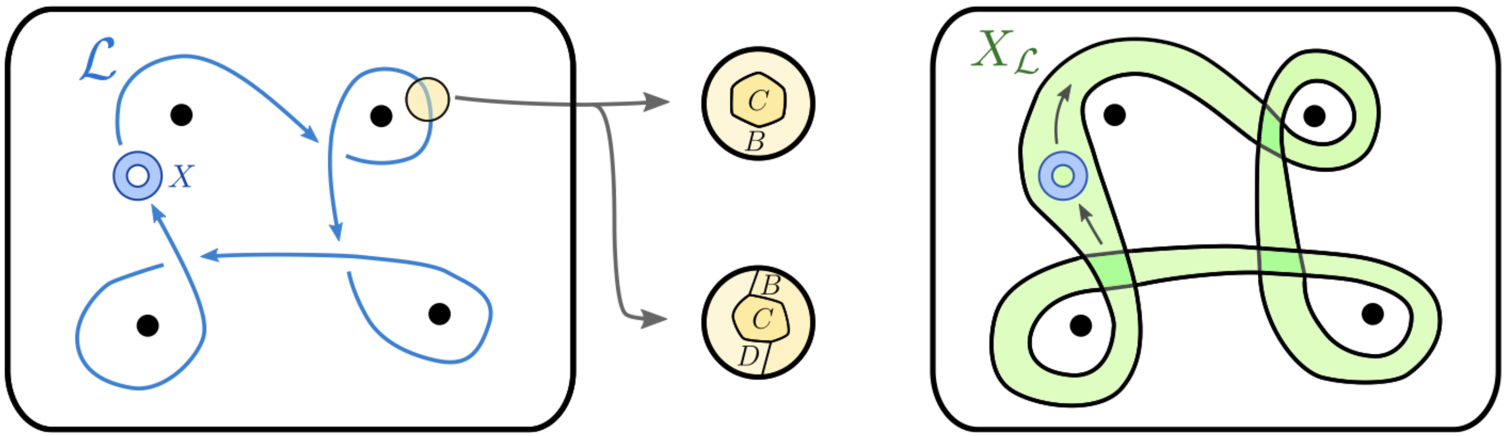
Homogeneity versus defect: an Entanglement Bootstrap view

Bowen Shi, UIUC

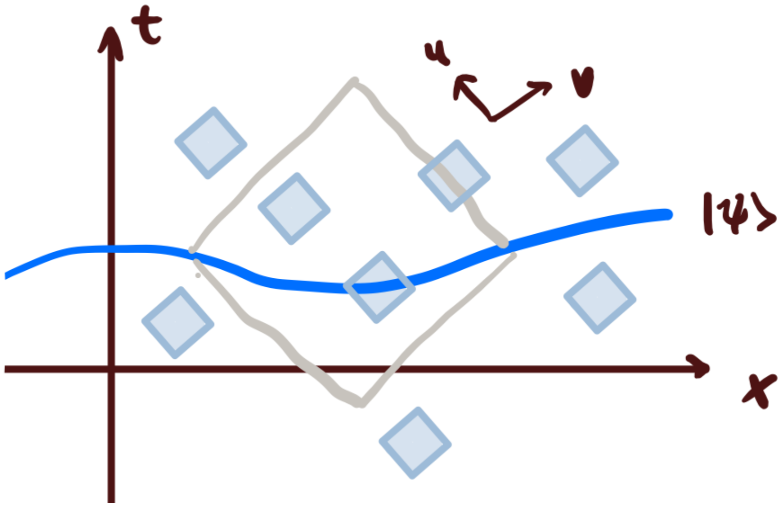
Workshop at the Yukawa Institute for Theoretical Physics, Kyoto University

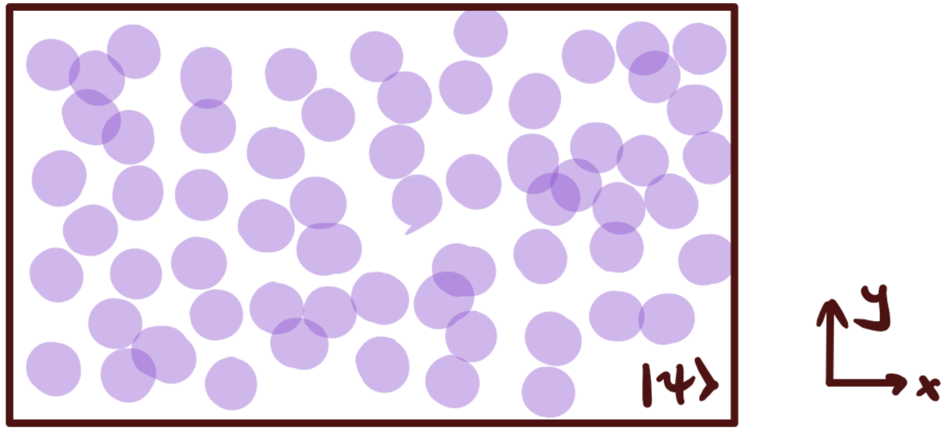
July 16 2025



Warm-up: QFT versus condensed matter

and a few remarks on Quantum Information.

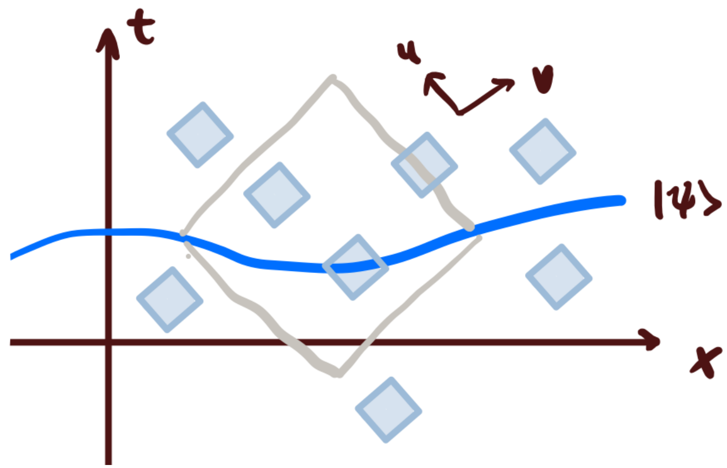




Many-body systems: in QFT and condensed matter

Quantum systems with many parts:

These are many-body systems!



Quantum field theories:

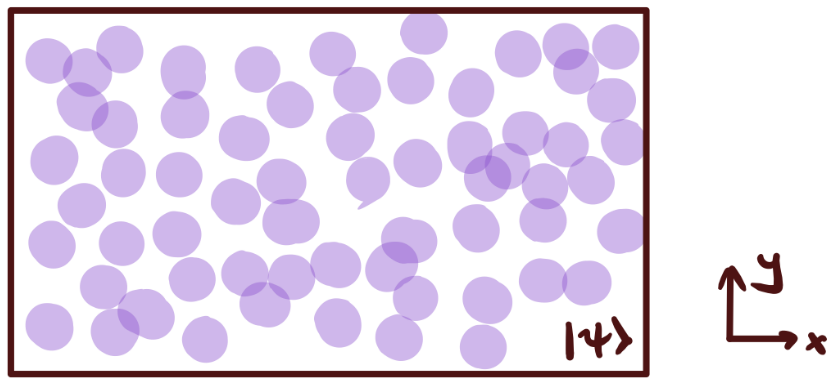
Space and time are often treated on equal footing.

Condensed matter systems:

Many particles are distributed in space.

There is no Lorentz symmetry for such systems.
However, relativistic QFT may **emerge**.

Not even translation symmetry.



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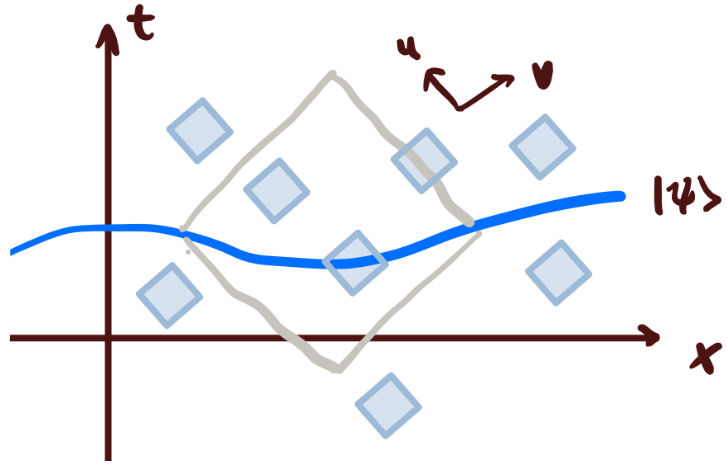

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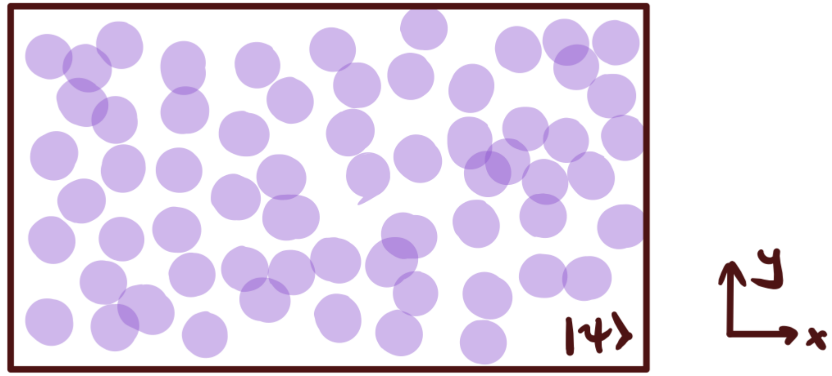

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In this talk, I will define **homogeneity** and **defects** in the framework of **entanglement bootstrap.** *BS, Kohtaro Kato, Isaac Kim 2019, …*

There is a shared interest in entanglement:

Entanglement entropy: (von Neumann entropy)

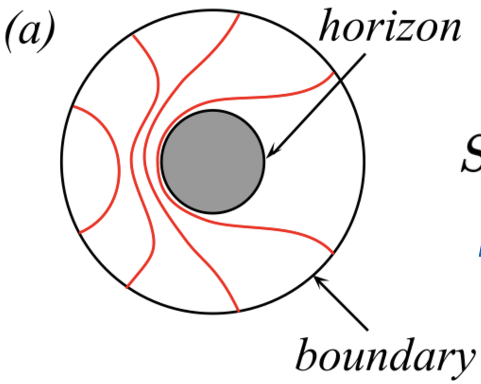
$$S(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A)$$

CFT single interval entropy:

$$S(\rho_A) = \frac{c}{3} \log\left(\frac{L_A}{\epsilon}\right)$$

Calabrese, Cardy 2004

Holographic entanglement entropy:



$$S_A = \frac{\text{Area of } \gamma_A}{4G}$$

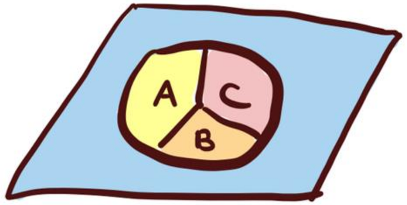
Ryu, Takayanagi 2006

covariant: *Hubeny, Rangamani, Takayanagi 2007*

Topological entanglement entropy:

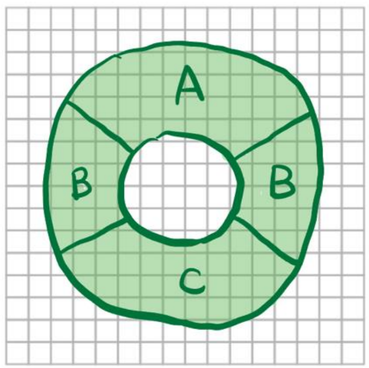
$$S(\rho_A) = \alpha|\partial A| - \gamma$$

$$\gamma = \log \sqrt{\textstyle\sum_a d_a^2}$$



$$S_{AB} + S_{BC} + S_{CA} - S_A - S_B - S_C - S_{ABC} = \gamma$$

Kitaev, Preskill 2005



Levin, Wen 2005

$$2\gamma = S_{AB} + S_{BC} - S_B - S_{ABC}$$

Quantum many-body systems on lattice:

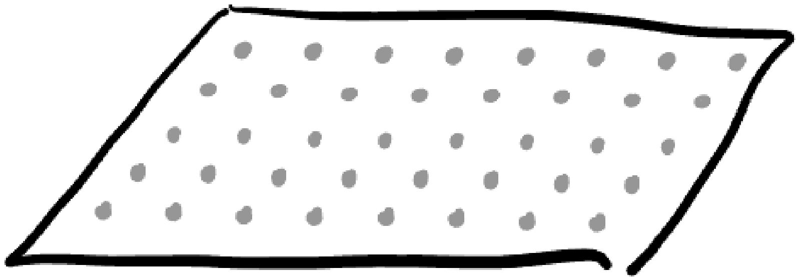
P. W. Anderson 1972

"More is different!"

Emergence!

Emergence! (interesting)

Interacting! (hard)



quantum many-body system

emergent TQFT behavior

fractional excitations: anyons

chiral transport: quantum Hall and quantum thermal Hall

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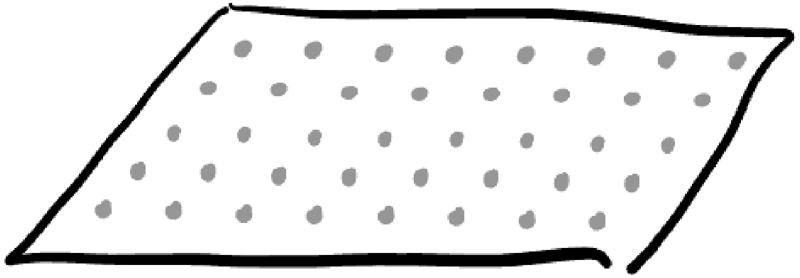


quantum many-body system

emergent TQFT behavior

fractional excitations: anyons

chiral transport: quantum Hall and quantum thermal Hall

Emergence in everyday biology:

pineapple

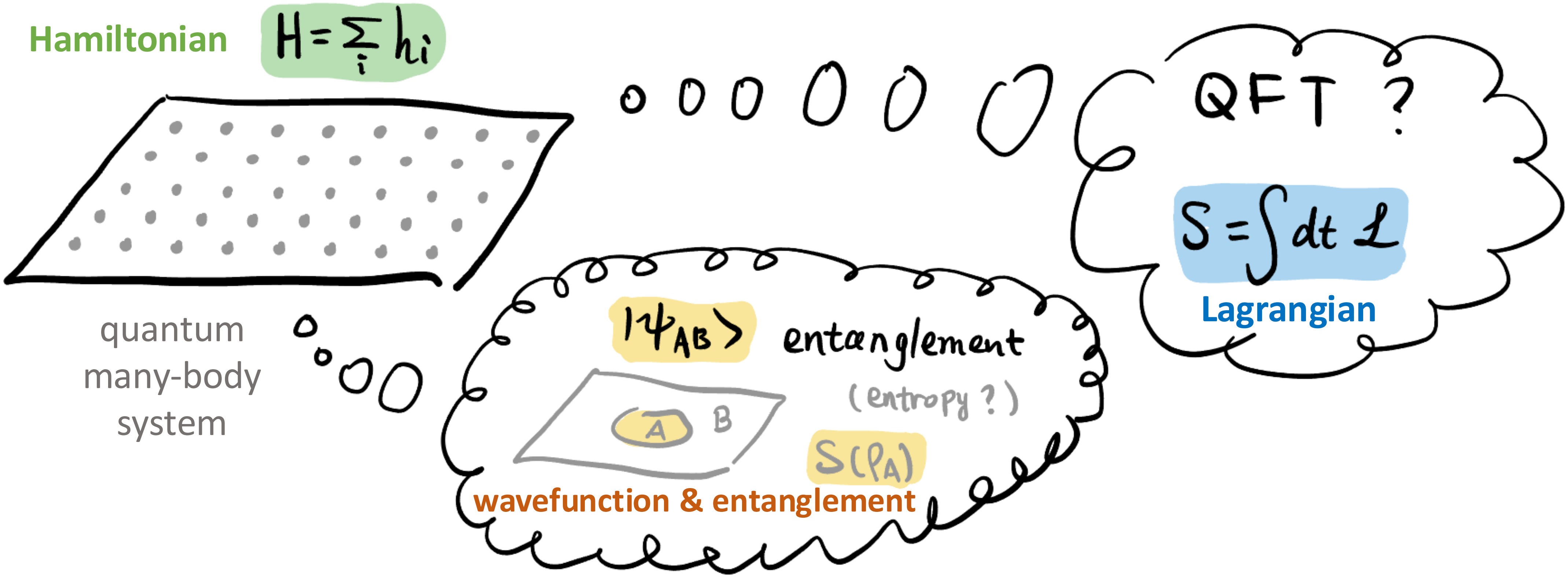
pinecone

Fibonacci numbers 3, 5, 8, 13, 21, ...
on plants. $F_n = F_{n-1} + F_{n-2}$

Quantum many-body systems on lattice:

Emergence! (interesting)

Interacting! (hard)



$$\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$$

$$S(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A).$$

Quantum information tools:

The conditional mutual information:

$$I(A:C|B) \equiv S_{AB} + S_{BC} - S_B - S_{ABC}$$

strong subadditivity: $I(A:C|B)_\rho \geq 0, \quad \forall \rho.$

Lieb, Ruskai 1973

Another frequently used:

$$\Delta(B,C,D) := S_{BC} + S_{CD} - S_B - S_D$$

$$\Delta(B,C,D) \geq I(A:C|B), \quad \rho_{ABCD}.$$

The mutual information is a special case, setting B empty.

$$I(A:C) = S_A + S_C - S_{AC}$$

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Quantum Markov states:

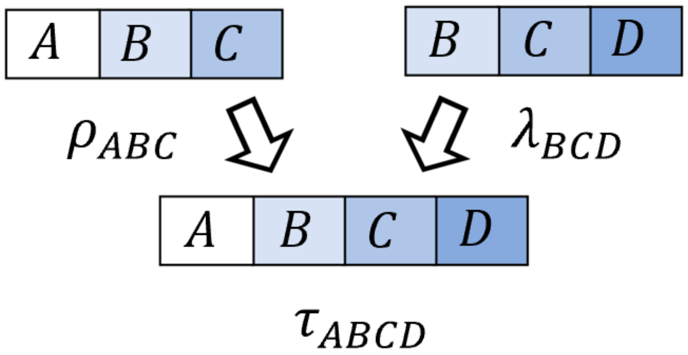
a state ρ_{ABC} such that $I(A:C|B)_\rho = 0$

Petz 2023

A quantum Markov state ρ_{ABC} is uniquely determined by ρ_{AB} and ρ_{BC}.

The merging lemma: *Kato, Furrer, Murao 2015*

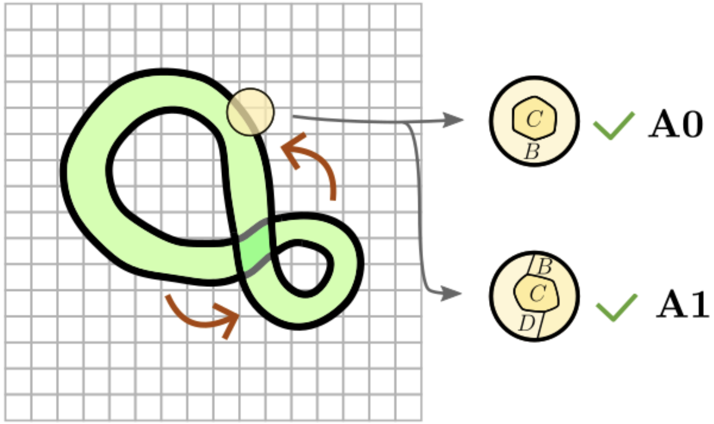
merge a pair of quantum Markov states:



Plan of the rest of the talk:

(1) homogeneity

uniform phase of matter
(obey a certain **law of entanglement**)



(2) defects

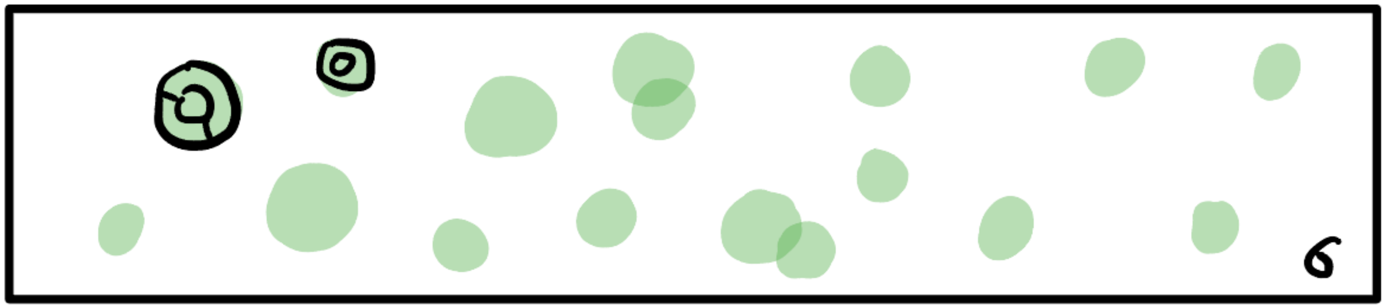
Defects break a certain "smoothness" of the entanglement pattern.

The violation can be shown to be quantized.

point-like (co-dimension 2)
domain walls (co-dimension 1)

For concreteness, I mostly discuss the entanglement bootstrap in 2+1D.
We do not assume global symmetries.
Many tools and conclusions generalize to higher dimensions.

(1) Emergence of topological homogeneity from entanglement



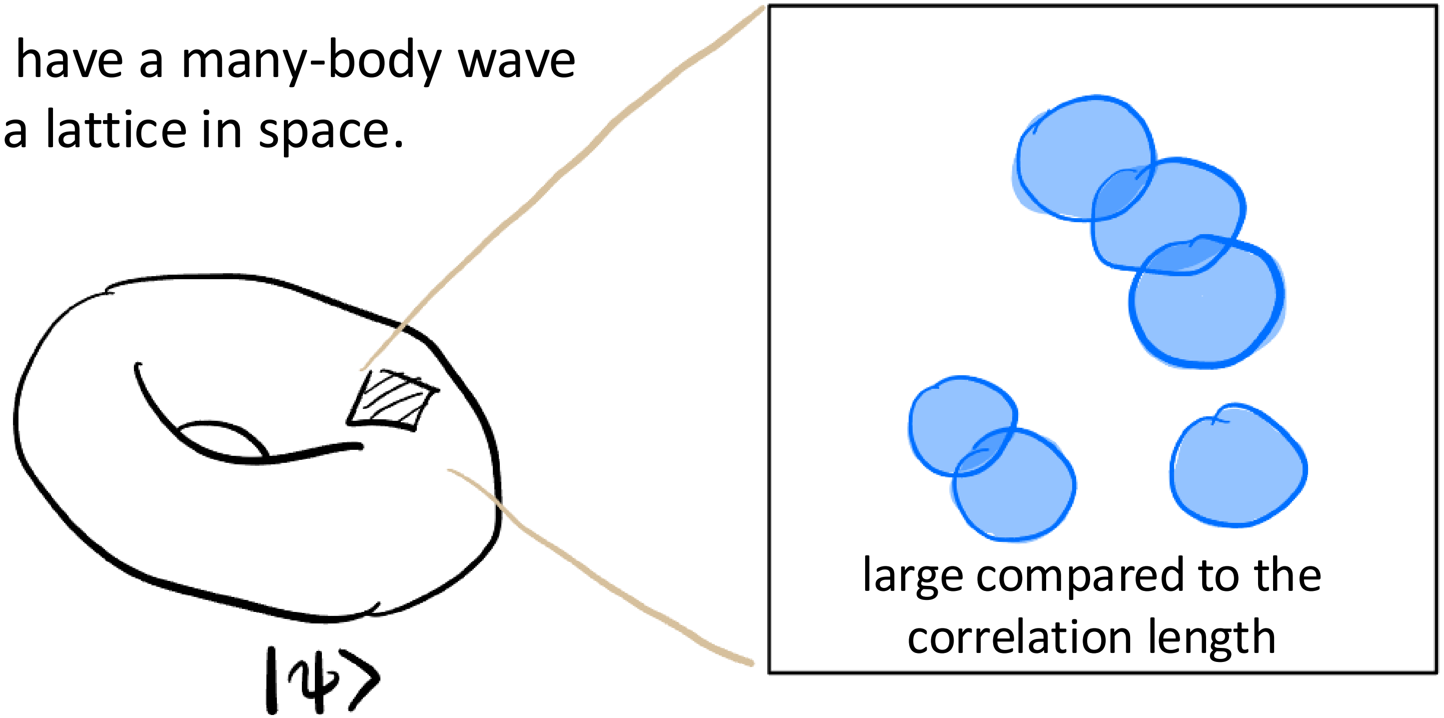
A0: for 

$(S_{BC}+S_{C}-S_{B})_{\sigma}=0$

A1: for 

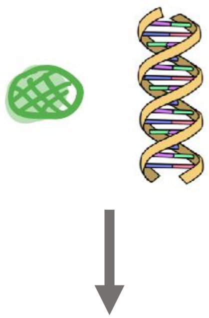
$(S_{BC}+S_{CD}-S_{B}-S_{D})_{\sigma}=0$

Suppose we have a many-body wave function on a lattice in space.

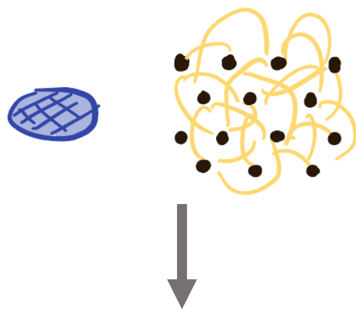


Want: Suppose the state is a ground state of a gapped system. We want to extract the **TQFT data** from a wave function on a ball (square).

DNA analog:

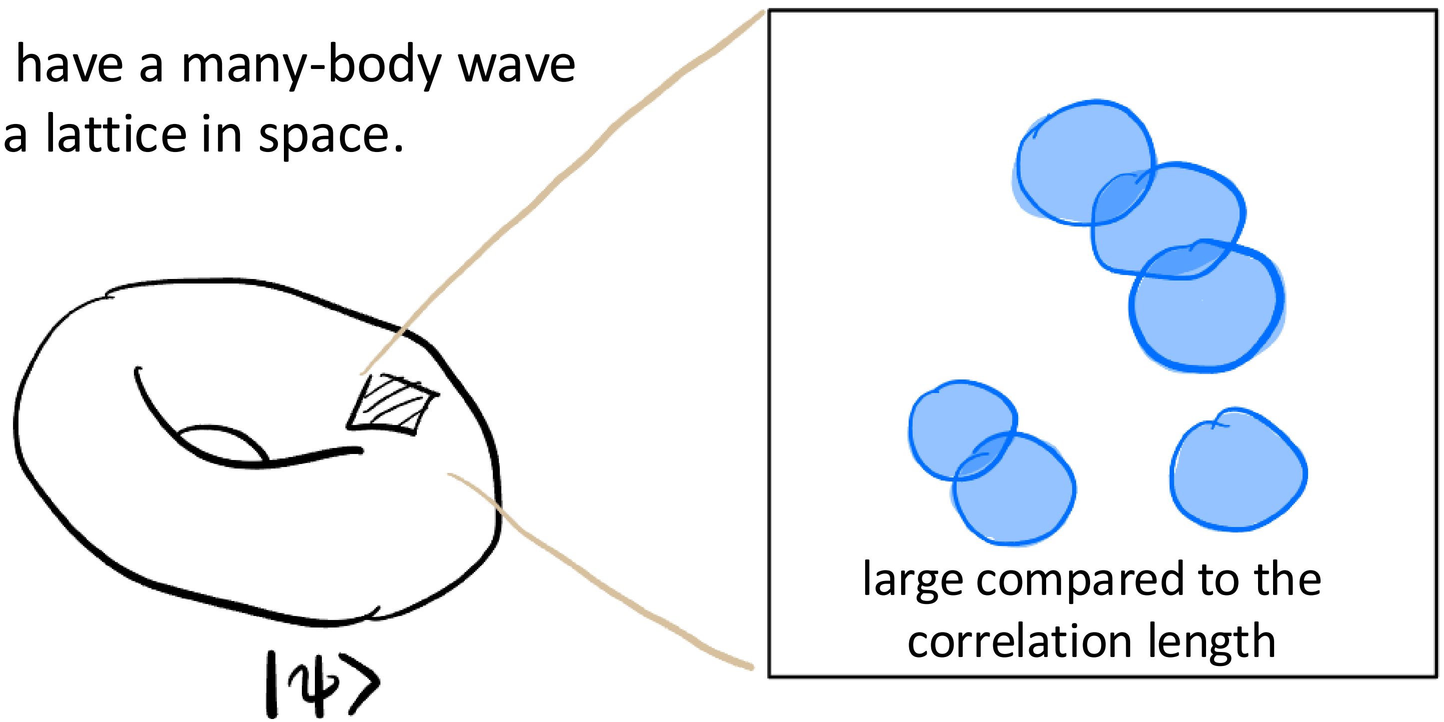


plant species



quantum phase type

Suppose we have a many-body wave function on a lattice in space.

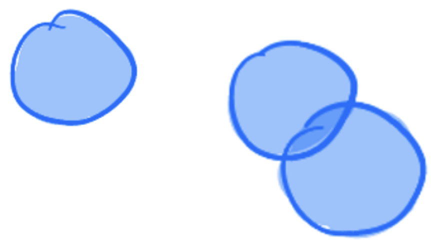


Want: Suppose the state is a ground state of a gapped system. We want to extract the **TQFT data** from a wave function on a ball (square).

Challenge 1: The Hilbert space is huge.

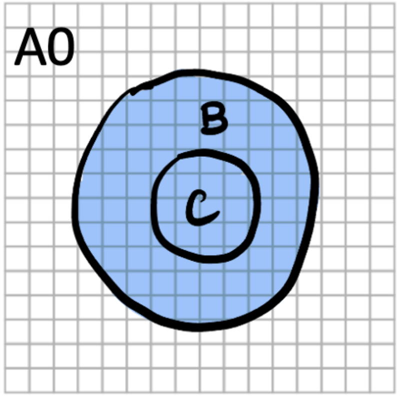
Challenge 2: There is no symmetry to use.

Idea: put **entanglement constraints** on local balls

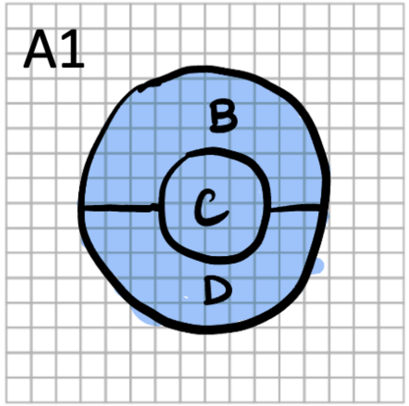


Entanglement bootstrap axioms and philosophy:

Two axioms, **A0** and **A1**, on a **reference quantum state.**



$$S_{BC} + S_C - S_B = 0$$



$$S_{BC} + S_{CD} - S_B - S_D = 0$$

BS, Kohtaro Kato, Isaac Kim 2019, Isaac Kim 2015

Ways to understand the axioms:

1. compact versions of the "entanglement area law"
2. "fixed point condition" in the RG sense.
3. "Nature likes extremes." Recall that:

$$\Delta(B, C, D) := S_{BC} + S_{CD} - S_B - S_D \geq 0$$

*For realistic many-body systems, the axioms are approximate. The errors typically decay with the length scale of the subsystems.

RG fixed point

The axioms, imposed on a finite scale, reproduce themselves on larger length scales.

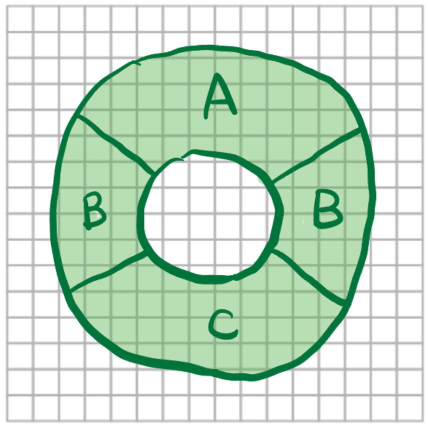
In this way, the reference state defines an **RG fixed point**.

Anyons, TEE, and information convex sets:

Topological entanglement entropy (TEE)

$$\gamma = (S_{AB} + S_{BC} - S_B - S_{ABC})/2$$

The physical meaning of γ is the **uncertainty** of states on the annulus ABC.



Levin, Wen 2005, Kitaev, Preskill 2005

	2γ
invertible	0
non-invertible	$\geq \log 2$

Thm: A convex set of "information" is a simplex

$$\Sigma(\circledcirc) = \left\{ \sum_a p_a \rho^a \;\middle|\; a \in \mathcal{C} \right\}$$

(1) $\rho^a \perp \rho^b$ for $a \neq b$.

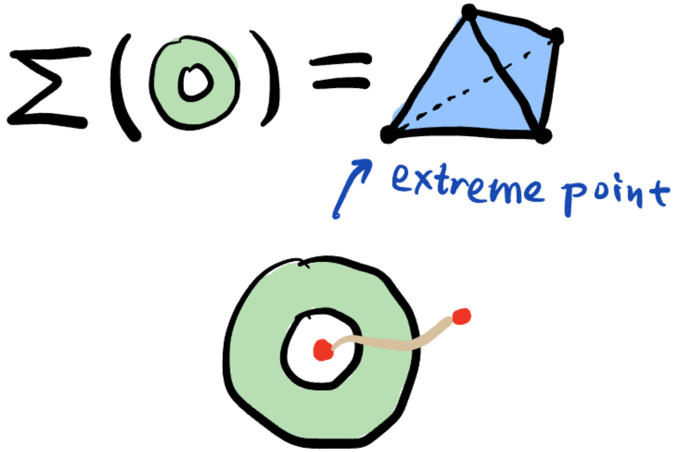
(2) $\max_{\rho,\lambda\in\Sigma}(S(\rho) - S(\lambda)) = 2\gamma$.

BS, Kato, Kim 2019

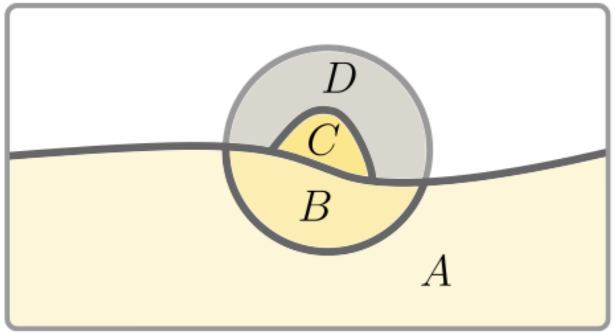
BS, Lu 2018

*The idea generalizes to higher dimensions.

intuition:



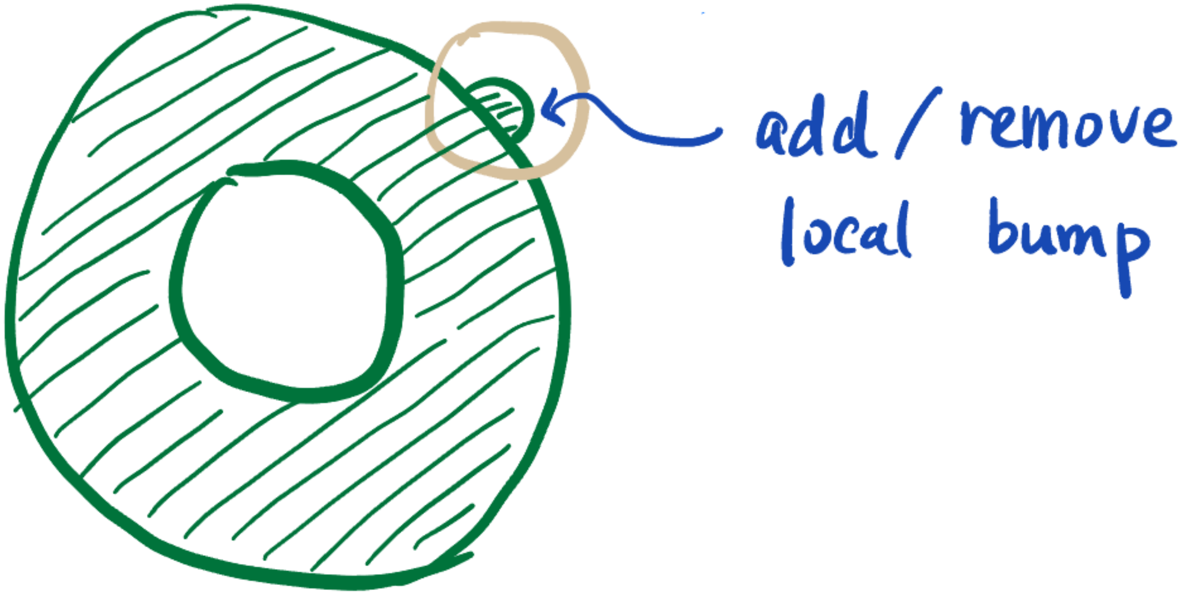
Isomorphism theorem of information convex set: BS, Kato, Kim 2019



adding or removing a local bump is allowed by the axioms

If $\Omega \sim \Omega'$, then $\Sigma(\Omega) \cong \Sigma(\Omega')$.

The convex sets are isomorphic under smooth deformation of the regions:



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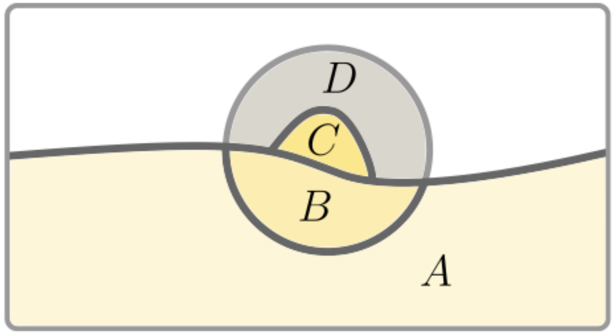

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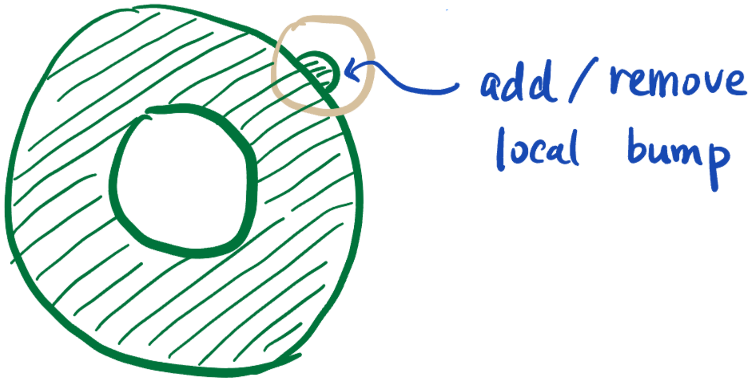

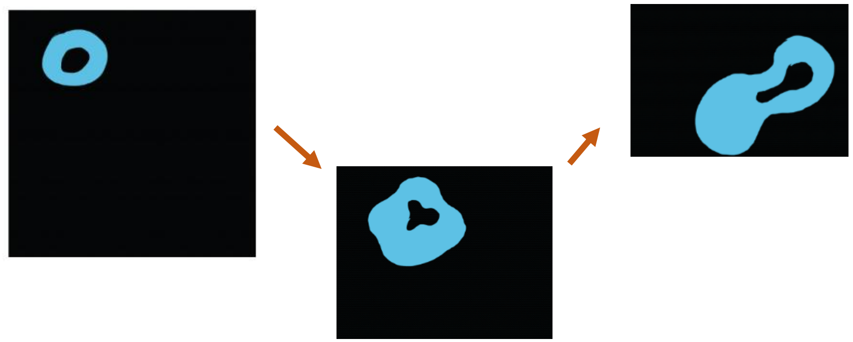

These few figures were in a GIF movie in the actual presentation.

Isomorphism theorem of information convex set:

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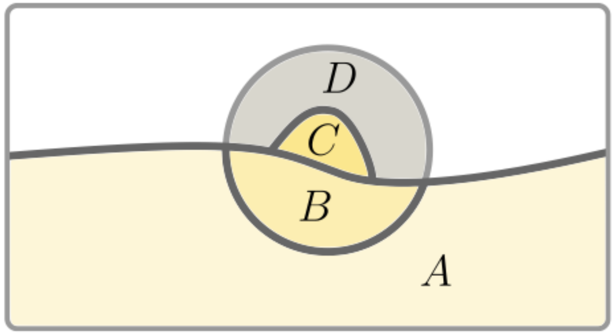
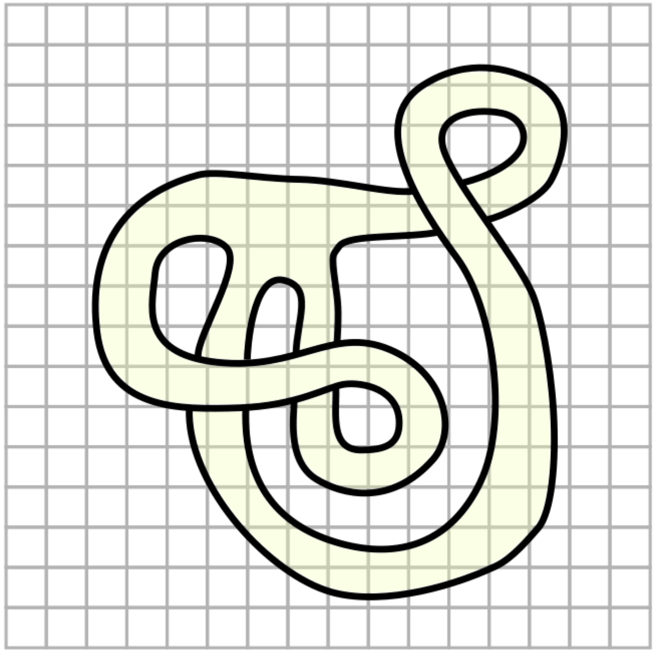


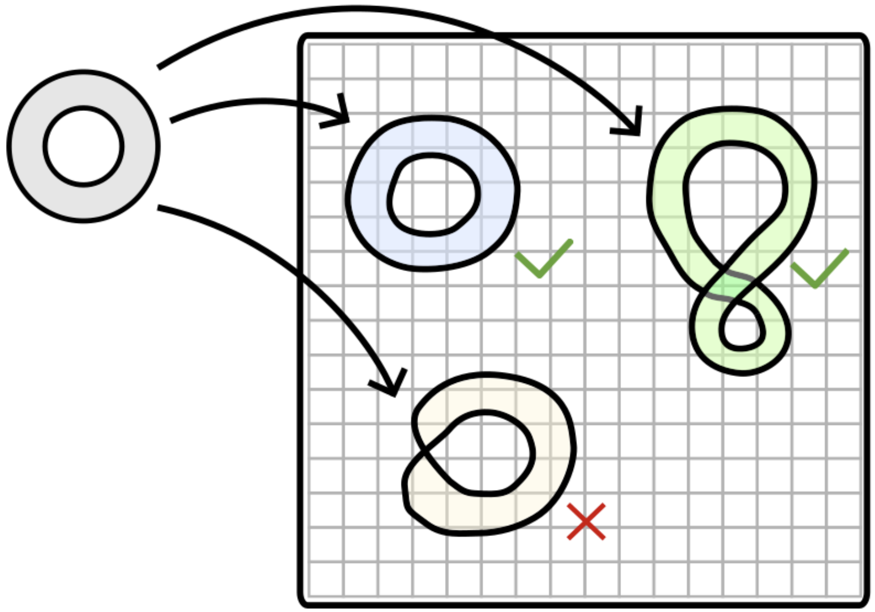
Applicable for **immersed regions:**



immersed punctured torus

$$T^2 \setminus \bullet \looparrowright \mathbb{R}^2$$

Immersion means local embedding:



Hastings 2013;
BS 2019; Huang, McGreevy, BS 2023; BS 2023

Isomorphism theorem of information convex set:

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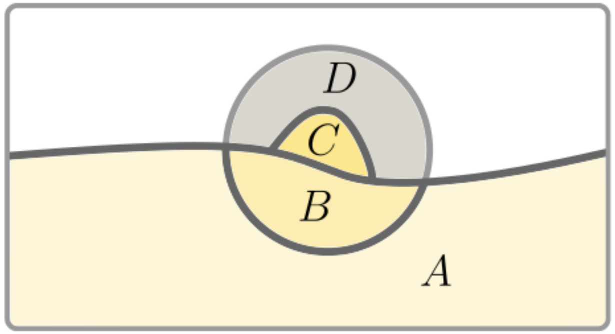

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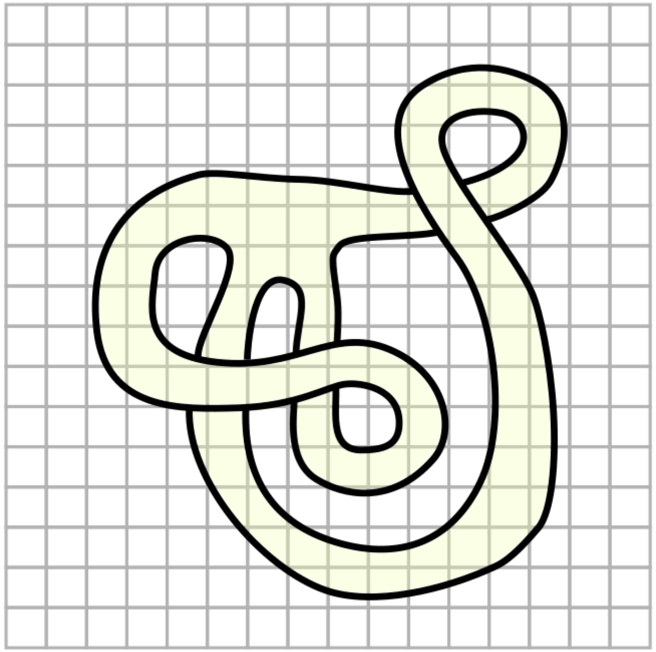

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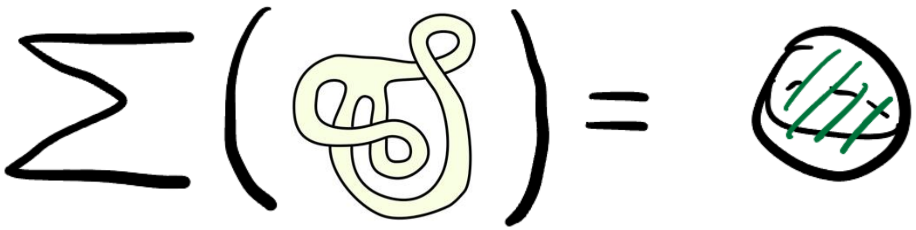
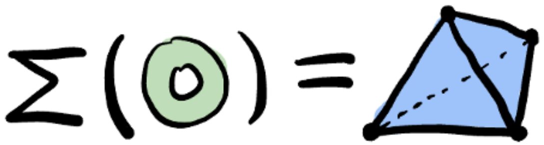


for abelian anyon theory

in contrast with

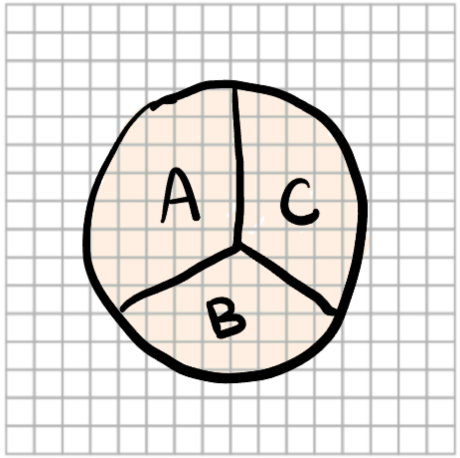


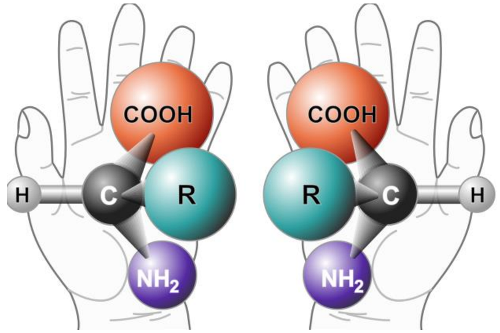
Hastings 2013 (invertible)

BS 2019; Huang, McGreevy, BS 2023; BS 2023

Chiral topological orders and chiral central charge:

The "modular commutator" formula for the chiral central charge, "argued" under area law:





$$J(A,B,C) := i\,\mathrm{Tr}([\ln \rho_{AB}, \ln \rho_{BC}]\rho_{ABC}).$$

$$J = \frac{\pi}{3} c_- .$$

Isaac Kim, BS, Kohtaro Kato, Vector Albert 2021

Main puzzle: Quantization of c_-? Puzzling even for invertible states.

For invertible 2D gapped systems, the conjecture is that $c_- = 0 \mod 8.$

A subtle fact: If we want J to be nonzero, we need to give up exact A1 or the finiteness of the Hilbert space dimension.

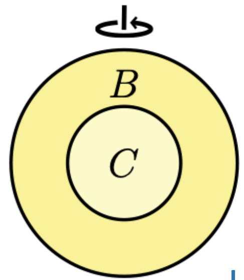
This motivates us to consider instantaneous modular flow and the associated Berry phases:

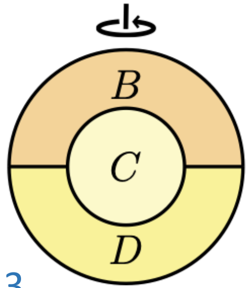
Xiang Li, Ting-Chun Lin, John McGreevy, BS 2024 + to appear

Generalizations of homogeneity into broader contexts:

Ways to appreciate the axioms:

1. ~~the "entanglement area law"~~
2. "fixed point" in the RG sense. **Yes.**
3. "Nature likes extremes." **Yes.**

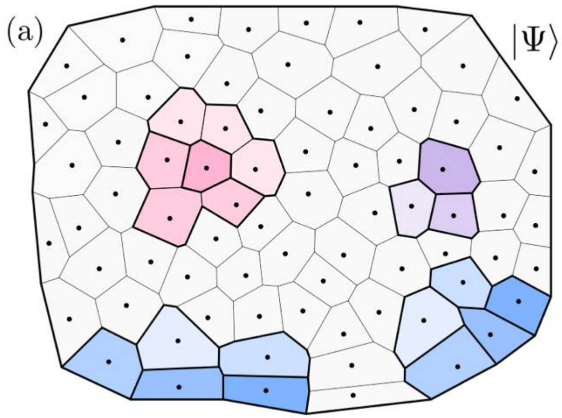




Huang, McGreevy, BS 2023

$\mathbf{A0}: \quad (S_{BC} + S_C - S_B)_\sigma = 0$ $\qquad$ $\mathbf{A1}: \quad (S_{BC} + S_{CD} - S_B - S_D)_\sigma = 0$

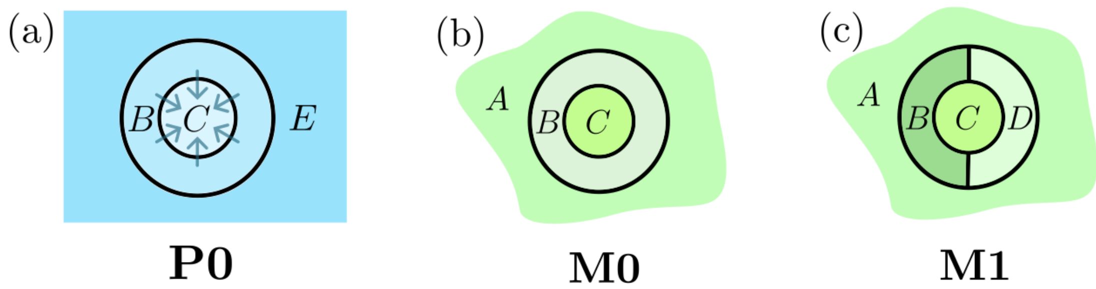
higher dimensional area law: topological homogeneity



critical systems and chiral edges: geometrical homogeneity

Lin, McGreevy 2023, Kim, Li, Lin, McGreevy, BS 2024, Xiang Li, ...

volume law state fixed points:



under decoherence: topological mixed state homogeneity

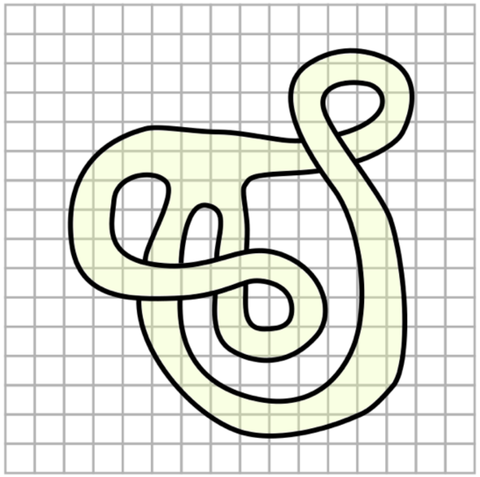
Yang, BS, Lee 2025

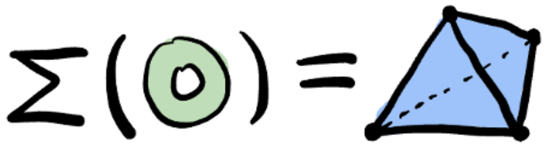
Summary:

$\overset{2+1D}{\approx}$ anyon theory

EB ≈ TQFT ⊇ invertible.

entanglement bootstrap





anyon types

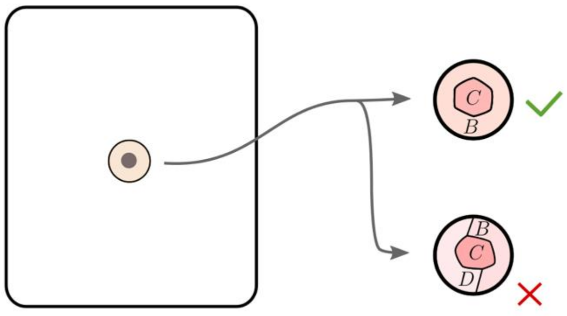
TQFT fusion spaces

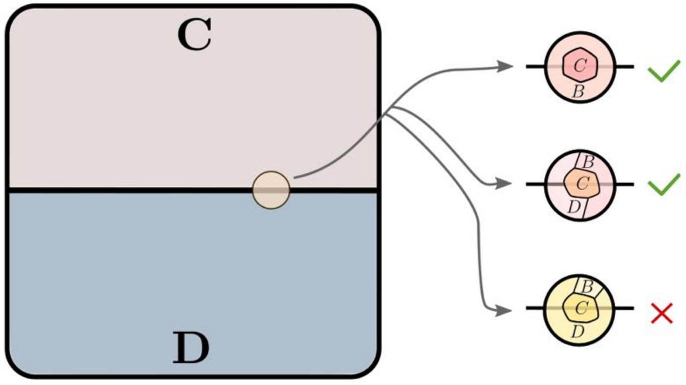
We only need a wave function on a disk (or ball).
Do not need a fancy topology to start with!

The entanglement-based axioms imply the isomorphism theorem, which establishes topological smoothness.

(2) breaking homogeneity

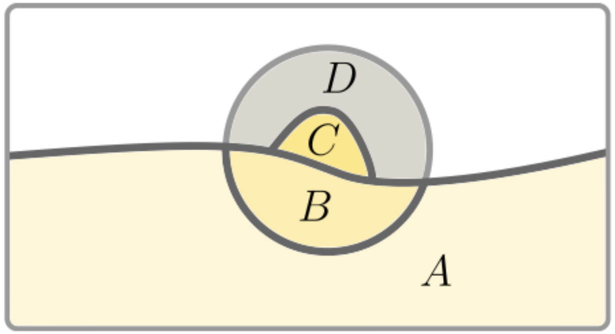
defects of different codimensions



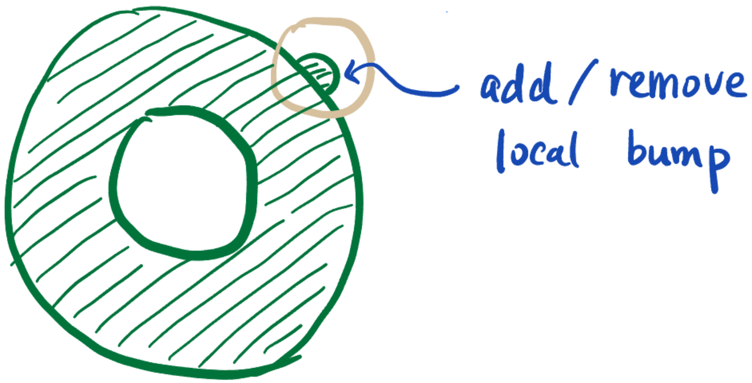


Recall that, isomorphism was the key to smoothness:

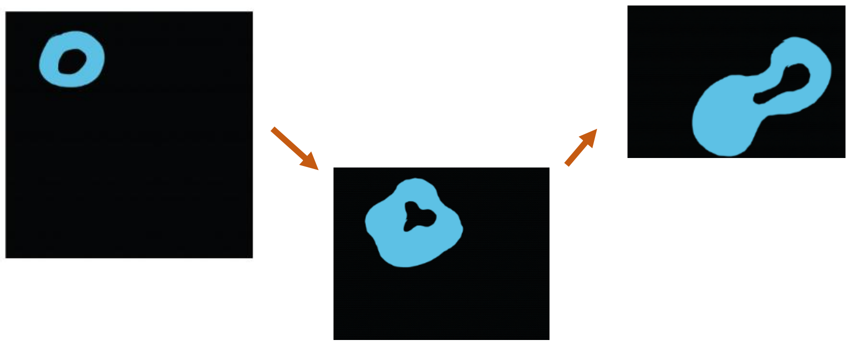
$$\text{If } \Omega \sim \Omega', \text{ then } \Sigma(\Omega) \cong \Sigma(\Omega').$$



The convex sets are isomorphic under smooth deformation of the regions:



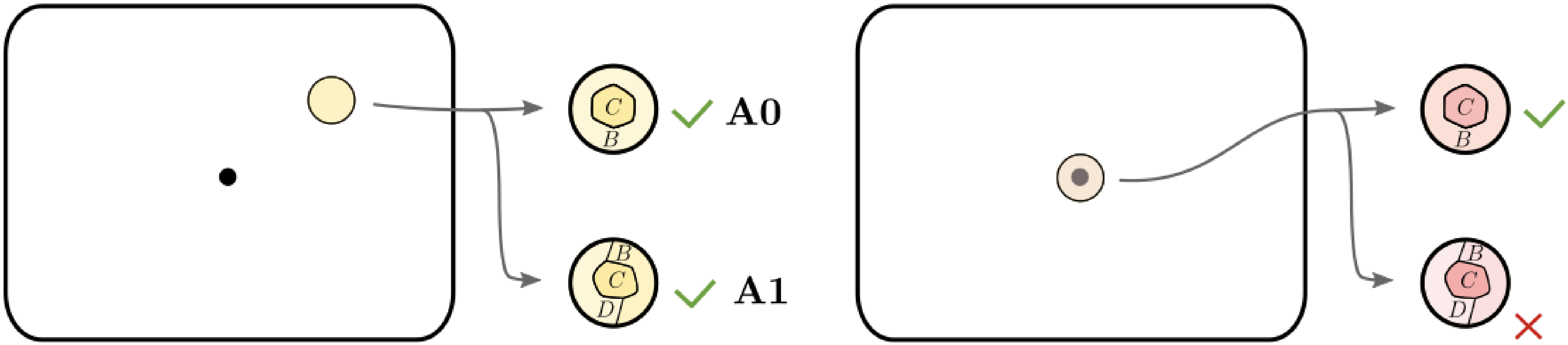
The move is allowed if axioms A0 and A1 hold on the small patch.



These few figures were in a GIF movie in the actual presentation.

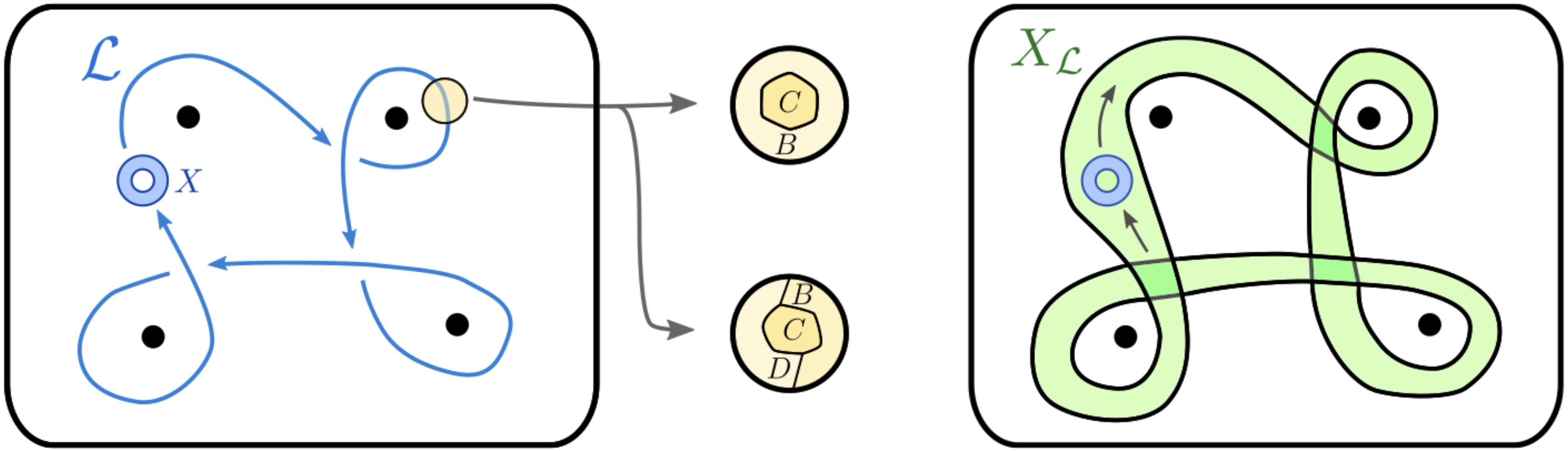
Codimension 2 defects:

defect points



Def.: A defect point is an isolated point in the 2D space where the axiom A1 is violated.

Transportation experiment:



Transport a **test annulus** $\boldsymbol{X}$ around a set of defects on the plane.

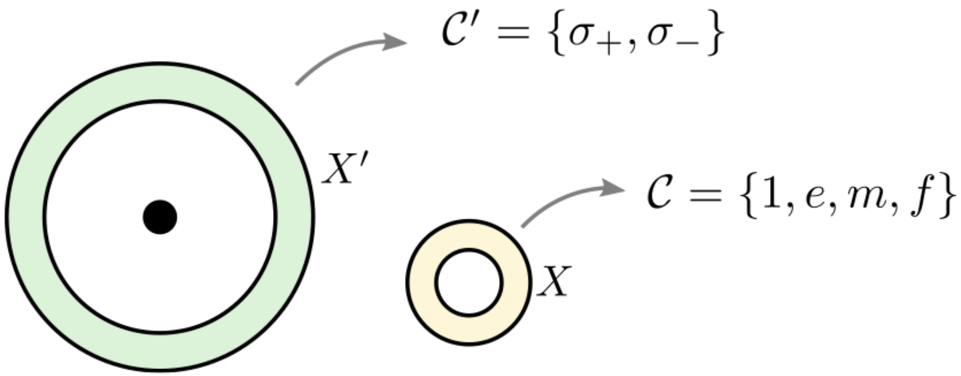
(1) The information convex set of X has an automorphism.

$$\Phi_{\mathcal{L}} : \Sigma(X) \to \Sigma(X) \quad \text{permute extreme p.t. ?}$$

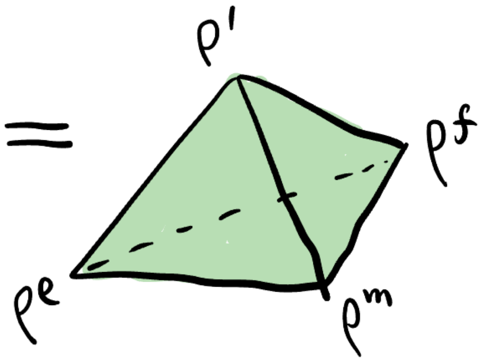
(2) The thickened loop $X_{\mathcal{L}}$ is an immersed annulus.

$$\Sigma(X_{\mathcal{L}}) \cong \Sigma(X) \, ?$$

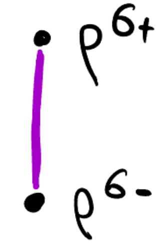
Example of the toric code: defect : $e \longleftrightarrow m$



$$\Sigma(X) = $$



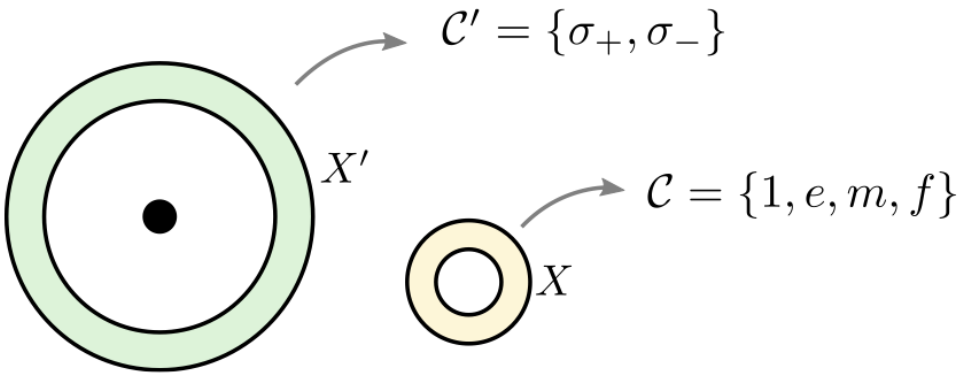
$$\Sigma(X') = $$



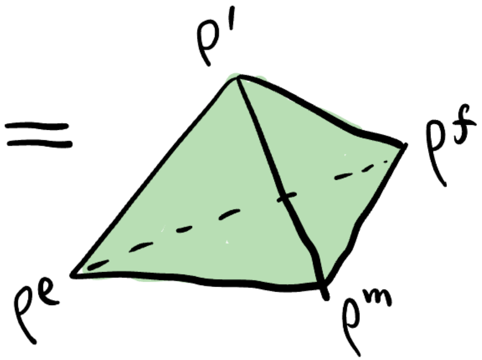
$$\Sigma(X') \not\cong \Sigma(X)$$

$$\Phi_{\circlearrowleft}(\rho^a) := \rho^{\varphi(a)}, \quad \text{where} \quad \begin{cases} \varphi(1) = 1, & \varphi(e) = m, \\ \varphi(m) = e, & \varphi(f) = f. \end{cases}$$

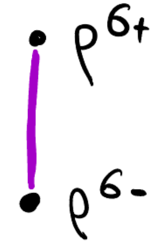
Example of the toric code: defect : $e \longleftrightarrow m$



$\Sigma(X) =$



$\Sigma(X') =$



$$\Sigma(X') \not\cong \Sigma(X)$$

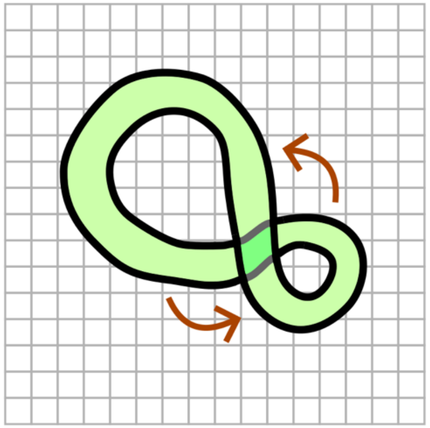
$$\Phi_{\circlearrowleft}(\rho^a) := \rho^{\varphi(a)}, \text{ where } \begin{cases} \varphi(1) = 1, & \varphi(e) = m, \\ \varphi(m) = e, & \varphi(f) = f. \end{cases}$$

Theorem (defect quantization): If **A1** is violated at an isolated point, **A0** is preserved everywhere. Then the violation of **A1** is at least $\log 2$.

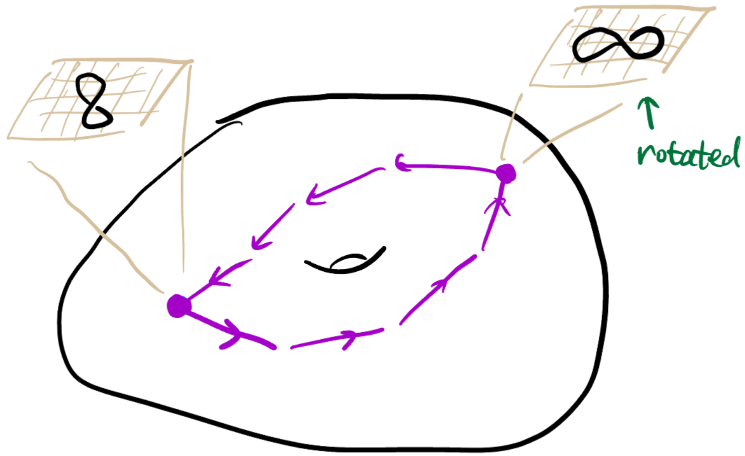
Another thought on defect: (I) back to a uniform system

A space of immersed regions and its homotopy group:

$$\mathfrak{m}(\Omega, N) := \{\Omega' \looparrowright N, \text{such that } \Omega' \sim \Omega\}.$$



Does the process of turning the figure-8 annulus by 180 degrees permute extreme points of $\Sigma(\Omega)$?

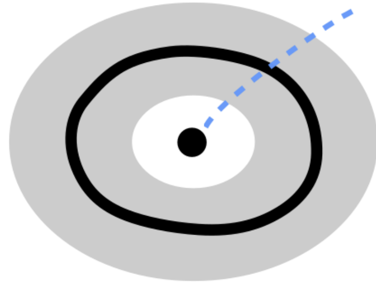


$\pi_1(\mathfrak{m}(8, S^2)) = ?$

"Immersed figure-8 annuli and anyons", BS 2023

(also see my CMSA talk on July 11th , 2025)

Another thought on defect: (II) with defects

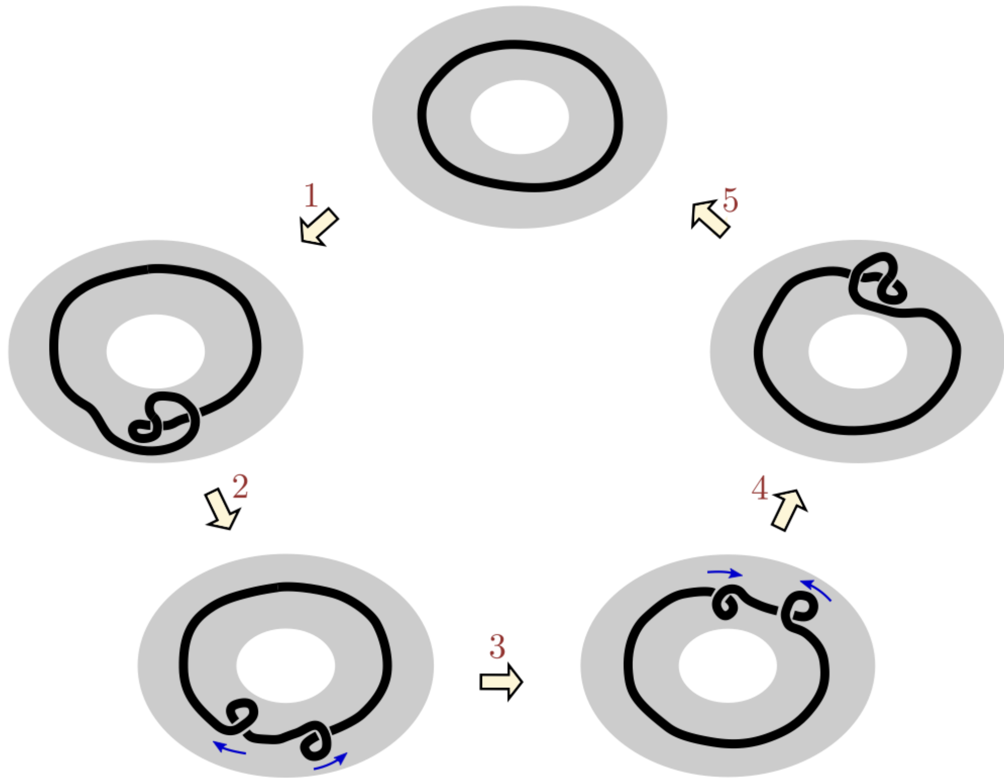


A space of immersed regions and its homotopy group:

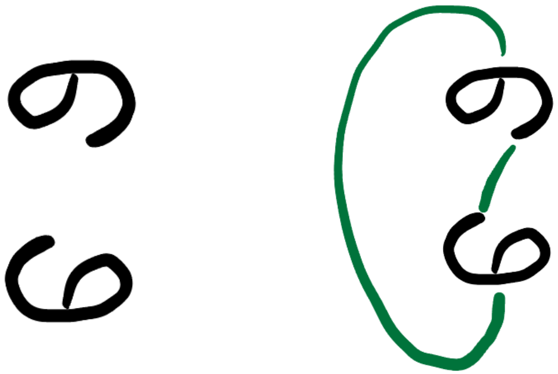
$$\mathfrak{m}(\Omega, N) := \{\Omega' \looparrowright N, \text{such that } \Omega' \sim \Omega\}.$$

Does the process of braiding around kinks permute extreme points?

A guess: the answer will depend on the defect type.



Remember the kinks with Japanese Hiragana:

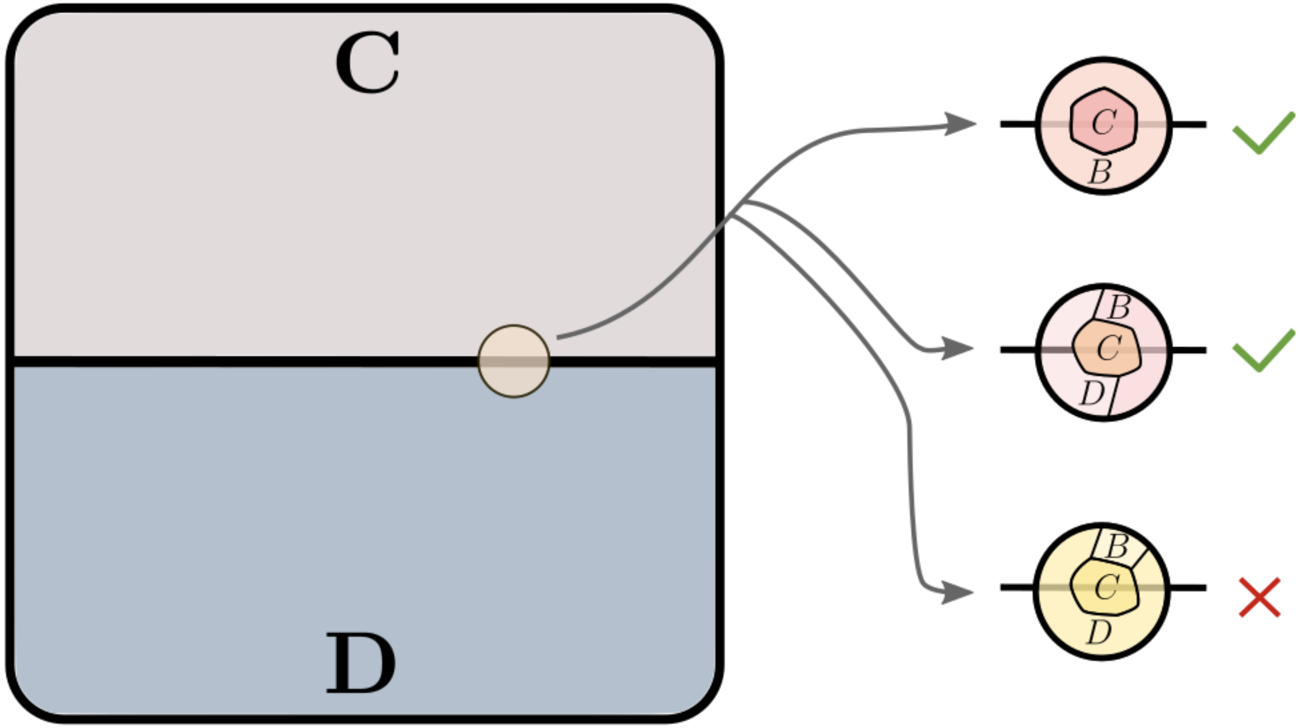


The related math question: $\pi_1(\mathfrak{m}(\Omega, N)) = ?$

Codimension 1 defects:

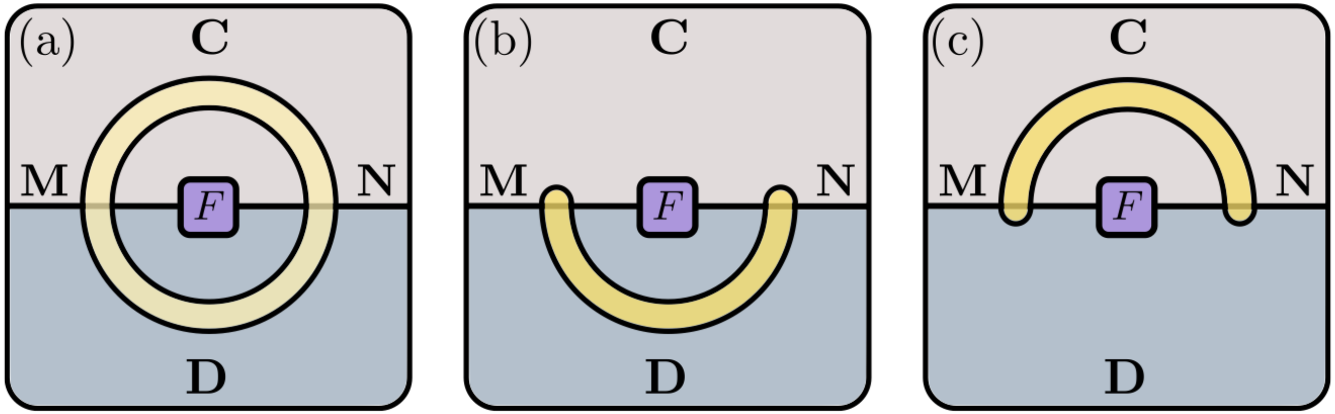
gapped domain walls and parton superselection sectors

BS, Kim 2020
Buican, Geiko, Moses, BS 2025



Def.: A gapped (i.e. , topological) domain wall in 2+1D is a line on which the axiom A1 is broken in the **attaching and detaching directions**.

Gapped domain walls:



A defect on the domain wall can be detected in multiple ways.

The traditional way is (a)

The more fundamental ways are (b) and (c), which gives us parton superselection sectors.

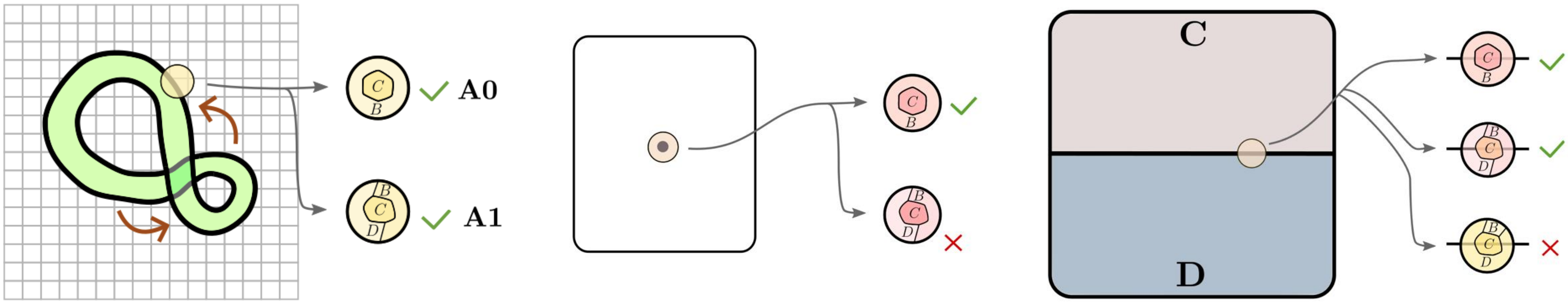
The original EB consideration: *BS, Kim 2020*
An algebraic theory for partons: *Buican, Geiko, Moses, BS 2025*

Summary & discussions

a) **Entanglement bootstrap** (EB) is aimed to be a **physical theory**, and it is promised to give new insight into the **emergence of TQFT** and more.

b) **Topological defects** of different codimensions are defined in EB, and hopefully, they will be classified within this framework.

-Question: How about instantons in the sense of 2403.07813?



Further topics to chat about (not included in the talk)

Why is **EB** (so far) formulated on lattice systems, whereas the **conformal bootstrap** was formulated in QFT?

I have omitted the beautiful story of relating EB to **quantum circuits**.

Thank you!

Questions please!