Reading between the special Kähler structures of coulomb branch geometries: $\mathcal{N} = 4$ sYM

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Motivation

Central Question: Understand what field theory data of a $4d$ $\mathcal{N}=2$ SCFT encodes the special Kähler structure (SKS) defining its Coulomb branch geometry (CBG)?

Coulomb branch is a universal component of the moduli space of vacua (conjecturally)

- Basic geometric components: fibration with base given by a singular space (algebraic variety) and fiber given by a special type of complex torus (polarized abelian variety)
- Base is target space for non-linear σ-model (nlσm) that represents allowed VEVs of massless vector multiplet scalars
- Torus fibers parametrize the couplings between $4d$ (pure) Maxwell theory and massive states; these couplings depend on the value of the VEVs, i.e. the location in the moduli space
- Special Kähler structure of fibration ties together nlσm + $4d$ Maxwell w/ charged matter in a way that's compatible with $\mathcal{N}=2$ SUSY

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An (S)CFT has two types of data associated to it: local and global data

- Local data determines correlation functions of local ops and is specific by spectrum of local operators and their OPEs
- Global data: additional data needed to specify the theory on arbitrary compact manifolds without boundary;
  - When such data is provided, the theory is said to be an absolute theory
- There can exist distinct sets of global data/absolute versions of the theory with the same local data;
  - Each ``consistent" set of (local, global) data is a global variant of the SCFT

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Local data of SCFT encodes part of CB geometry: chiral ops + OPE define coordinate ring of CBG as algebraic variety, but misses the fibration structure

Natural Question: What is the relation between CBG and global data of SCFT?

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Natural Question: **What is the relation between CBG and global data of SCFT?**

Additional motivation: This is particularly interesting for $4d$ $\mathcal{N}=2$ SCFTs because

- They often have Higgs branch components to their moduli space of vacua, and it's believed such components can be completely reconstructed from knowledge of their associated vertex operator algebra (VOA), and this VOA is believed to fully specify the local data of the theory

Natural Question: **Can we reconstruct the full CB geometry from SCFT data?**

Goal of This Project

Answer this question for $4d$ $\mathcal{N}{=}4$ sYM theories and $4d$ $\mathcal{N}{=}\,4$ CBGs

Why?

- $4d$ $\mathcal{N}{=}4$ sYM are gauge theories so we know what field theory data determines their global variants: consistent spectra of genuine, probe line operators [Aharony, Seiberg, Tachikawa; 1305.0318] $\equiv$ [RBTL]
- $\mathcal{N}{=}4$ SKSs are tightly constrained and we show in this work that they can be classified

Specific tasks:

- Classify the inequivalent SKSs of $4d$ $\mathcal{N}{=}4$ CBGs and compute their self-duality groups
- Compare with the global variants of $4d$ $\mathcal{N}{=}\,4$ sYM theories and their self-duality groups

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Answer this question for $4d$ $\mathcal{N}{=}4$ sYM theories and $4d$ $\mathcal{N}{=}4$ CBGs

Why should SKSs of CBGs capture global variants?

- Previous work demonstrated the line operator lattices (defining the global variants of a CB vacuum) of an $\mathcal{N}{=}4$ sYM CBG can capture the global variants of the low-energy theory on the CB [Argyres, Martone, Ray; 2204.09682]; and these are in 1-1 correspondence with the global variants of the sYM theory [del Zotto, Inaki; 2204.06495]

NB. Neither of these works compared the self-duality groups of these CB global variants with the global variants of the sYM theory

$\Rightarrow$ Our work implicitly fills this gap, and our results are more general because we do not assume the SKSs we classify correspond to those of sYM theories

Goal of This Project

Answer this question for $4d$ $\mathcal{N}$=4 sYM theories and $4d$ $\mathcal{N}$= 4 CBGs

What we learn:

- The extent to which SKSs encode the global variants of the SCFT, i.e. about the relationship between global data of the SCFT and the CBG
- If we confirm that SKSs can capture global variants of the SCFT, then we can apply these methods to infer the global variants of SCFTs that are not of a gauge theory/lagrangian type, i.e. such as $\mathcal{N}$=3 SCFTs
- Since the Coulomb branch is often an exactly computable observable of the SCFT even if it is strongly interacting and non-lagrangian, our methods provide a powerful tool to study the global variants of such theories

Moduli space of vacua for $\mathcal{N}{=}2$ SCFTs: space that parameterizes the inequivalent ground states (vacua) of the conformal theory preserving $\mathcal{N}{=}2$ SUSY but breaking scale invariance (i.e. dilatations),

- A low-energy effective action is associated with each (smooth) point that depends on the type of scalar ops getting VEVs
- Existence of conformal vacuum implies geometry is (locally) scale invariant (since dilatations are broken by VEVs) and distinct vacua are labeled by VEVs of scalar ops in SCFT

$4d$ $\mathcal{N} = 2$ SCFT: Moduli Space of Vacua

Scalar ops come in two types based on $4d$ $\mathcal{N} = 2$ superconformal algebra:

$$\mathfrak{su}(2,2|2) \supset \mathfrak{so}(4,2) \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_r$$

- Higgs branch ops ($\mathfrak{u}(1)_r$ invariant): VEVs imply existence of Higgs branch component of moduli space that is a singular hyperKahler variety
- Coulomb branch ops ($\mathfrak{su}(2)_R$ invariant): VEVs imply existence of the Coulomb branch component to be a singular special Kahler variety

SKS of $\mathcal{N}=2$ Coulomb branch geometry

A special Kähler structure describes the complex geometry of the CB as a complex lagrangian fibration consisting of the quadruple

$$(\pi : \mathcal{X} \to \mathcal{C}^*, J, \Lambda, \omega)$$

- $\mathcal{C}^*$ a smooth $\dim_{\mathbb{C}} = r$ Kähler manifold
- Generic fiber is a polarized abelian variety (PAV) $A_u \cong \pi^{-1}(u) \sim \mathbb{C}^r/\Lambda$ of rank$-r$ with polarization J
- ω is a holomorphic symplectic form with respect to which the fibers are lagrangian

SKS of $\mathcal{N}{=}2$ Coulomb branch geometry

Key physical features of the SKS specified by the lagrangian fibration $(\pi : \mathcal{X} \to \mathcal{C}^*, J, \Lambda, \omega)$:

- $\mathcal{C}^* = \mathcal{C}/\mathcal{D}$ is the CB $\mathcal{C}$ modulo the discriminant locus $\mathcal{D} \subset \mathcal{C}$ where singularities occur

- A_u is specified by a symplectic lattice (J, Λ) where the polarization $J : \Lambda \times \Lambda \to \mathbb{Z}$ is a non-degenerate, antisymmetric bilinear form

  - The polarization J represents the Dirac pairing between stable and probe charged states that live in the lattice Λ

  - The pair (J, Λ) satisfy compatibility conditions (so A_u is a PAV): in a symplectic basis of J, the period matrix Π_u of A_u takes the form

$$\Pi_u = (\Delta, \boldsymbol{\tau}(u))$$

    where $\boldsymbol{\tau}(u) \in \mathscr{H}_r$ is a symm, pos. def. $r \times r$ matrix valued in the Siegel upper half space

  - This $r \times r$ matrix $\boldsymbol{\tau}^{ij}(u)$ represents the electromagnetic coupling matrix between charged matter and the low-energy $U(1)^r$ gauge theory in each smooth vacua $u \in \mathcal{C}^*$

SKS of $\mathcal{N}=2$ Coulomb branch geometry

Symmetries of the physics that must be reflected in the SKS:

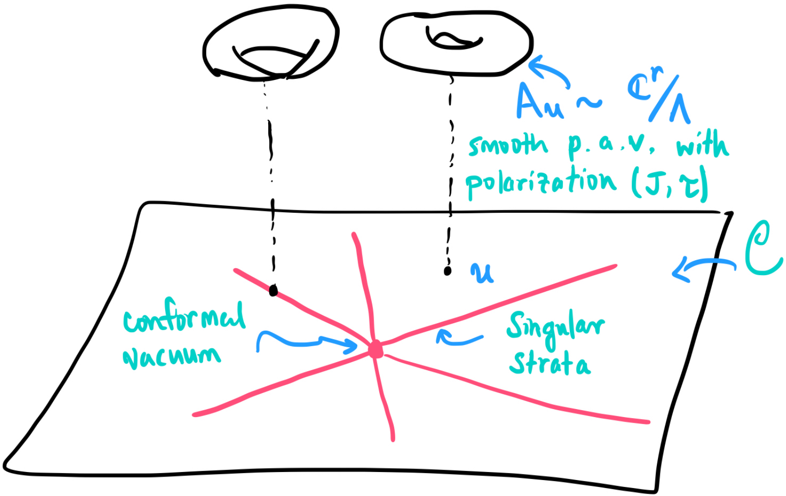
- This low-energy physics/SKS is invariant under EM-duality transformations: these are basis changes $M \in GL(2r,\mathbb{Z})$ of Λ that preserve the polarization in a (compatible) symplectic basis,

$$J \to MJM^{-t} = J \qquad \tau \to \tau' = M \circ \tau \quad (\circ \equiv \text{ Möbius action of } \mathrm{Sp}(2r,\mathbb{Z}) \text{ on } \mathscr{H}_r)$$

Such basis transformations form the EM-duality group $\mathrm{Sp}_J(2r,\mathbb{Z}) \ni M$.

- Upon dragging this linear system around a closed path $\gamma \in \pi_1(\mathcal{C}^*)$ linking $\mathcal{D}$, the physics must remain invariant
  - This allows the charge lattice Λ to undergo a monodromy $S(\gamma) \in \mathrm{Sp}_J(2r,\mathbb{Z})$ as it is dragged around γ
  - The monodromies $S(\gamma)$ define integral symplectic representations of $\pi_1(\mathcal{C}^*)$ in $\mathrm{Sp}_J(2r,\mathbb{Z})$
  - Two $\mathbb{Z}$-reps S_1 and S_2 are integrally equivalent $S_1 \cong_{\mathbb{Z}} S_2$ iff $\exists M \in \mathrm{Sp}_J(2r,\mathbb{Z})$ such that

$$S_1(\gamma) = MS_2(\gamma)M^{-1} \quad \forall \gamma \in \pi_1(\mathcal{C}^*)$$


$A_u \sim \mathbb{C}^r/\Lambda$
smooth p.a.v. with
polarization (J, τ)
u
$\mathcal{C}$
conformal
vacuum
Singular
Strata

SKS of $\mathcal{N}=4$ Coulomb branch geometry

Consider $\mathcal{N}=4$ sYM with compact gauge algebra $\mathfrak{g}$. The coulomb branch moduli space component is the flat orbifold

$$\mathcal{C}_{\mathfrak{g}} \cong_{\mathbb{C}} \mathbb{C}^r/(\mathcal{W}_{\mathfrak{g}} \otimes \mathbb{C}) \sim \mathbb{C}^r \qquad r = \mathrm{rank}(\mathfrak{g}) \qquad \mathcal{W}_{\mathfrak{g}} = \text{Weyl group of } \mathfrak{g}$$

- $\mathcal{C}_{\mathfrak{g}}$ reflects the local (chiral) data of the SCFT that is captured by the CBG
- At a generic point on $\mathcal{C}_{\mathfrak{g}}$, the gauge algebra is broken down to $\mathfrak{u}(1)^r$, and singularities (both metric and algebraic) occur on the loci of $\mathbb{C}^r$ that are fixed by some $w \in \mathcal{W}_{\mathfrak{g}}$

We want to classify the SKSs on the orbifold geometries $\mathcal{C}_{\mathfrak{g}}$.

SKS of $\mathcal{N}=4$ Coulomb branch geometry

Simplifying assumption: the polarization J is principal (so all invariant factors are 1 $\Delta = \mathbb{1}_{r\times r}$) and the EM-duality group is $\mathrm{Sp}_J(2r,\mathbb{Z}) = \mathrm{Sp}(2r,\mathbb{Z})$

This is physically well-motivated: it's been shown the line lattices $\Lambda^{\mathcal{L}}$ on the CB for $\mathcal{N}=4$ sYM theories are in $1-1$ correspondence with the global variants of the SCFT [Argyres, Martone, Ray; 2204.09682]

- These line lattices can serve as the lattice Λ of the PAV A_u iff the polarization on A_u is principal
- Therefore, to capture global variants of the low-energy theory and/or SCFT with SKSs, we should assume the polarization on A is principal

SKS of $\mathcal{N}=4$ Coulomb branch geometry

Some special features of $\mathcal{N}=4$ SKSs:

- $\mathcal{N}=4$ geometries are isotrivial
  - This means the PAV $A_u \cong_{\mathbb{C}} A$ is constant over the smooth locus $\mathcal{C}^*$, i.e. $\boldsymbol{\tau}$ is constant. Thus, a monodromy transformation $S(\gamma) \in \mathrm{Sp}(2r,\mathbb{Z})$ must fix $\boldsymbol{\tau}$ under the $\mathrm{Sp}(2r,\mathbb{Z})$ Möbius action:
  $$\boldsymbol{\tau} \to S(\gamma) \circ \boldsymbol{\tau} = \boldsymbol{\tau}$$
  which implies $\boldsymbol{\tau} \in \mathrm{Fix}(\mathrm{Im} S) \subset \mathscr{H}_r/\mathrm{Sp}_J(2r,\mathbb{Z})$

- This implies $\pi_1(\mathcal{C}^*) \cong \mathcal{W}_{\mathfrak{g}}$ so the monodromy map $S(\pi_1(\mathcal{C}^*)) \subset \mathrm{Sp}(2r,\mathbb{Z})$ defines an integral symplectic representation of the Weyl group $\mathcal{W}_{\mathfrak{g}}$ in $\mathrm{Sp}(2r,\mathbb{Z})$

- (Technical point) These facts allow for a refined notion of equivalence between the $\mathbb{Z}$-reps generated by monodromies: we must allow for reflection preserving outer automorphisms ϕ of $\mathcal{W}_{\mathfrak{g}}$ in the equivalence relation.

Two $\mathbb{Z}$-reps S_1 and S_2 of $\mathcal{W}_{\mathfrak{g}}$ are integrally equivalent $S_1 \cong_{\mathbb{Z}} S_2$ iff $\exists M \in \mathrm{Sp}(2r,\mathbb{Z})$ **and** $\phi \in \mathrm{Out}(\mathcal{W}_{\mathfrak{g}})$ such that
$$S_1(w) = M S_2(\phi(w)) M^{-1} \quad \forall w \in \mathcal{W}_{\mathfrak{g}}$$

With this setup, one can prove the following result for $\mathcal{N}$=4 SKS's:

An $\mathcal{N}$=4 SKS with principal polarization for semi-simple $\mathfrak{g}$ of rank r is an $\mathrm{Sp}(2r,\mathbb{Z})$-orbit of pairs $(S,\boldsymbol{\tau})$ under the action of $M \in \mathrm{Sp}(2r,\mathbb{Z})$, $M\cdot(S,\boldsymbol{\tau}) = (MSM^{-1}, M\circ\boldsymbol{\tau})$ where

- S is an integral, $2r$-dimensional symplectic representation $S : \mathcal{W}_\mathfrak{g} \to \mathrm{Sp}(2r,\mathbb{Z})$ which is $\mathbb{Q}$-equivalent to two copies of the reflection representation of $\mathcal{W}_\mathfrak{g}$,

$$S(w\in\mathcal{W}_\mathfrak{g}) \cong_\mathbb{Q} \rho_{ref}(w)\oplus\rho_{ref}(w)$$

- $\boldsymbol{\tau}$ is an $r\times r$ matrix in $\mathscr{H}_r/\mathrm{Sp}(2r,\mathbb{Z})$ such that $\boldsymbol{\tau}\in\mathrm{Fix}(\mathrm{Im}\,S)$, i.e. for all $w\in\mathcal{W}_\mathfrak{g}$, $S(w)\circ\boldsymbol{\tau}=\boldsymbol{\tau}$

$\mathcal{N}{=}4$ SKS: $\mathbb{Z}$-reps of $\mathcal{W}_\mathfrak{g}$ and self-duality groups

This says the $\mathcal{N}{=}4$ SKS is completely determined by the pair $(S, \boldsymbol{\tau})$ up to integral symplectic equivalence under $\mathrm{Sp}(2r, \mathbb{Z})$

Our task:

- Classify all inequivalent $\mathbb{Z}$-reps S of $\mathcal{W}_\mathfrak{g}$ in $\mathrm{Sp}(2r, \mathbb{Z})$ and their fixed point sets $\boldsymbol{\tau}$
- These equivalence classes of pairs $(S, \boldsymbol{\tau})$ will form an orbit under the EM duality group $\mathrm{Sp}(2r, \mathbb{Z})$
- Each orbit will have an associated **self-duality group**

  self-duality group $=$ the set of outer intertwiners $(M, \phi) \in \mathrm{Sp}(2r, \mathbb{Z}) \times \mathrm{Out}(\mathcal{W}_\mathfrak{g})$ that fix $S(w)$ for all $w \in \mathcal{W}_\mathfrak{g}$
- This group is always a congruence subgroup or a Hecke group (the latter holds for some non-simply laced $\mathfrak{g}$)

We did this for compact $\mathfrak{g} \in \{A_r, B_r, C_r, D_r, E_{6,7,8}, F_4, G_2\}$, and we compared these results with `[Aharony, Seiberg, Tachikawa; 1305.0318]` $\equiv$ `[RBTL]`

[RBTL] Reminder: Global variants of $4d$ non-abelian gauge theories

The global variants of a $4d$ gauge theory with compact gauge algebra $\mathfrak{g}$ are labeled by two equivalent sets of data [RBTL]

- Global forms of the gauge group along with a set of discrete parameters (discrete theta angles)
- Consistent spectra of probe, dyonic (Wilson-'t Hooft $\equiv$ WT) line operators [Kapustin; hep-th/0501015]

A WT line op is labeled by an orbit of pairs $(\mu, \nu) \in \Lambda_w \times \Lambda_{cw}$ under the Weyl group $\mathcal{W}_{\mathfrak{g}}$

[RBTL] showed consistent probe line ops must are subject to two consistency conditions:

- Integral Dirac pairing: all WT ops must be genuine/mutually local wrt one another
- A set of mutually local line ops must be maximal

Relation between the two: The global form + discrete θ-angles specifies the 1-form symmetry charges of the consistent sets of probe lines

[RBTL]: computed the S-duality orbits of these global variants, from which one can deduce their self-duality groups $=$ subgroup of S-duality trans. that fix a global structure

(NB. all global variants in the same orbit have the same self-duality group)

Important point: Relative to our results, we can only compare two pieces of data:

The **$\#$ of duality orbits** and the **self-duality group** of each orbit

- Why? This is (some) of the gauge invariant data on the conformal manifold in question; it implies we can't compare the number of ``global structures" in an orbit as was done in [RBTL] because this is not gauge invariant data
- Thus, we have no intrinsic analogue of the ``global structures" that fill out the S-duality orbits of the sYM theory we're comparing with

$4d\ \mathcal{N}=4$ SKSs versus $\mathcal{N}$=4 sYM global variants

Boring Cases: $\mathfrak{g} \in \{\, A_{n>1},\ BC_2 = \mathfrak{so}(5) \cong \mathfrak{sp}(4)\,,\ D_{odd} = \mathfrak{so}(4n+2),\ E_{6,7,8}\,,\ G_2\,,\ F_4\,\}$

- In all of these cases, we have a match between the **# of orbits** and the **self-duality group of each orbit**

Semi-Interesting Case: $\mathfrak{g} = D_{even} = \mathfrak{so}(4n)$

- This case appears to be in tension with [RBTL] since we find fewer **duality orbits** and, for some orbits, larger **self-duality groups** (i.e. smaller conformal manifolds)
- This tension is relieved once one takes into account the reflection outer automorphisms of $\mathcal{W}_{\mathfrak{g}}$ that generate additional equivalences between global structures beyond those generated by S-duality
- Additional equivalences are either between distinct orbits (so shrink the # of orbits) or within an orbit (so act as **self-duality transformation** $\Rightarrow$ enlarge the **conformal manifold**)

Interesting Cases: $\mathfrak{g} \in \{\, A_1 = \mathfrak{su}(2) \cong \mathfrak{sp}(2),\ BC_{n>2} = \mathfrak{so}(2n+1) \,\}$

Find the same **# of orbits**, but one orbit always has a **self-duality group** that is $3\times$ as large!

A_1: 1 orbit, $\mathcal{O}_1$

- **Self-dual grp:** SKS: $=\mathrm{SL}(2,\mathbb{Z})$ || [RBTL]: $\Gamma_0(2)$

BC_{2k}: 3 orbits, $\{\, \mathcal{O}_i | i = 1,2,3 \,\}$

- **Self-dual grp:** SKS: $\{\, \Gamma_0(2), \Gamma_0(2), \mathrm{SL}(2,\mathbb{Z}) \,\}$ || [RBTL]: $\{\, \Gamma_0(2), \Gamma_0(2), \Gamma_0(2) \,\}$

BC_{2k+1}: 2 orbits, $\{\, \mathcal{O}_i | i = 1,2 \,\}$

- **Self-dual grp:** SKS: $\{\, \Gamma_0(4), \mathrm{SL}(2,\mathbb{Z}) \,\}$ || [RBTL]: $\{\, \Gamma_0(4), \Gamma_0(2) \,\}$

Interesting Cases: $\mathfrak{g} \in \{ A_1 = \mathfrak{su}(2) \cong \mathfrak{sp}(2),\ BC_{n>2} = \mathfrak{so}(2n+1) \}$

What does this mean? It indicates the IR physics captured by the CBG is insufficiently sensitive to reconstruct the duality information of this orbit in the UV theory, i.e. the corresponding $\mathfrak{g}$ sYM theory with this choice of global variant orbit

- Equivalently, for each global variant in this orbit, their CB SKSs are all isomorphic
- This is not an inconsistency per say, because the SKS of the CBG is a low-energy observable, and there is no a priori reason it must capture all gauge invariant data of the UV SCFT (NB. We are currently thinking of a way to test for this!)

Future directions

We can readily extend these results to $\mathcal{N}=3$ theories because they are also isotrivial orbifold geometries (Easier because one doesn't have to worry about self-duality groups!)

- Work in progress w/ interesting preliminary results: some orbifold geometries do not admit principally polarized SKS ($\Rightarrow$ **Q:** Do such CBGs not admit any global structure??)
- Can also classify non-principally polarized SKSs (Work in progress)

Q: Can we extend these methods to more generic isotrivial $\mathcal{N}=2$ theories that aren't orbifolds?

- What about non-isotrivial theories? (Seems difficult...)

We can see probe line spectra on the CB and their 1-form symmetries from the SKSs
$\Rightarrow$ **Q:** can we see more generalized symmetry structure, e.g. non-invertible (duality) defects, 1-form symmetry gauging, $0, 2$-form symms?

More broadly, can we confirm our results in terms of the CB SymTFT ? (Work in progress)

Is there more data about the SCFT that can be extracted from moduli spaces of supersymmetric QFTs?

Thank You!