

Thermodynamic approach to cooling limit of Gaussian feedback

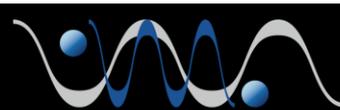
Ken Funo (The University of Tokyo)

Based on:

K. Kumasaki, Y. Yada, **KF** and T. Sagawa, PRR 7, 043147 (2025)

K. Kumasaki, K. Tojo, T. Sagawa, **KF**, arXiv:2508.06174

*Kyoto Workshop on Quantum
Thermodynamics and Stochastic
Thermodynamics 2025*
YITP, Kyoto University, 2025/12/8-12



ERATO Sagawa Information-to-Energy
Interconversion Project



Thermodynamics in small systems

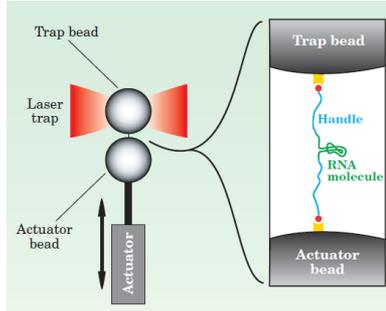
Classical thermodynamics



Steam engine

Wikipedia

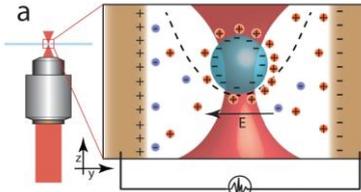
Stochastic thermodynamics



RNA molecules

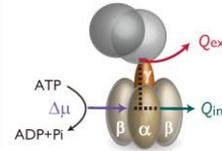
C. Bustamante *et al.*, *Physics Today* (2015)

Colloidal particles



S. Krishnamurthy, *et al.*, *Nat. Commun.* (2023)

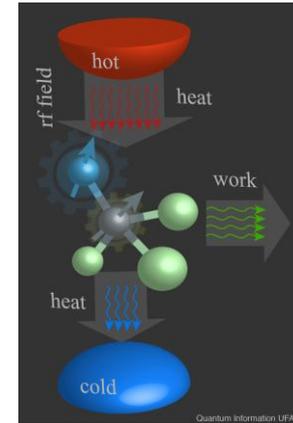
F1-ATPase



S. Toyabe *et al.*, *PRL* (2010)

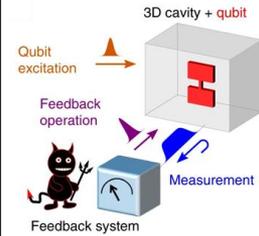
Quantum thermodynamics

NMR systems

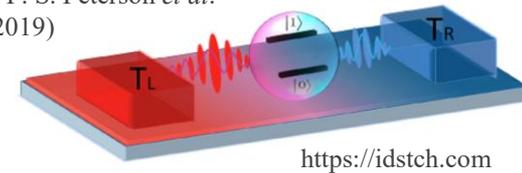


J. P. S. Peterson *et al.*, (2019)

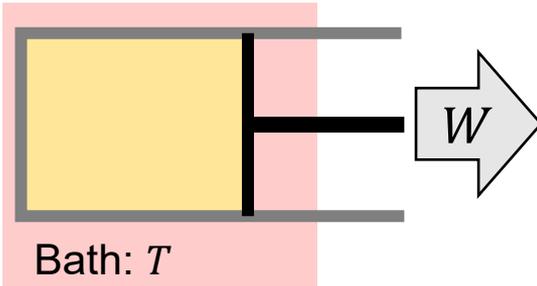
Superconducting qubits



Y. Masuyama, *et al.*, *Nat. Commun.* (2018)



<https://idstch.com>



Bath: T

$\sim m$

Thermal fluctuation

$\sim \mu m$

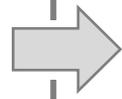
Quantum effects

$\mu m \sim nm$

Macroscopic

Microscopic

Equilibrium theory (established in 19th century)



Extended to **microscopic, out-of-equilibrium, and fluctuating** systems (late 1990s ~)

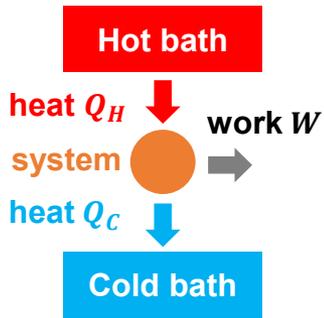
Thermodynamic laws out of equilibrium?

Thermodynamics and energetic costs

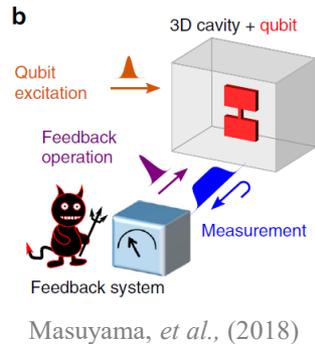
Stochastic thermodynamics & Quantum thermodynamics

✓ Fundamental energetic costs

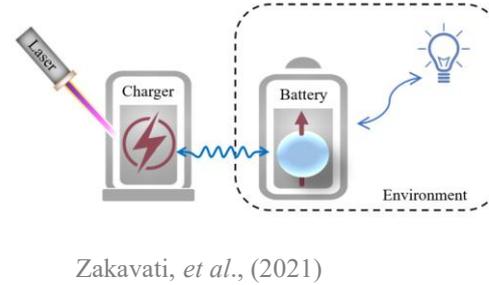
Heat engine



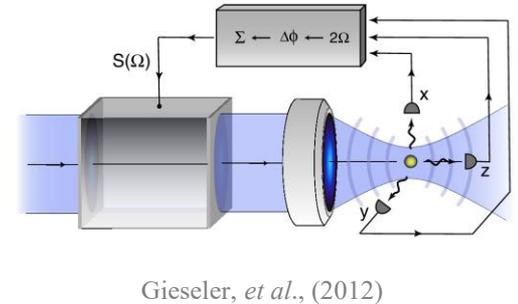
Information engine



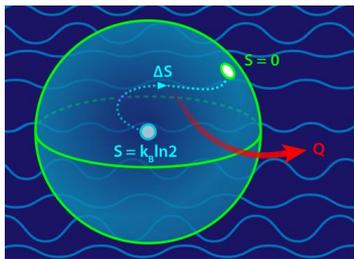
Quantum batteries



Feedback cooling

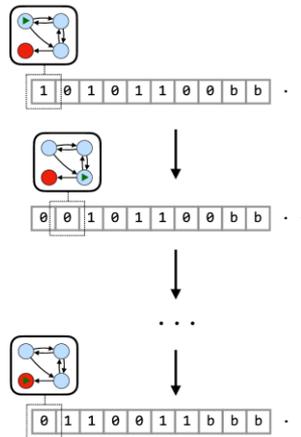


Information erasure



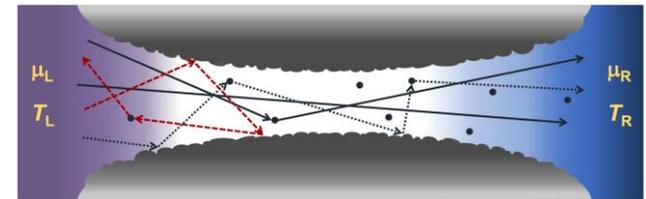
Physics, 11, 49, (2018)

Thermodynamics of computation



Kolchinsky and Wolpert (2020)

Transport



Pekola and Karimi (2021)

Information thermodynamics

Generalized second law

$$W_{\text{ext}} \leq -\Delta F + k_B T \Delta I$$

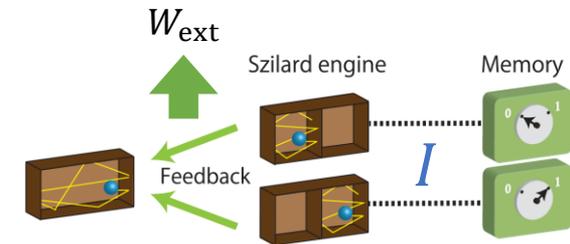
Conventional
second law

Mutual information

(single measurement & feedback)

Information flow, transfer entropy

(continuous measurement & feedback)

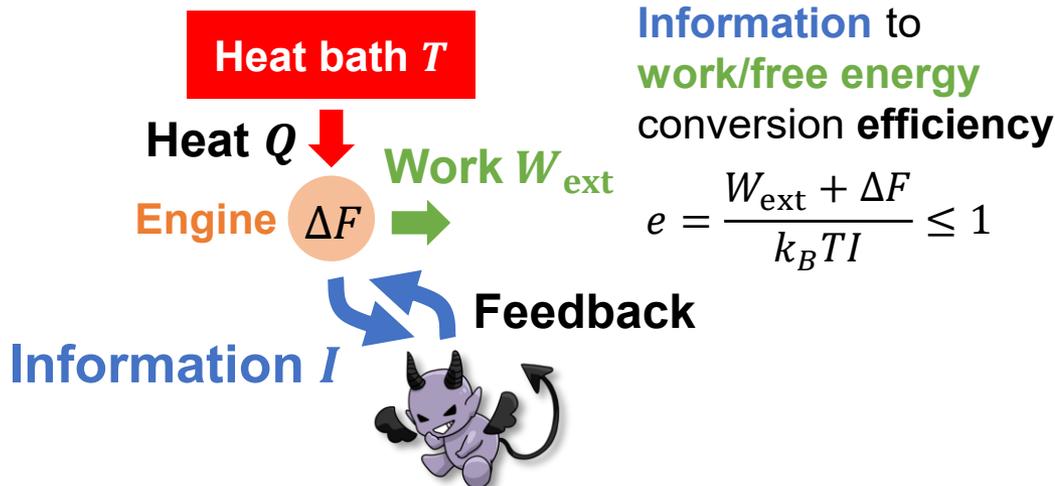


Sagawa & Ueda PRL (2008), (2012)

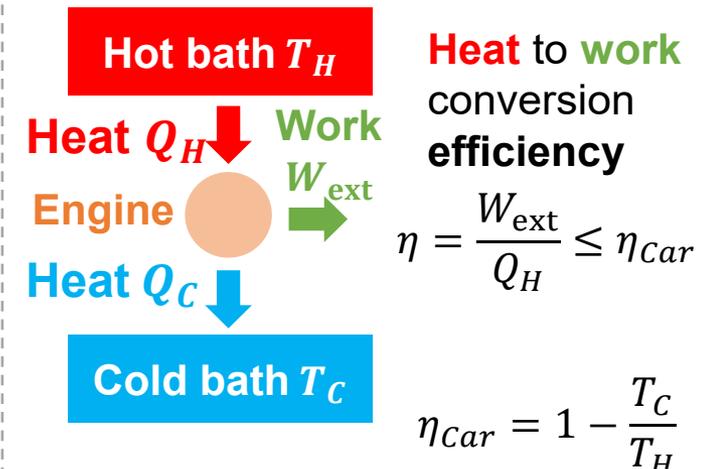
Parrondo, Horowitz, Sagawa, *Nat. Phys.* (2015)

Work can be extracted *beyond* the 2nd law up to the obtained **information**

Information heat engine



c.f. conventional heat engine



Quantum information thermodynamics

■ Single quantum measurement & feedback

$$W_{\text{ext}} \leq -\Delta F + k_B T I_{\text{QC}}$$

Quantum-Classical (QC) mutual information

$$I_{\text{QC}}(\rho; y) = S(\rho) - \sum_y p_y S(\rho_y)$$

additional information obtained via measurement = entropy before measurement - entropy after knowing measurement outcomes

$S(\rho) = -\text{Tr}[\rho \ln \rho]$: von Neumann entropy

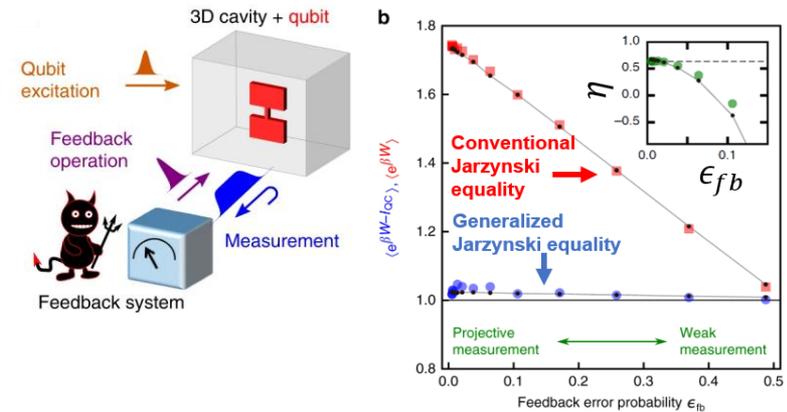
■ Repeated (continuous) quantum measurement & feedback

$$W_{\text{ext}} \leq -\Delta F + k_B T I_{\text{QCT}}$$

Quantum-Classical-Transfer (QCT) entropy

$$I_{\text{QCT}} = \sum_{n=0}^{N-1} E_{Y_n} [I_{\text{QC}}(\rho_{t_n}^{Y_n}; y_{n+1})]$$

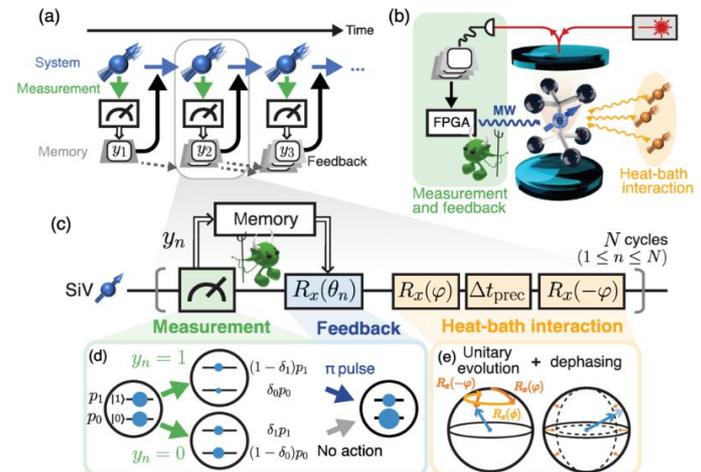
total amount of information obtained by N measurements



(Theory) Sagawa & Ueda PRL (2008)

(Experiment)

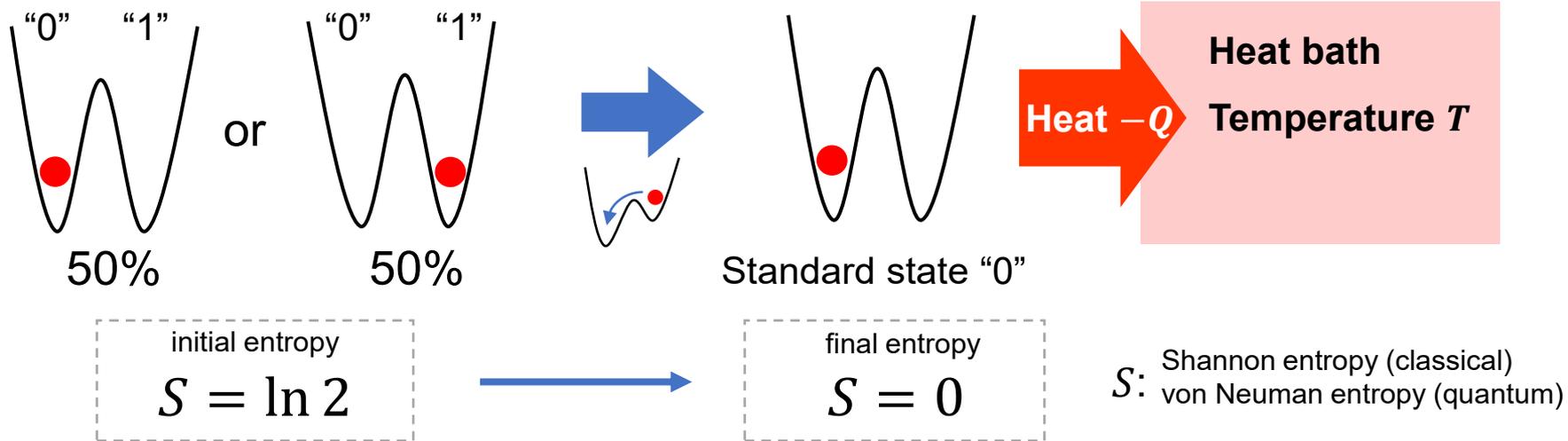
Masuyama, KF, et al., *Nat. Commun.* (2018). [Nakamura group]
M. Naghiloo, et al., PRL (2018) [Murch group] ...



(Theory) T. Yada et al., PRL (2022).

(Experiment) T. Yada P.-J. Stas, et al., PRX (2025). [Lukin group]

Information erasure (Memory reset)



Landauer's bound

R. Landauer (1961)



$$-Q \geq k_B T \Delta S = k_B T \ln 2$$

- **Heat dissipation is inevitable** for information erasure
- **Minimal energetic costs** required for *information processing and computation*

■ Second law

$$\sigma = \underbrace{\Delta S}_{\substack{\text{entropy} \\ \text{production}}} - \underbrace{\beta Q}_{\substack{\text{system} \\ \text{entropy} \\ \text{change}}} \underbrace{\geq}_{\substack{\text{bath} \\ \text{entropy} \\ \text{change}}} 0$$

equality is achieved by **quasi-static** (infinitely slow) processes

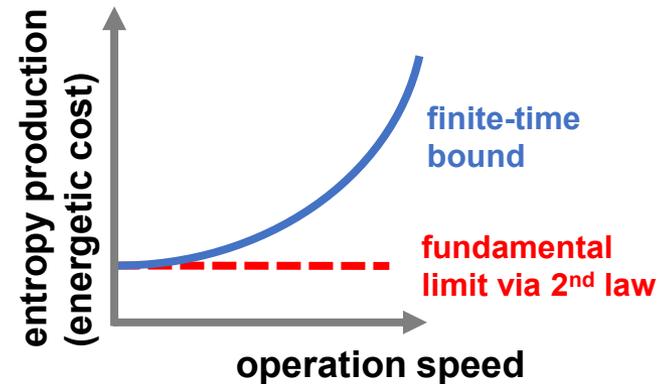
→ *fundamental limit in finite-time?*

■ Finite-time second law [Thermodynamic speed limits, Thermodynamic uncertainty relations (TUR)]

$$\frac{\substack{\text{L1 distance} \\ L(p_0, p_\tau)}}{\substack{\text{time duration} \\ \tau}} \leq \sqrt{\substack{\text{entropy production} \\ 2\sigma \bar{A}_{\text{act}}}}$$

activity (average transition rate)

Shiraishi, KF, Saito, PRL (2018)

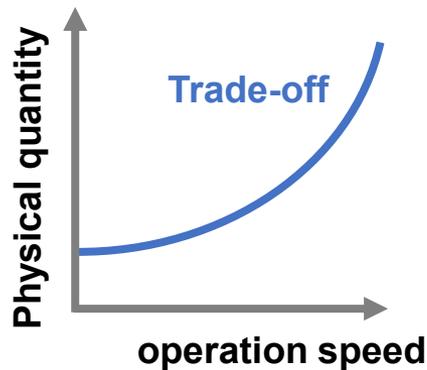


• fundamental *trade-off relation* between speed and energetic costs

- **Thermodynamic Speed limits:** Aurell, et al., PRL (2011), KF, Shiraishi, Saito, NJP (2019), Dechant, Sakurai (2019), Nakazato, Ito, PRR (2021), Van Vu, Saito, PRX (2023)...
- **Thermodynamic uncertainty relations:** Barato-Seifert PRL (2015), Gingrich, et al., PRL (2016), Shiraishi, Saito, Tasaki, PRL (2016)...

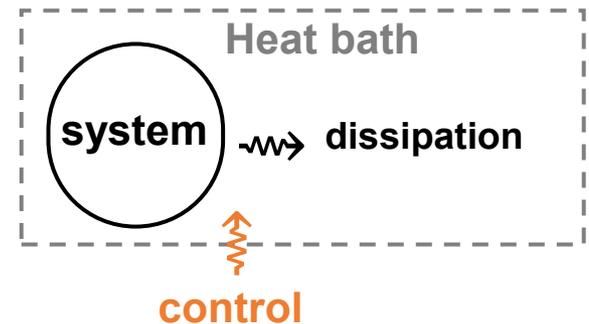
■ Method to control non-equilibrium and finite-time processes in open systems

□ Fundamental speed limits of physical processes



Quantum/thermodynamic speed limits
PRL (2016), PRL (2018), NJP (2019), ...

□ Design principles to counteract/utilize dissipation



Shortcuts to adiabaticity PRL (2020), PRL (2021)
Quantum dissipative systems PRL (2024), PRR (2025)

■ Realizing high-speed operation with low energetic costs

- To what extent quantum effects can be utilized?
- Thermodynamically optimal feedback cooling?

Tajima & KF, PRL 127, 190604 (2021)
KF & Tajima, PRL 134, 080401 (2025)

Kumasaki, Yada, KF and Sagawa, PRR (2025)
Kumasaki, Tojo, Sagawa, KF, arXiv:2508.06174

Thermodynamic approach to quantum cooling limit

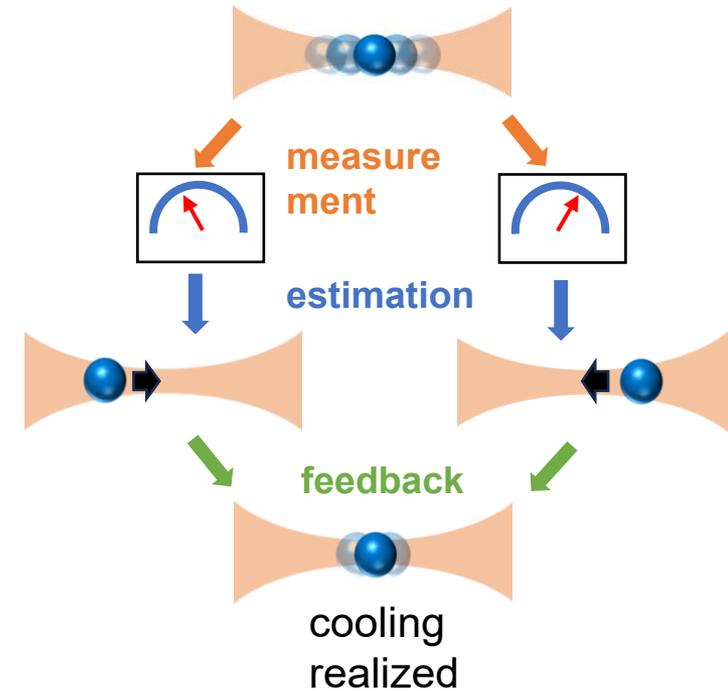
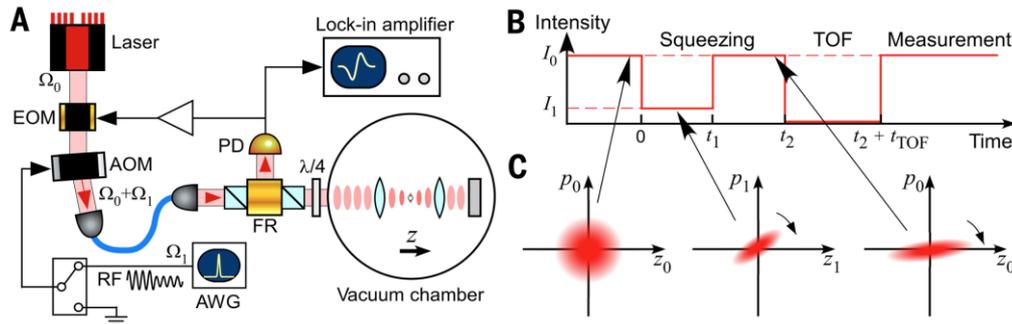
Kumasaki, Yada, **KF** and Sagawa, PRR (2025)

Feedback cooling

Feedback cooling

⇒ cooling close to the ground state is realized

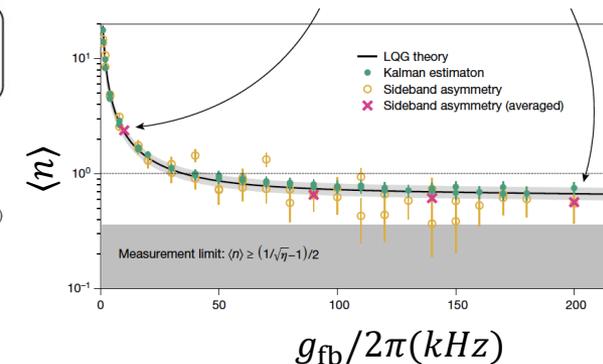
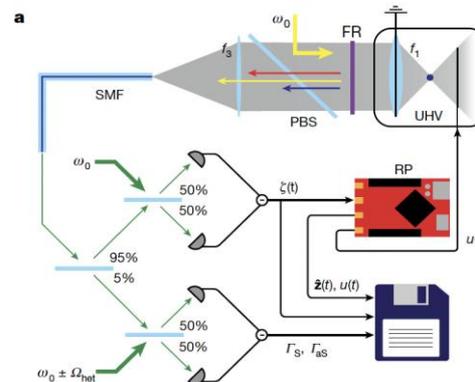
(quantum squeezing: Kamba *et al.*, Science 2025)



Kalman filter

optimal estimator for a linear (Gaussian) system

realized in nanoparticle systems:
L. Magrini, *et al. Nature* (2021)



Dynamics of continuous measurement and feedback ¹¹

■ Gaussian stochastic quantum master equation [c.f. Doherty and Jacobs, PRA 60, 2700 (1999)]

$$d\rho_c = -\frac{i}{\hbar} \left[\underbrace{\frac{\hbar\omega}{2} (\hat{x}^2 + \hat{p}^2)}_{\text{Harmonic potential}} + \underbrace{H_{FB}(t)}_{\text{Feedback}}, \rho_c \right] + \underbrace{(\gamma(N+1)\mathcal{D}[a] + \gamma N\mathcal{D}[a^\dagger])}_{\text{Effect of heat bath}} \rho_c - \underbrace{k[\hat{x}[\hat{x}, \rho_c]]}_{\text{Continuous position measurement}} dt + 8k\eta(\hat{x}\rho_c + \rho_c\hat{x} - 2\langle\hat{x}\rangle_c\rho_c)(dy - \langle\hat{x}\rangle_c dt)$$

$N = (e^{\hbar\beta\omega} - 1)^{-1}$

- Measurement outcome: $dy = \langle\hat{x}\rangle_c dt + dW/\sqrt{8k\eta}$

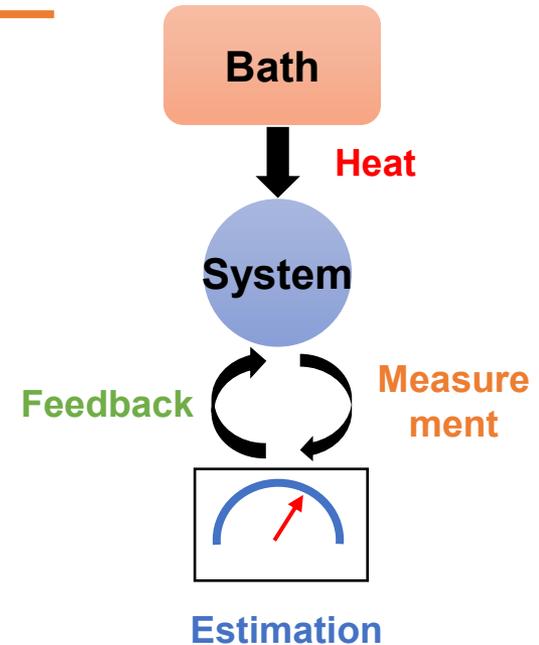
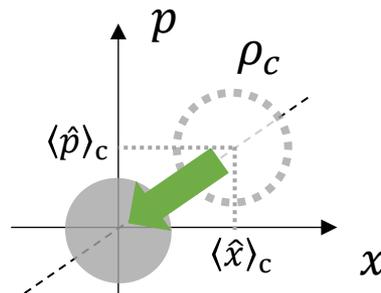
$$\langle\hat{x}\rangle_c = \text{Tr}[\hat{x}\rho_c] \quad \text{information of the position expectation value for given measurement outcome}$$

- ρ_c : state conditioned on $Y_t = \{dy\}_{[0,t]}$

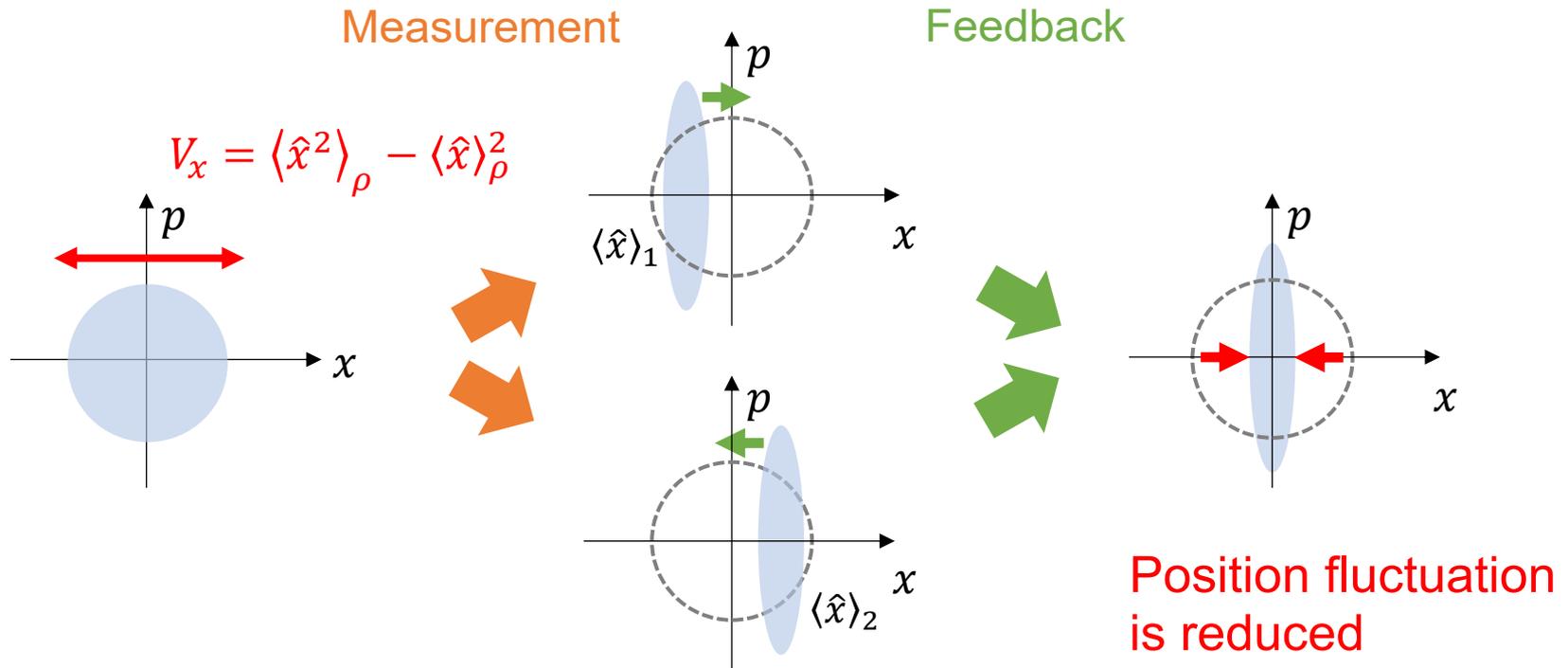
- Estimation:** $Y_t \rightarrow e_t$ **Quantum Kalman filter case:**
 $e_t = \{\langle\hat{x}\rangle_c, \langle\hat{p}\rangle_c\}$

- Linear feedback Hamiltonian**

$$H_{FB}(t) = c_x(e_t)\hat{x} - c_p(e_t)\hat{p}$$



□ Feedback cooling



- Momentum fluctuation can be reduced in a similar way
- Better cooling is achieved by an optimal **feedback control** (discussed later)
- Accurate estimation of $\langle \hat{x} \rangle_c, \langle \hat{p} \rangle_c$ is important

optimal estimator = quantum Kalman filter

Quantum Kalman filter

- Linear feedback Hamiltonian

$$H_{FB}(t) = \hbar(g_x x_c + g_p p_c) \hat{x} - \hbar(f_x x_c + f_p p_c) \hat{p}$$

- Update equation for the estimates ($x_c = \text{Tr}[\hat{x}\rho_c]$, $p_c = \text{Tr}[\hat{p}\rho_c]$)

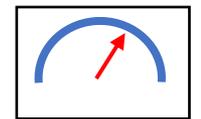
$$\begin{cases} dx_c = \frac{p_c}{M} dt - (f_x x_c + f_p p_c) dt - \frac{\gamma}{2} x_c dt + 8k\eta V_x^c (dy - x_c dt) \\ dp_c = -M\omega^2 x_c dt - (g_x x_c + g_p p_c) dt - \frac{\gamma}{2} p_c dt + 8k\eta C_{xp}^c (dy - x_c dt) \end{cases}$$

- Update equation for the covariance matrices (V_x^c, V_p^c, C_{xp}^c)

$$\begin{cases} \dot{V}_x^c = \frac{2}{M} C_{xp}^c - \gamma V_x^c + \frac{\gamma \hbar}{M\omega} (N + \frac{1}{2}) - 8k\eta (V_x^c)^2 \\ \dot{V}_p^c = -2M\omega^2 C_{xp}^c - \gamma V_p^c + \gamma \hbar M\omega (N + \frac{1}{2}) + 2k\hbar^2 - 8k\eta (C_{xp}^c)^2 \\ \dot{C}_{xp}^c = -M\omega^2 V_x^c + \frac{1}{M} V_p^c - \gamma C_{xp}^c - 8k\eta V_x^c C_{xp}^c \end{cases}$$

Feedback

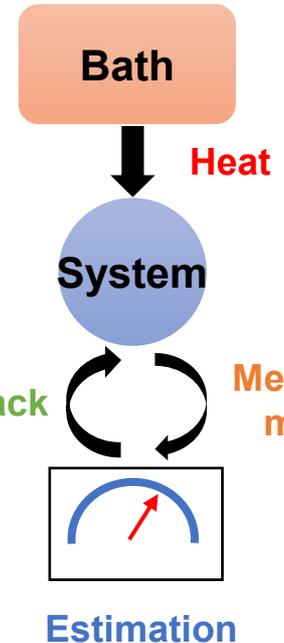
Measurement



Estimation

$$\begin{aligned} V_x^c &= \langle \hat{x}^2 \rangle_{\rho_c} - x_c^2 \\ V_p^c &= \langle \hat{p}^2 \rangle_{\rho_c} - p_c^2 \\ C_{xp}^c &= \frac{1}{2} \langle \{\hat{x}, \hat{p}\} \rangle_{\rho_c} - x_c p_c \end{aligned}$$

- Current values of $x_c(t), p_c(t), V_x^c(t), V_p^c(t), C_{xp}^c(t)$ & dy
 → next time-step values of those quantities can be obtained
 (Kalman filter)



Thermodynamic bound on feedback cooling

Generalized second law

$$W_{\text{ext}} \leq -\Delta F + k_B T \Delta I$$

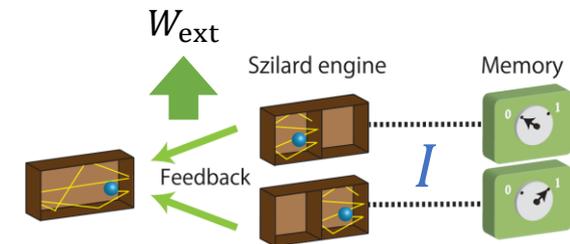
Bath temperature

Mutual information

(discrete measurement & feedback)

Information flow, transfer entropy

(continuous measurement & feedback)



Sagawa & Ueda PRL (2008), (2012)
Parrondo, Horowitz, Sagawa, *Nat. Phys.* (2015)

Work can be extracted *beyond* the 2nd law up to the obtained **information**

Second-law-like inequality for feedback cooling

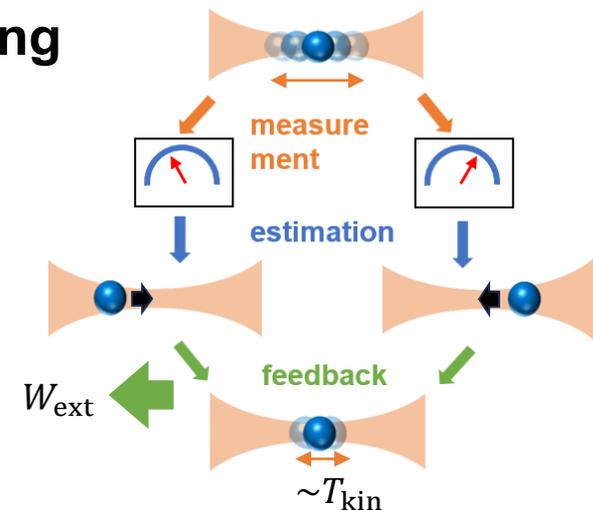
$$\dot{W}_{\text{ext}} \leq k_B T_{\text{kin}} \dot{I}_{\text{TE}}$$

Kinetic temperature ($= m\langle v^2 \rangle$) **Transfer entropy rate**

- derived for a classical free particle system in the steady-state

Horowitz and Sandberg, *NJP* 16 125007(2014)

→ **quantum systems?** (with trapping potential and transient regime?)



Definition of the work and heat

■ Gaussian stochastic quantum master equation

$$d\rho_c = \underbrace{-\frac{i}{\hbar} \left[\frac{\hbar\omega}{2} (\hat{x}^2 + \hat{p}^2) + H_{FB}(t), \rho_c \right]}_{\text{Harmonic potential}} + \underbrace{(\gamma(N+1)\mathcal{D}[a] + \gamma N\mathcal{D}[a^\dagger])}_{\text{Feedback}} \rho_c + \underbrace{(\gamma(N+1)\mathcal{D}[a] + \gamma N\mathcal{D}[a^\dagger])}_{\text{Effect of heat bath}} \rho_c - \underbrace{k[\hat{x}[\hat{x}, \rho_c]]}_{\text{Continuous position measurement}} dt + 8k\eta(\hat{x}\rho_c + \rho_c\hat{x} - 2\langle\hat{x}\rangle_c\rho_c)(dy - \langle\hat{x}\rangle_c dt)$$

$N = (e^{\hbar\beta\omega} - 1)^{-1}$

- $H_{FB}(t) = \hbar g(\langle\hat{p}\rangle_c\hat{x} - \langle\hat{x}\rangle_c\hat{p})$

■ Definition of work and heat

$$\dot{E}_{HO} = \dot{Q} - \dot{W}_{ext}$$

$$\dot{E}_{HO} = \frac{\hbar\omega}{2} \text{Tr}[(\hat{x}^2 + \hat{p}^2)\partial_t\rho]$$

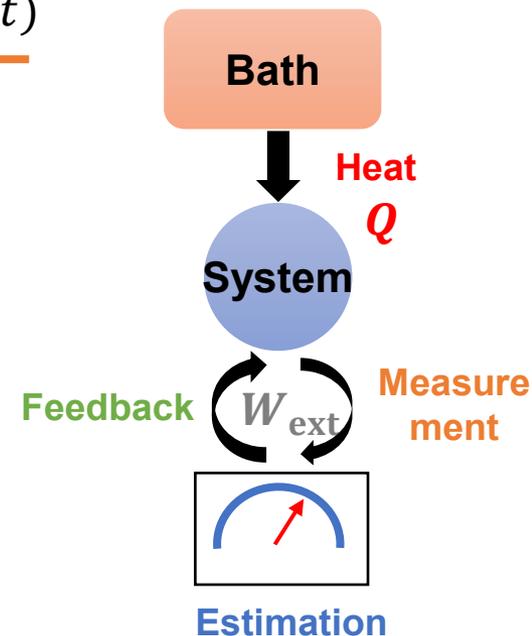
- Heat current: $\dot{Q} := \hbar\omega\gamma(N - \langle a^\dagger a \rangle)$

Energy exchange with the heat bath

- Work rate: $\dot{W}_{ext} := \frac{i}{\hbar} \mathbb{E}_{Y_t} \text{Tr}[H_{HO}[H_{FB}(t), \rho_c]] - k\hbar^2$

Energy change via feedback

Energy change via measurement



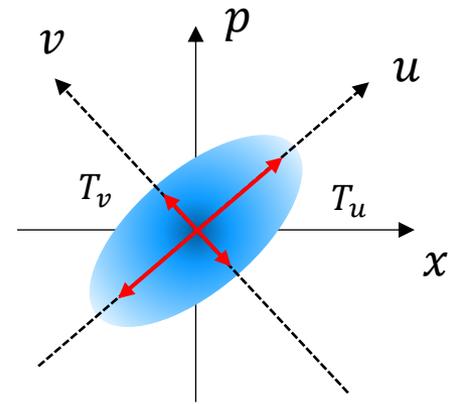
Kumasaki, Yada, Funo, Sagawa, PRR (2025)

Information-thermodynamic cooling limit in the steady-state

$$\frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v} \leq \dot{I}_{\text{QCT}}$$

Quantum Kalman filter + large feedback gain

- ✓ \dot{I}_{QCT} quantifies the limit of extracting work and reducing kinetic temperatures (=cooling)



T_u, T_v : Kinetic temperatures
 $\dot{W}_{\text{ext},u}, \dot{W}_{\text{ext},v}$: Extractable work

Quantum-Classical-Transfer (QCT) entropy rate

$$\dot{I}_{\text{QCT}} := \lim_{dt \rightarrow 0} \frac{1}{dt} E_{Y_t} [S(\hat{\rho}_c) - E_{dy} [S(\hat{\rho}_c^{\text{post}})]]$$

entropy reduction by knowing dy (=additionally obtained information)

- $\hat{\rho}_c^{\text{post}} := \hat{\rho}_c - k[\hat{x}[\hat{x}, \rho_c]]dt + \sqrt{8k\eta}(\{\hat{x}, \rho_c\} - 2x_c\rho_c)(dy - x_c dt)$: post measurement state

- ✓ Quantifies the upper bound of information available for multiple feedback

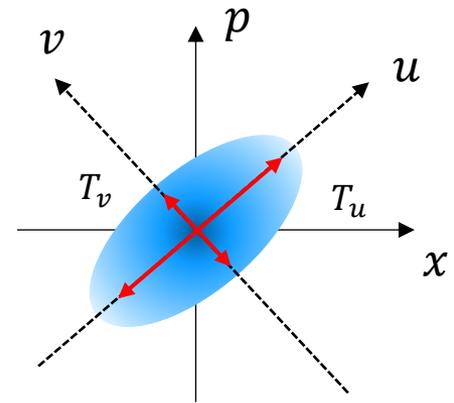
Kumasaki, Yada, Funo, Sagawa, PRR (2025)

Information-thermodynamic cooling limit in the steady-state

$$\frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v} \leq \dot{I}_{\text{QCT}}$$

Quantum Kalman filter + large feedback gain

- ✓ \dot{I}_{QCT} quantifies the limit of extracting work and reducing kinetic temperatures (=cooling)



T_u, T_v : Kinetic temperatures
 $\dot{W}_{\text{ext},u}, \dot{W}_{\text{ext},v}$: Extractable work

- Tighter than the generalized second law using the bath temperature

$$\frac{\dot{W}_{\text{ext}}}{T} \leq \frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v} \leq \dot{I}_{\text{QCT}}$$

- Generalization of the result for classical one-dimensional Langevin systems

$$\dot{W}_{\text{ext}} \leq T_{\text{kin}} \dot{I}_{\text{TE}}$$

J. M. Horowitz and H. Sandberg, *NJP* (2014)

Outline of the derivation

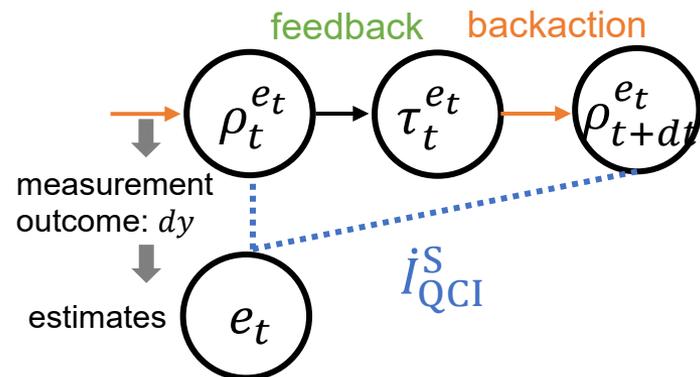
Information-thermodynamic bound applicable to non-steady states

$$\frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v} \leq \underbrace{-\dot{I}_{\text{QCI}}^S}_{\text{Quantum information flow}} - \underbrace{\dot{S}_{\text{BA}}}_{\text{Information-loss due to measurement back-action}}$$

Quantum information flow Information-loss due to measurement back-action

$$\dot{I}_{\text{QCI}}^S := \lim_{dt \rightarrow 0} \frac{1}{dt} [\chi(\rho_{t+dt}^{e_t}) - \chi(\rho_t^{e_t})]$$

Change in the correlation (Holevo information χ) between the system state and the estimate



Equality condition: $0 = \text{LHS} - \text{RHS} =$

$$\lim_{dt \rightarrow 0} \frac{1}{dt} E_{e_t} \left[S(\mathcal{E}_{\text{bath}}(\hat{\rho}_t^{e_t}) \parallel \mathcal{E}_{\text{bath}}(\hat{\rho}_t)) - S(\hat{\rho}_t^{e_t} \parallel \hat{\rho}_t) \right]$$

Relation between two information quantities in the steady-state

$$-\dot{I}_{\text{QCI}}^S - \dot{S}_{\text{BA}} \leq \dot{I}_{\text{QCT}}$$

Correlation used by feedback is bounded by the obtained information via measurement

Equality condition: $\dot{I}_{\text{QCT}} + \dot{I}_{\text{QCI}}^S + \dot{S}_{\text{BA}} =$
 $= \lim_{dt \rightarrow 0} E_{Y_t} [S(\hat{\rho}_c \parallel \hat{\rho}_t^{e_t}) - S(\hat{t}_c \parallel \hat{t}_t^{e_t})] = 0$

\Rightarrow Main result: $\frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v} \leq \dot{I}_{\text{QCT}}$

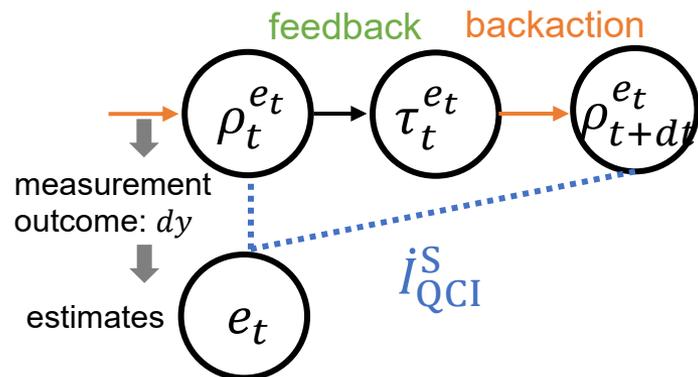
Roughly speaking, equality condition is given by

$$\hat{\rho}_t = \hat{\rho}_c = \hat{\rho}_t^{e_t}$$

Information-thermodynamic bound applicable to non-steady states

$$\frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v} \leq \underbrace{-\dot{i}_{\text{QCI}}^S}_{\text{Quantum information flow}} - \underbrace{\dot{S}_{\text{BA}}}_{\text{Information-loss due to measurement back-action}}$$

Quantum information flow Information-loss due to measurement back-action



$$i_{\text{QCI}}^S := \lim_{dt \rightarrow 0} \frac{1}{dt} [\chi(\rho_{t+dt}^{e_t}) - \chi(\rho_t^{e_t})]$$

Change in the correlation (Holevo information χ) between the system state and the estimate

Equality condition: $0 = \text{LHS} - \text{RHS} =$

$$\lim_{dt \rightarrow 0} \frac{1}{dt} [S(\rho_{t+dt}(\hat{\rho}^{e_t}) \| \rho_{t+dt}(\hat{\rho}_c)) - S(\rho_t(\hat{\rho}^{e_t}) \| \rho_t(\hat{\rho}_c))]$$

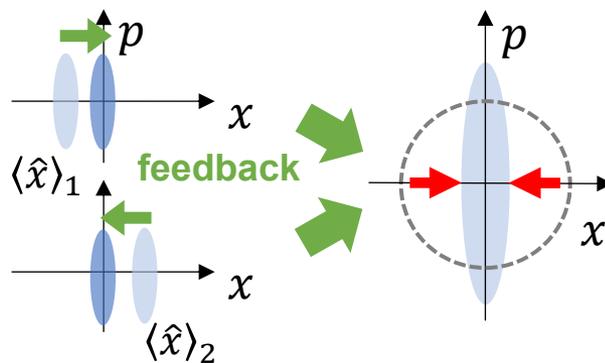
✓ Optimal feedback

$$H_{\text{FB}}^{x,p} = g(\langle \hat{p} \rangle_c \hat{x} - \langle \hat{x} \rangle_c \hat{p})$$

Feedback gain

$$g \rightarrow \infty$$

⇒ Reduction in all x-p directions



✓ Quantum Kalman filter

$$e_t = \{\langle \hat{x} \rangle_c, \langle \hat{p} \rangle_c\}$$

Optimal estimate

⇒ Main result: $\frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v} \leq \dot{i}_{\text{QCT}}$

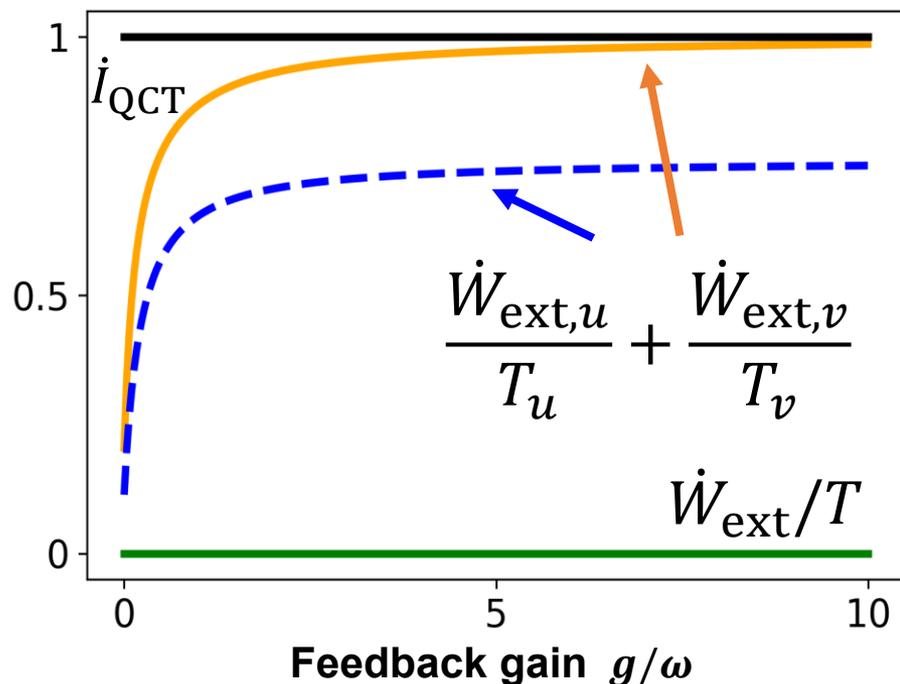
Roughly speaking, equality condition is given by

$$\hat{\rho}_t = \hat{\rho}_c = \hat{\rho}_t^{e_t}$$

Numerical demonstration

Information-thermodynamic limit

$$\checkmark \frac{\dot{W}_{\text{ext}}}{T} \leq \frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v} \leq \dot{I}_{\text{QCT}}$$

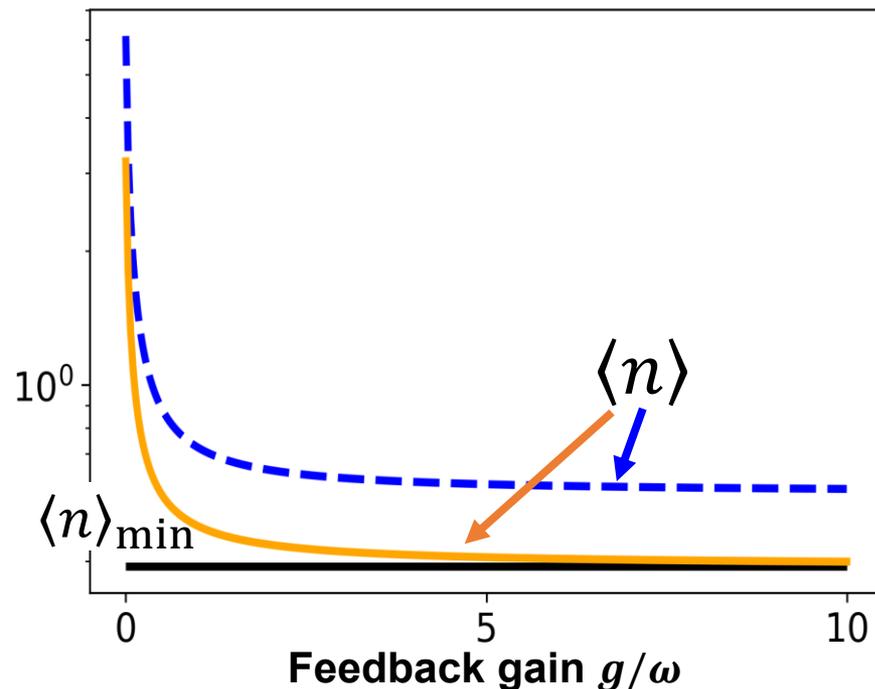


• Feedback Hamiltonian

□ optimal : $H_{FB}^{x,p} = g(\langle \hat{p} \rangle_c \hat{x} - \langle \hat{x} \rangle_c \hat{p})$

Energetic limit

$$\checkmark \langle n \rangle_{\text{min}} := \frac{1}{2} (V_x^c + V_p^c - 1) \leq \langle n \rangle$$



□ partial control : $H_{FB}^x = g(\langle p \rangle_c + \langle \hat{x} \rangle_c) \hat{x}$

Typically used in actual experiment

Thermodynamic uncertainty relation for feedback cooling (classical)

Kumasaki, Tojo, Sagawa, **KF**, arXiv:2508.06174

Thermodynamic uncertainty relation

■ Power-efficiency trade-off relation in heat engines

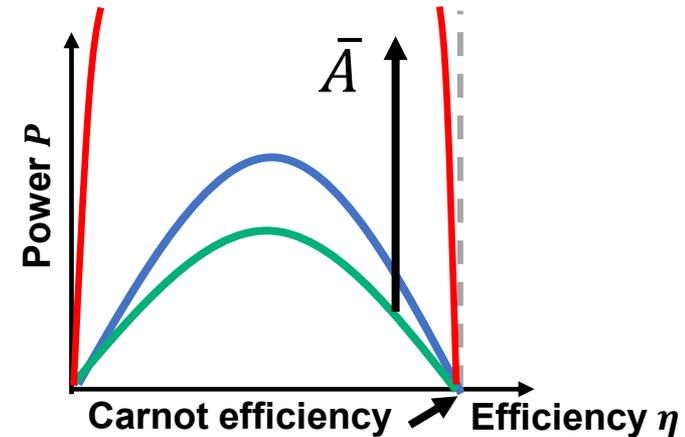
$$P \leq b \bar{A} \eta (\eta_{\text{Car}} - \eta)$$

Shiraishi, Saito, Tasaki, PRL (2016);
 Shiraishi, Saito, J. Stat. Phys (2019);
 Tajima & KF, PRL (2021);
 KF & Tajima, PRL (2025)

- constant factor: $b = 2(2 - \eta_{\text{Car}})^2 / \beta_C$
- $\bar{A} = \tau^{-1} \int_0^\tau dt A(\rho(t), \{L_{a,\omega}\})$

time-average **average jump rate**

- Finite-power and Carnot efficiency
 → \bar{A} needs to diverge



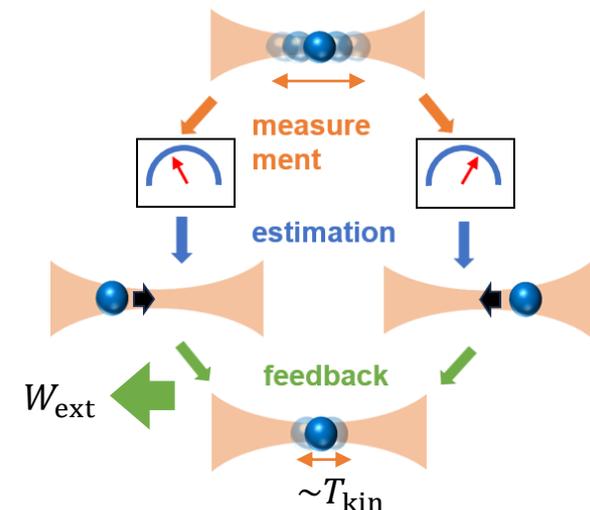
■ Feedback cooling scenario

$$\frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v} \leq \dot{I}_{\text{QCT}}$$

equality ⇒ ideal efficiency

**finite entropy reduction rate
(cooling power)**

- Finite cooling power and ideal efficiency?
- How is this observation consistent with the TUR (if it exists)?



Theoretical model

■ Classical underdamped Langevin equation

$$\begin{cases} dx_t = (\omega p_t + \underline{au_x(e_t)}) dt \\ dp_t = (\underline{-\gamma p_t} - \omega x_t + \underline{bu_p(e_t)}) dt + \sqrt{2\gamma T} dW_t \end{cases}$$

Feedback
Effect of heat bath

- measurement outcomes:

$$dy_t = (x_t dt + \sigma dW_t^y) \quad \longrightarrow \quad \text{estimates } e_t$$

■ Generalized second law

$$\begin{aligned} \dot{\Sigma}_s &= \dot{S} - \frac{\dot{Q}}{T} - \underline{\dot{I}_s} = -\frac{\dot{W}_{\text{ext}} + \dot{\mathcal{F}}}{T} - \dot{I}_s \\ &\quad \text{Information flow} \\ &= \frac{1}{\gamma T} \int dx dp de (\gamma p + \gamma T \partial_p \ln f)^2 f(x, p, e) \\ &= \|\mathbf{v}_s^{\text{irr}}\|^2 \geq 0 \end{aligned}$$

$$\dot{I} = \int dx dp de (\partial_t f(x, p, e)) \ln \frac{f(x, p, e)}{\tilde{f}(x, p)}$$

$\dot{\mathcal{F}}$: nonequilibrium free energy rate

□ mean local velocity

$$\mathbf{v}_s^{\text{irr}} := (0, -\gamma p - \gamma T \partial_p \ln f)^T$$

□ inner product

$$\langle A, B \rangle := \frac{1}{\gamma T} \int dx dp de A^T(x, p, e) B(x, p, e) f(x, p, e)$$

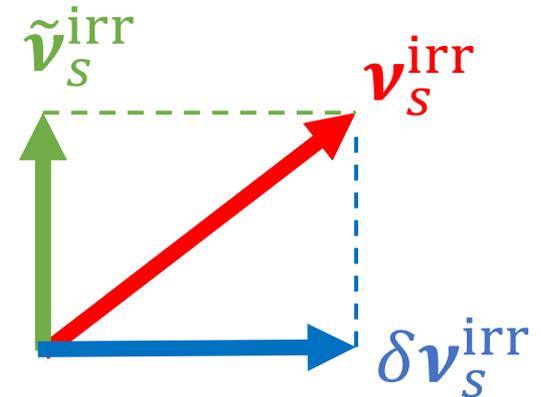
$$\|A\|^2 := \langle A, A \rangle$$

Orthogonal decomposition of the mean local velocity²⁴

- Orthogonal decomposition of the mean local velocity

$$\underline{\mathbf{v}}_s^{\text{irr}} = \underline{\tilde{\mathbf{v}}}_s^{\text{irr}} + \underline{\delta \mathbf{v}}_s^{\text{irr}}$$

orthogonal: $\langle \underline{\tilde{\mathbf{v}}}_s^{\text{irr}}, \underline{\delta \mathbf{v}}_s^{\text{irr}} \rangle = 0$



- $\underline{\tilde{\mathbf{v}}}_s^{\text{irr}} := \int de(\mathbf{v}_s^{\text{irr}} f) / \int de(f)$:
tracing out e (estimate)

- Orthogonal decomposition of the generalized second law

$$\dot{\Sigma}_s = \|\underline{\mathbf{v}}_s^{\text{irr}}\|^2 = -\frac{\dot{W}_{\text{ext}} + \dot{\mathcal{F}}}{T} - \dot{I}_s \geq 0$$

- Inequality between kinetic temperatures and bath temperature

$$\begin{aligned} \dot{\Sigma}_{\tilde{v}} &= \|\underline{\tilde{\mathbf{v}}}_s^{\text{irr}}\|^2 \\ &= \frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v} - \frac{\dot{W}_{\text{ext}} + \dot{\mathcal{F}}}{T} \geq 0 \end{aligned}$$

- Information-thermodynamic bound on cooling

$$\begin{aligned} \dot{\Sigma}_{\delta v} &= \|\underline{\delta \mathbf{v}}_s^{\text{irr}}\|^2 \\ &= -\frac{\dot{W}_{\text{ext},u}}{T_u} - \frac{\dot{W}_{\text{ext},v}}{T_v} - \dot{I}_s \geq 0 \end{aligned}$$

Orthogonal decomposition and TUR

■ Conventional TUR

□ General (irreversible) current

$$J_{\mathbf{v}_S}^{\mathbf{w}} := \gamma T \langle \mathbf{w}, \mathbf{v}_S^{\text{irr}} \rangle$$

w : weight function

Cauchy-Schwarz inequality

□ TUR

$$\left(J_{\mathbf{v}_S}^{\mathbf{w}} \right)^2 \leq (\gamma T)^2 \|\mathbf{w}\|^2 \dot{\Sigma}_S$$

■ Orthogonally decomposed currents \Rightarrow TUR

c.f. A. Dechant, S.-i. Sasa, and S. Ito, PRE (2022)

□ TUR for kinetic vs bath temperatures

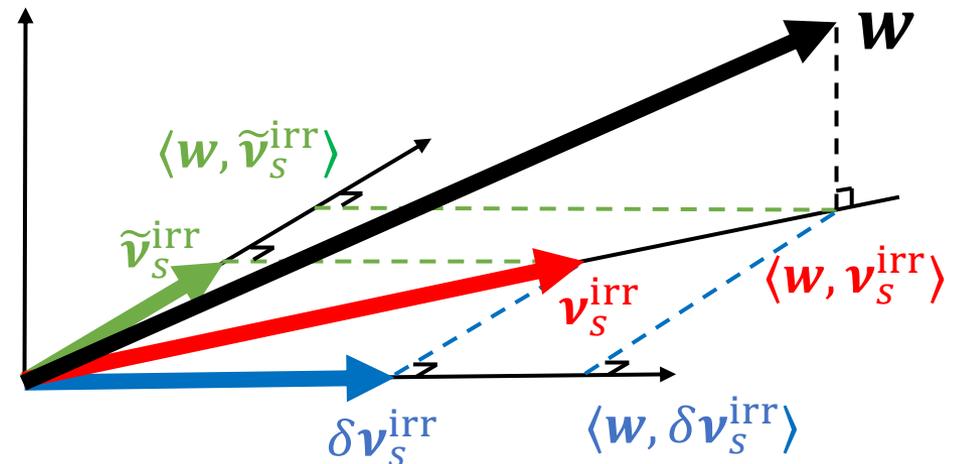
$$\left(J_{\tilde{\mathbf{v}}_S}^{\mathbf{w}} \right)^2 \leq (\gamma T)^2 \|\mathbf{w}\|^2 \dot{\Sigma}_{\tilde{v}}$$

□ TUR for feedback cooling

$$\left(J_{\delta \mathbf{v}_S}^{\mathbf{w}} \right)^2 \leq (\gamma T)^2 \|\mathbf{w}\|^2 \dot{\Sigma}_{\delta v}$$

includes information flow and entropy reduction rate

$$\dot{\Sigma}_{\delta v} = -\frac{\dot{W}_{\text{ext},u}}{T_u} - \frac{\dot{W}_{\text{ext},v}}{T_v} - \dot{I}_s \geq 0$$



Power-efficiency trade-off relation for feedback cooling ²⁶

■ Expressing information flow as a generalized current

$$\mathbf{w} = \delta \mathbf{v}_S^{\text{irr}} + \mathbf{v}_S^{\text{rev}} \longrightarrow$$

reversible ($H + H_{FB}$) part

$$\mathcal{J}_{\delta \mathbf{v}_S^{\text{irr}}}^{\mathbf{w}} = -\dot{I}_p$$

momentum part of
the information flow

$$\dot{I}_S = \dot{I}_x + \dot{I}_p$$

$$\dot{I}_p = \int dx dp de J_p \partial_p \ln \frac{f(x,p,e)}{\tilde{f}(x,p)}$$

■ Power-efficiency trade-off for feedback cooling

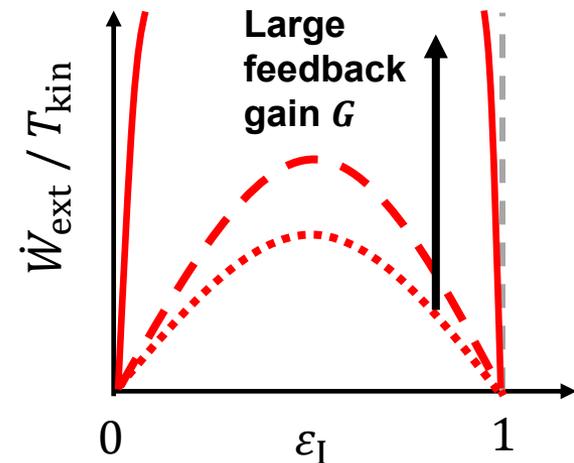
$$\left(\frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v} \right) \left(\frac{\dot{I}_p}{\dot{I}_S} \right)^2 \leq (\gamma T)^2 \underbrace{\|\delta \mathbf{v}_S^{\text{irr}} + \mathbf{v}_S^{\text{rev}}\|^2}_{\text{fluctuation (feedback gain)}} \underbrace{\varepsilon_I}_{\text{efficiency}} (1 - \varepsilon_I)$$

entropy reduction rate
(cooling power)

fluctuation
(feedback gain)

efficiency: $\varepsilon_I = \left(\frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v} \right) / \dot{I}_S$

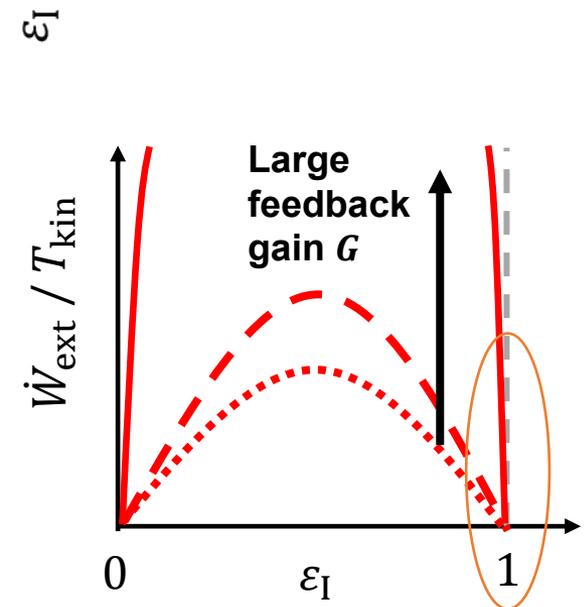
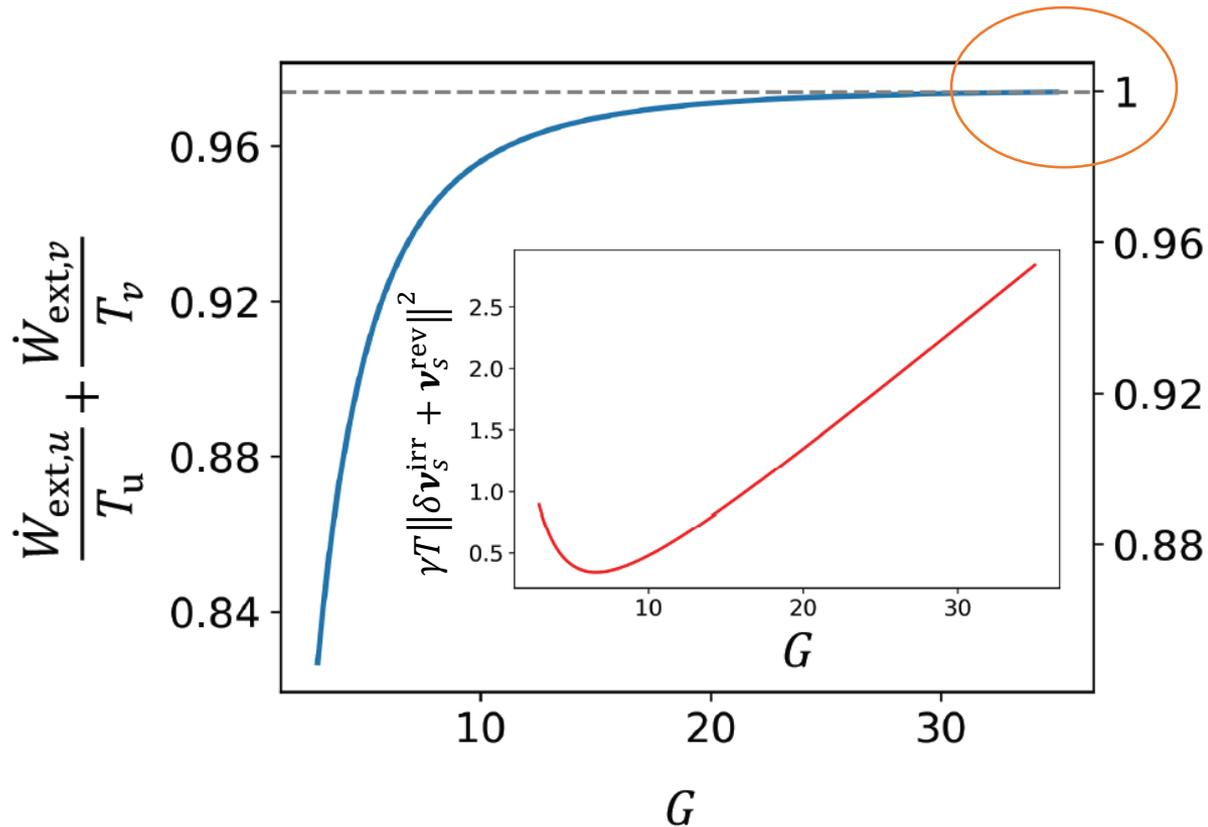
- Fluctuation term is related to the magnitude of the feedback gain G
- Finite cooling power & ideal efficiency is consistent with the TUR when feedback gain $G \rightarrow \infty$



Numerical demonstration

- Asymptotic achievement of finite cooling power & ideal efficiency

Kalman filter + large feedback gain G



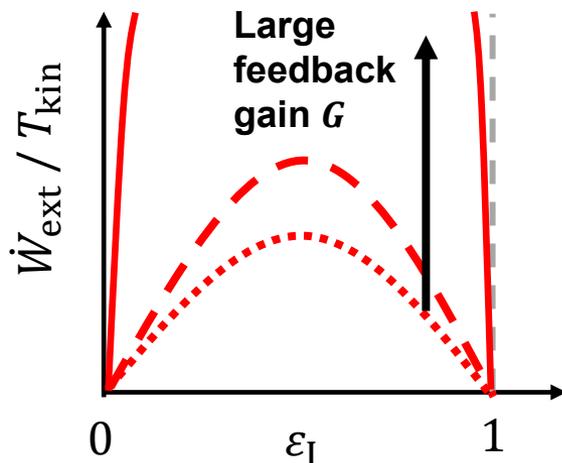
Comparison with trade-off relation for heat engines²⁸

■ Feedback cooling

$$\left(\frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v} \right) \left(\frac{\dot{i}_p}{\dot{i}_s} \right)^2 \leq (\gamma T)^2 \left\| \delta \mathbf{v}_s^{\text{irr}} + \mathbf{v}_s^{\text{rev}} \right\|^2 \varepsilon_I (1 - \varepsilon_I)$$

Fluctuation (feedback gain)

- Magnitude of the feedback gain G can be experimentally tuned
- **Further enhancement using quantum effects?**

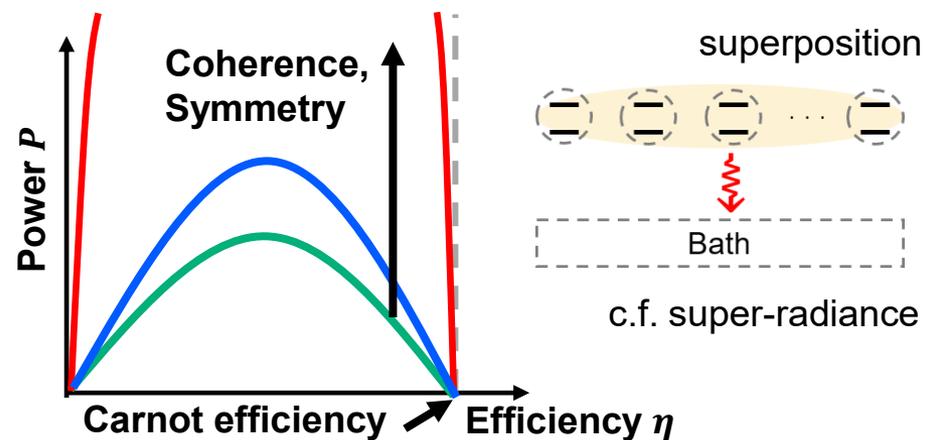


■ Heat engines

$$P \leq b(\bar{A}_{cl} + \bar{A}_{qm})\eta(\eta_{\text{Car}} - \eta)$$

Average jump rate (effective system-bath coupling strength)

- Coherent superposition of quantum states and jump operators allows increasing \bar{A}_{qm}
H. Tajima & KF PRL (2021)
- Fundamental limit of quantum enhancement of \bar{A}_{qm} & design principle based on symmetry
KF & H. Tajima, PRL (2025)



Conclusion

- Information-thermodynamic approach to the quantum cooling limit

$$\frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v} \leq \dot{I}_{\text{QCT}}$$

Quantum Kalman filter + large feedback gain

- TUR for feedback cooling (classical)

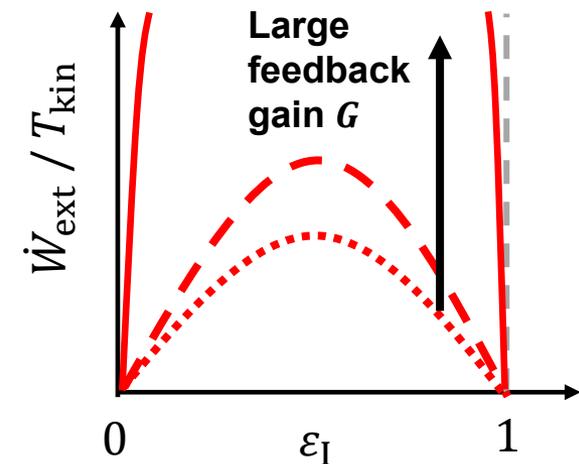
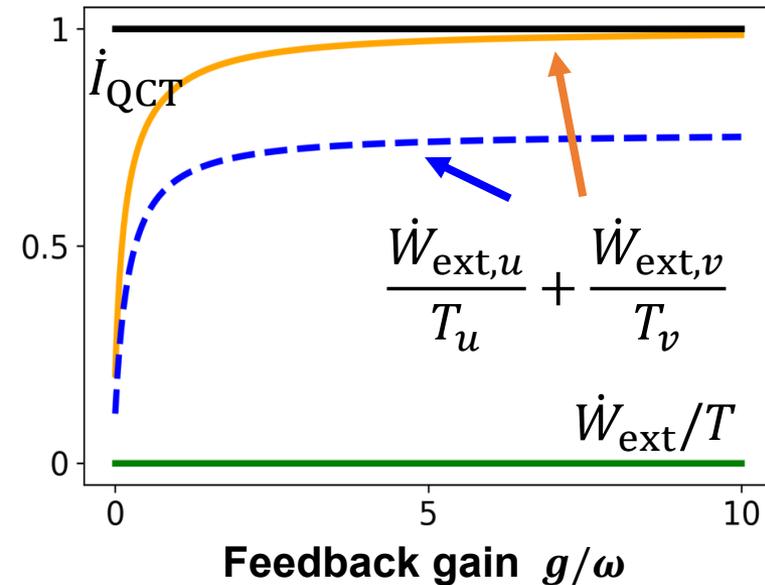
$$\left(\frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v} \right) \left(\frac{\dot{I}_p}{\dot{I}_s} \right)^2 \leq \gamma T \|\delta \mathbf{v}_s^{\text{irr}} + \mathbf{v}_s^{\text{rev}}\|^2 \varepsilon_I (1 - \varepsilon_I)$$

- Nonlinear (non-Gaussian) and quantum extensions?

- Definition of kinetic temperatures?

$$\Delta S_{\text{FB+BA}} \leq \dot{I}_{\text{QCT}} \quad (\text{Gaussian case: } \Delta S_{\text{FB+BA}} = \frac{\dot{W}_{\text{ext},u}}{T_u} + \frac{\dot{W}_{\text{ext},v}}{T_v})$$

- Quantum enhancement of cooling efficiency/power?



Cooling limit of Gaussian systems

- Gaussian state \rightarrow Completely characterized by $\{\langle \hat{x} \rangle_\rho, \langle \hat{p} \rangle_\rho, \underline{V_x}, \underline{V_p}, \underline{C_{xp}}\}$

Covariance matrices, e.g., $V_x = \langle \hat{x}^2 \rangle_\rho - \langle \hat{x} \rangle_\rho^2$

- Ensemble averaged state: $\rho = E_{Y_t}[\rho_c]$

- Decomposition property (assuming Gaussian stochastic master equation)

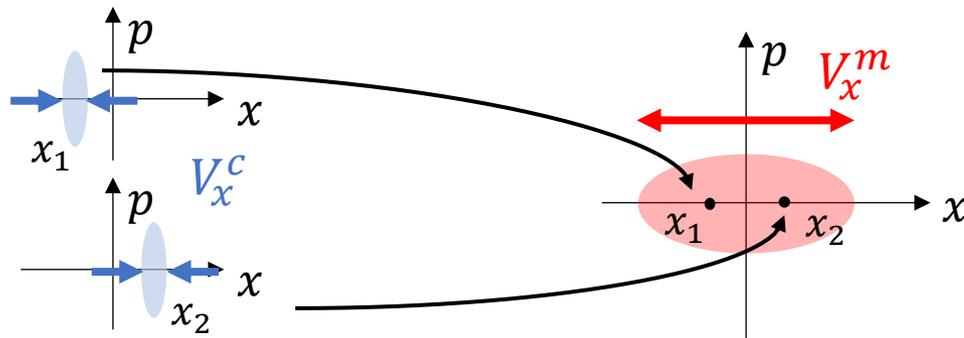
$$\underline{V_x} = \underline{V_x^c} + \underline{V_x^m}, \text{ etc.}$$

$$V_x = \langle \hat{x}^2 \rangle_\rho - \langle \hat{x} \rangle_\rho^2$$

$$V_x^c = \langle \hat{x}^2 \rangle_{\rho_c} - \langle \hat{x} \rangle_c^2$$

$$V_x^m = E_{Y_t}[\langle \hat{x} \rangle_c^2] - (E_{Y_t}[\langle \hat{x} \rangle_c])^2$$

Position variance of the averaged state ρ = Position variance of the conditional state ρ_c + Fluctuations of the mean position $\langle \hat{x} \rangle_c$ of the conditional state



Cooling limit of Gaussian systems

- Gaussian state → Completely characterized by $\{\langle \hat{x} \rangle_\rho, \langle \hat{p} \rangle_\rho, \underline{V_x}, V_p, C_{xp}\}$

Covariance matrices, e.g., $V_x = \langle \hat{x}^2 \rangle_\rho - \langle \hat{x} \rangle_\rho^2$

- Ensemble averaged state: $\rho = E_{Y_t}[\rho_c]$

- Decomposition property (assuming Gaussian stochastic master equation)

$$\underline{V_x} = \underline{V_x^c} + \underline{V_x^m}, \text{ etc.}$$

$$V_x = \langle \hat{x}^2 \rangle_\rho - \langle \hat{x} \rangle_\rho^2 \quad V_x^c = \langle \hat{x}^2 \rangle_{\rho_c} - \langle \hat{x} \rangle_c^2 \quad V_x^m = E_{Y_t}[\langle \hat{x} \rangle_c^2] - (E_{Y_t}[\langle \hat{x} \rangle_c])^2$$

Position variance of the averaged state ρ = Position variance of the conditional state ρ_c + Fluctuations of the mean position $\langle \hat{x} \rangle_c$ of the conditional state

Proof

$$V_x = E_{Y_t}[\text{Tr}[\hat{x}^2 \rho_c]] - (E_{Y_t}[\langle \hat{x} \rangle_c])^2 = E_{Y_t}[V_x^c] + V_x^m = V_x^c + V_x^m$$

By assuming Gaussian stochastic master equation,
we can show that $E_{Y_t}[V_x^c] = V_x^c$

(see next slide for example)

Cooling limit of Gaussian systems

[c.f. Doherty and Jacobs, PRA 60, 2700 (1999)]

Time-evolution equation of the conditional covariances (in dimensionless unit)

for simplicity, assume no heat bath ($\gamma = 0$)

$$\begin{cases} \dot{V}_x^c = 2\omega C_{xp}^c - 8k\eta(V_x^c)^2 \\ \dot{V}_p^c = -2\omega C_{xp}^c + 2k\hbar^2 - 8k\eta(C_{xp}^c)^2 \\ \dot{C}_{xp}^c = -\omega V_x^c + \omega V_p^c - 8k\eta V_x^c C_{xp}^c \end{cases}$$

Steady-state



$$\begin{aligned} V_x^{c*} &= \frac{\hbar}{\sqrt{2\eta}} \frac{1}{\sqrt{\xi+1}} \\ V_p^{c*} &= \frac{\hbar}{\sqrt{2\eta}} \frac{\xi}{\sqrt{\xi+1}} \\ C_{xp}^{c*} &= \frac{\hbar}{2\sqrt{\eta}} \frac{\sqrt{\xi-1}}{\sqrt{\xi+1}} \end{aligned}$$

parameter: $\xi = \sqrt{1 + 16\hbar^2\eta k^2}$

Do not depend on dy nor H_{FB}

Cooling limit of the average energy

$$\text{Tr}[H\rho] = \frac{\hbar\omega}{2} (V_x^c + V_p^c + \underbrace{V_x^m + V_p^m}_{\text{cooling limit}} + \underbrace{\langle x \rangle_\rho^2 + \langle p \rangle_\rho^2}_{\text{measurement limit}}) \geq \frac{\hbar\omega}{2} (V_x^c + V_p^c) \stackrel{\text{assume steady-state}}{=} \hbar\omega \sqrt{\frac{\xi+1}{8\eta}} \geq \frac{\hbar\omega}{2} \sqrt{\frac{1}{\eta}}$$

- Better cooling is achieved by reducing $V_x^m + V_p^m$ (and $\langle x \rangle_\rho, \langle p \rangle_\rho$) via **feedback control**
- Accurate estimation of $\langle \hat{x} \rangle_c, \langle \hat{p} \rangle_c$ is important
optimal estimator = quantum Kalman filter

