Conformal field theories from line defects and holography

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Based on work with L. Bianchi, D. Bonomi, G. Bliard, L. Griguolo, G. Peveri, D. Seminara

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Wilson loops as 1-dimensional defects

Wilson lines are fundamental observables in any gauge theories

$$\langle W \rangle = P \exp\left(- i \int_{t_1}^{t_2} dt\, \mathcal{L}(t) \right)$$

In a CFT, for instance $\mathcal{N} = 4$ SYM in $d = 4$ or ABJM in $d = 3$, a Wilson line can be viewed as a conformal defect.

[Giombi Roiban Tseytlin 17] [Giombi Beccaria Tseytlin 18]
[Bianchi, Bliard, Forini, Griguolo, Seminara 20]

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A straight line breaks the original conformal symmetry to

a) dilatations, translations and special conformal transformations along the line

b) rotations around the line

+ part of the R-symmetry + part of the supersymmetry

Thus the Wilson loop implicitly defines a defect CFT_1.

Can we study this “simpler” CFT?

A defect CFT$_1$

The set of correlators of operator insertions along the line

$$\langle \mathcal{O}(t_1)\mathcal{O}(t_2)\ldots\mathcal{O}(t_n)\rangle_W = \frac{\langle \mathrm{Tr}\mathcal{O}_1(t_1)W\mathcal{O}_2(t_2)\ldots\mathcal{O}_{n-1}(t_{n-1})W\mathcal{O}_n(t_n)\rangle}{\langle W\rangle}$$

where

$$\langle W\rangle = P\exp\big(-i\int_{t_1}^{t_2} dt\,\mathcal{L}(t)\big)$$

can be interpreted as characterizing a defect CFT$_1$.

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can be interpreted as characterizing a defect CFT$_1$.

It should be fully determined by its spectrum of dimensions and OPE coefficients.

Also: operator insertions are equivalent to deformations of the Wilson line

[Drukker, Kawamoto 2006]

Complete knowledge of these correlators would, in principle, allow to compute the expectation value of general Wilson loops which are deformations of the line or circle.

A defect CFT$_1$: the 1/2 BPS Wilson line in ABJM theory

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can be interpreted as characterizing a defect CFT$_1$.

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Consider the $\mathcal{N}=6$ superconformal Chern-Simons-matter theory in $d=3$ (ABJM).

Its original symmetry, $OSp(6|4)$, is broken by the 1/2 BPS Wilson line to $SU(1,1|3)$, the $\mathcal{N}=6$ superconformal group in $d=1$.

Its bosonic subgroup is $SO(2,1)\times U(1)_M\times SU(3)_R$.

Operator insertions along the Wilson line are labelled by $[\Delta;m;j_1,j_2]$.

The displacement supermultiplet

Among the possible operator insertions (defect operators), a special role is played by a set of “elementary excitations” with protected scaling dimension.

They fall into a short representation of the $SU(1,1|3)$ subalgebra

It is a chiral multiplet, the displacement supermultiplet

$$[\tfrac{1}{2}, \tfrac{3}{2}, 0, 0] \qquad \mathbb{F}, \ \bar{\mathbb{F}}$$

$$\searrow$$

$$[1, 2, 1, 0] \qquad \mathbb{O}^a, \ \bar{\mathbb{O}}_a \quad a = 1, 2, 3$$

$$\searrow$$

$$[\tfrac{3}{2}, \tfrac{5}{2}, 0, 1] \qquad \mathbb{\Lambda}^a, \ \bar{\mathbb{\Lambda}}_a \quad a = 1, 2, 3$$

$$\searrow$$

$$[2, 3, 0, 0] \qquad \mathbb{D}, \ \bar{\mathbb{D}}$$

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$$\partial_\mu T^{\mu m} = \delta^2(x^\perp)\, \mathbb{D}^m(t)$$

Translational invariance is broken, the stress tensor is no longer conserved and the usual conservation law needs to be modified by some additional terms localized on the defect.

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8F+8B
like the DOF of
transverse
string fluctuations

All operators in the supermultiplet can be related to broken symmetry generators.

The displacement supermultiplet

Their 2-point functions are particularly simple, e.g.

$$\langle \mathbb{D}(t_1)\bar{\mathbb{D}}(t_2)\rangle = \frac{C_{\mathbb{D}}}{t_{12}^4}$$

where the normalization constant $C_{\mathbb{D}} = 12\, B_{1/2}(\lambda)$ has a physical meaning: [Bianchi Lemos Meineri 18] [Bianchi Lemos 19]

it coincides with the Bremsstrahlung function, one of the few unprotected observables known to each order in AdS/CFT. [Correa Henn Maldacena Sever 12] [Bianchi Griguolo Preti Seminara17] [Bianchi Preti Vescovi 18]

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Their 3-point functions vanish by symmetry.

Their 4-point functions, on the other hand, have a less costrained form

$$\langle \mathcal{O}_\Delta(t_1)\mathcal{O}_\Delta(t_2)\mathcal{O}_\Delta(t_3)\mathcal{O}_\Delta(t_4)\rangle = \frac{1}{(t_{12}t_{34})^{2\Delta}} G(\chi) .$$

$G(\chi)$ has non-trivial dependence on the coupling and conformal cross ratio $\chi = \frac{t_{12}t_{34}}{t_{13}t_{24}}$

They encode in particular scaling dimensions and structure constants of unprotected operators appearing in the OPE.

The displacement supermultiplet

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superspace analysis

analytic bootstrap

direct (Witten) diagrammatics at strong coupling via AdS/CFT

Chiral correlators in superspace

The supermultiplet accomodating the displacement operator is the chiral one.

We consider the chiral superfield ($y = x - \theta_a \bar{\theta}^a$)

$$\Phi(y, \theta) = \mathbb{F}(y) + \theta_a \mathbb{O}^a(y) + \theta_a \theta_b \, \epsilon^{abc} \, \mathbb{\Lambda}_c(y) + \theta_a \theta_b \theta_c \, \epsilon^{abc} \, \mathbb{D}(y) \, ,$$

The two-point function reads

$$\langle \Phi(y_1, \theta_1) \bar{\Phi}(y_2, \bar{\theta}_2) \rangle = \frac{C_\Phi}{\langle 1\bar{2} \rangle^{2\Delta}} \, ,$$

in terms of the chiral distance $\langle i\bar{j} \rangle = y_i - y_j - 2\theta_i^a \bar{\theta}_{a\,j}$.

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The most general form for the 4-point function is

$$\langle \Phi(y_1,\theta_1)\bar{\Phi}(y_2,\bar{\theta}_2)\Phi(y_3,\theta_3)\bar{\Phi}(y_4,\bar{\theta}_4)\rangle = \frac{1}{(\langle 1\bar{2}\rangle\,\langle 3\bar{4}\rangle)^{2\Delta}}\,f(\mathcal{X})\,,$$

since the only superconformal invariant is

$$\mathcal{X} = \frac{\langle 1\bar{2}\rangle\,\langle 3\bar{4}\rangle}{\langle 1\bar{4}\rangle\,\langle 3\bar{2}\rangle}\,.$$

The corresponding bosonic cross-ratio $\quad \chi = \dfrac{x_{12}x_{34}}{x_{13}x_{24}}$

Four-point functions for the defect operators

Expanding in Graßmann variables we get a set of correlators for the elementary fields

$$z = \frac{\chi}{\chi - 1}$$

$$\langle \mathbb{F}(t_1)\bar{\mathbb{F}}(t_2)\mathbb{F}(t_3)\bar{\mathbb{F}}(t_4)\rangle = \frac{f(z)}{t_{12}t_{34}}$$

$$\langle \mathbb{O}^{a_1}(t_1)\bar{\mathbb{O}}_{a_2}(t_2)\mathbb{O}^{a_3}(t_3)\bar{\mathbb{O}}_{a_4}(t_4)\rangle = \frac{4}{t_{12}^2 t_{34}^2}\Big[\delta^{a_1}_{a_2}\delta^{a_3}_{a_4}\left(f(z) + zf'(z) + z^2 f''(z)\right) + \delta^{a_1}_{a_4}\delta^{a_3}_{a_2}\left(z^2 f'(z) - z^3 f''(z)\right)\Big]$$

$$\begin{aligned}\langle \mathbb{D}(t_1)\bar{\mathbb{D}}(t_2)\mathbb{D}(t_3)\bar{\mathbb{D}}(t_4)\rangle = \frac{64}{t_{12}^4 t_{34}^4}\Big[& z^6(1-z)^3 f^{(6)}(z) - 3\, f^{(5)}(z) z^5 (1-z)^2(7z+1) \\ & + 3\, f^{(4)}(z) z^4(-46z^3 + 63z^2 - 18z + 1) \\ & + 6\, f^{(3)}(z) z^3 (55z^3 - 39z^2 + 3z + 1) \\ & + 18\, f''(z)\,(-14z^5 + 3z^4 + z^2) - 36\, f'(z)\, z(1-z^3) + 36\, f(z)\Big]\end{aligned}$$

$$\begin{aligned}\langle \mathbb{D}(t_1)\bar{\mathbb{D}}(t_2)\mathbb{O}^{a_3}(t_3)\bar{\mathbb{O}}_{a_4}(t_4)\rangle = -\frac{16\,\delta^{a_3}_{a_4}}{t_{12}^4 t_{34}^4}\Big[& (1-z)\, z^4 f^{(4)}(z) + (3z+1)\, z^3 f^{(3)}(z) \\ & + 3z^2\, f''(z) + 6z f'(z) + 6 f(z)\Big]\end{aligned}$$

The correlation function $f(z)$ of the superconformal primary completely determines that of its superdescendants.

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$$\langle \mathbb{O}^{a_1}(t_1)\bar{\mathbb{O}}_{a_2}(t_2)\mathbb{O}^{a_3}(t_3)\bar{\mathbb{O}}_{a_4}(t_4)\rangle = \frac{4}{t_{12}^2 t_{34}^2}\Big[\delta^{a_1}_{a_2}\delta^{a_3}_{a_4}\left(f(z) + z f'(z) + z^2 f''(z)\right) + \delta^{a_1}_{a_4}\delta^{a_3}_{a_2}\left(z^2 f'(z) - z^3 f''(z)\right)\Big]$$

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The correlation function $f(z)$ of the superconformal primary completely determines that of its superdescendants.

→ non-trivial function of the coupling!

We can evaluate $f(z)$, and thus ALL the correlators, at strong coupling using string worldsheet worldsheet perturbation theory: via Witten diagrams.

String dual - AdS$_2$ minimal surface

In AdS/CFT dictionary, the Wilson loop operator is dual to a minimal string surface ending on the contour defining the operator at the boundary.
For ABJM, the dual is a fundamental Type IIA string in a $\mathrm{AdS}_4 \times \mathbb{C}\mathrm{P}^3$ background.
The bosonic part of the Nambu-Goto string action reads

$$S_B = T \int d^2\sigma \sqrt{\det \frac{1}{z^2} \left(\partial_\mu x^r \partial_\nu x^r + \partial_\mu z \partial_\nu z\right) + 4\Big(\frac{\partial_\mu \bar{w}_a \partial_\nu w^a}{1+|w|^2} - \frac{\partial_\mu \bar{w}_a w^a \bar{w}_b \partial_\nu w^b}{(1+|w|^2)^2} \Big)}$$

where T is the effective string tension

$$T = \frac{R^2}{2\pi\alpha'} = 2\sqrt{2\lambda}\,, \qquad \lambda = \frac{N}{k}\,.$$

The minimal surface dual to the 1/2-BPS Wilson line is given by

$$z = s\,, \qquad x^0 = t\,, \qquad x^i = 0\,, \qquad w^a = 0$$

The induced metric is just that of AdS_2

$$ds^2 = \frac{1}{s^2}(dt^2 + ds^2)$$

String dual - AdS_2 minimal surface

This setup preserves same superconformal symmetry $SU(1,1|3)$ of our defect CFT_1
In particular, the isometry of AdS_2 is the conformal group in $d=1$,

Fluctuation modes over the minimal surface are scalar fields over AdS_2
and their dynamics is governed by the fluctuation Lagrangian

$$S_B \equiv T \int d^2\sigma \sqrt{g}\ L_B\,, \qquad L_B = \ L_2 + L_{4X} + L_{2X,2w} + L_{4w} + \ldots\,,$$

$$L_2 = g^{\mu\nu}\partial_\mu X \partial_\nu \bar{X} + 2|X|^2 + g^{\mu\nu}\partial_\mu w^a \partial_\nu \bar{w}_a\ ,$$

$$L_{4X} = 2|X|^4 + |X|^2\left(g^{\mu\nu}\partial_\mu X\partial_\nu \bar{X}\right) - \tfrac{1}{2}\left(g^{\mu\nu}\partial_\mu X \partial_\nu X\right)\left(g^{\rho\kappa}\partial_\rho \bar{X}\partial_\kappa \bar{X}\right),$$

$$L_{2X,2w} = \left(g^{\mu\nu}\partial_\mu X\partial_\nu \bar{X}\right)\left(g^{\rho\kappa}\partial_\rho w^a \partial_\kappa \bar{w}_a\right) - 2\left(g^{\mu\nu}\partial_\mu X \partial_\nu w^a\right)\left(g^{\rho\kappa}\partial_\rho \bar{X}\partial_\kappa \bar{w}_a\right)\ ,$$

$$L_{4w} = -\tfrac{1}{2}(w^a\bar{w}_a)(g^{\mu\nu}\partial_\mu w^b \partial_\nu \bar{w}_b) - \tfrac{1}{2}(w^a\bar{w}_b)(g^{\mu\nu}\partial_\mu w^b\partial_\nu \bar{w}_a) + \tfrac{1}{2}\left(g^{\mu\nu}\partial_\mu w^a \partial_\nu \bar{w}_a\right)^2$$
$$- \tfrac{1}{2}(g^{\mu\nu}\partial_\mu w^a \partial_\nu \bar{w}_b)\left(g^{\rho\kappa}\partial_\rho \bar{w}_a \partial_\kappa w^b\right) - \tfrac{1}{2}(g^{\mu\nu}\partial_\mu w^a\partial_\nu w^b)\left(g^{\rho\kappa}\partial_\rho \bar{w}_a \partial_\kappa \bar{w}_b\right)\ .$$

Effective 2d field theory of **1+3 complex scalars** in AdS_2 geometry

String dual - AdS_2 minimal surface

This setup preserves same superconformal symmetry $SU(1,1|3)$ of our defect CFT_1!
In particular, the isometry of AdS_2 is the conformal group in $d = 1$,

Fluctuation modes over the minimal surface are scalar fields over AdS_2

Then AdS_2/CFT_1 states that they should be dual to operators inserted at the $d = 1$ boundary with dimensions

$$\Delta(\Delta - 1) = m^2 \qquad \text{bosons} \qquad \Delta = \frac{1}{2} + |m| \qquad \text{spinors}$$

Hence, we recover the eight bosonic operators in the super-displacement multiplet

$\Delta = \frac{1}{2}$	$\mathbb{F},\ \bar{\mathbb{F}}$	$\Longleftrightarrow$	$\psi\,,\bar{\psi}$	$m^2 = 0$
$\Delta = 1$	$\mathbb{O}^a,\ \bar{\mathbb{O}}_a \quad a = 1,2,3$	$\Longleftrightarrow$	$w^a\,,\bar{w}_a$	$m^2 = 0$
$\Delta = \frac{3}{2}$	$\mathbb{\Lambda}^a,\ \bar{\mathbb{\Lambda}}_a \quad a = 1,2,3$	$\Longleftrightarrow$	$\psi^a\,,\bar{\psi}_a$	$m_F = \pm 1$
$\Delta = 2$	$\mathbb{D},\ \bar{\mathbb{D}}$	$\Longleftrightarrow$	$X\,,\bar{X}$	$m^2 = 2$

Witten diagrams in AdS_2

The four-point functions of the dual operators at strong coupling can then be obtained from familiar AdS/CFT techniques by computing Witten diagrams in AdS_2.

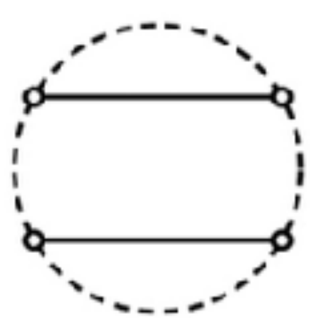
For the 4-point function of fields e.g. in AdS

$$\langle X(t_1)\,\bar{X}(t_2)\,X(t_3)\,\bar{X}(t_4)\rangle = \frac{1}{t_{12}^2 t_{34}^2} G(z)\ ,$$

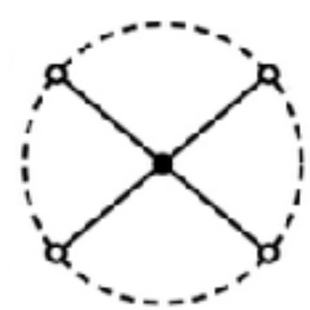
where $G(z)$ has the strong coupling expansion

$$G(z) = G^{(0)}(z) + \frac{1}{T} G^{(1)}(z) + \ldots$$

disconnected contribution (diagrams with 2 "boundary-to-boundary" propagators)



tree-level contact diagrams (4-vertices with 4 bulk-to-boundary propagators attached)



Summary of 4-point function results

The correlators of string worldsheet excitations read

$$\langle X(t_1)\,\bar{X}(t_2)\,X(t_3)\,\bar{X}(t_4)\rangle = \frac{1}{t_{12}^4 t_{34}^4}\Big[1+z^4+\tfrac{1}{T}\big[-8z^4-(3-8z)z^4(\ln z-\ln(1-z)) - z^3-\tfrac{7}{6}z^2-z-(8-3z)\tfrac{\ln(1-z)}{z}-8\big]\Big]$$

$$\langle w^{a_1}(t_1)\,\bar{w}_{a_2}(t_2)\,w^{a_3}(t_3)\,\bar{w}_{a_4}(t_4)\rangle = \frac{1}{t_{12}^2 t_{34}^2}\Big[\delta^{a_1}_{a_2}\delta^{a_3}_{a_4}\big[1+\tfrac{1}{2\,T}\big(z^2\ln z-\big(z^2-\tfrac{4}{z}+3\big)\ln(1-z)-z+4\big)\big] +\delta^{a_1}_{a_4}\delta^{a_3}_{a_2}\big[z^2+\tfrac{1}{2\,T}\big((3-4z)z^2\ln z+(4z^3-3z^2-1)\ln(1-z)+(4z-1)z\big)\big]\Big]$$

$$\langle X(t_1)\,\bar{X}(t_2)\,w^{a_3}(t_3)\,\bar{w}_{a_4}(t_4)\rangle = \frac{1}{t_{12}^4 t_{34}^2}\,\delta^{a_3}_{a_4}\big[1+\tfrac{1}{T}\big(2(z-2)\tfrac{\ln(1-z)}{z}-4\big)\big]$$

Summary of 4-point function results

The correlators of string worldsheet excitations read

The superspace analysis of correlators for defect operators gives

$$\langle X(t_1)\,\bar X(t_2)\,X(t_3)\,\bar X(t_4)\rangle = \frac{1}{t_{12}^4 t_{34}^4}\Big[1+z^4+\tfrac{1}{T}\big[-8z^4-(3-8z)z^4(\ln z-\ln(1-z)) - z^3-\tfrac{7}{6}z^2-z-(8-3z)\tfrac{\ln(1-z)}{z}-8\big]\Big]$$

$\Updownarrow$

$$\langle \mathbb{D}(t_1)\bar{\mathbb{D}}(t_2)\mathbb{D}(t_3)\bar{\mathbb{D}}(t_4)\rangle = \frac{64}{t_{12}^4 t_{34}^4}\Big[z^6(1-z)^3 f^{(6)}(z)-3\,f^{(5)}(z)z^5(1-z)^2(7z+1) + 3\,f^{(4)}(z)z^4(-46z^3+63z^2-18z+1) + 6\,f^{(3)}(z)z^3(55z^3-39z^2+3z+1) + 18\,f''(z)\,(-14z^5+3z^4+z^2)-36\,f'(z)\,z(1-z^3)+36\,f(z)\Big]$$

$$\langle w^{a_1}(t_1)\,\bar w_{a_2}(t_2)\,w^{a_3}(t_3)\,\bar w_{a_4}(t_4)\rangle = \frac{1}{t_{12}^2 t_{34}^2}\Big[\delta^{a_1}_{a_2}\delta^{a_3}_{a_4}\big[1+\tfrac{1}{2T}\big(z^2\ln z-\big(z^2-\tfrac{4}{z}+3\big)\ln(1-z)-z+4\big)\big] + \delta^{a_1}_{a_4}\delta^{a_3}_{a_2}\big[z^2+\tfrac{1}{2T}\big((3-4z)z^2\ln z+(4z^3-3z^2-1)\ln(1-z)+(4z-1)z\big)\big]\Big]$$

$\Updownarrow$

$$\langle \mathbb{O}^{a_1}(t_1)\bar{\mathbb{O}}_{a_2}(t_2)\mathbb{O}^{a_3}(t_3)\bar{\mathbb{O}}_{a_4}(t_4)\rangle = \frac{4}{t_{12}^2 t_{34}^2}\Big[\delta^{a_1}_{a_2}\delta^{a_3}_{a_4}\,\big(f(z)+zf'(z)+z^2f''(z)\big) + \delta^{a_1}_{a_4}\delta^{a_3}_{a_2}\,\big(z^2f'(z)-z^3f''(z)\big)\Big]$$

$$\langle X(t_1)\,\bar X(t_2)\,w^{a_3}(t_3)\,\bar w_{a_4}(t_4)\rangle = \frac{1}{t_{12}^4 t_{34}^2}\,\delta^{a_3}_{a_4}\big[1+\tfrac{1}{T}\big(2(z-2)\tfrac{\ln(1-z)}{z}-4\big)\big]$$

$\Updownarrow$

$$\langle \mathbb{D}(t_1)\bar{\mathbb{D}}(t_2)\mathbb{O}^{a_3}(t_3)\bar{\mathbb{O}}_{a_4}(t_4)\rangle = -\frac{16\,\delta^{a_3}_{a_4}}{t_{12}^4 t_{34}^4}\Big[(1-z)\,z^4 f^{(4)}(z)+(3z+1)\,z^3 f^{(3)}(z)+3z^2\,f''(z)+6zf'(z)+6f(z)\Big]$$

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The correlators of string worldsheet excitations read

The superspace analysis of correlators for defect operators gives

$$\langle X(t_1)\,\bar{X}(t_2)\,X(t_3)\,\bar{X}(t_4)\rangle = \frac{1}{t_{12}^4 t_{34}^4}\Big[1+z^4+\tfrac{1}{T}\big[-8z^4-(3-8z)z^4(\ln z-\ln(1-z)) - z^3-\tfrac{7}{6}z^2-z-(8-3z)\tfrac{\ln(1-z)}{z}-8\big]\Big]$$

$\Updownarrow$

$$\langle \mathbb{D}(t_1)\bar{\mathbb{D}}(t_2)\mathbb{D}(t_3)\bar{\mathbb{D}}(t_4)\rangle = \frac{64}{t_{12}^4 t_{34}^4}\Big[z^6(1-z)^3 f^{(6)}(z) - 3\,f^{(5)}(z)z^5(1-z)^2(7z+1) + 3\,f^{(4)}(z)z^4(-46z^3+63z^2-18z+1) + 6\,f^{(3)}(z)z^3(55z^3-39z^2+3z+1) + 18\,f''(z)\,(-14z^5+3z^4+z^2) - 36\,f'(z)\,z(1-z^3)+36\,f(z)\Big]$$

$$\langle w^{a_1}(t_1)\,\bar{w}_{a_2}(t_2)\,w^{a_3}(t_3)\,\bar{w}_{a_4}(t_4)\rangle = \frac{1}{t_{12}^2 t_{34}^2}\Big[\delta^{a_1}_{a_2}\delta^{a_3}_{a_4}\big[1+\tfrac{1}{2T}\big(z^2\ln z-\big(z^2-\tfrac{4}{z}+3\big)\ln(1-z)-z+4\big)\big] + \delta^{a_1}_{a_4}\delta^{a_3}_{a_2}\big[z^2+\tfrac{1}{2T}\big((3-4z)z^2\ln z+(4z^3-3z^2-1)\ln(1-z)+(4z-1)z\big)\big]\Big]$$

$\Updownarrow$

$$\langle \mathbb{O}^{a_1}(t_1)\bar{\mathbb{O}}_{a_2}(t_2)\mathbb{O}^{a_3}(t_3)\bar{\mathbb{O}}_{a_4}(t_4)\rangle = \frac{4}{t_{12}^2 t_{34}^2}\Big[\delta^{a_1}_{a_2}\delta^{a_3}_{a_4}\,\big(f(z)+zf'(z)+z^2f''(z)\big) + \delta^{a_1}_{a_4}\delta^{a_3}_{a_2}\,\big(z^2f'(z)-z^3f''(z)\big)\Big]$$

$$\langle X(t_1)\,\bar{X}(t_2)\,w^{a_3}(t_3)\,\bar{w}_{a_4}(t_4)\rangle = \frac{1}{t_{12}^4 t_{34}^2}\,\delta^{a_3}_{a_4}\big[1+\tfrac{1}{T}\big(2(z-2)\tfrac{\ln(1-z)}{z}-4\big)\big]$$

$\Updownarrow$

$$\langle \mathbb{D}(t_1)\bar{\mathbb{D}}(t_2)\mathbb{O}^{a_3}(t_3)\bar{\mathbb{O}}_{a_4}(t_4)\rangle = -\frac{16\,\delta^{a_3}_{a_4}}{t_{12}^4 t_{34}^4}\Big[(1-z)\,z^4 f^{(4)}(z)+(3z+1)\,z^3 f^{(3)}(z)+3z^2\,f''(z)+6zf'(z)+6f(z)\Big]$$

Is there a single f(z) solving simultaneously these non-trivial ODEs?

Summary of 4-point function results

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These differential equations are all solved by the simple function

$$f(z) = 1 - z + \frac{1}{T}\left(1 - z - (3-z)z\ln z + \frac{(1-z)^3}{z}\ln(1-z)\right) + \mathcal{O}\left(\frac{1}{T^2}\right),$$

This is the strong coupling expansion of the function governing all correlation functions of operators in the displacement supermultiplet.

Also derived using analytic bootstrap

CFT data at strong coupling

The four-point function has an OPE expansion in superblocks

$$\langle \mathbb{F}(t_1)\bar{\mathbb{F}}(t_2)\mathbb{F}(t_3)\bar{\mathbb{F}}(t_4)\rangle = \frac{1}{t_{12}t_{34}} f(z) = \frac{1}{t_{12}t_{34}} \sum_h c_h (-z)^h \, {}_2F_1(h,h,2h+3,z)\,.$$

eigenfunctions of the super-Casimir of *N*=6 algebra in d=1

[Dolan, Osborn 2011]

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eigenfunctions of the super-Casimir of *N*=6 algebra in d=1

At this order, operators exchanged in the OPE are just the identity and the tower of operators $\mathcal{O}\partial^n\mathcal{O}$, built out of the elementary excitations. Therefore,

$$h = 1 + n + \frac{1}{T}\gamma_n^{(1)} \qquad\qquad c_n = c_n^{(0)} + \frac{1}{T}c_n^{(1)}$$

$$f(z) = 1 - z + \frac{1}{T}\left(1 - z - (3-z)z\ln z + \frac{(1-z)^3}{z}\ln(1-z)\right) + \mathcal{O}\left(\frac{1}{T^2}\right),$$

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$$h = 1 + n + \frac{1}{T}\gamma_n^{(1)} \qquad c_n = c_n^{(0)} + \frac{1}{T} c_n^{(1)}$$

$$c_n^{(1)} = \partial_n \left(c_n^{(0)}\,\gamma_n^{(1)}\right)$$

$$\gamma_n^{(1)} = 3 + 4n + n^2$$

$$c_n = \sqrt{\pi}2^{-2n-3}(n+3)\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{5}{2}\right)}\Big[(n+2) + \frac{1}{T}\big[4n^2 - 2\left(n^3+6n^2+11n+6\right)\ln 2 + 15n$$
$$+(n+1)(n+2)(n+3)\psi^{(0)}(n+1) - (n+1)(n+2)(n+3)\psi^{(0)}\left(n+\tfrac{5}{2}\right) + 13\big]\Big]$$

"inverting" for the coefficients in the sum, namely using orthogonality relations for the hypergeometric functions.

Intermediate conclusions

- We have considered a class of four-point correlators in the CFT_1 defined on the 1/2-BPS Wilson line in the 3d superconformal ABJM theory.
- Superconformal symmetry determines four-point correlators of the displacement supermultiplet in terms of a single function, that we evaluate at strong coupling using holography and Witten diagrams and the analytic boostrap. We can extract CFT data.
- Further progress on the ABJM Wilson line: topological sector (kinematical defect) [Gorini, Griguolo, Guerrini, Penati, Seminara, Soresina 22], integrability for the cusp-deformed WL [Correa, Giraldo-Rivera, Lagares 23] three-loop (in AdS) correlators via analytic bootstrap [Bliard, Ferrero, to appear]

Intermediate conclusions and questions

- We have considered a class of four-point correlators in the CFT_1 defined on the 1/2-BPS Wilson line in the 3d superconformal ABJM theory.
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- What happens beyond tree-level? Witten diagrams with loops in AdS should be well-defined, since the 2d theory is supposed to be UV finite. However, issues of regularization appear.

  Is there a representation ("momentum space") in which these computations simplify and the scattering nature of the correlator becomes transparent?

Conformal correlators and Mellin space

Higher dimensions

[Mack 2009] [Penedones 2010]

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{x_{13}^{2\Delta} x_{24}^{2\Delta}}\, F(u,v)\,, \qquad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

$$F(u,v) = \int_{\mathcal{C}} d\gamma_{12} d\gamma_{14}\, M(\gamma_{12},\gamma_{14})\, \Gamma^2(\gamma_{12})\, \Gamma^2(\gamma_{14}) \Gamma^2(\Delta - \gamma_{12} - \gamma_{14})\, u^{-\gamma_{12}} v^{-\gamma_{14}}$$

$M(\gamma_{12},\gamma_{14})$ has the properties of a scattering amplitude:

- Crossing symmetry
- Poles corresponding to operators exchanged in the OPE
- Asymptotic behavior compatible with the Regge limit
- Simple expression for Witten diagrams

Conformal correlators and Mellin space in $d = 1$

In $d = 1$, just one independent cross ratio and thus one independent Mellin variable

Reduction from higher dimensions is subtle -> inherently one-dimensional formulation inspired by same guiding principles

Notice that, in fact, a **family** of Mellin amplitudes can be defined $t = \frac{x_{12}\, x_{34}}{x_{14} x_{23}} > 0$

$$\mathcal{M}_a(s) = \int_0^\infty dt\, f(t) \left(\frac{t}{1+t}\right)^a t^{-1-s} \qquad \mathcal{M}_a^{-1}[\mathcal{M}_a(s)] = \int_{\mathcal{C}} \frac{ds}{2\pi i}\, f(t)\, t^s \left(\frac{t}{1+t}\right)^{-a} \mathcal{M}_a(s)$$

$a = 0$ $\rightarrow$ Mellin transform of $f(t)$ in $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{(x_{12}\, x_{34})^{2\Delta_\phi}} f(t)$

$a = -2\Delta_\phi$ $\rightarrow$ Mellin transform of the crossing-symmetric $g(t) = \left(\frac{t}{1+t}\right)^{2\Delta_\phi} f(t)$

$a = -2\Delta_\phi + 1$ leads to simple results in a perturbative expansion around GFF

Definition and properties

$$M(s) = \frac{1}{\Gamma(s)\Gamma(2\Delta - s)} \int_0^\infty dt\, t^{-1-s}\, f(t) \qquad t = \frac{x_{12}\, x_{34}}{x_{14} x_{23}} > 0$$

with inverse

$$f(t) = \int_{\mathcal{C}} \frac{ds}{2\pi i}\, \Gamma(s)\, \Gamma(2\Delta - s)\, M(s)\, t^s$$

where $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}}\, f(t)\,.$

Crossing $f(t) = t^{2\Delta_\phi} f(1/t)$ translates to $M(s) = M(2\Delta - s)$

reminiscent of the crossing $S(\mathrm{s}) = S(4m^2 - \mathrm{s})$ in two (flat) dimensions.

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To obtain a definition on the whole s-complex plane an **analytic continuation** of the region of convergence ($2\Delta_\phi - \Delta_0 < \text{Re}(s) < \Delta_0$) is required.

(Δ_0 : dimension of the lightest operator exchanged)

$$f_0(t) = f(t) - \left(\frac{t}{1+t}\right)^{-2\Delta} \sum_{\Delta+k=\Delta_0}^{\Delta_\phi} c_\Delta C_{\Delta,k} t^{\Delta+k}$$

Subtraction procedure

[Costa, Penedones, Zhiboedov 2021]

$$\psi_0(s) = \int_0^1 dt t^{-1-s} f_0(t) + \sum_{\Delta+k=\Delta_0}^{\Delta_\phi} c_\Delta C_{\Delta,k} \frac{1}{s - \Delta - k}$$

$$M(s) = \frac{\psi_0(s) + \psi_\infty(s)}{\Gamma(s)\Gamma(2\Delta_\phi - s)}$$

Nonperturbative Mellin amplitude in $d = 1$

Adding more and more poles we can further extend the area of analyticity obtaining a representation valid in the whole complex plane

$$M(s) = \frac{\psi_0(s) + \psi_\infty(s)}{\Gamma(s)\Gamma(2\Delta_\phi - s)}$$

$$\psi_0(s) = \sum_\Delta \sum_{k=0}^{\infty} c_\Delta \frac{(-1)^{k+1}\Gamma(\Delta+k)^2\Gamma(2\Delta)}{k!\Gamma(\Delta)^2\Gamma(2\Delta+k)} \frac{1}{s-\Delta-k}\,,$$

Right poles

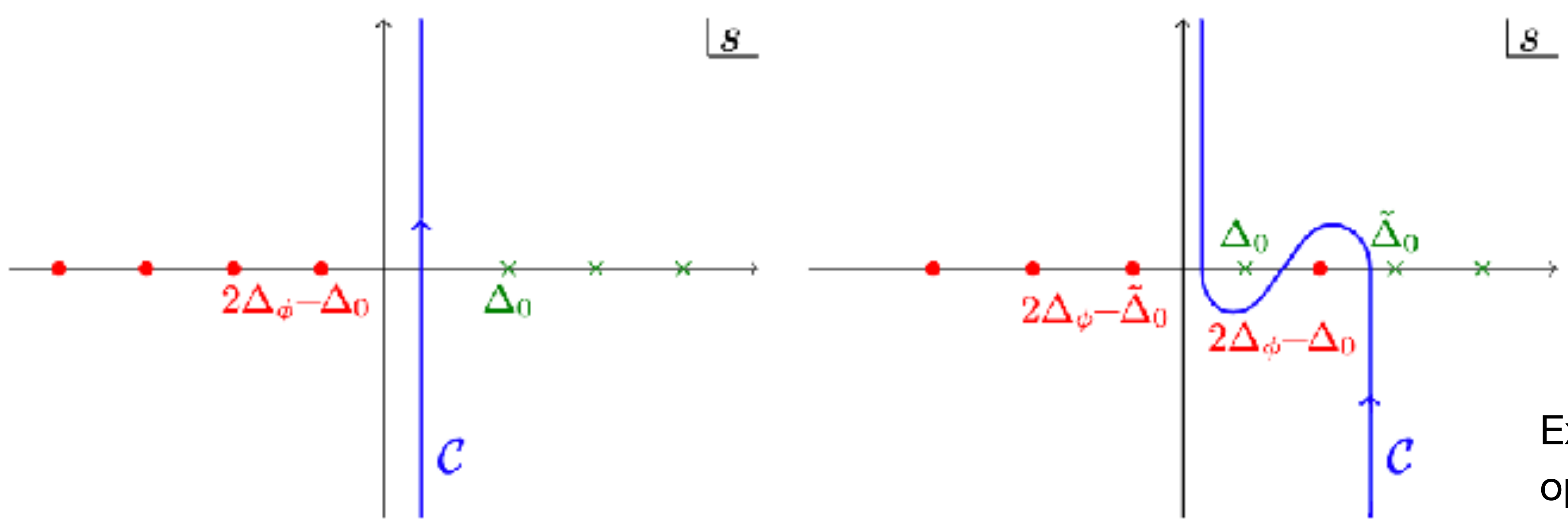
$$s_R = \Delta + k\,, \qquad k = 0, 1, 2, \ldots$$

$$\psi_\infty(s) = \sum_\Delta \sum_{k=0}^{\infty} c_\Delta \frac{(-1)^{k}\Gamma(\Delta+k)^2\Gamma(2\Delta)}{k!\Gamma(\Delta)^2\Gamma(2\Delta+k)} \frac{1}{s-2\Delta_\phi+\Delta+k}$$

Left poles

$$s_L = 2\Delta_\phi - \Delta - k\,, \qquad k = 0, 1, 2, \ldots$$

and the contour $\mathcal{C}$ is chosen so to leave *right* poles on its right and *left* poles on its left.



Ex. of one light operator exchanged

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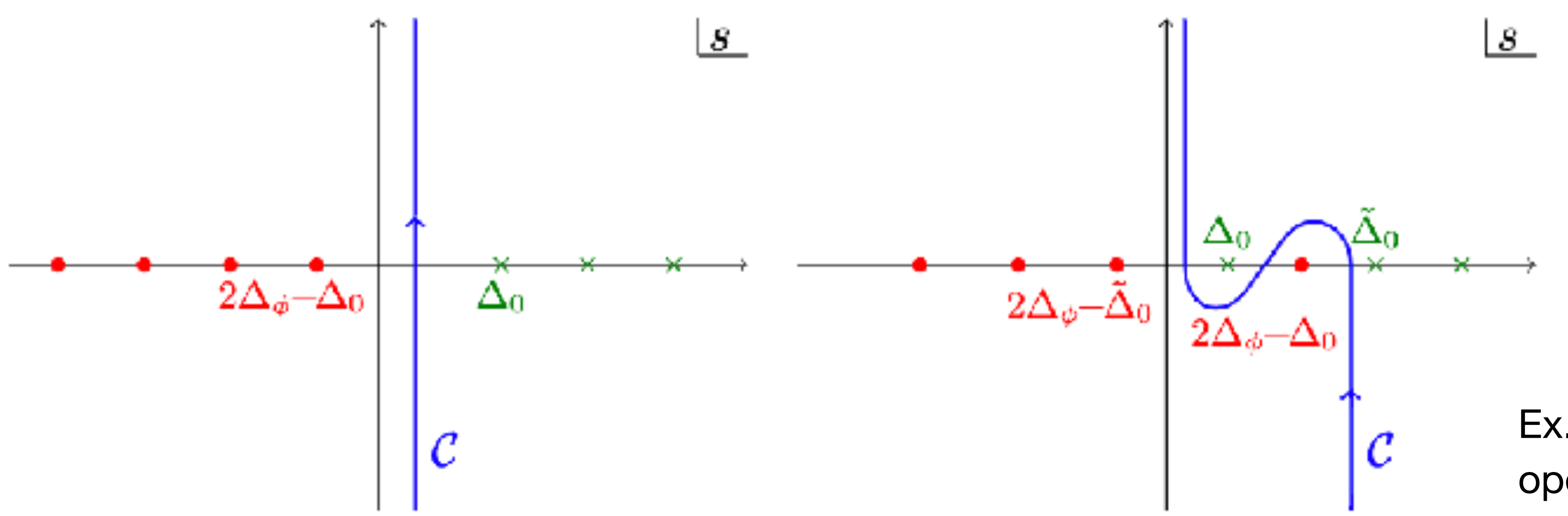

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Nonperturbative Mellin amplitude in $d = 1$

Adding more and more poles we can further extend the area of analyticity obtaining a representation valid in the whole complex plane

$$M(s) = \frac{1}{\Gamma(s)\,\Gamma(2\Delta_\phi - s)} \sum_{\Delta,k} a_\Delta\, C_{\Delta,k} \left[\frac{1}{s-k-\Delta} + \frac{1}{2\Delta_\phi - s - k - \Delta}\right]$$

Summing over k gives the Mellin counterpart of the conformal block expansion

$$M(s) = \sum_{\Delta} \frac{G_\Delta(s) + G_\Delta(2\Delta_\phi - s)}{\Gamma(s)\,\Gamma(2\Delta_\phi - s)} \qquad G_\Delta(s) = \frac{{}_3F_2(\Delta, \Delta, \Delta - s; 2\Delta, 1 + \Delta - s; 1)}{\Delta - s}$$

- $M(s)$ is crossing-invariant
- Asymptotic behavior: $M(s) \sim \frac{1}{s^a}\,, a > 1$

  (controlled by the Regge limit of the correlator and ensured by the prefactor)

- $M(s)$ has poles for physical exchanged operators
- $M(s)$ has zeros (generically) at $s = 2\Delta_\phi + k\,,\ k = 0, 1, 2, ..$

  (canceling unwanted OPE contributions)

From this bounded, meromorphic function and its properties some nonperturbative sum rules can be derived. However the most efficient use of this Mellin formalism happens at **perturbative level.**

Perturbation theory: quartic interactions with derivatives in AdS_2

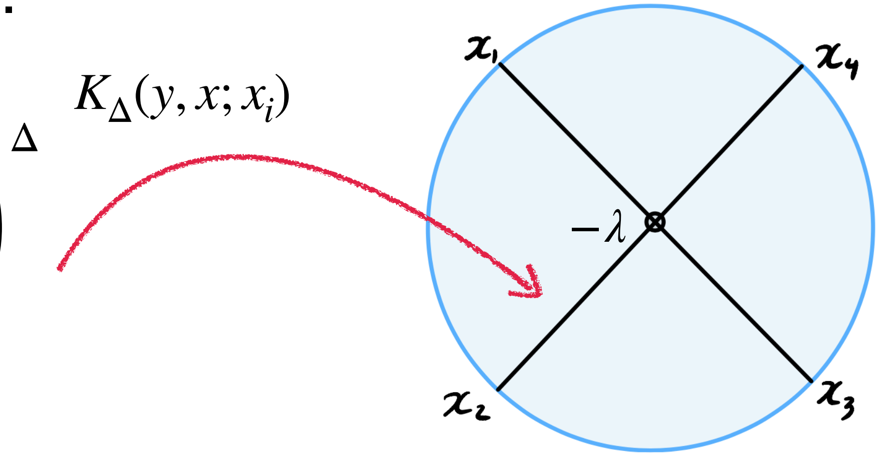
$$S = \int dxdz\,\sqrt{g}\,\left[\,g^{\mu\nu}\,\partial_\mu\Phi\,\partial_\nu\Phi + m^2_{\Delta_\phi}\Phi^2 + g_L\,(\partial^L\Phi)^4\,\right], \qquad L = 0, 1, \ldots$$

where $ds^2 = \frac{1}{z^2}(dx^2 + dz^2)$. Here $(\partial^L\Phi)^4$ denotes a complete and independent set of quartic vertices with four fields and up to $4L$ derivatives.

For $L = 0$, this is ϕ^4 theory: correlators are $\bar{D}$-functions.

$$< \phi_\Delta(x_1)\phi_\Delta(x_2)\phi_\Delta(x_2)\phi_\Delta(x_2) > = -\lambda \int \frac{dydx}{y^2}\prod_{i=1}^{4}\left(\frac{y}{y^2 + (x - x_i)^2}\right)^{\Delta}$$

$$= \frac{C_\Delta}{(x_{12}x_{34})^{2\Delta}}\bar{D}_\Delta(z)$$



No closed from expression is known, in cross ratio space, for general Δ.

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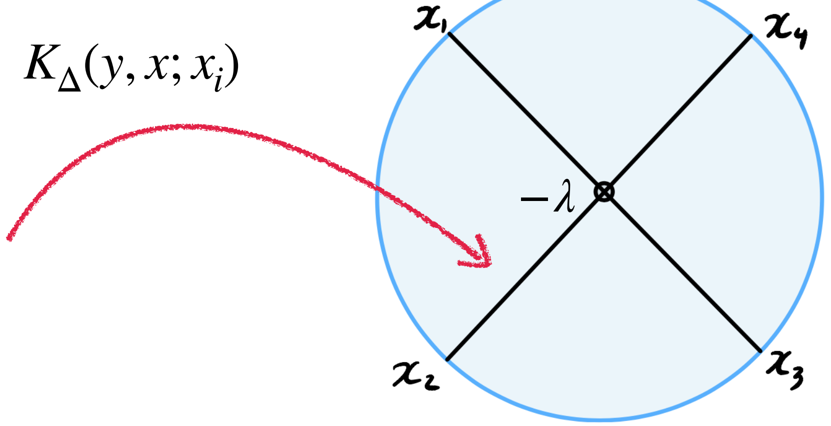

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In Mellin space their explicit expressions are simpler

$$\bar{D}_{1,1,1,1} = -\frac{2\log(1-\chi)}{\chi} - \frac{2\log(\chi)}{1-\chi} \longrightarrow M_{1111}(s) = 2\,\Gamma(s-1)\Gamma(-s)$$

$$\bar{D}_{2,2,2,2} = -\frac{2\,(\chi^2-\chi+1)}{15(1-\chi)^2\chi^2} + \frac{(2\chi^2-5\chi+5)\log(\chi)}{15(\chi-1)^3} - \frac{(2\chi^2+\chi+2)\log(1-\chi)}{15\chi^3} \longrightarrow M_{2222}(s) = 2(2-s+s^2)\,\Gamma(s-3)\Gamma(-2-s)$$

Perturbation theory: quartic interactions with derivatives in AdS_2

$$S = \int dx dz\, \sqrt{g}\, \left[\, g^{\mu\nu}\, \partial_\mu \Phi\, \partial_\nu \Phi + m^2_{\Delta_\phi} \Phi^2 + g_L\, (\partial^L \Phi)^4 \,\right], \qquad L = 0, 1, \ldots$$

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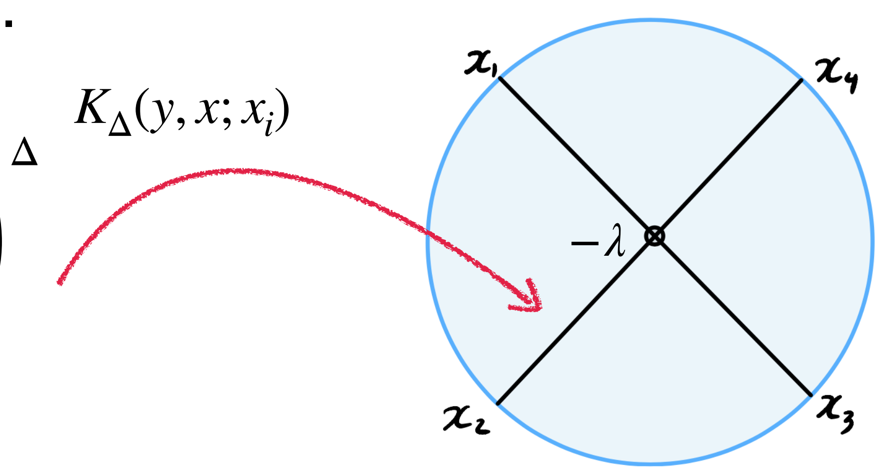
$$= \frac{C_\Delta}{(x_{12}x_{34})^{2\Delta}} \bar{D}_\Delta(z)$$



In Mellin space their explicit expressions are simpler and closed expressions can be found

$$\hat{M}_{\Delta_\phi}(s) = \pi \csc(\pi s) \left(\pi \cot(\pi s) P_{\Delta_\phi}(s) - \sum_{k=1}^{2\Delta_\phi - 1} \frac{P_{\Delta_\phi}(k)}{s - k} \right)$$
$$P_{\Delta_\phi}(s) = 2 \frac{\Gamma(\Delta_\phi)^4}{\Gamma(2\Delta_\phi)}\, {}_4F_3(\{\tfrac{1}{2}, s, 1 - \Delta_\phi, 2\Delta_\phi - s\}; \{1, 1, \Delta_\phi + \tfrac{1}{2}\}; 1)$$

Perturbation theory: quartic interactions with derivatives in AdS_2

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For quartic bulk interactions of the kind $(\partial^L \phi)^4$

$$M_{(\partial^L \phi)^4} = \sum_{l=0}^{L} a_l \sum_{k=0}^{2l} c_{k,l} M_{\Delta+l}(s+k) \qquad 2c_{k,l} = \frac{\Gamma(l+1)}{\Gamma(k+1)\Gamma(l-k+1)} + \delta_{k,0} + \delta_{k,2l}$$

With such **closed** formulas we can successfully extract **new** CFT data in closed form.

$$\hat{\gamma}^{(1)}_{L,n}(\Delta_\phi) = \hat{\mathcal{G}}_{L,n}(\Delta_\phi)\hat{\mathcal{P}}_{L,n}(\Delta_\phi)$$

$$\hat{\mathcal{G}}_{L,n}(\Delta_\phi) = \frac{\sqrt{\pi}4^{-2\Delta-L+1}\Gamma(2\Delta)^2\Gamma(L+\frac{1}{2})\Gamma(L+\Delta)^4\Gamma(L+2\Delta-\frac{1}{2})\Gamma(n+\Delta+\frac{1}{2})\Gamma(L-n+\Delta)}{\Gamma(L+1)\Gamma(L+\Delta+\frac{1}{2})^2\Gamma(L+2\Delta)\Gamma(n+\Delta)^3\Gamma(2n+2\Delta-\frac{1}{2})\Gamma(L+n+\Delta+\frac{1}{2})}$$

$\hat{\mathcal{P}}_{L,n}(\Delta_\phi)$ is a polynomial in n and in Δ_ϕ of degree $6L$.

Verified in [Knop, Mazac 22]

Obtained comparing residues at poles of. $M_{(\partial^L \phi)^4}$ with those of Mellin block expansion

From OPE inversion formula to dispersion relation

- The OPE expresses a four-point correlator as a discrete sum of conformal blocks

$$\mathcal{G}(z) = \sum_{\Delta} a_\Delta G_\Delta(z) \qquad\qquad z = \frac{x_{12}\, x_{34}}{x_{13}\, x_{24}}$$

- Another expansion - the conformal partial wave decomposition - is in terms of a complete basis of orthonormal functions (principal series, $\Delta \in \frac{1}{2} + i\mathbb{R}$ and discrete series).

$$\mathcal{G}(z) = \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} \frac{d\Delta}{2\pi i} \frac{I_\Delta}{n_\Delta} \Psi_\Delta(z) + \sum_{m=0}^{\infty} \frac{4m-1}{4\pi^2} \tilde{I}_{2m} \Psi_{2m}(z)$$

$$\Psi_\Delta(z) = \kappa_{1-\Delta} G_\Delta(z) + \kappa_\Delta G_{1-\Delta}(z)\,, \qquad \kappa_\Delta = \frac{\sqrt{\pi}\,\Gamma(\Delta - \frac{1}{2})\Gamma(\frac{1-\Delta}{2})^2}{\Gamma(1-\Delta)\Gamma(\frac{\Delta}{2})^2}$$

$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\Delta}{2\pi i} \frac{I_\Delta}{2\kappa_\Delta} G_\Delta(z) + \sum_{m=0}^{\infty} \frac{\Gamma^2(2m+2)}{2\pi^2\Gamma(4m+3)} \tilde{I}_{2m+2} G_{2m+2}(z)$$

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From the poles of the coefficients one recovers the OPE expansion

$$a_{\Delta} = -\text{Res}\left[\frac{I_{\Delta'}}{2\kappa_{\Delta'}}\right]_{\Delta'=\Delta}$$

From OPE inversion formula to dispersion relation

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$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\Delta}{2\pi i}\frac{I_\Delta}{2\kappa_\Delta}G_\Delta(z) + \sum_{m=0}^{\infty}\frac{\Gamma^2(2m+2)}{2\pi^2\Gamma(4m+3)}\tilde{I}_{2m+2}G_{2m+2}(z)$$

- Because of the orthonormality one can perform a (trivial) inversion

$$I_\Delta = \int_{-\infty}^{\infty} dz\, z^{-2}\,\Psi_\Delta(z)\,\mathcal{G}(z) \quad \text{for } \Delta \in \frac{1}{2} + i\mathbb{R}\,, \qquad \tilde{I}_\Delta = \int_{-\infty}^{\infty} dz\, z^{-2}\,\Psi_\Delta(z)\,\mathcal{G}(z) \quad \text{for } \Delta \in 2\mathbb{N}$$

From OPE inversion formula to dispersion relation

A more powerful inversion can be derived from a contour-deformation argument based on the analytic structure of the correlator and its (Regge) behavior at infinity

[Caron-Huot 17] [Simmons-Duffin, Stanford, Witten 2017] [Mazac 2018]

$$I_\Delta = 2\int_0^1 dw\, w^{-2} H_\Delta(w)\, \mathrm{dDisc}[\mathcal{G}(w)]$$

$$\tilde{I}_m = \frac{4\Gamma^2(m)}{\Gamma(2m)} \int_0^1 dw\, w^{-2} G_m(w)\, \mathrm{dDisc}[\mathcal{G}(w)]$$

$H_\Delta(w)$: known explicitly for all integer (bos) and half-integers (ferm) dimensions Δ_ϕ of the external operators.

$G_m(w)$: sl(2,R) conformal block

makes use of the **double discontinuity** of the correlator

$$\mathrm{dDisc}[\mathcal{G}(z)] = \mathcal{G}(z) - \frac{\mathcal{G}^{\circlearrowleft}(z) + \mathcal{G}^{\circlearrowright}(z)}{2} \quad \text{for } z \in (0,1)$$

$\mathcal{G}^{\circlearrowleft}(z)$: value of *G*(z) moving counterclockwise around the branch cut at z=1, vv for $\mathcal{G}^{\circlearrowright}(z)$.

A less trivial inversion

A less trivial inversion extracts CFT data and reconstruct correlators from their singularities using complex analysis and their behavior at infinity.

Sketch of the argument:

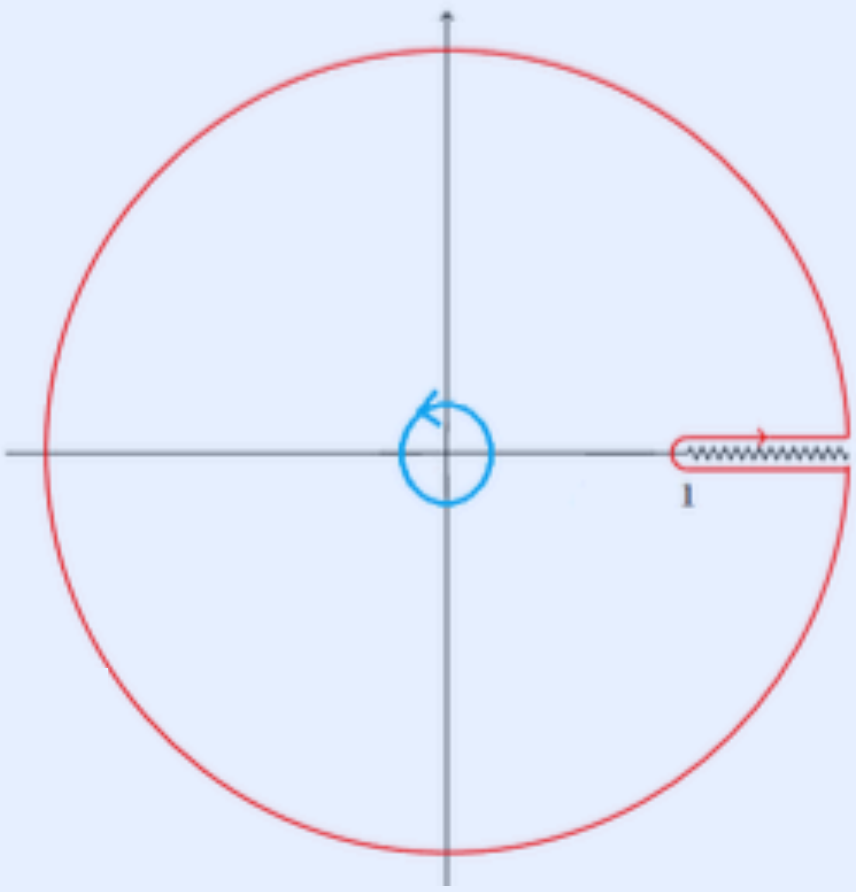
Consider $f(z) = \sum_{j=1}^{\infty} f_j z^j$ such thath

- $f(z)$ has a branch cut for $z > 1$.
- $|f(z)/z| \to 0$ as $|z| \to \infty$.

Then $f_j = \frac{1}{2\pi i} \oint \frac{dz}{z} z^{-j} f(z)$

and by deforming the contour

$$f_j = \frac{1}{2\pi} \int_1^{\infty} \frac{dz}{z} z^{-j} \underbrace{\text{Disc}\, f(z)}_{\text{simpler than } f(z)}$$



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[Simmons-Duffin, Stanford, Witten 2017] [Mazac 2018]

$$\tilde{I}_m = \frac{4\Gamma^2(m)}{\Gamma(2m)}\int_0^1 dw\, w^{-2} G_m(w)\, \mathrm{dDisc}[\mathcal{G}(w)]$$

$G_m(w)$: sl(2,R) conformal block

makes use of the **double discontinuity** of the correlator

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$\mathcal{G}^{\circlearrowleft}(z)$: value of *G*(z) moving counterclockwise around the branch cut at z=1, vv for $\mathcal{G}^{\circlearrowright}(z)$.

It provides an analytic continuation of the coefficients (**in higher d, this means we can think of spin as a expansion parameter).**

The dDisc of a correlator is **much simpler** than the correlator itself, in perturbation theory. Crucially **can be computed at any order from lower order data**!

Dispersion relation for CFT1 correlators

The double discontinuity can then be taken as the starting point to reconstruct the full correlator

$$\mathcal{G}(z) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\Delta}{2\pi i} \frac{I_\Delta}{2\kappa_\Delta} G_\Delta(z) + \sum_{m=0}^{\infty} \frac{\Gamma^2(2m+2)}{2\pi^2\Gamma(4m+3)} \tilde{I}_{2m+2} G_{2m+2}(z)$$

$$= \int_0^1 dw\, w^{-2} \mathrm{dDisc}[\mathcal{G}(w)] \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\Delta}{2\pi i} \frac{H_\Delta^{B/F}(w)}{\kappa_\Delta} G_\Delta(z)$$

$$+ \int_0^1 dw\, w^{-2} \mathrm{dDisc}[\mathcal{G}(w)] \sum_{m=0}^{\infty} \frac{2\Gamma(2m+2)^4}{\pi^2\Gamma(4m+4)\Gamma(4m+3)} G_{2m+2}(w) G_{2m+2}(z)$$

$$\equiv \int_0^1 dw\, w^{-2} \mathrm{dDisc}[\mathcal{G}(w)] K_{\Delta_\phi}(z,w)\,,$$

$$K_{\Delta_\phi}(z,w) = \frac{w\, z^2(w-2)\log(1-w)}{\pi^2(w-z)(w+z-wz)} - \frac{z\, w^2(z-2)\log(1-z)}{\pi^2(w-z)(w+z-wz)}$$

$$\pm \frac{z^2}{\pi^2}\left[\log(1-w)\frac{(1-2w)w^{2-2\Delta_\phi}}{(w-1)wz^2+z-1} + \frac{\log(1-z)}{z}\frac{w^{2-2\Delta_\phi}}{wz-1} + \log(z)\frac{(1-2w)w^{2-2\Delta_\phi}}{(w-1)wz^2+z-1} + (w \to \tfrac{w}{w-1})\right]$$

$$- w^{2-2\Delta_\phi} \sum_{n=0}^{2\Delta_\phi-4} a_n^{\Delta_\phi}(w)\, \mathcal{C}^n \left[\frac{2}{\pi^2}\left(\frac{z^2\log(z)}{1-z} + z\log(1-z)\right)\right]$$

$\mathcal{C}$ ↓ sl(2, R) Casimir

The kernel of the integral can be evaluated explicitly at each given integer and half-integer dimension Δ_ϕ of the external identical operators.

Dispersion relation for CFT1 correlators

The double discontinuity can then be taken as the starting point to reconstruct the full correlator

$$\mathcal{G}(z) = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\Delta}{2\pi i} \frac{I_\Delta}{2\kappa_\Delta} G_\Delta(z) + \sum_{m=0}^{\infty} \frac{\Gamma^2(2m+2)}{2\pi^2\Gamma(4m+3)} \tilde{I}_{2m+2} G_{2m+2}(z)$$

$$= \int_0^1 dw\, w^{-2} \mathrm{dDisc}[\mathcal{G}(w)] \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\Delta}{2\pi i} \frac{H_\Delta^{B/F}(w)}{\kappa_\Delta} G_\Delta(z)$$

$$+ \int_0^1 dw\, w^{-2} \mathrm{dDisc}[\mathcal{G}(w)] \sum_{m=0}^{\infty} \frac{2\Gamma(2m+2)^4}{\pi^2\Gamma(4m+4)\Gamma(4m+3)} G_{2m+2}(w) G_{2m+2}(z)$$

$$\equiv \int_0^1 dw\, w^{-2} \mathrm{dDisc}[\mathcal{G}(w)] K_{\Delta_\phi}(z,w),$$

$$K_{\Delta_\phi}(z,w) = \frac{w\, z^2(w-2)\log(1-w)}{\pi^2(w-z)(w+z-wz)} - \frac{z\, w^2(z-2)\log(1-z)}{\pi^2(w-z)(w+z-wz)}$$
$$\pm \frac{z^2}{\pi^2}\left[\log(1-w)\frac{(1-2w)w^{2-2\Delta_\phi}}{(w-1)wz^2+z-1} + \frac{\log(1-z)}{z}\frac{w^{2-2\Delta_\phi}}{wz-1} + \log(z)\frac{(1-2w)w^{2-2\Delta_\phi}}{(w-1)wz^2+z-1} + (w \to \tfrac{w}{w-1})\right]$$
$$- w^{2-2\Delta_\phi} \sum_{n=0}^{2\Delta_\phi-4} a_n^{\Delta_\phi}(w)\, \mathcal{C}^n \left[\frac{2}{\pi^2}\left(\frac{z^2\log(z)}{1-z} + z\log(1-z)\right)\right]$$

The kernel, crossing symmetric (in z), Regge bounded, and definite positive, explicitly depends on the dimension Δ_ϕ of the external operators (≠ **from higher d**).

Dispersion relation in perturbation theory

The double discontinuity can then be taken as the starting point to reconstruct the full correlator,

$$\mathcal{G}(z) = \int_0^1 dw\, w^{-2}\, \mathrm{dDisc}[\mathcal{G}(w)]\, K_{\Delta_\phi}(z,w)\,,$$

$$\mathrm{dDisc}_t[\mathcal{G}(z)] = \mathcal{G}(z) - \frac{\mathcal{G}^{\curvearrowleft}(z) + \mathcal{G}^{\curvearrowright}(z)}{2}$$

- Much simpler than correlator!

$$\mathrm{dDisc}[\log(1-z)] = 0\;,$$
$$\mathrm{dDisc}[\log^2(1-z)] = 4\pi^2$$

- On conformal blocks, dDisc acts as

$$\mathrm{dDisc}\Big[\tfrac{z^{2\Delta_\phi}}{(1-z)^{2\Delta_\phi}} G_\Delta(1-z)\Big] = 2\sin^2 \tfrac{\pi}{2}(\Delta - 2\Delta_\phi)\, \tfrac{z^{2\Delta_\phi}}{(1-z)^{2\Delta_\phi}} G_\Delta(1-z)$$

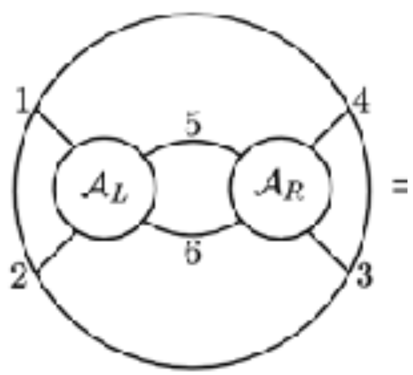
If the correlator is evaluated in a perturbative expansion about generalised free theory, this implies that each given order dDisc is given in terms of lower order data

E.g. $$\mathrm{dDisc}[\mathcal{G}^{(2)}(z)] = \pi^2 \sum_n \frac{1}{2} a_n^{(0)} \left(\gamma_n^{(1)}\right)^2 \frac{z^{2\Delta_\phi}}{(1-z)^{2\Delta_\phi}} G_{2\Delta_\phi+2n}(1-z)$$

Double discontinuity in perturbation theory

Direct connection of Ddisc with “unitarity” cut operators in AdS, which act on bulk amplitudes putting virtual lines on shell [Alday, Caron-Huot 17] [Meltzer Perlmutter Sivaramakrishan 19]

$$\mathcal{A} =$$



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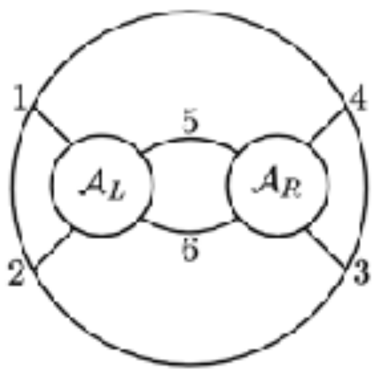
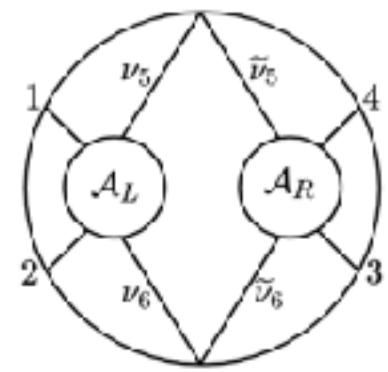
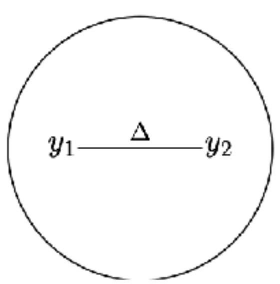


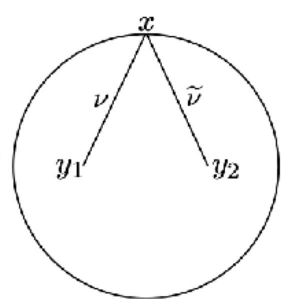
$= \int d\nu_{5,6} \int d^d x_{5,6} P(\nu_5, \Delta_5) P(\nu_6, \Delta_6)$



Split representation of bulk-to-bulk propagator in terms of two bulk-to-boundary propagators



$= \int d\nu d^d x P(\nu, \Delta)$



$$G_\Delta(y_1, y_2) = \int_{-\infty}^{\infty} d\nu\, P(\nu, \Delta) \int_{\partial \mathrm{AdS}} d^d x K_{\frac{d}{2}+i\nu}(x, y_1) K_{\frac{d}{2}-i\nu}(x, y_2),$$

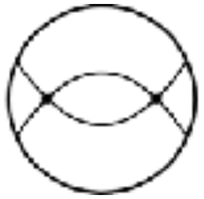
A propagator goes on-shell when localised onto a pole of $P(\nu, \Delta)$

A “Cut operator” can be defined, effect same as Ddisc (vanishes on contact diagrams, etc)

Effective to 4 point, 1-loop (no general unitarity) - to be developed.

Correlators from dispersion in perturbation theory

- Checked at one-loop for the $\lambda\phi^4$ theory in AdS2



- dCFT1 defined by 1/2 BPS Wilson line in N=4 sYM: state of the art is 4th order in strong coupling (=3 loops in AdS) obtained with perturbative Ansatz [Ferrero, Meneghelli 23]

rational functions

$$G(z) = \sum_{\ell=0}^{\infty} G^{(\ell)}(z) \quad \text{where} \quad G^{(\ell)}(z) = \sum_{i=1}^{N(\ell)} r_i(z)\, \mathcal{T}_i(z)$$

$$\mathcal{T}_i(z) \in \{\text{HPLs of transcendentality } \mathtt{t} \leq \mathtt{t}_{\mathtt{max}}(\ell)\} \qquad N(\ell) = \sum_{\mathtt{t}=0}^{\mathtt{t}_{\mathtt{max}}(\ell)} 2^{\mathtt{t}} = 2^{1+\mathtt{t}_{\mathtt{max}}(\ell)} - 1$$

$$\mathtt{t}_{\mathtt{max}}(\ell) = \ell$$

Unknowns are some coefficients in an educated guess for the rational functions $r_i(z)$

Ansatz constrained by: [Aharony, Alday, Bissi, Perlmutter 16]

a) AdS unitarity (highest logarithmic singularities fixed in terms of lower order ones)

b) Crossing symmetry, Regge bound and supersymmetric localization
fix the remaining terms ~ 1, log(z), log(1-z).

Correlators from dispersion in perturbation theory

The dispersion relation bypasses the need of an Ansatz incorporating all constraints!

$$\mathcal{G}(z) = \int_0^1 dw\, w^{-2} \mathrm{dDisc}[\mathcal{G}(w)] K_{\Delta_\phi}(z, w)$$

with a **caveat:** the **regularization** procedure necessary order by order in perturbation theory (where the Regge behaviour is worse than in the full nonperturbative correlator) implies subtractions which depend on a few **unknown** OPE data (i.e. data at same pert. order).

@ 1 loop: $a_0^{(2)},\ \gamma_0^{(2)}$

@ 2 loops: $a_0^{(3)}, a_1^{(3)}\ \ \gamma_0^{(3)}\ \gamma_1^{(3)}$

@ 3 loops: $a_0^{(4)}, a_1^{(4)}\ \ \gamma_0^{(4)}\ \gamma_1^{(4)}$

These **can be fixed**, **in the N=4 SYM case**, using inputs from supersymmetric localization or constraints from integrated correlators. [Cavaglia'Gromov Julius Preti 22] [Drukker, Kong, Sakkas 22]

This kind of leftover ambiguity is not surprising in this context,
e.g. in higher d there is a low spin ambiguity.

Quick look at the fourth order correlator

[Ferrero Meneghelli 23]

$$\mathcal{G}^{(4)}(z) = -\frac{3\left(46z^8-175z^7+287z^6-271z^5+254z^4-271z^3+287z^2-175z+46\right)\mathrm{Li}_2(z)}{16(z-1)^3z^2}$$
$$-\frac{3\left(96z^{11}-423z^{10}+785z^9-799z^8+477z^7-137z^6-137z^5+477z^4-799z^3+785z^2-423z+96\right)\mathrm{Li}_2(z)\log^2(1-z)}{8(z-1)^3z^3}$$
$$+\frac{3\left(264z^{11}-1572z^{10}+4097z^9-6169z^8+5713z^7-3107z^6+190z^5+1460z^4-1466z^3+796z^2-236z+30\right)\mathrm{Li}_2(z)\log(z)}{16(z-1)^5z^2}$$
$$-\frac{3\left(96z^{11}-633z^{10}+1835z^9-3071z^8+3265z^7-2292z^6+980z^5-262z^4-30z^3+25z^2-11z+2\right)\mathrm{Li}_2(z)\log^2(z)}{8(z-1)^5z}$$
$$+\log(1-z)\left[-\frac{3\mathrm{Li}_2(z)\left(264z^{10}-1086z^9-1893z^8-1797z^7+740z^6+456z^5-1293z^4+1255z^3-580z^2+30z+48\right)}{16(z-1)^3z^3}\right.$$
$$\left.-\frac{3(1-z)\left(96z^{11}-528z^{10}+1252z^9-1674z^8+1376z^7-700z^6+228z^5-86z^4+100z^3-90z^2+44z-9\right)\mathrm{Li}_2(z)\log(z)}{4(z-1)^5z^2}\right]$$
$$-\frac{3\left(z^2-z+1\right)\left(46z^8-175z^7+241z^6-142z^5+142z^4-241z^3+175z^2-46z\right)\mathrm{Li}_2\left(\frac{z}{z-1}\right)}{16(z-1)^4z^3}$$
$$-\frac{3\left(z^2-z+1\right)\left(24z^9-93z^8+136z^7-90z^6+26z^5-26z^4+90z^3-136z^2+93z-24\right)\mathrm{Li}_2\left(\frac{z}{z-1}\right)\log(1-z)}{8(z-1)^4z^3}$$
$$+\frac{3\left(z^2-z+1\right)\left(24z^8-108z^7+188z^6-156z^5+58z^4+32z^3-66z^2+52z-15\right)\mathrm{Li}_2\left(\frac{z}{z-1}\right)\log(z)}{8(z-1)^4z^2}$$
$$-\frac{3\left(72z^{11}-396z^{10}+900z^9-1080z^8+94z^7+1939z^6-4005z^5+4625z^4-3193z^3+1260z^2-216z\right)\mathrm{Li}_3(z)}{16(z-1)^5z^3}-$$
$$\frac{3\left(27z^{12}-163z^{11}+424z^{10}-620z^9+570z^8-226z^7-388z^6+1320z^5-2240z^4+2368z^3-1564z^2+588z-96\right)\mathrm{Li}_3(z)\log(1-z)}{8(z-1)^5z^3}$$
$$+\frac{3\left(27z^{11}-161z^{10}+413z^9-595z^8+540z^7-488z^6+592z^5-972z^4+1025z^3-703z^2+271z-45\right)\mathrm{Li}_3(z)\log(z)}{8(z-1)^5z^2}$$
$$+\frac{\left(648z^{11}-3348z^{10}+7419z^9-9156z^8+6099z^7-6099z^5+9156z^4-7419z^3+3348z^2-648z\right)\mathrm{Li}_3\left(\frac{z}{z-1}\right)}{16(z-1)^4z^3}+$$
$$\frac{3\left(96z^{12}-519z^{11}+1208z^{10}-1584z^9+1276z^8-614z^7+614z^5-1276z^4+1584z^3-1208z^2+519z-96\right)\mathrm{Li}_3\left(\frac{z}{z-1}\right)\log(1-z)}{8(z-1)^4z^3}$$

$$-\frac{3\left(96z^{11}-564z^{10}+1432z^{9}-2052z^{8}+1808z^{7}-1024z^{6}+444z^{5}-410z^{4}+532z^{3}-468z^{2}+224z-45\right)\mathrm{Li}_3\left(\frac{z}{z-1}\right)\log(z)}{8(z-1)^4z^2}$$

$$\frac{\left(1800z^{13}-9144z^{12}+20115z^{11}-25093z^{10}+19259z^{9}-9009z^{8}+2403z^{7}-1581z^{6}\right)\log^4(1-z)}{48(z-1)^3z^4}$$

$$\frac{\left(6147z^{5}-14465z^{4}+20383z^{3}-17577z^{2}+8568z-1800\right)\log^4(1-z)}{48(z-1)^3z^4}$$

$$+\frac{\left(1728z^{8}-8446z^{7}+17973z^{6}-21752z^{5}+17642z^{4}-8178z^{3}+2824z^{2}-84z+21\right)\log^2(z)}{16(z-1)^4}$$

$$+\frac{\left(-3024z^{10}+19629z^{9}-56379z^{8}+94369z^{7}-101549z^{6}+71379z^{5}-34041z^{4}+8808z^{3}-2232z^{2}+20z-4\right)\log^3(z)}{24(z-1)^5}$$

$$+\frac{z^2\left(900z^{10}-7272z^{9}+26487z^{8}-57436z^{7}+82346z^{6}-81720z^{5}+56934z^{4}-26952z^{3}+8613z^{2}-1250z+250\right)\log^4(z)}{24(z-1)^6}$$

$$+\log^3(1-z)\left[\frac{12096z^{11}-55620z^{10}+110010z^{9}-122387z^{8}+82023z^{7}-35200z^{6}}{96(z-1)^3z^3}\right.$$

$$\frac{31084z^{5}-71961z^{4}+112931z^{3}-105816z^{2}+55566z-12528}{96(z-1)^3z^3}$$

$$+\frac{\left(-7200z^{12}+42912z^{11}-112473z^{10}+170512z^{9}-164340z^{8}+102980z^{7}\right)\log(z)}{48(z-1)^4z^2}$$

$$\left.+\frac{\left(-40586z^{6}+9300z^{5}-1996z^{4}+1800z^{3}-1620z^{2}+792z-162\right)\log(z)}{48(z-1)^4z^2}\right]$$

$$\log^2(1-z)\left[\frac{3456z^{9}-14350z^{8}+25751z^{7}-26307z^{6}}{32(z-1)^3z^2}\right.$$

$$+\frac{19443z^{5}-19392z^{4}+26259z^{3}-26087z^{2}+14737z-3594}{32(z-1)^3z^2}$$

$$+\frac{\left(-12096z^{11}+66492z^{10}-159516z^{9}+219295z^{8}-188716z^{7}+102443z^{6}\right)\log(z)}{32(z-1)^4z^2}$$

$$+\frac{\left(-32956z^{5}+6317z^{4}-2608z^{3}+2251z^{2}-1050z+198\right)\log(z)}{32(z-1)^4z^2}+\frac{\left(3600z^{12}-21312z^{11}+55404z^{10}-83184z^{9}\right)\log^2(z)}{16(z-1)^4z^2}$$

$$\left.+\frac{\left(79272z^{8}-49008z^{7}+18976z^{6}-4116z^{5}+602z^{4}-332z^{3}+288z^{2}-136z+27\right)\log^2(z)}{16(z-1)^4z^2}\right]$$

$$
\begin{aligned}
&+\frac{-648z^8\zeta_3+3348z^7\zeta_3+z^6(26-7451\zeta_3)+4z^5(2321\zeta_3-26)+z^4(208-7369\zeta_3)+z^3(3362\zeta_3-260)}{16(z-1)^4}\\
&+\frac{z^2(208-1270\zeta_3)+8z(16\zeta_3-13)-32\zeta_3+26}{16(z-1)^4}\\
&+\frac{(2304z^{10}\zeta_3-15840z^9\zeta_3+152z^8(316\zeta_3-7)+z^7(5583-84320\zeta_3)+z^6(94240\zeta_3-14343))\log(z)}{64(z-1)^5}\\
&+\frac{(z^5(22559-68768\zeta_3)+z^4(33376\zeta_3-23629)+z^3(17036-8960\zeta_3)+z^2(2240\zeta_3-8294)+2690z-538)\log(z)}{64(z-1)^5}\\
&+\log(1-z)\Bigg[\frac{\left(-3456z^8+13824z^7-23847z^6+23157z^5-13699z^4+4931z^3-2289z^2+1379z-388\right)\log(z)}{16(z-1)^3z}\\
&+\frac{-2304z^{10}\zeta_3+12456z^9\zeta_3-56z^8(520\zeta_3-19)+z^7(38592\zeta_3-5057)-8z^6(3956\zeta_3-1497)+z^5(16752\zeta_3-18877)}{64(z-1)^4z}\\
&+\frac{z^4(21788-5088\zeta_3)+z^3(2016\zeta_3-18877)-8z^2(128\zeta_3-1497)+z(576\zeta_3-5057)-128\zeta_3+1064}{64(z-1)^4z}\\
&+\frac{\left(6048z^{10}-33030z^9+78696z^8-107437z^7+91780z^6-49499z^5+15706z^4-2762z^3+532z^2-274z+60\right)\log^2(z)}{16(z-1)^4z}\\
&+\frac{\left(-1800z^{12}+12528z^{11}-38799z^{10}+70469z^9-83015z^8+65845z^7\right)\log^3(z)}{12(z-1)^5z}\\
&+\frac{\left(-34966z^6+11946z^5-2256z^4+280z^3-25z^2+11z-2\right)\log^3(z)}{12(z-1)^5z}\Bigg]\,.
\end{aligned}
$$

Simple: three-loop result only hastranscendentality four, same as for **one loop** results in a toy model with no supersymmetry; it can be rewritten in a way in which all polylogarithms of even weight are absent.

Conclusions

- We have considered a class of four-point correlators in the CFT_1 defined on the 1/2-BPS Wilson line in the 3d superconformal ABJM theory.

- Superconformal symmetry determines four-point correlators of the displacement supermultiplet in terms of a single function, that we evaluate at strong coupling using holography and Witten diagrams and the analytic boostrap. We can extract CFT data.

- We defined a Mellin amplitude for CFT_1 four-point functions; bounded, meromorphic function of a single complex variable, whose analytical properties are inferred from physical requirements on the correlator.

  Closed-form expressions for Mellin transform of tree-level contact interactions with an arbitrary number of derivatives in a bulk AdS2 field theory, and for first correction to the scaling dimension of “two-particle” operators exchanged.

- Derived from the inversion formula a dispersion relation for CFT_1 four-point functions, an integral over the double discontinuity of the correlator.

Outlook

- Higher-order analysis, multi-point correlators, non-identical in the same setup

- Organising principles/hidden symmetries?

  Recent observation of integrable structure underlying contact Witten diagrams
  If generalizes to other classes of diagrams this would open a playground of applications of integrability in AdS spaces. [Rigatos, Zhou 22]

- Despite/with the help of these analytic bootstrap tools, a motivation to develop technology for Witten diagrams remains, thanks to the general observation that (for a class of boundary correlators related to inflationary correlators)

  perturbation theory in rigid de Sitter -> Witten diagrams in **E**AdS

  [Sleight, Taronna 20,21] [Di Pietro, Gorbenko, Komatsu 21]

  It would be great to develop loop-technology for AdS_2 models with derivative interactions (some hints in [Chen, Gimenez-Grau, Hynek, Zhou 24]) [Stemplowski Castellani **VF**]

- Wilson lines in N=4 SYM are a great setup for "fusion rules" recently formulated: the "generalized cusp anomalous dimension" is reinterpreted as fusion energy.

  [Kravchuk, Radcliffe, Sinha 2024] [Cuomo, He, Komargodski 2024] [Diatlyk, Khanchandani, Popov, Wang 2024]

  Special features appear in the "fusion algebra" when enriching the setup with large charge insertions [Bonomi VF Griguolo Giangreco Seminara, in progress]

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Boundaries, impurities, and defects (BIDs) are crucial for understanding many of the most important systems in modern physics. Prominent examples include protected edge modes of topological insulators, impurities in high-temperature superconductors, graphene as a co-dimension one defect, Wilson and ‘t Hooft (defect) lines classifying infrared phases, and quantum error correcting codes created by controlling boundaries of quantum spin systems.

While quantum field theory (QFT) has had enormous success in describing many such systems, important questions about BIDs remain out of reach due to strong coupling. Fortunately, in the last few decades, many promising physical and mathematical tools have emerged. On the physics side, duality, supersymmetric localization, conformal bootstrap, integrability, and entanglement entropy have undergone rapid development. On the mathematics side, (higher) category theory, associated rigorous notions of duality, like categorical Morita equivalence, and representation theory relevant for non-rational conformal field theories have seen striking advances. Physicists have begun to reinterpret and extend these results through generalized symmetries, tensor networks, and by twisting and deforming (supersymmetric) quantum field theories and their boundaries.

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EXTRA-SLIDES

Correlators from dispersion in perturbation theory: STRATEGY

1) Compute dDisc$[\mathcal{G}^{(\ell)}(z)]$ from lower order CFT data.

$$= \sum_{k=0}^{\ell} G^{(\ell)}_{\log^k}(z) \log^k z \qquad z \to 0 \text{ OPE limit}$$

≡ compute terms proportional to $\log^k z$ with $k > 1$ from lower order data

$$\text{e.g. } \mathrm{dDisc}[\mathcal{G}^{(2)}] = \pi^2 \sum_n \frac{1}{2} a_n^{(0)} (\gamma_n^{(1)})^2 \frac{z^{2\Delta_\phi}}{(1-z)^{2\Delta_\phi}} G_{2\Delta_\phi + 2n}(1-z)$$

MIXING: because of degeneracy between operators in the free theory this is rather an average

$$\langle a_n^{(0)} (\gamma_n^{(1)})^2 \rangle := \sum_{\mathcal{O}} (\mu_{\mathcal{O}}^{(0)})^2 (\gamma_{\mathcal{O}}^{(1)})^2$$

eigenstates of dil. operator — OPE coeff with external operators

However, at previous order, from one correlator it is only possible to extract

$$\langle a_n^{(0)} \gamma_n^{(1)} \rangle := \sum_{\mathcal{O}} (\mu_{\mathcal{O}}^{(0)})^2 \gamma_{\mathcal{O}}^{(1)}$$

For a solution of the mixing problem in this setup, Ferrero Meneghelli 2023

Correlators from dispersion in perturbation theory: STRATEGY

Usually solved considering more correlators (since μ_σ depend on external operators, the result of the average depends on the four-point function one is considering

⇒ enough correlators, enough inequivalent averages $\langle a_n^{(0)} \gamma_n^{(1)} \rangle$ to calculate the actual $\gamma_n^{(1)}$.

2) Regge boundedness of correlator in Regge limit $\left(\frac{1}{2}+it\right)^{-2\Delta_\phi} \mathcal{G}\left(\frac{1}{2}+it\right) < \infty$ for $t \to \infty$ is broken perturbatively. Assume mild $\gamma_n^{(\ell)} \sim n^{\ell+1}$ ⇒ $\mathcal{G}\left(\frac{1}{2}+it\right) \sim t^{\ell}$

⇒ Reflected in the inversion formula $H_\Delta^{\text{imm.}}(\omega)$

⇒ In the dispersion relation $K^{\text{unbounded}}$ (with extra poles that may spoil convergence of the integral defining the correlator)

⇒ Subtraction at level of correlator $\mathcal{G}^{\text{reg}}(z) = \mathcal{G}^{(\ell)} -$ subtractions

⇒ 'dDisc$[\mathcal{G}^{\text{reg}}]$ enters the dispersion relation multiplying $K^{\text{unbounded}}$ and one then just demands that the integral of the dispersion relation converges

3) the subtractions then depend on specific unknown at the given order

⇒ to fix these unknown, use constraints from localization/integrated correlators of Cavaglià Gromov Julius Preti

3.2 Generalized free field OPE coefficients

It will be useful for what follows to collect some results for the OPE coefficients of generalized free fields (see, e.g., [55–58, 12]). In the case of the 4-point function of identical operators of dimension Δ, the generalized free field 4-point function has $\mathcal{G}(u,v) = 1 + u^{\Delta} + (u/v)^{\Delta}$, i.e. in $d = 1$ (3.3) is given by

$$\langle O_\Delta(t_1) O_\Delta(t_2) O_\Delta(t_3) O_\Delta(t_4)\rangle = \frac{1}{(t_{12}t_{34})^{2\Delta}}\left[1 + \chi^{2\Delta} + \frac{\chi^{2\Delta}}{(1-\chi)^{2\Delta}}\right], \tag{3.8}$$

where we assumed unit normalization of the 2-point function. The operators exchanged in the OPE are just the identity and the tower of "two-particle" operators

$$[O_\Delta O_\Delta]_{2n} \sim O_\Delta \partial_t^{2n} O_\Delta \tag{3.9}$$

of dimension $2\Delta + 2n$, $n = 0, 1, \ldots$. The corresponding OPE coefficients are given explicitly by

as one can verify from the identity

$$\sum_{n=0}^{\infty} c_{\Delta,\Delta;2\Delta+2n}\, \chi^{2\Delta+2n}\ {}_2F_1(2\Delta+2n, 2\Delta+2n, 4\Delta+4n, \chi) = \chi^{2\Delta} + \frac{\chi^{2\Delta}}{(1-\chi)^{2\Delta}}\,. \tag{3.11}$$

"AdS unitarity"

$$G(\chi) = \sum_h a_h \, g_h(\chi)\,, \tag{3.21}$$

where a_h are squared OPE coefficients and the sum runs over the operators $\mathcal{O}_h$ exchanged in the $\varphi \times \varphi$ OPE, with dimension h. In a perturbative theory both the dimensions of the exchanged operators and the squared OPE coefficients can be expanded in powers of g

$$h = \Delta + \sum_{\ell=1}^{\infty} g^\ell \, \gamma_\Delta^{(\ell)}\,, \qquad a_h = \sum_{\ell=0}^{\infty} g^\ell \, a_\Delta^{(\ell)}\,, \tag{3.22}$$

where we will use the convention that Δ denotes the free-theory value of the conformal dimensions of the operators exchanged by a certain four-point function, and therefore works as a label for the exchanged operators. We will be interested in the case when the set of allowed values for Δ are all integers (or half-integers for fermionic operators). While the hypergeometric function appearing in the conformal blocks (B.3) is analytic at $\chi = 0$, the overall power of χ is not and produces logarithmic singularities via

$$\chi^h = \chi^\Delta \left(1 + g\,\gamma_\Delta^{(1)} \log\chi + g^2(\gamma_\Delta^{(2)} \log\chi + \frac{1}{2}(\gamma_\Delta^{(1)})^2 \log^2\chi) + \ldots\right)\,, \tag{3.23}$$

where due to the assumption that $\Delta \in \mathbb{Z}$ the factor of χ^Δ is regular at $\chi = 0$ and the only non-analytic terms are explicit powers of $\log\chi$, which are higher at higher perturbative order. This leads to a natural decomposition of four-point functions at each order as

$$G^{(\ell)}(\chi) = \sum_{k=0}^{\ell} G^{(\ell)}_{\log^k}(\chi)\, \log^k \chi\,, \tag{3.24}$$

where each $G^{(\ell)}_{\log^k}(\chi)$ is analytic at $\chi = 0$ and admits an expansion in conformal blocks and their derivatives[19]. For the leading logarithmic singularities, such expansions read[20]

$$\begin{aligned}
G^{(\ell)}_{\log^\ell}(\chi) =& \frac{1}{\ell!} \sum_\Delta a_\Delta^{(0)} \, (\gamma_\Delta^{(1)})^\ell \, g_\Delta(\chi)\,,\\
G^{(\ell)}_{\log^{\ell-1}}(\chi) =& \frac{1}{(\ell-1)!} \sum_\Delta \left[\left(a_\Delta^{(1)} (\gamma_\Delta^{(1)})^{\ell-1} + (\ell-1)\, a_\Delta^{(0)}\, \gamma_\Delta^{(2)}\, (\gamma_\Delta^{(1)})^{\ell-2}\right) g_\Delta(\chi) + a_\Delta^{(0)} \, (\gamma_\Delta^{(1)})^\ell \, g_\Delta^{(1)}(\chi)\right]\,,\\
G^{(\ell)}_{\log^{\ell-2}}(\chi) =& \frac{1}{(\ell-2)!} \sum_\Delta \Bigg[\Big(a_\Delta^{(2)} (\gamma_\Delta^{(1)})^{\ell-2} + (\ell-2)\, (a_\Delta^{(0)}\, \gamma_\Delta^{(3)} + a_\Delta^{(1)}\, \gamma_\Delta^{(2)})\, (\gamma_\Delta^{(1)})^{\ell-3}\\
&+ \frac{1}{2}(\ell-2)(\ell-3) a_\Delta^{(0)}\, (\gamma_\Delta^{(2)})^2\, (\gamma_\Delta^{(1)})^{\ell-4}\Big)\, g_\Delta(\chi)\\
&+ \left(a_\Delta^{(1)} (\gamma_\Delta^{(1)})^{\ell-1} + (\ell-1)\, a_\Delta^{(0)}\, \gamma_\Delta^{(2)}\, (\gamma_\Delta^{(1)})^{\ell-2}\right) g_\Delta^{(1)}(\chi) + \frac{1}{2} a_\Delta^{(0)}\, (\gamma_\Delta^{(1)})^\ell \, g_\Delta^{(2)}(\chi)\Bigg]\,,
\end{aligned}$$

HPL's

The harmonic polylogarithms (HPL) $H(a_1,\ldots,a_k;x)$ are functions of one variable x labeled by a vector $a=(a_1,\ldots,a_k)$. The dimension k of the vector a is called the weight of the HPL. We define the functions

$$
\begin{aligned}
f_1(x) &= \frac{1}{1-x} \\
f_0(x) &= \frac{1}{x} \\
f_{-1}(x) &= \frac{1}{1+x}
\end{aligned}
\tag{1}
$$

The HPL's are defined recursively through integration of these functions. For weight one we have

$$
\begin{aligned}
H(1;x) &= \int_0^x f_1(t)\mathrm{d}t = \int_0^x \frac{1}{1-t}\mathrm{d}t = -\log(1-x) \\
H(0;x) &= \log(x) \\
H(-1;x) &= \int_0^x f_{-1}(t)\mathrm{d}t = \int_0^x \frac{1}{1+t}\mathrm{d}t = \log(1+x),
\end{aligned}
\tag{2}
$$

and for higher weights

$$
\begin{aligned}
H(^n0;x) &= \frac{1}{n!}\log^n x \\
H(a,a_{1,\ldots,k};x) &= \int_0^x f_a(t)H(a_{1,\ldots,k};t)\mathrm{d}t\,,
\end{aligned}
\tag{3}
$$

where we used the notations

$$
{}^n i = \underbrace{i,\ldots,i}_{n}, \quad \text{and} \quad a_{1,\ldots,k} = a_1,\ldots,a_k.
$$

Dispersion relation in higher dimensions

[Carmi, Caron-Huot 19]

The correlator is a function of two cross-ratios, the kernel is a function of two pairs of cross-ratios, one pair being integrated over

$$\mathcal{G}^t(z,\bar{z}) = \int_0^1 dw d\bar{w} K(z,\bar{z},w,\bar{w}) \mathrm{dDisc}[\mathcal{G}(w,\bar{w})]$$

$$K(z,\bar{z},w,\bar{w}) = K_B \theta(\rho_z \bar{\rho}_z \bar{\rho}_w - \rho_w) + K_C \frac{d\rho_w}{dw} \delta(\rho_w - \rho_z \bar{\rho}_z \bar{\rho}_w)$$

For identical external operators the explicit form reads

$$K_B = -\frac{1}{64\pi} \left(\frac{z\bar{z}}{w\bar{w}}\right)^{3/2} \frac{(\bar{w}-w)(\frac{1}{w}+\frac{1}{\bar{w}}+\frac{1}{z}+\frac{1}{\bar{z}}-2)}{((1-z)(1-\bar{z})(1-w)(1-\bar{w}))^{\frac{3}{4}}} \, x^{\frac{3}{2}} {}_2F_1(\tfrac{1}{2},\tfrac{3}{2},2,1-x),$$

$$K_C = \frac{4}{\pi} \frac{1}{\bar{w}^2} \left(\frac{1-\rho_z^2 \bar{\rho}_z^2 \bar{\rho}_w^2}{(1-\rho_z^2)(1-\bar{\rho}_z^2)(1-\bar{\rho}_w^2)}\right)^{1/2} \frac{1-\rho_z \bar{\rho}_z \bar{\rho}_w^2}{(1-\rho_z \bar{\rho}_w)(1-\bar{\rho}_z \bar{\rho}_w)}.$$

$$x \equiv \frac{\rho_z \bar{\rho}_z \rho_w \bar{\rho}_w (1-\rho_z^2)(1-\bar{\rho}_z^2)(1-\rho_w^2)(1-\bar{\rho}_w^2)}{(\bar{\rho}_z \bar{\rho}_w - \rho_w \rho_z)(\rho_z \bar{\rho}_w - \rho_w \bar{\rho}_z)(\rho_z \bar{\rho}_z - \rho_w \bar{\rho}_w)(1-\rho_w \rho_z \bar{\rho}_w \bar{\rho}_z)}$$

$\rho_z \equiv \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}$, and similarly for $\bar{z}, w, \bar{w}$.

The only explicit calculation is GFF (disconnected contributions), and numerical.

real axis, which we have to introduce to avoid the pole generated by $\widehat{H}^{B}_{0,2}(w)$. The strategy used to obtain the Regge-bounded inversion kernel can be obviously generalized if one needs, as we do in this paper, to a kernel for correlators *unbounded* in the Regge limit, such as those that usually appear in perturbation theory [12]. Given a certain behaviour in the Regge limit, one can obtain the corresponding inversion kernel by defining

$$H^{\mathrm{unbd}}_{\Delta}(w) \equiv H^{B}_{\Delta}(w) - \sum_{m,n} A_{m,n}\, \widehat{H}^{B}_{m,2}(w)\, \frac{2\pi\, \Delta^{n}(\Delta-1)^{n}}{\sin(\pi\Delta)} - \sum_{m,n} B_{m,n}\, \widehat{H}^{B}_{m,1}(w)\, \frac{2\pi\, \Delta^{n}(\Delta-1)^{n}}{\sin(\pi\Delta)} \qquad (2.36)$$

and fixing the unknown coefficients $A_{m,n}, B_{m,n}$ by imposing the desired behaviour at small w as in (2.29). The more $\mathcal{G}(z)$ diverges in the Regge limit, the more subtractions will be needed and the stronger the singularities at $w = 1$ will be[13]. For example, for a correlator that diverges linearly in the Regge limit two subtractions are necessary. We shall see related examples in Section 4.

CFT data at strong coupling

The four-point function has an OPE expansion in superblocks

$$\langle \mathbb{F}(t_1)\bar{\mathbb{F}}(t_2)\mathbb{F}(t_3)\bar{\mathbb{F}}(t_4)\rangle = \frac{1}{t_{12}t_{34}} f(z) = \frac{1}{t_{12}t_{34}} \sum_h c_h (-z)^h \, 2F_1(h,h,2h+3,z)\,.$$

Mixing Problem? Mixing between two-particle operators $\mathbb{F}\partial_t^n\bar{\mathbb{F}}$ with multi-particle operators. For instance, for n=3 $\mathbb{F}\partial^3\bar{\mathbb{F}}$ potentially mixes with the four-particle operator $\mathbb{F}\bar{\mathbb{F}}\partial\mathbb{F}\partial\bar{\mathbb{F}}$.

$$h = 1 + n + \frac{1}{T}\gamma_n^{(1)} \qquad c_n = c_n^{(0)} + \frac{1}{T} c_n^{(1)}$$

$$\gamma_n^{(1)} = -3 - 4n - n^2$$

$$c_n = \sqrt{\pi}2^{-2n-3}(n+3)\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{5}{2}\right)}\Big[(n+2) + \frac{1}{T}\big[4n^2 - 2\left(n^3+6n^2+11n+6\right)\ln 2 + 15n$$
$$+(n+1)(n+2)(n+3)\psi^{(0)}(n+1) - (n+1)(n+2)(n+3)\psi^{(0)}\left(n+\tfrac{5}{2}\right) + 13\big]\Big]$$

γ_n **only** a **linear combination** of the anomalous dimensions of the operators appearing in the mixing? Not yet at this order.

See also [Ferrero, Meneghelli, 21 and 24]

1d CFT defined by supersymmetric Wilson line in d=4 and perturbative Ansatz for correlators

Appendix B: The basis of transcendental functions

The analytic structure of four-point functions in a 1d CFT in terms of the unique cross-ratio χ was discussed in detail in [40]: in the complex χ-plane, such correlators are analytic functions with branch points at $\chi = 0, 1$. Thus, when making an ansatz in terms of harmonic polylogarithms, only this type of singularities is allowed: such functions can be obtained using the symbol map as "words" made of the two "letters" χ and $1-\chi$. The number of such words is $2^{\mathtt{t}}$, which then gives us the dimension of our basis for weight $\mathtt{t}$. Using functional relations between polylogarithms, we choose the following basis for $\mathtt{t} \leq 4$:

- $\mathtt{t} = 0$: $\{1\}$.
- $\mathtt{t} = 1$: $\{\log(\chi), \log(1-\chi)\}$.
- $\mathtt{t} = 2$: $\{\log^2(\chi), \log(\chi)\log(1-\chi), \log^2(1-\chi), \mathrm{Li}_2(\chi)\}$.
- $\mathtt{t} = 3$: $\{\log^3(\chi), \log^2(\chi)\log(1-\chi), \log(\chi)\log^2(1-\chi), \log^3(\chi), \mathrm{Li}_2(\chi)\log(\chi), \mathrm{Li}_2(\chi)\log(1-\chi), \mathrm{Li}_3(\chi), \mathrm{Li}_3\left(\frac{\chi}{\chi-1}\right)\}$.
- $\mathtt{t} = 4$: $\{\log^4(\chi), \log^3(\chi)\log(1-\chi), \log(\chi)^2\log^2(1-\chi), \log(\chi)\log^3(1-\chi), \log^4(\chi), \mathrm{Li}_2(\chi)\log^2(\chi), \mathrm{Li}_2(\chi)\log(\chi)\log(1-\chi), \mathrm{Li}_2(\chi)\log^2(1-\chi), \mathrm{Li}_3(\chi)\log(\chi), \mathrm{Li}_3(\chi)\log(1-\chi), \mathrm{Li}_3\left(\frac{\chi}{\chi-1}\right)\log(\chi), \mathrm{Li}_3\left(\frac{\chi}{\chi-1}\right)\log(1-\chi), \mathrm{Li}_4(\chi), \mathrm{Li}_4(1-\chi), \mathrm{Li}_4\left(\frac{\chi}{\chi-1}\right)\}$.

Sum rules

Define a family of functionals

$$\omega_p = \oint_{\mathbb{C}|_\infty} \frac{ds}{2\pi i} \frac{M(s)}{s - 2\Delta - p} = 0$$

Then

$$\omega_p = \sum_{s^*} \frac{\text{Res}_{s=s^*}(M(s))}{s^* - 2\Delta - p} + \cancelto{0}{M(2\Delta - p)}$$

and the sum rules read

$$\sum_{\Delta,k} c_\Delta \frac{(-1)^{k+1}\Gamma(2\Delta)\Gamma(\Delta+k)}{\Gamma(\Delta)^2\Gamma(2\Delta+k)\Gamma(2\Delta_\phi-\Delta-k)\Gamma(k+1)} \frac{2(\Delta+k-\Delta_\phi)(p_1+p_2+2\Delta_\phi)}{(\Delta+k+p_1)(\Delta+k+p_2)(2\Delta_\phi-\Delta-k+p_1)(2\Delta_\phi-\Delta-k+p_2)} = 0\,.$$

Sum rules

Define a family of functionals

$$\omega_p = \oint_{\mathbb{C}|_\infty} \frac{ds}{2\pi i} \frac{M(s)}{s - 2\Delta - p} = 0$$

Then

0

$$\omega_p = \sum_{s^*} \frac{\text{Res}_{s=s^*}(M(s))}{s^* - 2\Delta - p} + M(2\Delta - p)$$

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$$\sum_{\Delta,k} c_\Delta \frac{(-1)^{k+1}\Gamma(2\Delta)\Gamma(\Delta+k)}{\Gamma(\Delta)^2\Gamma(2\Delta+k)\Gamma(2\Delta_\phi-\Delta-k)\Gamma(k+1)} \frac{2(\Delta+k-\Delta_\phi)(p_1+p_2+2\Delta_\phi)}{(\Delta+k+p_1)(\Delta+k+p_2)(2\Delta_\phi-\Delta-k+p_1)(2\Delta_\phi-\Delta-k+p_2)} = 0 .$$

Tested on generalised free field theory and perturbatively on ϕ^4 model.

Back to

Integrated correlators from integrability

Constraints on the 4-point function of identical scalar operators G(x)
arising from integrable deformations of the straight line.

They take the form of integrals over the cross ratio for the amplitude G(x)
or, equivalently, f(x) (a piece of G(x)).
They are two integral identities involving the "Bremsstrahlung" and "Curvature" functions
evaluated via localization and integrability.

$$\text{Constraint 1:} \qquad \int_0^1 \delta G(x) \frac{1+\log x}{x^2} dx = \frac{3\mathbb{C}-\mathbb{B}}{8\,\mathbb{B}^2}\,,$$

$$\text{Constraint 2:} \qquad \int_0^1 dx \frac{\delta f(x)}{x} = \frac{\mathbb{C}}{4\,\mathbb{B}^2} + \mathbb{F} - 3\,,$$

First, as we have pointed out in several places along the way, although our results might naively appear complicated, they are actually much simpler than they could potentially be given our initial ansatz. In particular, while at order ℓ we are allowing for the presence of $2^{\ell+1}-1$ independent HPLs (including the identity), we found that with a convenient choice of basis one can actually express the results in terms of fewer functions. A summary is contained in table 1. Note that this feature

ℓ	1	2	3	4
$2^{\ell+1}-1$	3	7	15	31
# functions	3	6	12	21

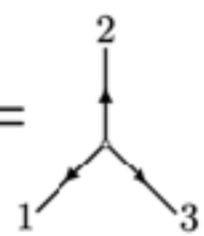
Table 1: Number of independent functions used to express our results for $f^{(\ell)}(\chi)$ in (4.41), (4.63), (4.82) and (4.105), compared to the number of independent HPLs of transcendentality $\mathtt{t} \leq \ell$.

of our results only becomes apparent once a particular basis is adopted, which in our case up to three loops only involves the functions log and L_3 (introduced in (4.80)), evaluated at arguments χ and $1-\chi$. It is then straightforward to identify the reason for the mismatch between the last two lines in table 1: it is due to the fact that in the chosen basis the functions Li_2 and Li_4 never appear in the final result, although they are a priori allowed by our ansatz following the observations of Section 3.

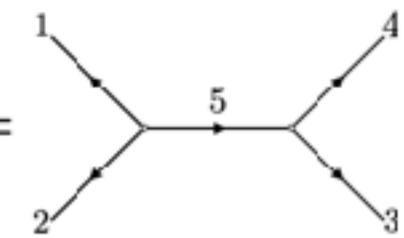
CPWs can be expressed as a conformal integral over two conformally invariant three-point functions

$$\Psi_5^{1234}(x_1,x_2,x_3,x_4)=\int d^dx_5\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_5(x_5)\rangle\langle\widetilde{\mathcal{O}}_5(x_5)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle,$$

$\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle =$



$\Psi_5^{1234} =$



In principle, one could perform the OPE and find an expression for the CFT data $\langle \gamma^{(4)} \rangle_\Delta$ and $\langle a^{(4)} \rangle_\Delta$ at three loops. However, due to the presence of mixing, rather than the former from $f^{(4)}_{\log^1}(\chi)$ one can only extract the combination $\langle a^{(3)}\, \gamma^{(1)} + a^{(2)}\, \gamma^{(2)} + a^{(1)}\, \gamma^{(3)} + a^{(0)}\, \gamma^{(4)} \rangle_\Delta$ (see (4.87)), for which we were not able to find a closed-form expression. On the other hand, to extract $\langle a^{(4)} \rangle_\Delta$ from the OPE one should study the mixing problem to compute quantities such as $\langle a^{(2)}\, \gamma^{(2)} + a^{(1)}\, \gamma^{(3)} \rangle_\Delta$, which is a problem that we have not attempted to solve. However, since for $\Delta = 2$ there is no degeneracy, we can drop the average symbol around CFT data and use the first terms in the small χ expansion of $f^{(4)}(\chi)$ and (4.87) to obtain

$$\gamma^{(4)}_{\Delta=2} = \frac{351845}{13824} - \frac{75}{2}\, \zeta(3)\,, \qquad a^{(4)}_{\Delta=2} = -\frac{1705}{96} - \frac{1613}{24}\, \zeta(3)\,. \tag{4.106}$$

We can now to summarize the CFT data that we have computed analytically for the superconformal primary of the multiplet $\mathcal{L}^{\Delta=2}_{0,[0,0]}$, which we refer to schematically as φ^2. Its dimension at strong coupling is

$$h_{\varphi^2} = 2 - \frac{5}{\lambda^{1/2}} + \frac{295}{24}\frac{1}{\lambda} - \frac{305}{16}\frac{1}{\lambda^{3/2}} + \left(\frac{351845}{13824} - \frac{75}{2}\, \zeta(3)\right)\frac{1}{\lambda^2} + \mathcal{O}(\lambda^{-5/2})\,, \tag{4.107}$$

while the OPE coefficient $\mu_{11\varphi^2}$, such that $\mu^2_{11\varphi^2} = a_{\Delta=2}$, has the expansion

$$\begin{aligned} \mu_{11\varphi^2} = \sqrt{\frac{2}{5}}\, \Bigg[1 &- \frac{43}{24}\frac{1}{\lambda^{1/2}} - \frac{649}{1152}\frac{1}{\lambda} + \left(\frac{7259}{1024} + 5\, \zeta(3)\right)\frac{1}{\lambda^{3/2}} \\ &- \left(\frac{25635205}{2654208} + \frac{7205}{96}\, \zeta(3)\right)\frac{1}{\lambda^2}\Bigg] + \mathcal{O}(\lambda^{-5/2})\,. \end{aligned} \tag{4.108}$$

The explicit form of such inversion kernels is only known in the case of identical bosons (fermions) with integer (half integer) conformal dimension [11], and reads [8]

$$H_\Delta^{B/F}(w) = \pm\frac{2\pi}{\sin(\pi\Delta)}\left[w^{2-2\Delta_\phi}p_\Delta(w) + \left(\tfrac{w}{w-1}\right)^{2-2\Delta_\phi}p_\Delta(\tfrac{w}{w-1}) + q_\Delta^{\Delta_\phi}(w)\right]\,, \tag{2.22}$$

$$p_\Delta(w) = {}_2F_1(\Delta, 1-\Delta, 1, w)\,, \tag{2.23}$$

$$q_\Delta^{\Delta_\phi}(w) = a_\Delta^{\Delta_\phi}(w) + b_\Delta^{\Delta_\phi}(w)\log(1-w)\,. \tag{2.24}$$

In (2.24), $a_\Delta^{\Delta_\phi}(w)$ and $b_\Delta^{\Delta_\phi}(w)$ are polynomials in Δ[9] and w,

$$\begin{aligned} a_\Delta^{\Delta_\phi}(w) &= \sum_{m=0}^{2\Delta_\phi-2}\sum_{n=0}^{2\Delta_\phi-4}\alpha_{m,n}\,w^{m+2-2\Delta_\phi}\Delta^n(\Delta-1)^n\,,\\ b_\Delta^{\Delta_\phi}(w) &= \sum_{m=0}^{2\Delta_\phi-2}\sum_{n=0}^{2\Delta_\phi-4}\beta_{m,n}\,w^{m+2-2\Delta_\phi}\Delta^n(\Delta-1)^n\,. \end{aligned} \tag{2.25}$$

The coefficients $\alpha_{m,n}$ and $\beta_{m,n}$ above must be determined, for each given Δ_ϕ, from the requirement that $H_\Delta^{B/F}(w)$ has no pole at $w=0$. The first few examples read [11]

$$\begin{aligned} &a_\Delta^1(w) = 0\,, \quad b_\Delta^1(w) = 0\,,\\ &a_\Delta^2(w) = w^2+2w-2\,, \quad b_\Delta^2(w) = 0\,,\\ &a_\Delta^{1/2}(w) = 0\,, \quad b_\Delta^{1/2}(w) = 0\,,\\ &a_\Delta^{3/2}(w) = \left(2\Delta^2-2\Delta-1\right)w\,, \quad b_\Delta^{3/2}(w) = 0\,. \end{aligned} \tag{2.26}$$

It is worth noticing that in higher dimensions the inversion kernel does not depend on the external dimensions Δ_ϕ and it is simply a conformal block [3]. On the other hand, in the present $d = 1$ case the inversion formula – which does depend on Δ_ϕ – is manifestly crossing symmetric [11], meaning that the coefficient function I_Δ obtained from a single t-channel conformal block of dimension Δ encodes the OPE data of the crossing-symmetric sum of exchange Witten diagrams in AdS_2 with the same dimension[10].

Let us consider a function

$$f(x) = \sum_{j=1}^{\infty} f_j x^j \ ,$$

with the properties that

- $f(x)$ is analytic in the whole complex plane, except for the branch cuts at real $x > 1$,
- $\left|\frac{f(x)}{x}\right| \to 0$ as $x \to \infty$.

This allows us to use Cauchy's theorem to extract the coefficients f_j as

$$f_j = \frac{1}{2\pi i} \oint \frac{dx}{x} x^{-j} f(x) \ .$$

By deforming the contour and using the second property above we can write

$$f_j = \frac{1}{2\pi} \int_1^{\infty} \frac{dx}{x} x^{-j} \mathrm{Disc} f(x)$$

Mellin amplitudes in higher dimensions

Constraints on the Mellin variables

$$\sum_{j=1}^{n} \gamma_{ij} = 0\,, \qquad \gamma_{ij} = \gamma_{ji}\,, \qquad \gamma_{ii} = -\Delta_i$$

It is convenient to introduce fictitious moments p_i such that $\gamma_{ij} = p_i \cdot p_j$. Imposing momentum conservation $\sum_{i=1}^{n} p_i = 0$ and the on-shell condition $p_i^2 = -\Delta_i$ automatically leads to the constraints

In the case of the four-point function it is convenient to write the Mellin amplitude in terms of 'Mandelstam invariants'

$$s = -(p_1+p_2)^2 = \Delta_1 + \Delta_2 - 2\gamma_{12}$$
$$t = -(p_1+p_3)^2 = \Delta_1 + \Delta_3 - 2\gamma_{13}$$

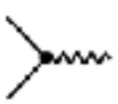
Strings in flat spacetime	**CFT_d or Strings in AdS_{d+1}**
Scattering amplitude $\mathcal{T}(s,t)$	Correlation function or Mellin amplitude $M(s,t)$
Tree-level: $g_s \to 0$	Planar level: $N \to \infty$
Finite string length $l_s = \sqrt{\alpha'}$	Finite 't Hooft coupling[1] $g^2 \sim g_{YM}^2 N = \frac{R^4}{\alpha'^2}$
Partial wave expansion $\mathcal{T}(s,t) = \sum_J a_J(t) \underbrace{P_J(\cos\theta)}_{\text{partial wave}}$	Conformal partial wave expansion $M(s,t) = \sum_J \int d\nu\, b_J(\nu^2) \underbrace{M_{\nu,J}(s,t)}_{\text{partial wave}}$
On-shell poles $a_J(t) \sim \frac{C^2(J)}{t - m^2(J)}$	On-shell poles $b_J(\nu^2) \sim \frac{C^2(J)}{\nu^2 + \left(\Delta(J) - \frac{d}{2}\right)^2}$
Leading Regge trajectory $m^2(J) = \frac{2}{\alpha'}(J-2)$	Leading twist operators $\Delta(J) = d - 2 + J + \underbrace{\gamma(J,g^2)}_{\text{anomalous dimension}}$
Cubic couplings $C(J) \sim$ 	3-pt functions or OPE coefficients $C(J) \sim$ 
Regge limit: $s \to \infty$ with fixed t $P_J(\cos\theta) \approx \left(\frac{2s}{t}\right)^J$ $\mathcal{T}(s,t) \approx \beta(t)\, s^{j(t)}$	Regge limit: $s \to \infty$ with fixed t $M_{\nu,J}(s,t) \approx \omega_{\nu,J}(t)\, s^J$ $M(s,t) \approx \int d\nu\, \omega_{\nu,j(\nu)}(t)\, \beta(\nu)\, s^{j(\nu)}$
Regge pole and residue $t - m^2(J) = 0 \;\Rightarrow\; J = j(t)$ $\beta(t) \sim C^2(j(t))$	Regge pole and residue $\left(\Delta(J) - \frac{d}{2}\right)^2 + \nu^2 = 0 \;\Rightarrow\; J = j(\nu)$ $\beta(\nu) \sim C^2(j(\nu))$

Table 1: Analogy between standard Regge theory for scattering amplitudes in flat spacetime and conformal Regge theory. The notation will be explained later, but the analogy should already be clear for readers familiarized with Regge theory (and AdS/CFT).