From Weyl Anomaly to Universal Surface Defect Entropy and Casimir Energy

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Based on [ZXH-Yuan-Zhou, arXiv: 2501.09498]

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Weyl anomaly of CFT

Weyl anomaly (central charge) governs the **entanglement structure** and **ground state energy** of CFTs

- **Casimir energy** (low-temperature asymptotics of partition function)

$$d = 2 \qquad\qquad E = -\frac{c}{12}$$

- **Rényi entropy** (generalization of the entanglement entropy)

$$d = 2 \qquad\qquad S_n^{\mathrm{2d\,CFT}} = \frac{c}{6}\left(\frac{1}{n}+1\right)\log\frac{l}{\epsilon}$$

$n \to 1$: entanglement entropy

Defects in QFT

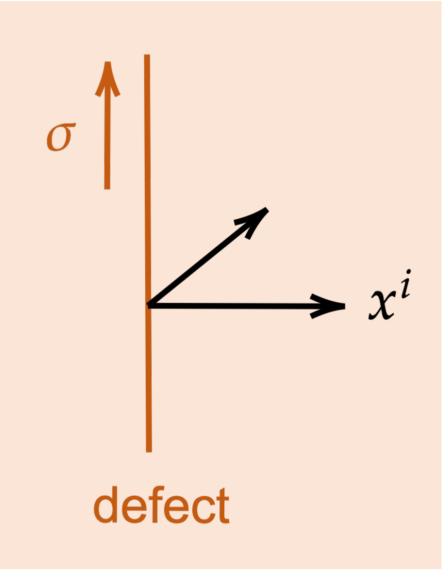
- Defects are usually defined by non-local operators/B.C. in QFT

- Line defects, surface defects, boundaries…

  e.g. Wilson loops, Wilson surfaces, conformal boundary

- Nontrivial interaction with the bulk D.o.F.

$$I_{\text{DCFT}} = \int d^D x \sqrt{g}\, \mathcal{L}_{\text{CFT}}[\phi] + \int d^p \sigma \sqrt{\gamma}\, \mathcal{L}_{\text{defect}}[\phi, \psi]$$



Defects in QFT

Recall: Weyl anomaly (central charge) governs the entanglement structure and ground state energy of CFTs

$$d = 2 \qquad E = -\frac{c}{12} \qquad S_n^{\text{2d CFT}} = \frac{c}{6}\left(\frac{1}{n} + 1\right) \log \frac{l}{\epsilon}$$

- Is this relation still valid in the presence of defects?
- Any non-perturbative way to study defects?

Surface Defects in 6d (2,0) SCFTs

We study ½-BPS surface defect in 6d A_{N-1} (2,0) SCFT

- *Strongly coupled*
- *No Lagrangian description*

Surface defect Weyl anomaly

(coefficients of the trace of the defect localized stress tensor)

$$\langle t^{\mu}_{\mu} \rangle = -\frac{1}{24\pi}\left[b\, R^{\Sigma} + d_1 \Pi^{\mu}_{ab}\Pi^{ab}_{\mu} - d_2 W^{ab}_{ab} \right]$$

Labelled by representation

$$b = 24(\lambda, \rho) + 3(\lambda, \lambda) \qquad d_2 = 24(\lambda, \rho) + 6(\lambda, \lambda)$$

[Estes–Krym–O'Bannon–Robinson–Rodgers, 2018] [Rodgers, 2018]

Surface Defects in 6d (2,0) SCFTs

- Defect contribution to the SUSY Rényi entropy (dSRE)

$$S_n = \frac{1}{1-n} \log \frac{Z_n}{(Z_1)^n}$$

Z_n: SUSY partition function on the n-replica space with defect insertion

- Defect contribution to SUSY Casimir energy (dSCE)

$$\log \langle D \rangle = -\beta E_{\mathcal{R}} + O(\beta^0), \qquad \beta \to \infty$$

$\langle D \rangle$: Defect expectation value

(partition function with the defect insertion normalized by the original bulk one)

Surface Defects in 6d (2,0) SCFTs

- Defect Weyl anomaly governs defect contribution to the SUSY Renyi entropy (dSRE) and Casimir energy (dSCE)

$$S_n = \frac{2b - d_2}{12}\left[r_1 r_2\left(\frac{1}{n} - 1\right) + 2\right]\log\frac{l}{\epsilon}$$

$$E_{\mathcal{R}} = -\frac{1}{\omega_1}\left[\frac{d_2 - b}{6}\omega_2\omega_3 + \frac{2b - d_2}{24}\sigma_1\sigma_2\right]$$

b, d_2 : surface defect Weyl anomaly

$\omega_{1,2,3}$	chemical potentials conjugate to the 6d angular momentum
$\sigma_{1,2}$	chemical potentials conjugate to the 6d R-symmetry Cartans
$r_{1,2}$	weights of the R-symmetry chemical potentials

$$Z = \mathrm{Tr}\left[e^{-\beta\sum_{i=1}^{3}\omega_i J_i - \beta\sum_{a=1}^{2}\sigma_a R_a}\right]$$

Surface Defects in 6d (2,0) SCFTs

- No Lagrangian description.
- Non-perturbative method combining:

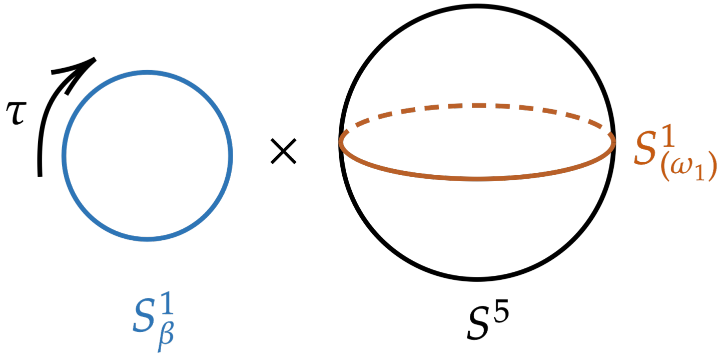
✓ **Localization**

✓ **Holography**

✓ **Anomaly polynomial**

Localization

The large N behavior of dSCE



Surface defect wrapping $S^1_\beta \times S^1_{(\omega_1)} \subset S^1_\beta \times S^5$

Dimensional reduction on $S^1_\beta \implies$ 5d MSYM on S^5 with Wilson loop on $S^1_{(\omega_1)} \subset S^5$

$$\langle D \rangle = \frac{1}{Z} \int \mathrm{d}v \exp\left[\frac{2\pi}{\omega_1\omega_2\omega_3}\left(-\frac{\pi}{\beta}\sum_{i=1}^{N} v_i^2 + \frac{\sigma_1\sigma_2}{2}\sum_{j<i}(v_i - v_j)\right)\right] \mathrm{Tr}_{\mathcal{R}} \exp\left[\frac{2\pi v}{\omega_1}\right]$$

Saddle point approximation $\qquad \log\langle D \rangle = -\beta E_{\mathcal{R}} + O(\beta^0), \qquad \beta \to \infty$

$$E_{(k)} = -\frac{1}{2\omega_1}\left[\, k^2\, \omega_2\omega_3 + k(N-1)\, \sigma_1\sigma_2\right]$$

$$E_{[k]} = -\frac{1}{2\omega_1}\left[\, k\, \omega_2\omega_3 + k(N-k)\, \sigma_1\sigma_2\right]$$

$$E_{\mathcal{R}} = -\frac{1}{\omega_1}\left[\frac{d_2 - b}{6}\omega_2\omega_3 + \frac{2b - d_2}{24}\sigma_1\sigma_2\right]$$

Localization

The $n \to 0$ behavior of dSRE

dSCE on

$S^1_\beta \times S^1_n \subset S^1_\beta \times S^5_n$

governs $n \to 0$ behavior of

(via geometric relation)

dSRE on

$\mathbb{H}^1 \times S^1_n \subset \mathbb{H}^5 \times S^1_n$

$$E_{\mathcal{R}} = -\frac{1}{\omega_1}\left[\frac{d_2 - b}{6}\omega_2\omega_3 + \frac{2b - d_2}{24}\sigma_1\sigma_2\right]$$

$$\Rightarrow \quad S_{n\to 0} = -\frac{\mathrm{Vol}[\mathbb{H}^1]}{\mathrm{Vol}[S^1_\beta]}\beta E_{\mathcal{R}}\Big|_{n\to 0} = \frac{2b - d_2}{12}\,\frac{r_1 r_2}{n}\log\frac{l}{\epsilon}$$

$$\omega_1 = 1/n$$

$$\omega_2 = \omega_3 = 1$$

$$\sigma_I\Big|_{n\to 0} = \frac{r_I}{n}$$

Holography

The large N & exact n behavior of dSRE

For details, see Ma-Ke's talk.

Surface defect in fundamental representation $\overset{\mathrm{AdS}_7/\mathrm{CFT}_6}{\longleftrightarrow}$ Probe M2-brane in 7d 2-charged topological black hole

$$\log \langle D \rangle = -I_{M2} = \pi n T_2 r_H^2 \,\mathrm{Vol}[\mathbb{H}^1]$$

$$S_n(N \to \infty) = \frac{\log \langle D \rangle - n \log \langle D \rangle|_{n=1}}{1-n} = N \left[r_1 r_2 \left(\frac{1}{n} - 1 \right) + 2 \right] \log \frac{l}{\epsilon}$$

**Localization:*

$$S_{n \to 0} = \frac{2b - d_2}{12} \frac{r_1 r_2}{n} \log \frac{l}{\epsilon}$$

$$\frac{2b-d_2}{12} = (N-1) \sim N \quad \text{at large } N$$

Anomaly polynomial

Exact results & consistency check

- (D+2)-form encoding the information of anomaly of a D-dimensional QFT
- Depends on symmetries
- Constructed by characteristic classes of b.g. gauge field of the symmetries

Gravitational anomaly ‘t Hooft anomaly Mixed anomaly

$$A_4 = -\frac{k_g}{24} p_1(T) + \frac{k_I}{2} c_2(F_I) + \frac{k_{IJ}}{2} c_1(F_I) c_1(F_J)$$

first Pontryagin class of the tangent bundle of the spacetime manifold

Chern classes of the background gauge field strength F

Anomaly polynomial

Exact results & consistency check

Even d-dim SCFT $$E = -\int_{\mathbb{R}^d} A_{d+2}$$ [Bobev–Bullimore–Kim, 2015]

Generalize to surface defect $$E_{\mathcal{R}} = -\int_{\mathbb{R}^2} A_4$$

Surface defect anomaly polynomial

$$A_4 = \frac{1}{2}(\lambda,\lambda)\big(c_2(L) - c_2(R)\big) + (\lambda,\rho)\big(c_2(I) - c_2(F)\big)$$ [Shimizu–Tachikawa, 2016]

$$\Rightarrow \quad E_{\mathcal{R}} = -\frac{1}{\omega_1}\left[\frac{d_2 - b}{6}\omega_2\omega_3 + \frac{2b - d_2}{24}\sigma_1\sigma_2\right]$$

Anomaly polynomial

Exact results & consistency check

Extend with an additional U(1) symmetry

$$\tilde{E} = -\int_{\mathbb{R}^d \times \mathbb{R}^2} \tilde{A}_{d+2}$$

SUSY Rényi entropy is given by the "extended Casimir energy"

$$S_n = \frac{n\tilde{E}|_{n=1} - \tilde{E}|_n}{1-n} \mathrm{Vol}[\mathbb{H}^1]$$

[Yankielowicz-Zhou, 2017]

Generalize to surface defect

$$\tilde{E}_{\mathcal{R}} = -\int_{\mathbb{R}^2 \times \mathbb{R}^2} \tilde{A}_4$$

$$\Rightarrow \quad S_n = \frac{2b - d_2}{12} \left[r_1 r_2 \left(\frac{1}{n} - 1 \right) + 2 \right] \log \frac{l}{\epsilon}$$

Conclusion

- **Closed formulae** for surface defect contribution to twisted Rényi entropy and Casimir energy, controlled by defect Weyl anomaly

$$S_n = \frac{2b - d_2}{12}\left[r_1 r_2\left(\frac{1}{n} - 1\right) + 2\right]\log\frac{l}{\epsilon}$$

$$E_{\mathcal{R}} = -\frac{1}{\omega_1}\left[\frac{d_2 - b}{6}\omega_2\omega_3 + \frac{2b - d_2}{24}\sigma_1\sigma_2\right]$$

- **Non-perturbative methods** combining localization, holography and anomaly polynomial

Outlook

- Generalize our approach to defects of other co-dimensions (e.g., co-dimension 2).
- Defects in bulk theories with different spacetime dimensions and symmetries, e.g., those in 6d (1,0) SCFTs and 4d SCFTs.
- High temperature universality of the surface defect contribution to the partition function (Cardy formula in the presence of defect).