak{q})^{-\frac{1}{2}}$ 

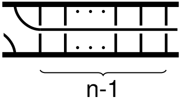
$+ (-\mathfrak{q})^{\frac{1}{2}}$ 

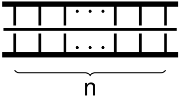
$=$ 

.

# The central element $Y = -(\mathfrak{q} b + \mathfrak{q}^{-1} b^{-1})$

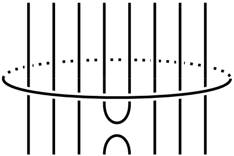
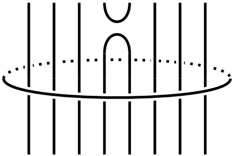
$Y = (-\mathfrak{q})^{-\frac{1}{2}}$ 

$+ (-\mathfrak{q})^{\frac{1}{2}}$ 

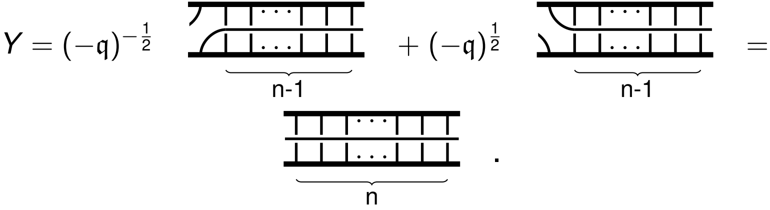
$=$ 

.

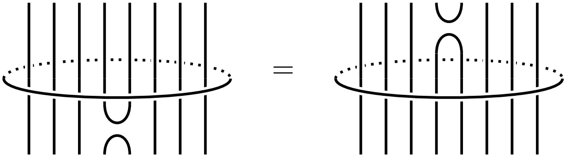
- This makes it clear that $[Y, \mathrm{aTL}_n(\mathfrak{q})] = 0$:


$=$


# The central element $Y = -(\mathfrak{q}b + \mathfrak{q}^{-1}b^{-1})$




- This makes it clear that $[Y, \mathrm{aTL}_n(\mathfrak{q})] = 0$:

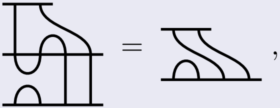


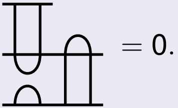
- Similarly define barred quantities via underpassings (using algebra automorphism $b \to b^{-1}$ and $\mathfrak{q} \to \mathfrak{q}^{-1}$).

# Standard modules

## Case of regular Temperley-Lieb algebra $\mathrm{TL}_n(\mathfrak{q})$

- Diagrams with $n$ nodes on the bottom, and $k$ nodes (= defects) on the top.
- Action via diagram stacking (on the bottom):



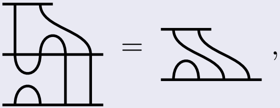
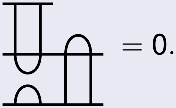




- These standard modules are denoted $\mathrm{S}_k(n)$.

# Standard modules

## Case of regular Temperley-Lieb algebra $\mathrm{TL}_n(\mathfrak{q})$

- Diagrams with $n$ nodes on the bottom, and $k$ nodes (= defects) on the top.
- Action via diagram stacking (on the bottom):

- These standard modules are denoted $\mathrm{S}_k(n)$.

## Case of affine Temperley-Lieb algebra $\mathrm{aTL}_n(\mathfrak{q})$

- This becomes $\infty$-dimensional, due to winding of defect lines.
- To get a finite-dimensional module $\mathrm{W}_{k,z}(n)$, fix the eigenvalues of central elements $-Y = \mathfrak{q}b + \mathfrak{q}^{-1}b^{-1}$ and $-\bar{Y} = \mathfrak{q}\bar{b} + \mathfrak{q}^{-1}\bar{b}^{-1}$

- Attribute weight $z$ (or $z^{-1}$) to each winding defect line.
- This fixes

$$Y = -(\mathfrak{q} b + \mathfrak{q}^{-1} b^{-1}) = z(-\mathfrak{q})^k + z^{-1}(-\mathfrak{q})^{-k},$$
$$\bar{Y} = -(\mathfrak{q} \bar{b} + \mathfrak{q}^{-1} \bar{b}^{-1}) = z(-\mathfrak{q})^{-k} + z^{-1}(-\mathfrak{q})^{k}.$$

- Attribute weight $z$ (or $z^{-1}$) to each winding defect line.
- This fixes

$$Y = -(\mathfrak{q} b + \mathfrak{q}^{-1} b^{-1}) = z(-\mathfrak{q})^{k} + z^{-1}(-\mathfrak{q})^{-k},$$
$$\bar{Y} = -(\mathfrak{q} \bar{b} + \mathfrak{q}^{-1} \bar{b}^{-1}) = z(-\mathfrak{q})^{-k} + z^{-1}(-\mathfrak{q})^{k}.$$

## A convenient notation

$$\mathsf{W}^{o}_{\pm|k|,\delta}(n) \equiv \mathsf{W}_{|k|,\delta^{\pm 1}(-\mathfrak{q})^{-k}}(n), \qquad \mathsf{W}^{u}_{\pm|k|,\mu}(n) \equiv \mathsf{W}_{|k|,\mu^{\pm 1}(-\mathfrak{q})^{k}}(n).$$

- This fixes the eigenvalues of $Y = \delta + \delta^{-1}$ and $\bar{Y} = \mu + \mu^{-1}$.
- Superscript $o/u$ stands for over / under.

# Structure of modules for generic $\mathfrak{q}$ [Graham-Lehrer]

- There exists a non-zero morphism $f\colon \mathsf{W}_{s,w}(n) \to \mathsf{W}_{r,z}(n)$ iff $s \geq r$ and $Y, \bar{Y}$ have the same eigenvalues on both modules.
- This morphism is proportional to the identity (for $s = r$) or to a unique injective map.
- Equality of eigenvalues gives conditions

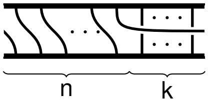
$$z = \begin{cases} w(-\mathfrak{q})^{r-s} & \text{if } (-\mathfrak{q})^{2(r-s)} = 1 \text{ or } w^2 = (-\mathfrak{q})^{2r}, \\ w^{-1}(-\mathfrak{q})^{r+s} & \text{if } (-\mathfrak{q})^{2(r+s)} = 1 \text{ or } w^2 = (-\mathfrak{q})^{-2r}. \end{cases}$$

- Each standard module has a unique simple quotient $\overline{\mathsf{W}}_{r,z}(n)$.
- These form a complete set of irreducible modules.

# Tower structure

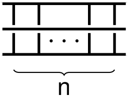
- Inclusions $\mathrm{aTL}_n \subset \mathrm{aTL}_{n+1} \subset \mathrm{aTL}_{n+2} \subset \ldots$.
- Define morphism $\phi^u_{n,k}:\ \mathrm{aTL}_n \to \mathrm{aTL}_{n+k}$ from action on generators.
- We have simply $\left(b^{(n)}\right)^{\pm 1} \mapsto \left(b^{(n+k)}\right)^{\pm 1}$ and $e_i^{(n)} \mapsto e_i^{(n+k)}$.
- Less trivially:

$$\phi^u_{n,k}: \quad u^{(n)} \mapsto u^{(n+k)} g^{(n+k)}_{n+k-1} g^{(n+k)}_{n+k-2} \cdots g^{(n+k)}_{n} =$$



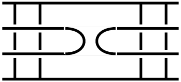

- Similar definition of $\phi^o_{n,k}:\ \mathrm{aTL}_n \to \mathrm{aTL}_{n+k}$ using overpasses

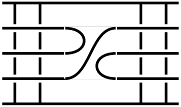
# Lattice topological defects in the crossed channel

- Recall $Y = -(\mathfrak{q} b + \mathfrak{q}^{-1} b^{-1}) =$  .



- $Y^m$ similarly generates a defect of width $m$.
- TL operators can act on this *horizontally*:

$$Y^2(e_1) =$$  $$= (\mathfrak{q} + \mathfrak{q}^{-1}) 1_{\mathrm{aTL}_n},$$

$$Y^3(e_1 e_2) =$$  $$= Y.$$

- Corresponds to taking a Markov trace in the horizontal direction.
- Map from $\mathrm{TL}_m$ to the ring of endomorphisms of $\mathrm{aTL}_n$:
  $Y^m: \ \mathrm{TL}_m \to \mathrm{End}_{\mathrm{aTL}_n}(\mathrm{aTL}_n)$.

- Now act instead on an entire ideal $\mathsf{S}_j(m) = \mathsf{TL}_m P_j$, where $P_j$ is an idempotent of spin $j$.
- For $j = m/2$, use Jones-Wenzl projectors $P_{m/2} = W_1^{m+1}$ defined recursively via

$$W_i^1(n) \equiv W_i^2(n) \equiv 1_{\mathsf{TL}_n},$$
$$W_i^m(n) \equiv W_{i+1}^{m-1}(n) \left( 1_{\mathsf{TL}_n} - \frac{\mathfrak{q}^{m-2} - \mathfrak{q}^{2-m}}{\mathfrak{q}^{m-1} - \mathfrak{q}^{1-m}} e_i \right) W_{i+1}^{m-1}(n).$$

- It follows that $Y^m(\mathsf{S}_j(m)) = \mathbb{C} Y_j$ with the central element $Y_j := Y^{2j}(W_1^{2j+1})$.
- More precisely $Y_j = \mathsf{U}_{2j}\left(\frac{1}{2} Y\right)$ with $\mathsf{U}_k(x)$ the order-$k$ Chebyshev polynomial of the second kind.

# Fusion of defects

- Using functional relation for $\mathrm{U}_k(x)$, we obtain *the fusion property of defects*:

$$Y_j \cdot Y_k = \sum_{r=|j-k|}^{j+k} Y_r .$$

- Recall $Y$ acts on $\mathrm{W}^{o}_{k,\delta}$ as $(\delta + \delta^{-1})$.
- Writing $\delta = e^{i\theta}$, the higher-spin operator eigevalues are thus

$$Y_j = \frac{\sin((2j+1)\theta)}{\sin\theta} .$$

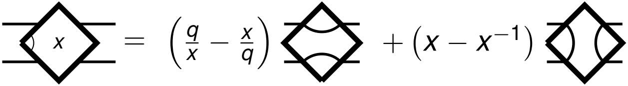
- Similarly for $\bar{Y}_j$.

# Properties

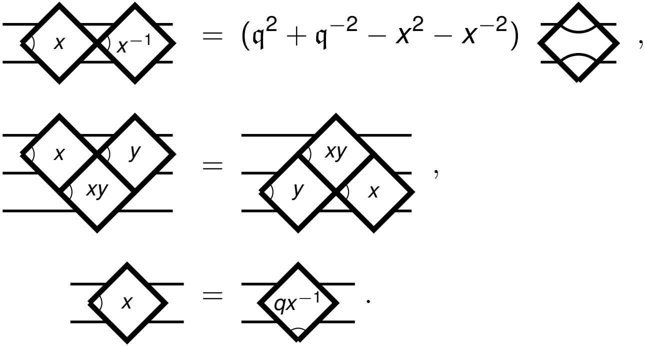
- Define the *symmetric centre* $\mathrm{Z_{sym}}$ as the algebra generated by $Y_j$ and $\bar{Y}_k$.
- Products of Chebyshev polynomials in $Y$ and $\bar{Y}$ provide a “canoncical" basis in $\mathrm{Z_{sym}}$ with *non-negative integer* structure constants.
- We interpret these as fusion rules of the defects.
- We can write $Y_j$ as symmetric functions of Jucys-Murphy elements.
- Not so clear if $\mathrm{Z_{sym}}$ is the whole centre of $\mathrm{aTL}_n(\mathfrak{q})$.

# Lattice topological defects in the direct channel

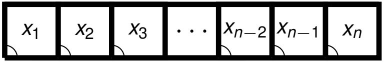
- Integrable formulation in terms of planar tiles




- Inversion relation, Yang-Baxter relation, and crossing symmetry:

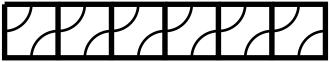


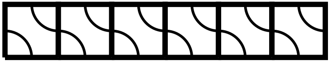
- Transfer matrix

$$T_n(\vec{x}) =$$ 

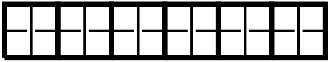
$$, \qquad \vec{x} = \{x_1, x_2, \ldots, x_n\} .$$

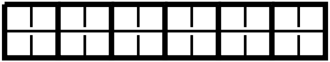
- Shift generators

$$T_n(1) = (\mathfrak{q} - \mathfrak{q}^{-1})^n$$  $$= (\mathfrak{q} - \mathfrak{q}^{-1})^n u^{-1},$$

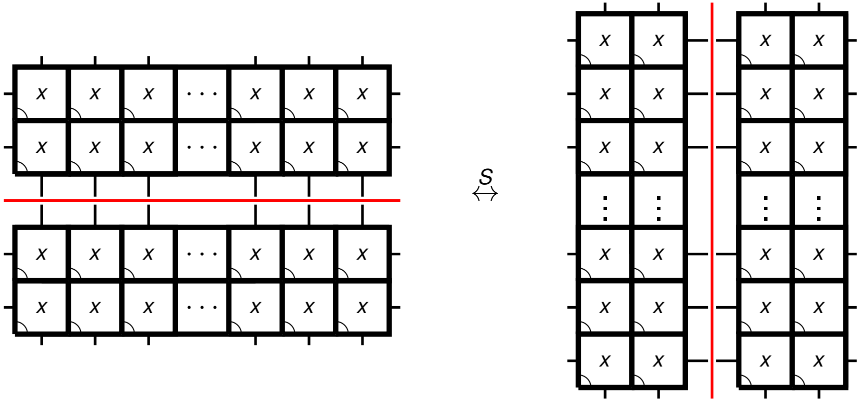
$$T_n(\mathfrak{q}) = (\mathfrak{q} - \mathfrak{q}^{-1})^n$$  $$= (\mathfrak{q} - \mathfrak{q}^{-1})^n u.$$

- Hoop operators

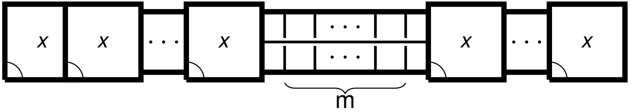
$$\lim_{x \to 0} ((-(-\mathfrak{q})^{-\frac{1}{2}} x)^n T_n(x)) =$$  $$= \bar{Y},$$

$$\lim_{x \to \infty} (((-\mathfrak{q})^{-\frac{1}{2}} x)^{-n} T_n(x)) =$$  $$= Y.$$

Relate crossed and direct channel defects by modular $S$-transformation:



- We obtain a transfer matrix $T_n(x;m)$ carrying a defect of width $m$ going under the other lines:

$T_n(x;k) =$ 


- Its logarithmic derivative at $x = 1$ gives a Hamiltonian on $n + m$ sites:

$$H^u = \sum_{j=1}^{n-1} e_j^{(n+m)} + \mu_{n,m}^{-1} e_n^{(n+m)} \mu_{n,m} \rho,$$

where $\mu_{n,m} = g_n g_{n+1} \dots g_{n+m}$ and $\rho$ is the JW idempotent $W_1^{m+1}$.

- This can be written as

$$H^u = \phi_{n,m}^u \Big( \sum_{j=1}^{n} e_j^{(n)} \Big),$$

# Task: Solve the spectral problem of $H^u$

## Two algebraic formulations

- Fusion product: Add new strands carrying the defect to the module. For example

$$\mathsf{W}^u_{k,\delta} \times^o_f \mathsf{S}_T \simeq \bigoplus_{i=k-t}^{k+t} \mathsf{W}^u_{i,\delta} \simeq \bigoplus_{i=k-T}^{k+T} \mathsf{W}_{i,(-q)^{(i-k)}\delta}.$$

- Fusion quotient: Impose the defect on an existing part of the module.

# Task: Solve the spectral problem of $H^u$

## Two algebraic formulations

- Fusion product: Add new strands carrying the defect to the module. For example

$$\mathsf{W}^u_{k,\delta} \times^o_f \mathsf{S}_T \simeq \bigoplus_{i=k-t}^{k+t} \mathsf{W}^u_{i,\delta} \simeq \bigoplus_{i=k-T}^{k+T} \mathsf{W}_{i,(-q)^{(i-k)}\delta}.$$

- Fusion quotient: Impose the defect on an existing part of the module.

## Main theorem

Let $\rho \in \mathsf{TL}_m$ be an idempotent such that $\mathsf{TL}_m\rho \simeq V$, then for any $\mathsf{aTL}_{n+m}$-module $M$ the Hamiltonian $H_n^{u/o}$ is similar (as a matrix) to the direct sum of the classical Hamiltonian $H_n$ acting on $M \div_f^{u/d} V$ and a zero matrix of dimension $\dim((1-\rho)M)$.

# Conclusions

- Defined lattice operators $Y$ and $\bar{Y}$ in $\mathrm{aTL}_n$ that commuted with local interactions (generators).
- In the crossed channel, they generate an algebra spanned by $Y_j$, $\bar{Y}_j$ and their products.
- Their fusion rules resemble the chiral and anti-chiral fusion rules of Virasoro Kac modules of type $(1, s)$ where $s = 2j + 1$.

# Conclusions

- Defined lattice operators $Y$ and $\bar{Y}$ in $\mathrm{aTL}_n$ that commuted with local interactions (generators).
- In the crossed channel, they generate an algebra spanned by $Y_j$, $\bar{Y}_j$ and their products.
- Their fusion rules resemble the chiral and anti-chiral fusion rules of Virasoro Kac modules of type $(1, s)$ where $s = 2j + 1$.

## Evidence of second kind of topological defect

- Depends on a spectral parameter.
- Not central in $\mathrm{aTL}_n$, but possibly topological in the continuum limit.