Non-invertible symmetries in 2+1d TQFTs

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Motivation

- Gauging symmetries relates quantum field theories.
- Obstructions to gauging symmetries provide constraints on RG flows.
- Non-invertible symmetries $\implies$ new connections between theories and a generalized notion of obstructions.

Punchline

What we do in the paper:

1. show how to invert anyon condensation in 2+1d TQFTs.
2. provide intuitive (and precise) explanation of gauging using topological domain walls.
3. use theory of 1 and 2-categories to prove general theorems about gauging...

In the talk

Gauging finite symmetries is equivalent the decorating the spacetime with topological defects and summing over the triangulations.

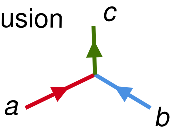
In the talk, we will discuss three types of topological extended objects in 3d TQFTs,

1. Condensation surfaces.
2. Twisted sectors.
3. Domain walls.

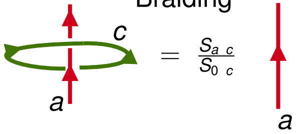
3d TQFTs

Operators are topological lines.

Fusion



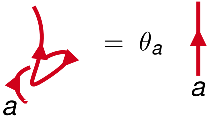
Braiding



$$a \times b = \oplus_c N^c_{a\ b}\, c$$

$$d_a\, d_b = \sum_c N^c_{a\ b}\, d_c$$

Topological spin



Their algebra is given by fusion, braiding and topological spins.

Example: MTC

Toric code : $D(\mathbb{Z}_2)$, $S = \frac{2}{2\pi} \int a\, d\, \tilde{a}$.

Lines: $1 \equiv 1$, $e = e^{\mathbf{i} \int a}$, $m = e^{\mathbf{i} \int \tilde{a}}$, $f = e^{\mathbf{i} \int a + \tilde{a}}$.

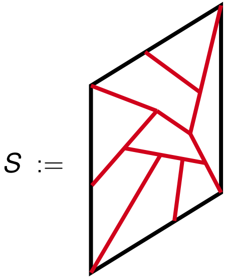
Fusion: $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $d_1 = d_e = d_m = d_f = 1$.

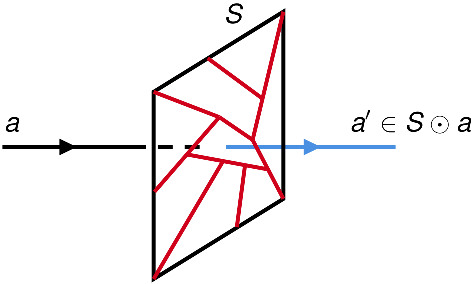
Braiding and spin:

$$S = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} , \quad T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Object 1: A surface operator

A condensation surface: Take a surface and triangulate with lines [Roumpedakis et al. 2022]. This is called higher gauging.





Such condensation defects generate global symmetries of the TQFT.

Example in toric code

Toric code has an invertible $\mathbb{Z}_2$ symmetry that swaps $e \leftrightarrow m$, lines $1, f$ are fixed. As a condensation surface, this is obtained by higher gauging $\mathbb{Z}_2 \simeq \{1, f\}$.

There is also an interesting non-invertible symmetry S_e, which obtained by higher gauging $\mathbb{Z}_2 \simeq \{1, e\}$. Its action,

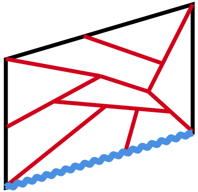
$$S_e \odot 1 = 1 \oplus e \ , \quad S_e \odot e = 1 \oplus e$$

$$S_e \odot m = 0 \ , \quad S_e \odot f = 0.$$

Object II: Twisted sectors

Twisted sectors are surface operators with boundary conditions.

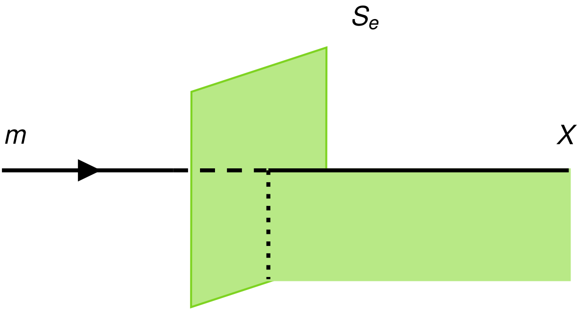
These are also called non-genuine lines.



Twisted sectors become genuine lines in the gauged theory.

Example

S_e in Toric code has two twisted sectors $\{X, Y\}$. In fact S_e, being a non-invertible symmetry, maps genuine lines to twisted sectors.



This is what we computed as $S_e \odot m = 0$, as twisted sectors are not part of the Hilbert space.

Towards gauging S_e symmetry in toric code

The lines (genuine and twisted) relevant to gauging S_e symmetry are,

$$\mathrm{Obj}(\mathcal{C}) = \{1, e, m, f\} \oplus \{X, Y\}$$

With some work, we can show that $d_X = d_Y = 2$ and compute the fusion rules of these lines to be,

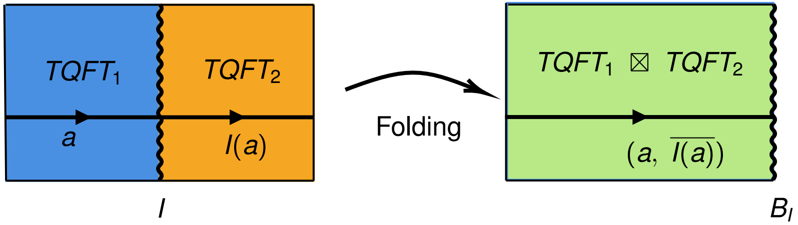
$$e \times X = X \quad , \quad m \times X = Y$$
$$X \times X = 1 + e + X$$

The fusion category formed by these lines,

$$\mathcal{C} \simeq \mathrm{Vec}_{\mathbb{Z}_2} \boxtimes \mathrm{Rep}(S_3)^{\mathrm{op}}$$

Object 3: Domain walls

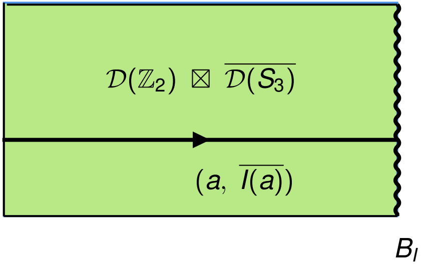
Topological domain walls connect topological field theories. They preserve the modular data of lines being transported [Lan et al. 2014].



Theorem: Two 3d TQFTs are connected by a non-trivial topological domain wall if and only if they are related by gauging of 0 and/or 1-form symmetries [Davydov et al. 2010].

Example

We can consider the Domain wall between toric code and S_3-finite group gauge theory. We apply folding trick to get,



We can derive that B_I as a fusion category is $\mathrm{Vec}_{\mathbb{Z}_2} \boxtimes \mathrm{Rep}(S_3)^{\mathrm{op}}$.

Putting things together

This is exactly the fusion category formed by the object $\{1, e, m, f, X, Y\}$, where $\{X, Y\}$ are twisted sectors of S_e. So, the interface I is equivalent to $\mathcal{C}$ as fusion categories!

1. This suggests that S_e global symmetry in toric code is gauged to get $\mathcal{D}(S_3)$.
2. We create the interface I by taking toric code on a manifold and gauging S_e symmetry on one half.
3. $\{1, e, m, f, X, Y\}$ are precisely the lines that live on the domain wall...

A full proof of these statements requires category theory, and is presented in our paper.

Takeaways

We study gauging of non-invertible symmetries using topological domain walls and twisted sectors.

A complete understanding of such generalized gauging requires a rigorous study of twisted sectors and their fusion rules.

A first principle derivation of modular data for generalized gauging remains out of reach...

Thank you!