Quantum Fisher information as a measure of symmetry breaking in quantum many-body systems [arXiv:2509.07468]

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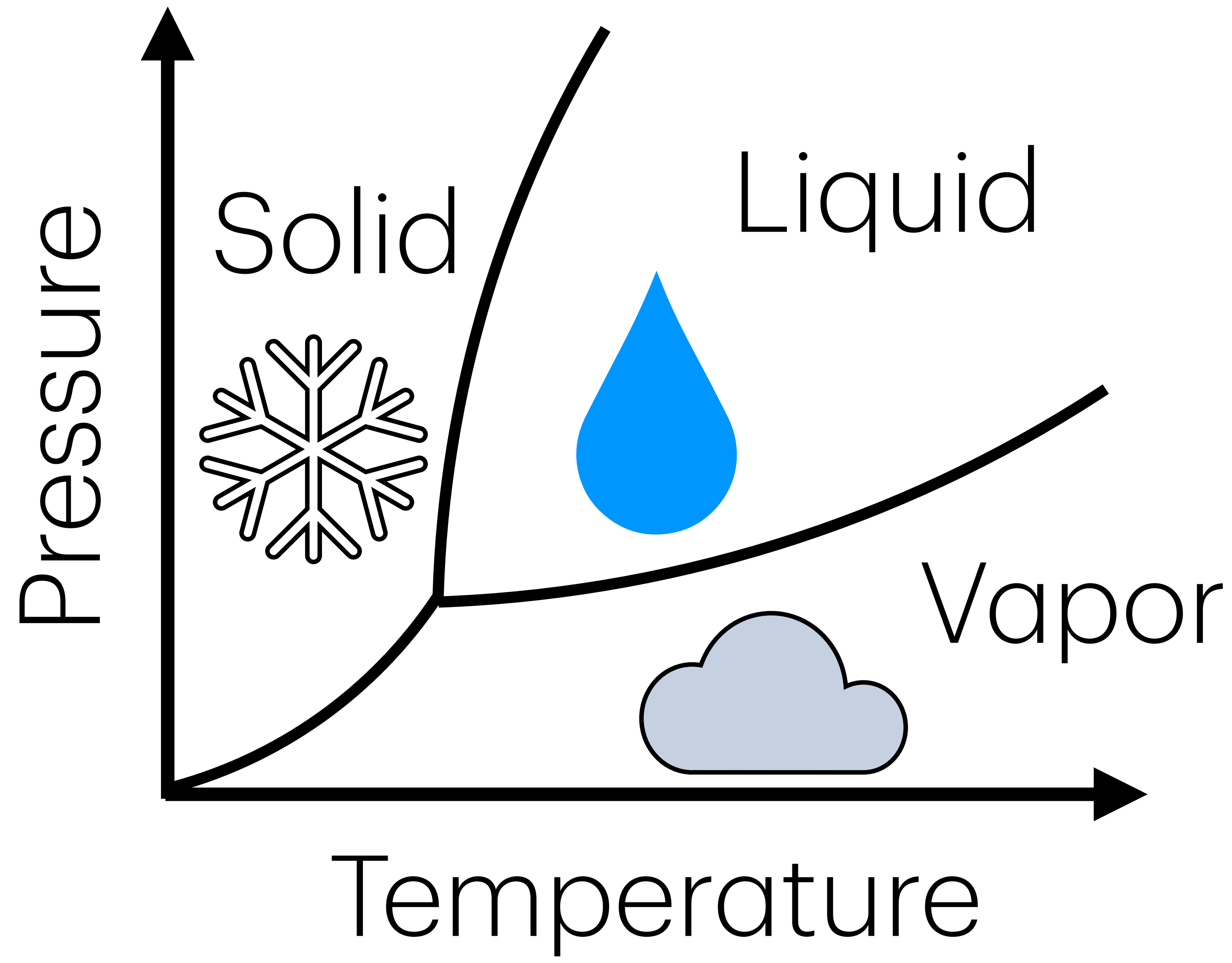
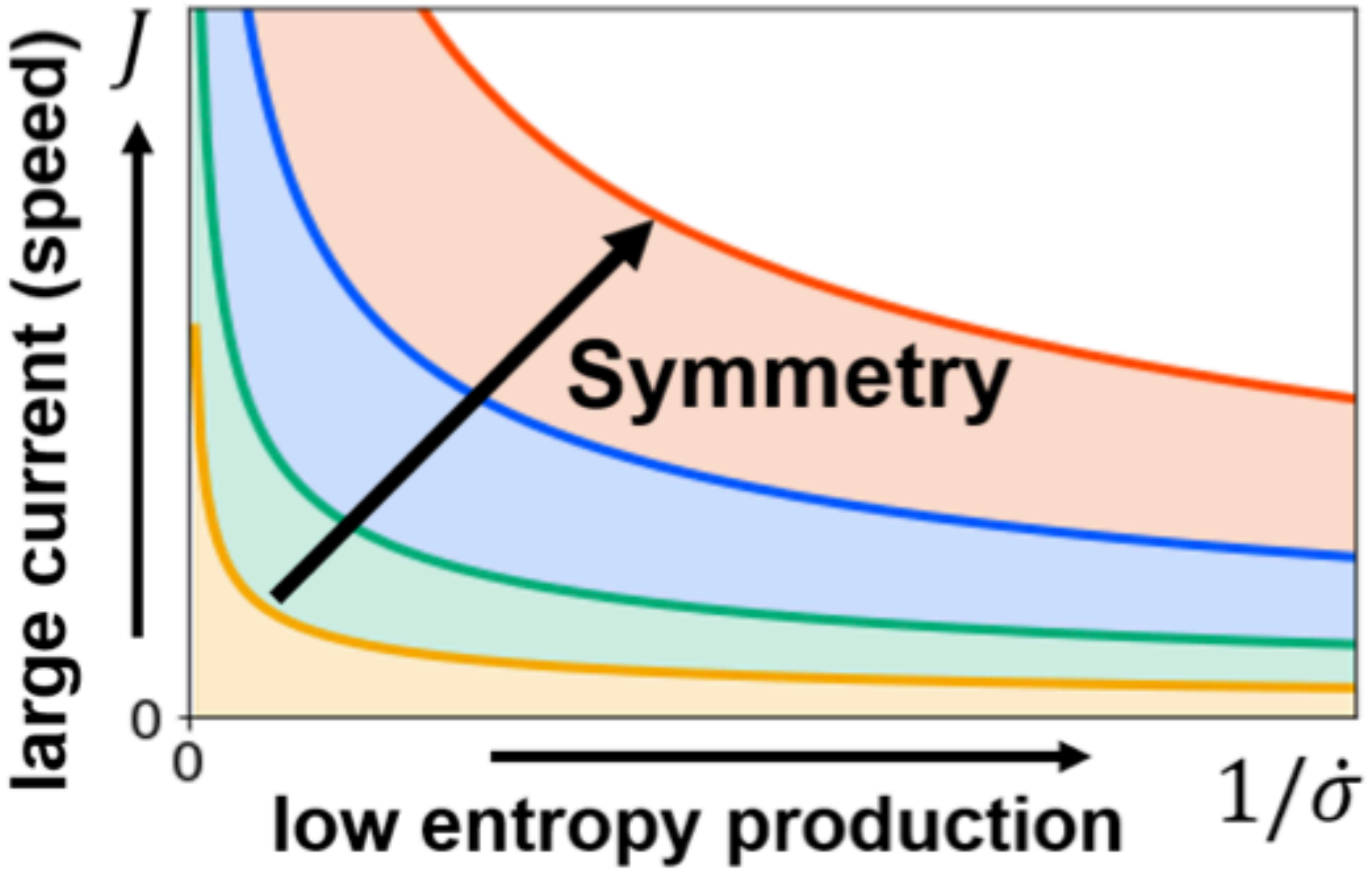
Introduction



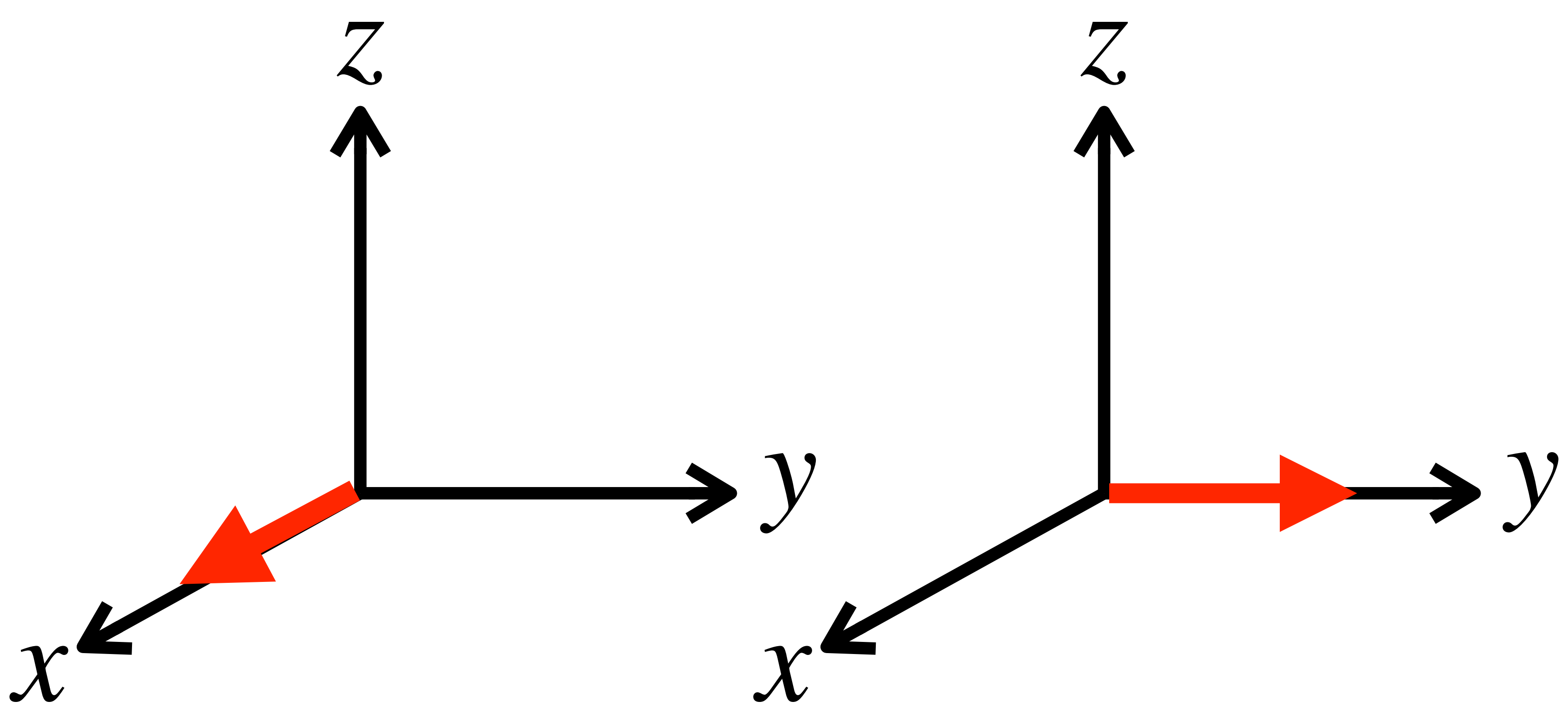


[K. Funo, H. Tajima, PRL **134**, 080401 (2025)]

How do we quantify the degree of symmetry breaking?

Introduction

$$\langle \sigma^x \rangle \neq 0 \qquad \langle \sigma^x \rangle = 0$$



- **Landau order parameter**
  expectation value of observable that nontrivially changes under the symmetry transformation
  : system-dependent, not faithful

- **Entanglement asymmetry** [F. Ares, et al., Nature Communications **14**, 2036 (2023)]

$$\Delta S = \mathrm{tr}[\rho\{\log(\rho) - \log(\mathscr{T}[\rho])\}]$$

: faithful, but logarithmically scales in system size
(ill-defined in thermodynamic limit)
difficult to measure in experiments

$$\lim_{V\to\infty} \frac{\Delta S}{V} = 0$$

Quantum Fisher information overcomes these limitations!

Quantum Fisher information

QFI of ρ with respect to U(1) symmetry G generated by X

$$F_\rho(X) = 2\sum_{i,j} \frac{(p_i - p_j)^2}{p_i + p_j} |\langle i|X|j\rangle|^2 \qquad \rho = \sum_i p_i |i\rangle\langle i|$$

- ***Faithfulness***: $F_\rho(X) \geq 0$ and $F_\rho(X) = 0$ iff $[\rho, X] = 0$
- ***Monotonicity***: $F_\rho(X) \geq F_{\mathscr{E}(\rho)}(X)$ $\forall \mathscr{E}$ s.t. $\mathscr{E}(e^{i\theta X} \cdot e^{-i\theta X}) = e^{i\theta X}\mathscr{E}(\,\cdot\,)e^{-i\theta X}$
- ***Additivity***: $F_{\rho^{\otimes n}}(X^{\otimes n}) = nF_\rho(X)$ (extensive and well defined in thermodynamic limit)

Quantum Fisher information

- $F_\rho(X) = -4\partial_\theta^2 \mathrm{Tr}[(\rho[e^{-i\theta X}\rho e^{i\theta X}]^{1/2}\rho)^{1/2}]$
- $F_\rho(X) = 4[\langle X^2\rangle - \langle X\rangle^2]$ (if ρ is pure)
- $F_\rho(X) = \frac{4}{\pi}\int_0^\infty d\omega\chi''(\omega,t)\tanh\left(\frac{\omega}{2T}\right)$ (if ρ is in thermal equilibrium)

$\chi(\omega,T) = i\int_0^\infty dt e^{-i\omega t}\mathrm{Tr}(\rho[X(t),X])$: dynamical susceptibility

QFI is computable and measurable

Applications

Application 1: BCS superconductors

$$H_{\rm BCS} = \sum_{\mathbf{k},\sigma} (\xi_{\mathbf{k}} - \mu) c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} (c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \Delta + \mathrm{H.c.})$$

$$\Delta = U \sum_{\mathbf{k}} \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle$$

- Ground state: $|\mathrm{BCS}\rangle = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}) |0\rangle$

  : U(1) particle-number symmetry generated by $N = \sum_{\mathbf{k}\sigma} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}$ is broken

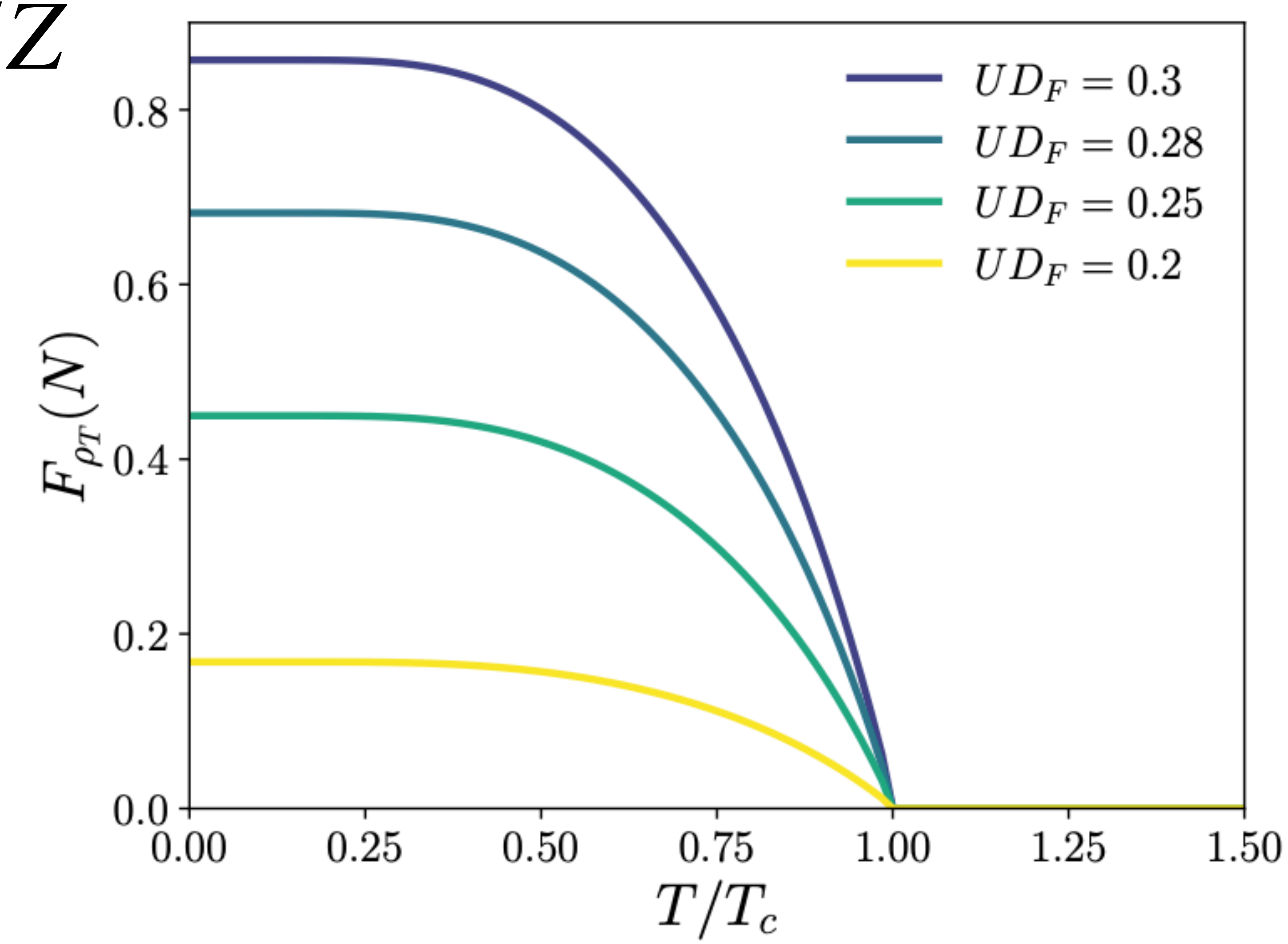
$$F_{|\mathrm{BCS}\rangle}(N) = 16 \sum_{\mathbf{k}} |\langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle|^2$$

QFI = number of Cooper pairs

Application 1: BCS superconductors

- Finite temperature → $\rho(T) = e^{-H_{\mathrm{BCS}}/T}/Z$

$$F_{\rho(T)}(N) = \sum_{\mathbf{k}} \frac{16\,|\langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}\rangle|^2}{\sqrt{P_{\mathbf{k}}(T)}}$$

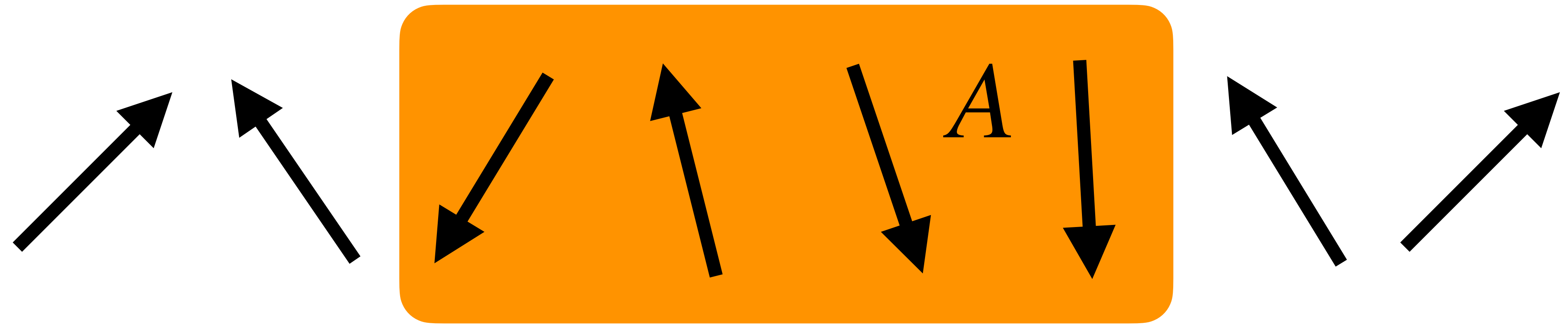


- $P_{\mathbf{k}}(T)$: purity of each mode ($\Pi_{\mathbf{k}} P_{\mathbf{k}}(T) = \mathrm{Tr}[\rho(T)^2]$)

→ ***QFI takes into account the effect of thermal decoherence on the symmetry breaking***

Application 2: QFI out of equilibrium

$$H(\eta) = -\sum_i \left[\frac{1+\eta}{4}\sigma_i^x\sigma_{i+1}^x + \frac{1-\eta}{4}\sigma_i^y\sigma_{i+1}^y + \frac{h}{2}\sigma_i^z\right]$$



- $\eta \neq 0 \rightarrow$ spin-rotational symmetry around z-axis is broken
- Quench dynamics from ground state with $\eta \neq 0$ into $\eta = 0$
- $|\psi(t)\rangle = e^{-itH(0)}|g(\eta)\rangle$: no relaxation, no symmetry restoration
- $\rho_A(t) = \mathrm{Tr}_{\bar{A}}(|\psi(t)\rangle\langle\psi(t)|)$ relaxes into the (generalized) Gibbs ensemble of $H(\eta = 0)$, which respects the rotational symmetry around z-axis
  $\rightarrow$ ***The U(1) rotational symmetry is restored at the subsystem level***

Application 2: QFI out of equilibrium

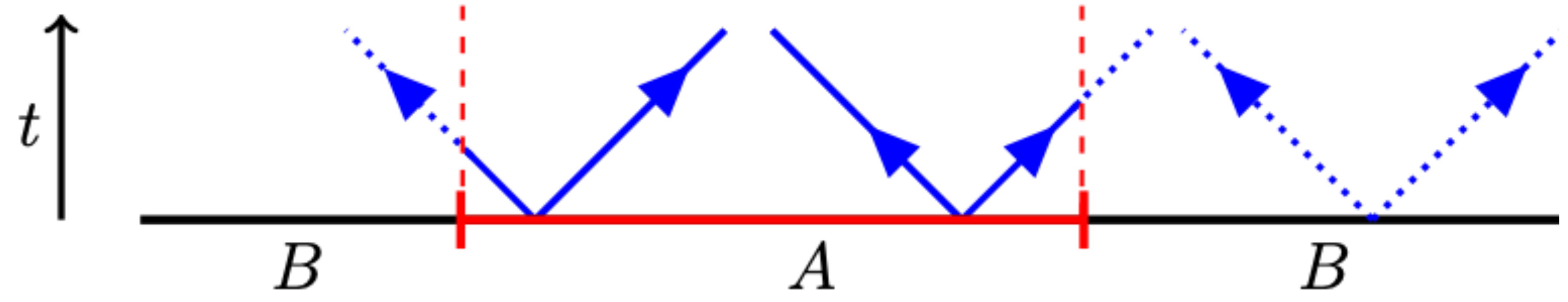
- Jordan-Wigner transform: $c_i = (\Pi_{j=1}^{i-1}\sigma_j^z)(\sigma_i^x + i\sigma_i^y)/2$

$$f_{\rho_A}(M_A) = 8\int_{-\pi}^{\pi}\frac{dk}{2\pi}|\langle g(\eta)|c_{-k}c_k|g(\eta)\rangle|^2 \max(1-2|v_k|t/L_A, 0)$$

($v_k = \sin k$: group velocity of fermions)

Mode occupation of Cooper pairs in the initial state

Fraction of the propagating Cooper pairs inside A



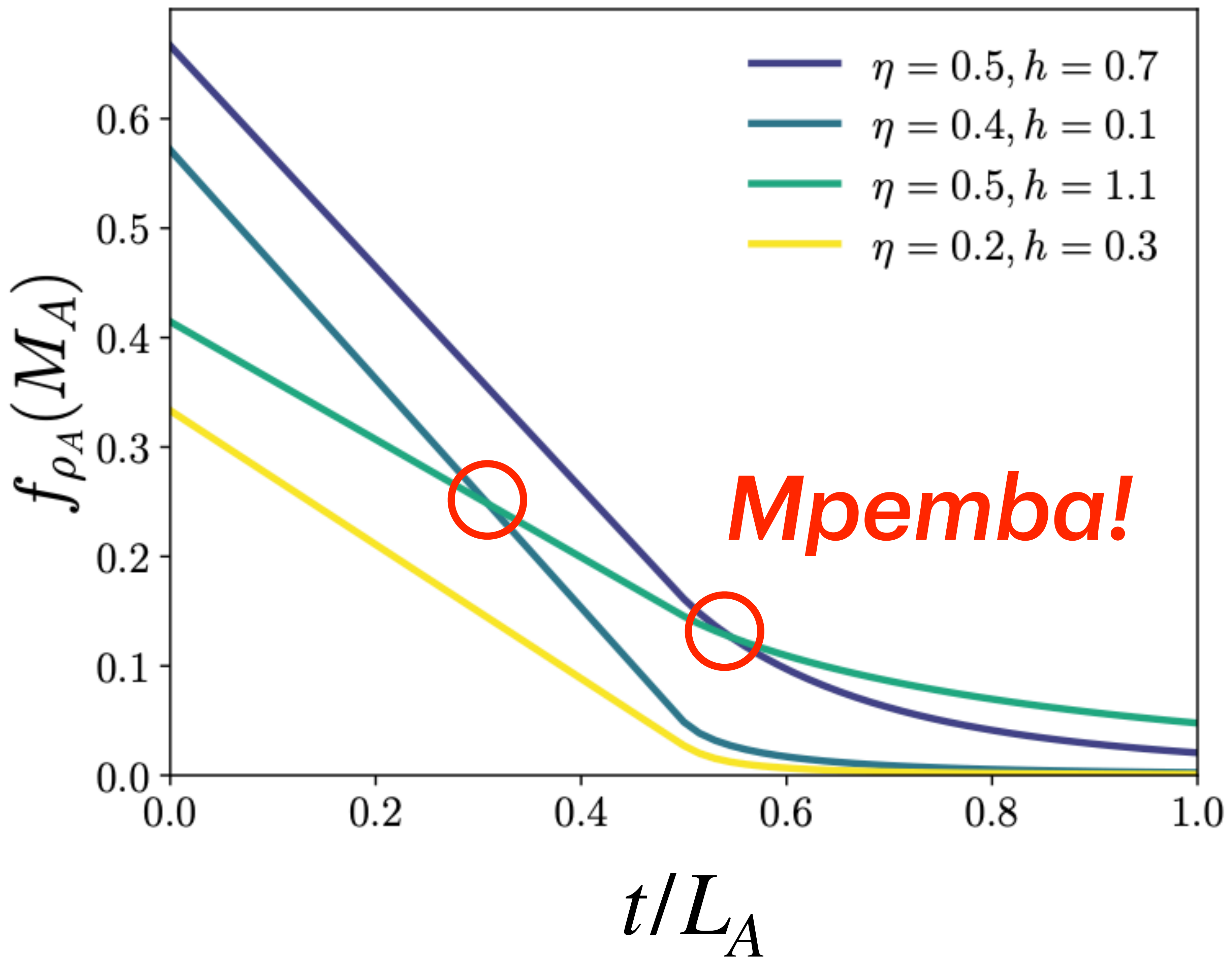
- After the quench, Cooper pairs propagate with velocities $\pm v_k$

- Only the pairs within A contribute to the symmetry breaking

→ ***QFI = # of Cooper pairs in the system of interest, even out of equilibrium!***

Application 2: QFI out of equilibrium

- The QFI decreases as the pairs escape from A
- The initial state that breaks more the symmetry can restore it faster (Quantum Mpemba effect)
- More the total number of Cooper pairs, larger the QFI at $t = 0$
- Smaller the slow-moving pairs, lower the QFI at large times



→QFI tells us that the Quantum Mpemba effect occurs when the state has more Cooper pairs but smaller fraction of slow-moving pairs

Summary

- The QFI has desired properties as a measure of symmetry breaking: (Faithful, computable, measurable, extensive, etc.)
- We applied the QFI to symmetry breaking phenomena such as BCS superconductors and quantum Mpemba effect in spin chains.

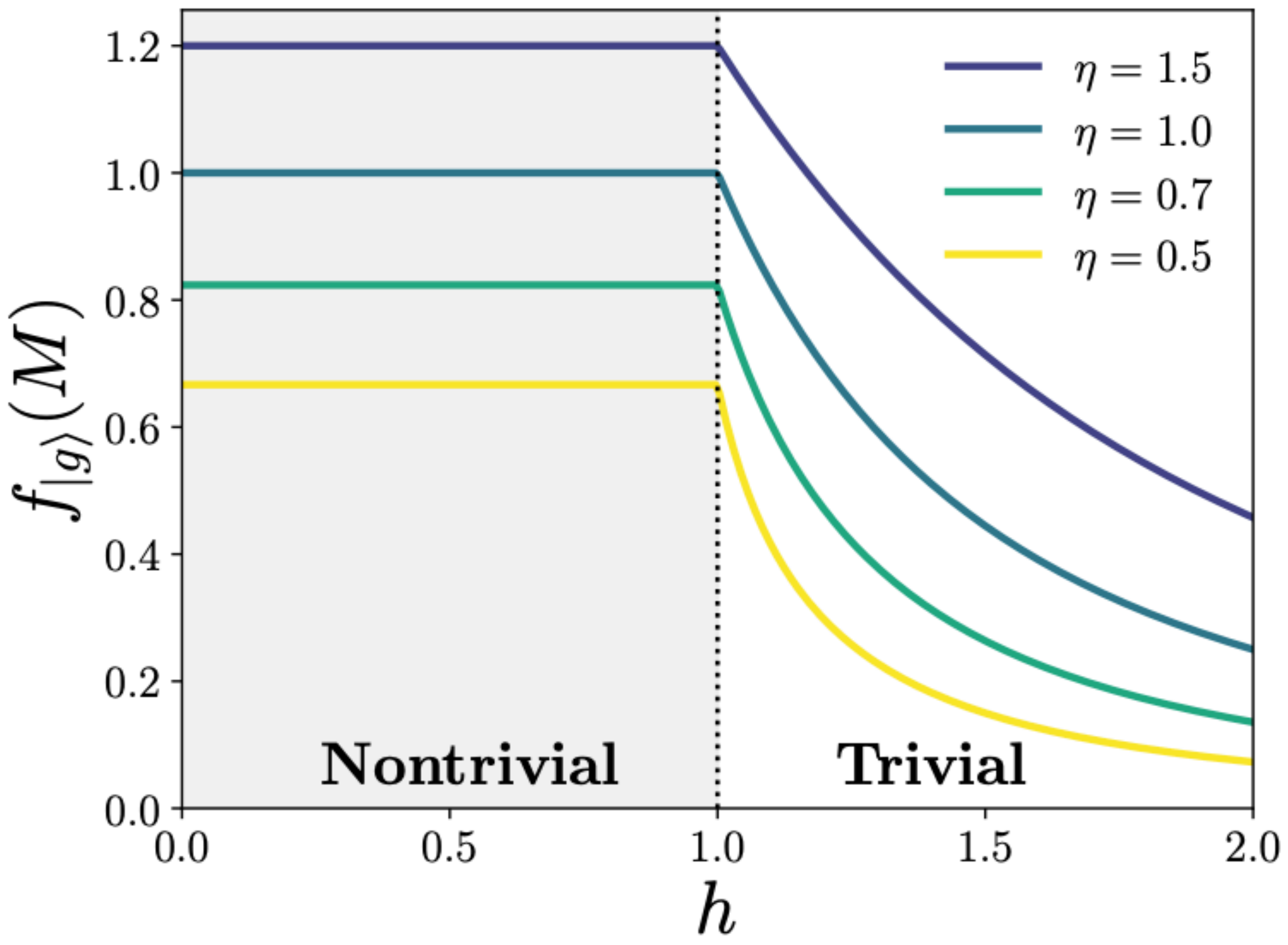
Thank you very much for your attentions!

Application 2: Topological transition

$$H_{\mathrm{XY}} = -\sum_i \left[\frac{1+\eta}{4}\sigma_i^x\sigma_{i+1}^x + \frac{1-\eta}{4}\sigma_i^y\sigma_{i+1}^y + \frac{h}{2}\sigma_i^z\right]$$

- Topological phase transition at $|h| = 1$ (no Landau order parameter)
- QFI of g.s. for rotational symmetry generated by $M = \Sigma_i \sigma_i^z$

$$\lim_{L\to\infty}\frac{F_{|g_{\mathrm{XY}}\rangle}(M)}{L} = \begin{cases} \dfrac{2\eta}{1+\eta} & |h| \leq 1 \\ \dfrac{2\eta}{1-\eta^2}\left(\dfrac{|h|}{h^2+\eta^2-1} - 1\right) & |h| > 1 \end{cases}$$



→ ***QFI is sensitive to topological phase transition***

Introduction

We want a measure of symmetry breaking that satisfies

- ***Faithful***
- ***System-independent***
- ***Computable***
- ***Measurable***
- ***Well defined in thermodynamic limit***

A. Quantum Fisher Information