Rényi Entropy with Surface Defects in Six Dimensions

Ma-Ke Yuan (袁马轲), Fudan university

New advancements on defects and their applications, YITP

Based on: [with Yang Zhou, arXiv:2310.02096] [with Zi-Xiao Huang and Yang Zhou, arXiv:2501.09498]

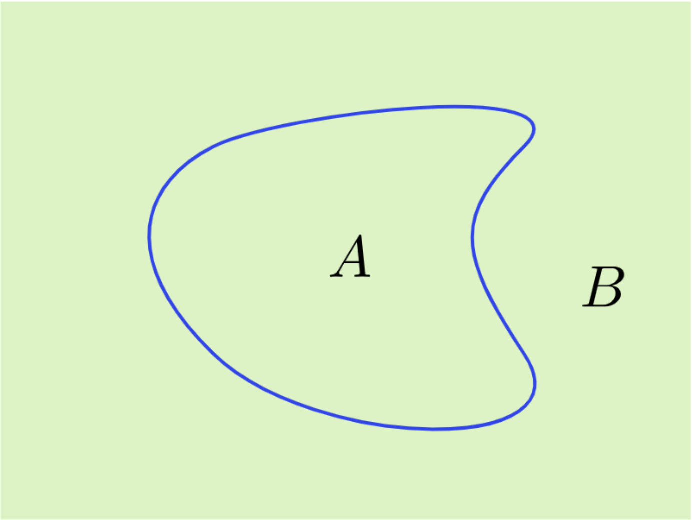
Contents

- EE, (susy) Rényi entropy
- Motivation
- Free field calculation, Large N calculation, Closed formula
- Summary and outlook

Entanglement entropy

- For a bipartite pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, EE between A and B is defined as the von-Neumann entropy of $\rho_A = \mathrm{Tr}_B|\psi\rangle\langle\psi|$

$$S_{\mathrm{EE}}(A{:}B) = -\mathrm{Tr}(\rho_A \log \rho_A)$$



- EE in large *c* CFT: Ryu-Takayanagi formula

- $\log\rho_A$ term in EE is hard to deal with
  → introduce Rényi entropy

(susy) Rényi entropy

- Rényi entropy (n: Rényi index, $S_1 = S_{\mathrm{EE}}$)

$$S_n = \frac{1}{1-n} \log \mathrm{Tr} \rho_A^n = \frac{1}{1-n} \log \frac{Z_n}{Z_1^n}$$

- conical singularity breaks susy
- open background gauge field to preserve susy

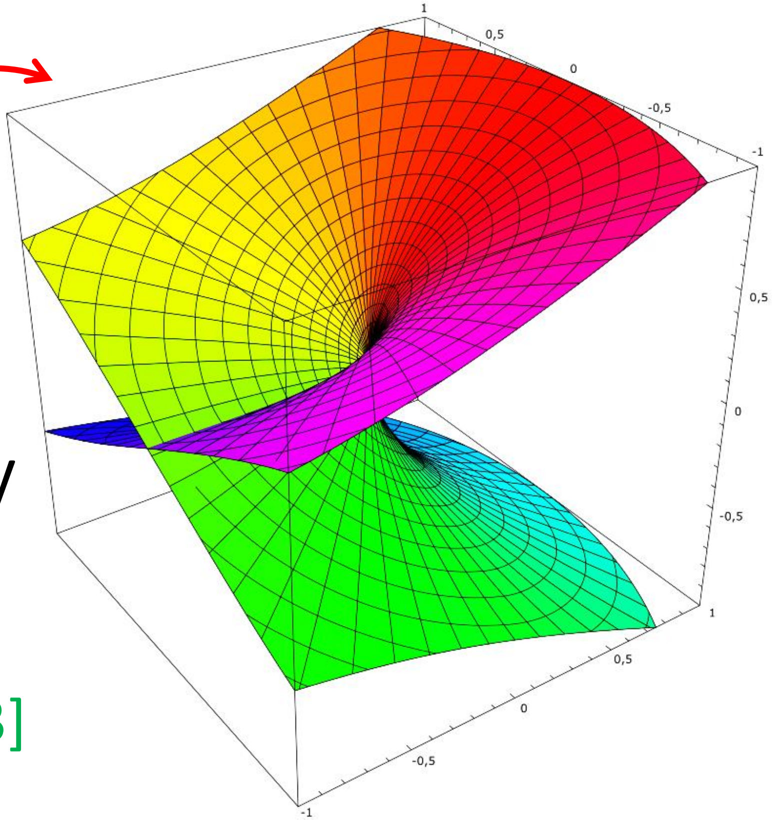
→ susy Rényi entropy $S_n = \frac{1}{1-n} \log \frac{Z_n^{\mathrm{susy}}}{Z_1^n}$

[Nishioka-Yaakov, 13]

- Application: Entanglement spectrum

[Baiguera-Chapman-Northe-Policastro-Schwartzman, 24]

2-sheet Riemann surface
Figure from Wikipedia



Motivation

- EE of surface defect: linear combination of its Weyl anomalies

[Jensen-O'Bannon-Robinson-Rodgers, 18]

$$S = \frac{1}{3}\left(b - \frac{d-3}{d-1}d_2\right)\log \ell/\epsilon$$

$$S = \frac{F-E}{T}$$

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- Surface defect Weyl anomaly:

$$\langle\langle \hat{T}^a_a \rangle\rangle = -\frac{1}{12\pi}\left[bR^\Sigma + d_1 \tilde{\Pi}^\mu_{ab}\tilde{\Pi}^{ab}_\mu - d_2 W^{ab}_{ab}\right]$$

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defect free energy: $\log \langle \mathcal{D}^{(2)} \rangle = \cdots + \frac{b}{3}\log\frac{l}{\epsilon} + \cdots$

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stress tensor 1pt of background theory:

$$\langle\langle T^{ab} \rangle\rangle = -\frac{d-p-1}{d}\frac{h}{|x^i|^d}\delta^{ab}\,, \quad \langle\langle T^{ai} \rangle\rangle = 0\,, \quad \langle\langle T^{ij} \rangle\rangle = \frac{h}{|x^i|^d}\left(\frac{p+1}{d}\delta^{ij} - \frac{x^i x^j}{|x^i|^2}\right)$$

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- Susy-Rényi entropy of <u>6d (2,0) theory</u> $(\gamma = 1/n)$ [Zhou, 15]

$$S_\gamma[\mathfrak{g}] = \frac{r_1^2 r_2^2}{48}(7\bar{a}_\mathfrak{g} - 3\bar{c}_\mathfrak{g})(\gamma-1)^3 + \frac{r_1 r_2}{12}\bar{c}_\mathfrak{g}(\gamma-1)^2 + \frac{1+2r_1 r_2}{12}\bar{c}_\mathfrak{g}(\gamma-1) + \frac{7}{12}\bar{a}_\mathfrak{g}$$

Motivation

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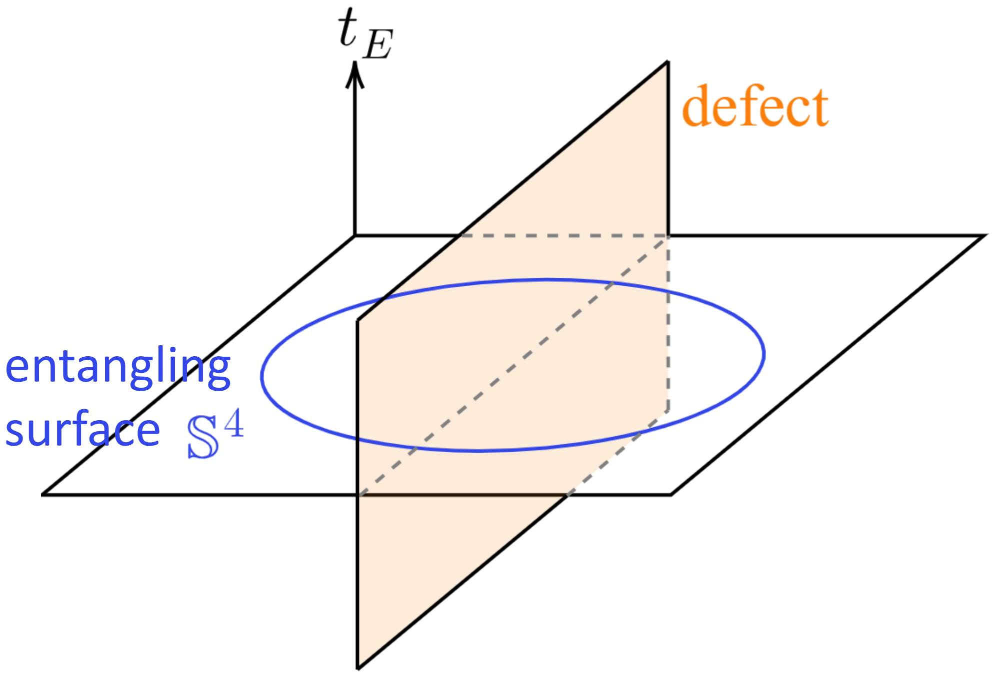
- Will a similar structure appear in the susy-Rényi entropy of surface defect in 6d (2,0) theory?

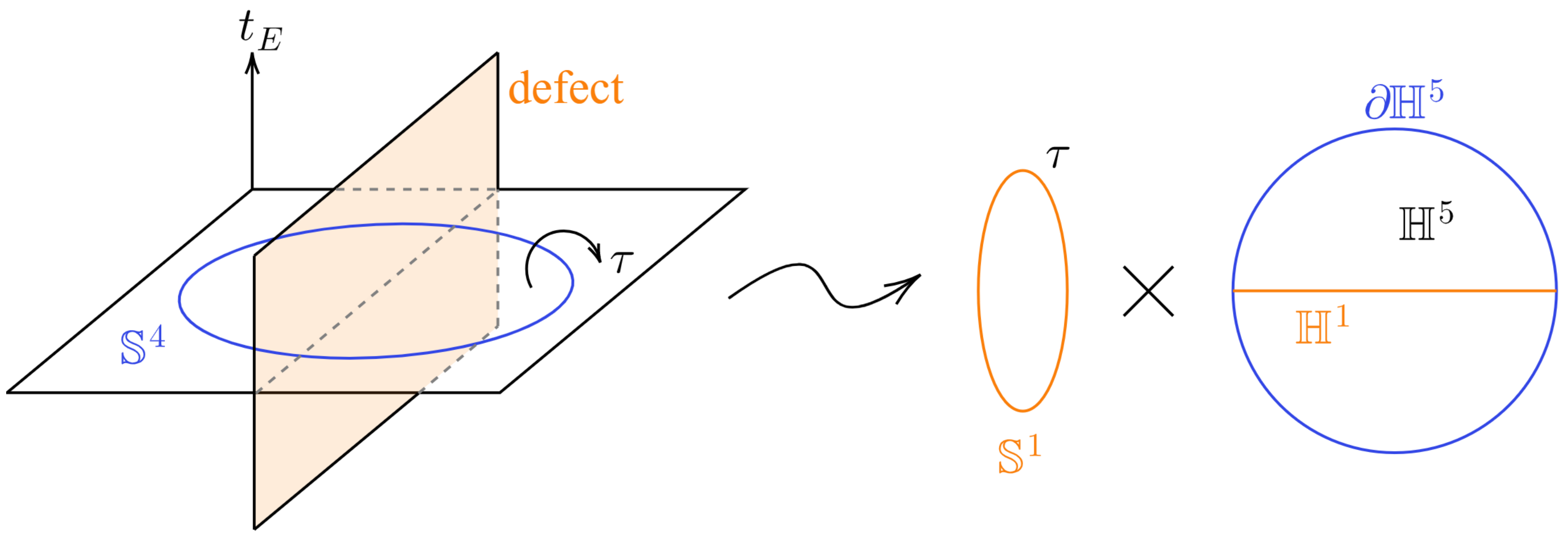
Warm-up: free field setup

- Background $$L_\phi = \frac{1}{2}(\partial\phi)^2 + \frac{1}{10}R\phi^2$$
$$L_B = \frac{1}{12}F_{\mu_1\mu_2\mu_3}F^{\mu_1\mu_2\mu_3}$$

- Defects $$D_\phi = \exp\left(\int_\Sigma \mathrm{d}^2\sigma\,\phi(\sigma)\right)$$
$$D_B = \exp\left(i\int_\Sigma B\right)$$



Casini-Huerta-Myers map



$\mathbb{R}^6$	$\longrightarrow$	$\mathbb{S}^1_{2\pi} \times \mathbb{H}^5$
n-replica space	$\longrightarrow$	$\mathbb{S}^1_{\beta=2\pi n} \times \mathbb{H}^5$
Rényi entropy	$\longrightarrow$	Thermal entropy

Bulk contribution [MKY-Zhou, 23]

- Surface defect expectation value in free field theory

$$\log\langle D_\phi\rangle_n = \frac{1}{2}\int_{\mathbb{S}^1_\beta\times\mathbb{H}^1} \mathrm{d}^2x_1 \int_{\mathbb{S}^1_\beta\times\mathbb{H}^1} \mathrm{d}^2x_2 \langle\phi(x_1)\phi(x_2)\rangle$$
$$= \left[\frac{1}{8\pi^2\epsilon^2} + \mathcal{O}\left(\epsilon^2\right)\right]\beta\log\ell/\epsilon$$

- proportional to β, does not contribute to Rényi entropy

Boundary contribution [MKY-Zhou, 23]

- Curvature coupling in free scalar Lagrangian

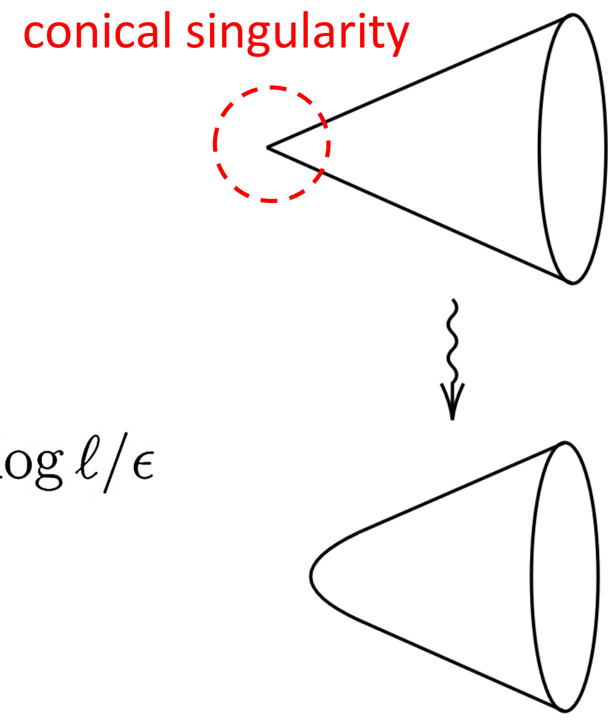
$$L_\phi = \frac{1}{2}(\partial\phi)^2 + \frac{1}{10}R\phi^2$$

- Ricci scalar $R \propto (1-n)\delta_{\partial\mathbb{H}^5}$

- Final result (non-susy) $S_\phi = \frac{nF_{\text{boundary}}[1] - F_{\text{boundary}}[n]}{1-n} = -\frac{1}{10\pi}\log \ell/\epsilon$

- Check: $S = \frac{1}{3}\left(b - \frac{d-3}{d-1}d_2\right)\log \ell/\epsilon = -\frac{1}{10\pi}\log \ell/\epsilon$

$b = 0, d_2 = 1/(2\pi)$ [Drukker-Probst-Trépanier, 20] [Billò-Gonçalves-Lauria-Meineri, 16]



Large N calculation

- Holographic dual of 6d (2,0): M-theory in $\mathrm{AdS}_7 \times S^4$

$$\mathrm{d}s_{11}^2 = L^2\left(\mathrm{d}s^2_{\mathrm{AdS}_7} + \frac{1}{4}\mathrm{d}s^2_{S^4}\right), \quad F_4 = \mathrm{d}C_3 = \pi^2 L^3 \mathrm{vol}_{S^4}$$

- Solution asymptotic to n-replica space $\mathbb{S}^1_{2\pi n} \times \mathbb{H}^5$

$$\mathrm{d}s_7^2 = -(H_1 H_2)^{-4/5} f \mathrm{d}t^2 + (H_1 H_2)^{1/5}\left(f^{-1}\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2_{5,k}\right),$$

$$f(r) = k - \frac{m}{r^4} + \frac{r^2}{L^2} H_1 H_2, \qquad H_i = 1 + \frac{q_i}{r^4},$$

[Cvetic-Duff-Hoxha-Liu-Lv-Lu-Martinez-Acosta-Pope-Sati-Tran, 99]

Large N calculation [MKY-Zhou, 23] [Huang-MKY-Zhou, 25]

- M2 brane action $$I_{\mathrm{M2}} = T_2 \int \mathrm{d}^3\sigma \sqrt{-\det[g]}$$

- M2 embedding $$\sigma_0 = \tau\,,\ \ \sigma_1 = \rho\,,\ \ \sigma_2 = r$$

- M2 on-shell action $$I_{\mathrm{M2}} = T_2 \int_0^\beta \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\rho \int_{r_H}^{\Lambda} \mathrm{d}r\, r (H_1 H_2)^{-1/5} \sqrt{\Delta}$$

- Susy-Rényi entropy $$S_n^{\mathrm{susy}} = \frac{nF_1 - F_n}{1-n} = \frac{\pi V_{\mathbb{H}^1} T_2}{4} \left(r_1 r_2 (\gamma - 1) + 2\right)$$

Surface defect susy-Rényi entropy [Huang-MKY-Zhou, 25]

$$S_{\gamma=1/n} = \left[\boxed{\#1}\,(\gamma - 1) + \boxed{\#0} \right] \log\frac{\ell}{\epsilon}$$

Large *N* result: $$S^{\text{susy}}_{\gamma=1/n} = \frac{\pi V_{\mathbb{H}^1} T_2}{4}\left(r_1 r_2 (\gamma - 1) + 2\right)$$

Surface defect susy-Rényi entropy [Huang-MKY-Zhou, 25]

defect Casimir energy on $\mathbb{S}^5_{n\to 0} \times \mathbb{S}^1$

Localization

Anomaly polynomial

[Zi-Xiao's talk]

$$S_{\gamma=1/n} = \left[\boxed{\frac{2b-d_2}{12} r_1 r_2}(\gamma - 1) + \boxed{\#0}\right] \log\frac{\ell}{\epsilon}$$

Large *N* result:

$$S^{\text{susy}}_{\gamma=1/n} = \frac{\pi V_{\mathbb{H}^1} T_2}{4}\left(r_1 r_2(\gamma-1)+2\right)$$

Surface defect susy-Rényi entropy [Huang-MKY-Zhou, 25]

defect Casimir energy on $\mathbb{S}^5_{n\to 0} \times \mathbb{S}^1$ $\left\{ \begin{array}{l} \text{Localization} \\ \text{Anomaly polynomial} \end{array} \right.$ [Zi-Xiao's talk]

$$S_{\gamma=1/n} = \left[\frac{2b - d_2}{12} r_1 r_2 (\gamma - 1) + \frac{b}{3} - \frac{d_2}{6}\right] \log\frac{\ell}{\epsilon}$$

Defect susy EE

(recall non-susy EE: $S = \frac{1}{3}\left(b - \frac{d-3}{d-1} d_2\right) \log \ell/\epsilon$)

Large *N* result:

$$S^{\text{susy}}_{\gamma=1/n} = \frac{\pi V_{\mathbb{H}^1} T_2}{4}(r_1 r_2(\gamma - 1) + 2)$$

Surface defect susy-Rényi entropy [Huang-MKY-Zhou, 25]

defect Casimir energy on $\mathbb{S}^5_{n\to 0} \times \mathbb{S}^1$ { Localization / Anomaly polynomial } [Zi-Xiao's talk]

$$S_{\gamma=1/n} = \left[\boxed{\frac{2b-d_2}{12} r_1 r_2}(\gamma-1) + \boxed{\frac{b}{3} - \frac{d_2}{6}}\right] \log\frac{\ell}{\epsilon}$$

Defect susy EE

⇓ Large N limit ✓

Large N result:

$$S^{\text{susy}}_{\gamma=1/n} = \frac{\pi V_{\mathbb{H}^1} T_2}{4}\left(r_1 r_2(\gamma-1)+2\right)$$

Free field non-susy result: $S_n = -\frac{1}{10\pi}\log \ell/\epsilon \xrightarrow{\text{supersymmetrize}} ???$

Free limit? ↑

[Zi-Xiao's talk]

defect Casimir energy on $\mathbb{S}^5_{n\to 0}\times\mathbb{S}^1$ { Localization / Anomaly polynomial }

$$S_{\gamma=1/n} = \left[\frac{2b-d_2}{12} r_1 r_2(\gamma-1) + \frac{b}{3} - \frac{d_2}{6}\right]\log\frac{\ell}{\epsilon}$$

Defect susy EE

Large N limit ✓ ↓

Large N result: $S^{\text{susy}}_{\gamma=1/n} = \frac{\pi V_{\mathbb{H}^1} T_2}{4}\left(r_1 r_2(\gamma-1)+2\right)$

Summary

- Free limit, surface defect Rényi entropy: $S_n = -\frac{1}{10\pi}\log\ell/\epsilon$
- Large *N* limit, surface defect susy-Rényi entropy

$$S^{\text{susy}}_{\gamma=1/n} = \frac{\pi V_{\mathbb{H}^1} T_2}{4}(r_1 r_2(\gamma-1)+2)$$

- A closed formula

$$S^{\text{susy}}_{\gamma=1/n} = \left[\frac{2b-d_2}{12} r_1 r_2(\gamma-1) + \frac{b}{3} - \frac{d_2}{6}\right]\log\frac{\ell}{\epsilon}$$

Outlook

- Free field calculation of surface defect susy-Rényi entropy (combine with monodromy defect)

[Giombi-Helfenberger-Ji-Khanchandani, 21]
[Bianchi-Chalabi-Procházka-Robinsone-Sistic, 21]

- Quantum information inequalities ($H_n \equiv S_n^{\text{susy}}/S_1^{\text{susy}}$)

$$\partial_n H_n \leq 0, \qquad \partial_n\big((1-n)H_n/n\big) \geq 0,$$

$$\partial_n\big((n-1)H_n\big) \geq 0, \qquad \partial_n^2\big((n-1)H_n\big) \leq 0.$$

closed formula $+$ these inequalities $\Rightarrow$

non-trivial constraints on defect Weyl anomalies, unitary bounds...

Thank you!

Q & A