Mpemba effect in bistable continuous potentials

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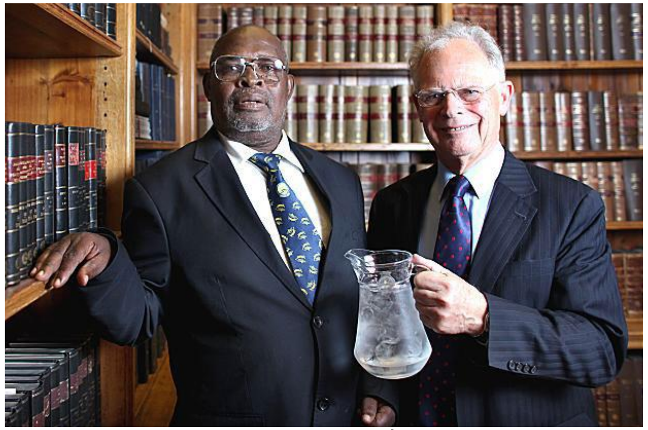
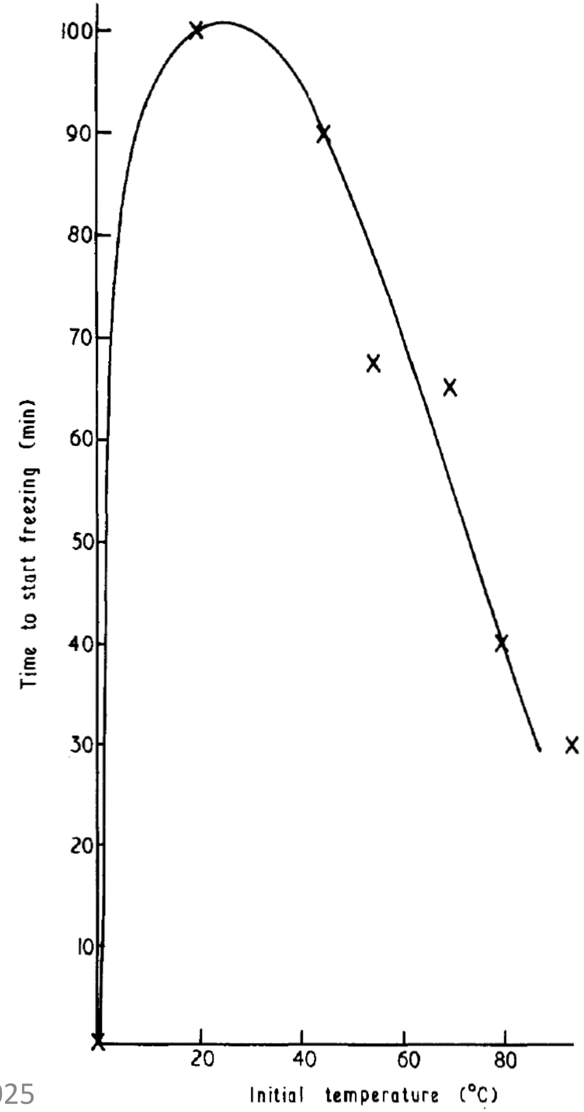
What is the Mpemba effect?

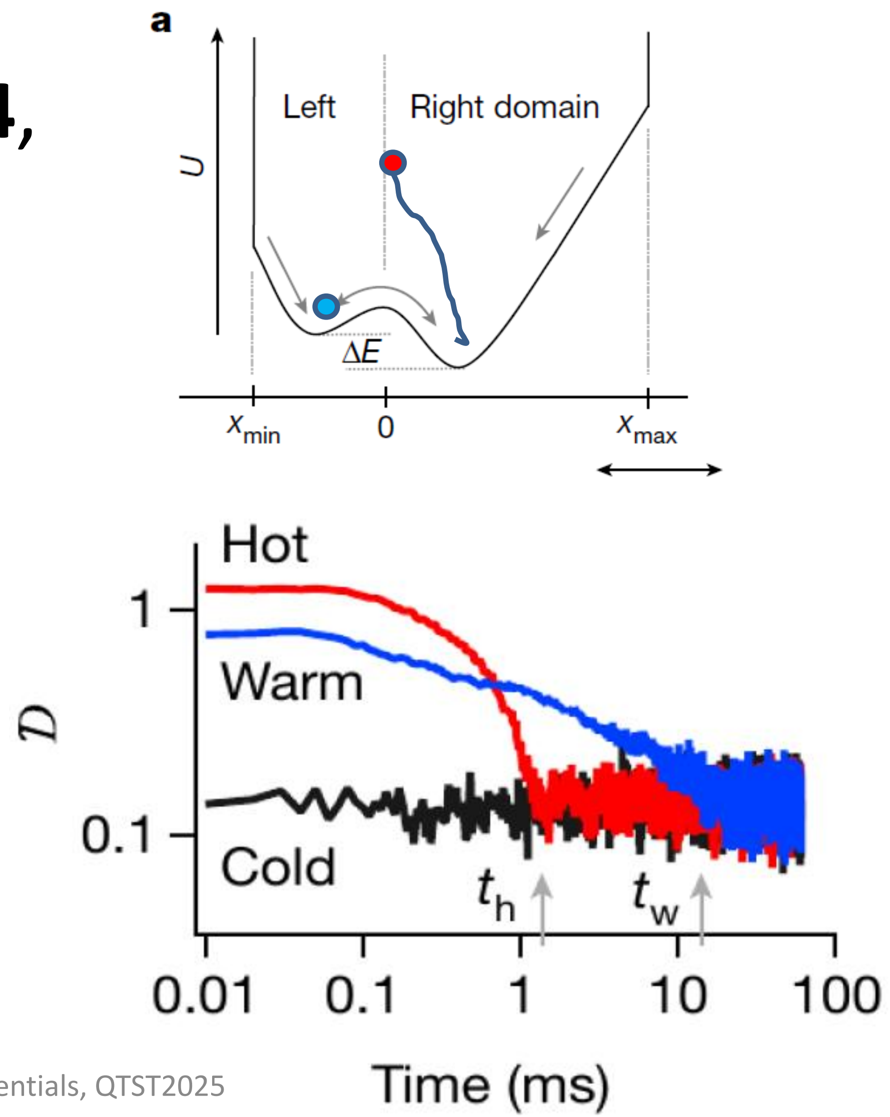
- Erasto B. Mpemba found that some hot suspensions (*ice cream mix*) can freeze faster than cold (1963).
- Mpemba & D. G. Osborne published a paper (1969).



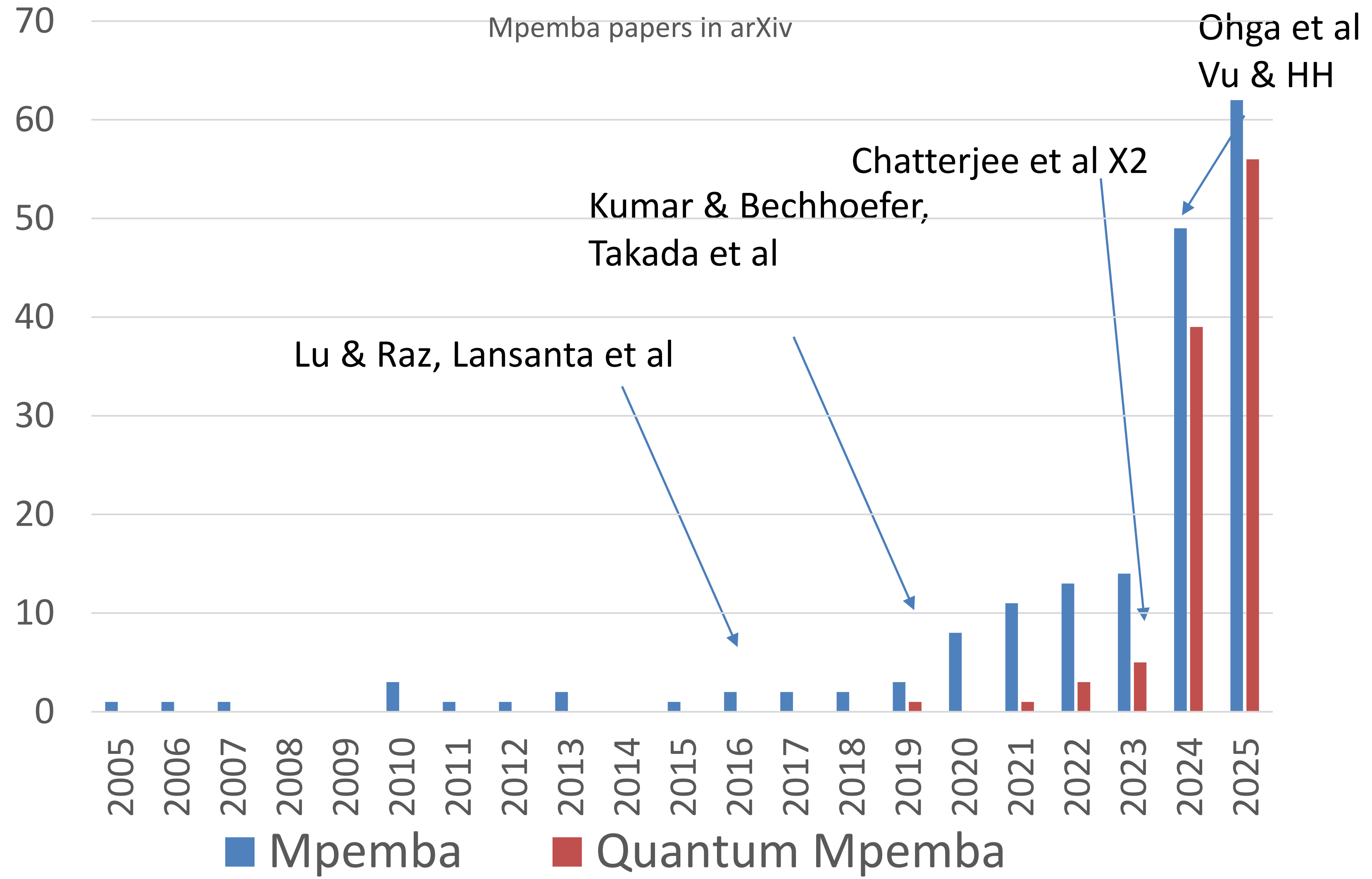


Experimental confirmation

- Kumar & Bechhoefer, Nature **584**, 64 (2020).
- They have analyzed trapped colloids in a double well potential.
- They observed the distance between the distribution and equilibrium one.



Papers on Mpemba effect in arXiv (December 5th)



Analysis of the Mpemba

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	Two equilirium initial conditions	Control of initial conditions
Classical	✓ Vu & Hayakawa Ohga et al	✓ Takada et al
Quantum	?	✓ Yamashika's talk (Chatterjee et al x2)

The second scenario (Lu & Raz 2017)

- We use the Fokker-Planck equation in a potential $V(x)$:

$$\partial_t p = \mathbb{W}(\beta)p, \qquad \mathbb{W}(\beta) := \partial_x(V'(x) + T\partial_x)$$

- Because FP eq. with detailed balance does not have any degeneracy, we can write

$$p(x,\beta;t) = \sum_n \mathrm{e}^{-\lambda_n t} r_n(x,\beta) a_n$$

$$\boxed{a_n := \int \mathrm{d}x' \ell_n(x',\beta) p_i(x',\beta_i)}$$

l_n and r_n are the left and right eigenvectors.

$$p(x,t) \approx p_{\mathrm{eq}}(x,\beta) + \mathrm{e}^{-\lambda_2 t} r_2(x) a_2(\beta_i,\beta), \qquad a_2(\beta_i,\beta) := \int \mathrm{d}x' \ell_2(x',\beta) p_i(x')$$

$$p_i := e^{-\beta_i V}/Z(\beta_i)$$

Continuous potential

- We can use the mapping onto imaginary time Schrödinger equation with

$$g(x,t) := e^{-\beta V(x)/2} p(x,t)$$

$$T\partial_t g(x,t) = [T^2\partial_x^2 - U(x)]g(x,t) = T\mathbb{L}g(x,t)$$

Effective potential $U(x) := \frac{1}{4}V'(x)^2 - \frac{1}{2}TV''(x)$

$$\mathbb{L}\varphi_n(x) = -\lambda_n\varphi_n(x)$$

$$r_n(x) = e^{-\frac{\beta V(x)}{2}}\varphi_n(x)$$

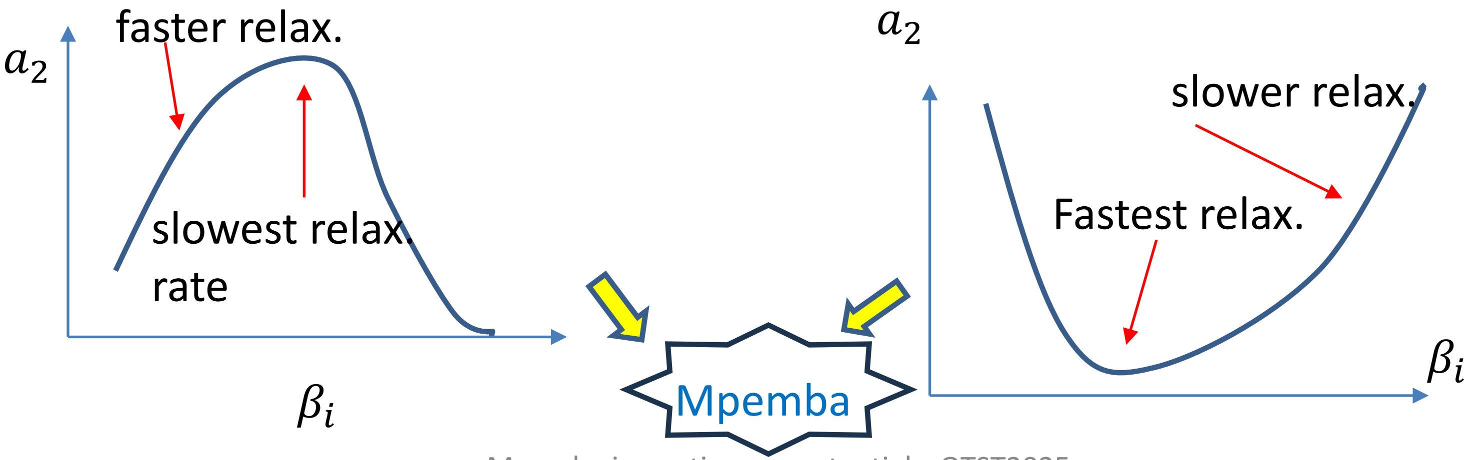
$$l_n(x) = e^{\frac{\beta V(x)}{2}}\varphi_n(x)$$

Observable dynamics & condition of Mpemba effect

- An observable A obeys:

$$\langle A\rangle(t) = \langle A\rangle_{\mathrm{eq}} + \sum_n \mathrm{e}^{-\lambda_n t} a_n(\beta_i, \beta) \int \mathrm{d}x r_n(x) A(x)$$

- To observe the Mpemba condition a_2 with small n should have a maximum or a minimum.



Triple harmonic potential

- If we adopt a triple-harmonic potential, the effective potential is still harmonic.

$$V(x) = \begin{cases} V_{\mathrm{I}}(x) := \frac{k_{\mathrm{I}}}{2}(x+1)^2, & \text{if } x < x_- \\ V_{\mathrm{II}}(x) := V_M - \frac{k_{\mathrm{II}}}{2}x^2, & \text{if } x_- \leq x \leq x_+, \\ V_{\mathrm{III}}(x) := -V_m + \frac{k_{\mathrm{III}}}{2}(x-\alpha)^2, & \text{if } x > x_+, \end{cases}$$

⇩

$$U(x) = \begin{cases} U_{\mathrm{I}}(x) := \frac{k_{\mathrm{I}}^2}{4}(x+1)^2 - T\frac{k_{\mathrm{I}}}{2}, & \text{if } x < x_- \\ U_{\mathrm{II}}(x) := \frac{k_{\mathrm{II}}^2}{4}x^2 + \frac{T}{2}k_{\mathrm{II}}, & \text{if } x_- \leq x \leq x_+, \\ U_{\mathrm{III}}(x) := \frac{k_{\mathrm{III}}^2}{4}(x-\alpha)^2 - \frac{T}{2}k_{\mathrm{III}}, & \text{if } x > x_+, \end{cases}$$

Solution of the triple harmonic model

- Eigenvalues and approximate eigenfunctions

$$\varphi_n^{\mathrm{I}}(x) = c_1^{\mathrm{I}} H_{n-1}(\zeta_{\mathrm{I}}) e^{-\zeta_{\mathrm{I}}^2/2} \quad (x < x_-; \quad n \geq 1),$$

$$\varphi_n^{\mathrm{II}}(x) = c_1^{\mathrm{II}} H_{n-1}(\zeta_{\mathrm{II}}) e^{-\zeta_{\mathrm{II}}^2/2} + c_2^{\mathrm{II}} e^{-\zeta_{\mathrm{II}}^2/2} {}_1F_1\left[\frac{n}{2}, \frac{1}{2}; \zeta_{\mathrm{II}}^2\right], \qquad (x_- \leq x \leq x_+; \quad n$$

$$\varphi_n^{\mathrm{III}}(x) = c_1^{\mathrm{III}} H_{n-1}(\zeta_{\mathrm{III}}) e^{-\zeta_{\mathrm{III}}^2/2} \quad (x > x_+; \quad n \geq 1),$$

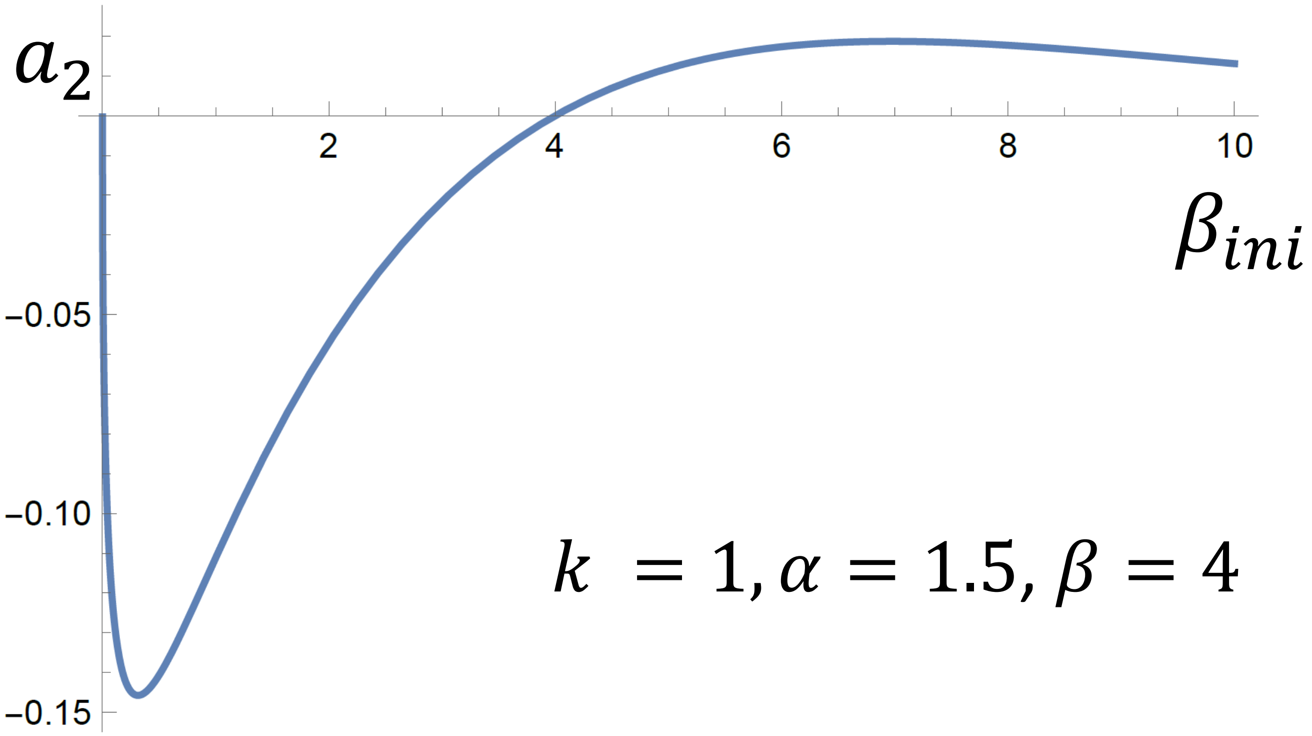
$$\lambda_n^{\mathrm{I}} \to k_{\mathrm{I}}(n-1), \quad \lambda_n^{\mathrm{II}} \to k_{\mathrm{II}}(n-1), \quad \lambda_n^{\mathrm{III}} \to k_{\mathrm{III}}(n-1),$$ ⇨ $k_I = k_{II} = k_{III}$

$\xi_{\mathrm{I}} := \sqrt{k_{\mathrm{I}}/T}(x+1)$, $\xi_{\mathrm{II}} := \sqrt{k_{\mathrm{II}}/T}x$, and $\xi_{\mathrm{III}} := \sqrt{k_{\mathrm{III}}/T}(x-\alpha)$

$H_m(x)$ is the Hermite polynomial

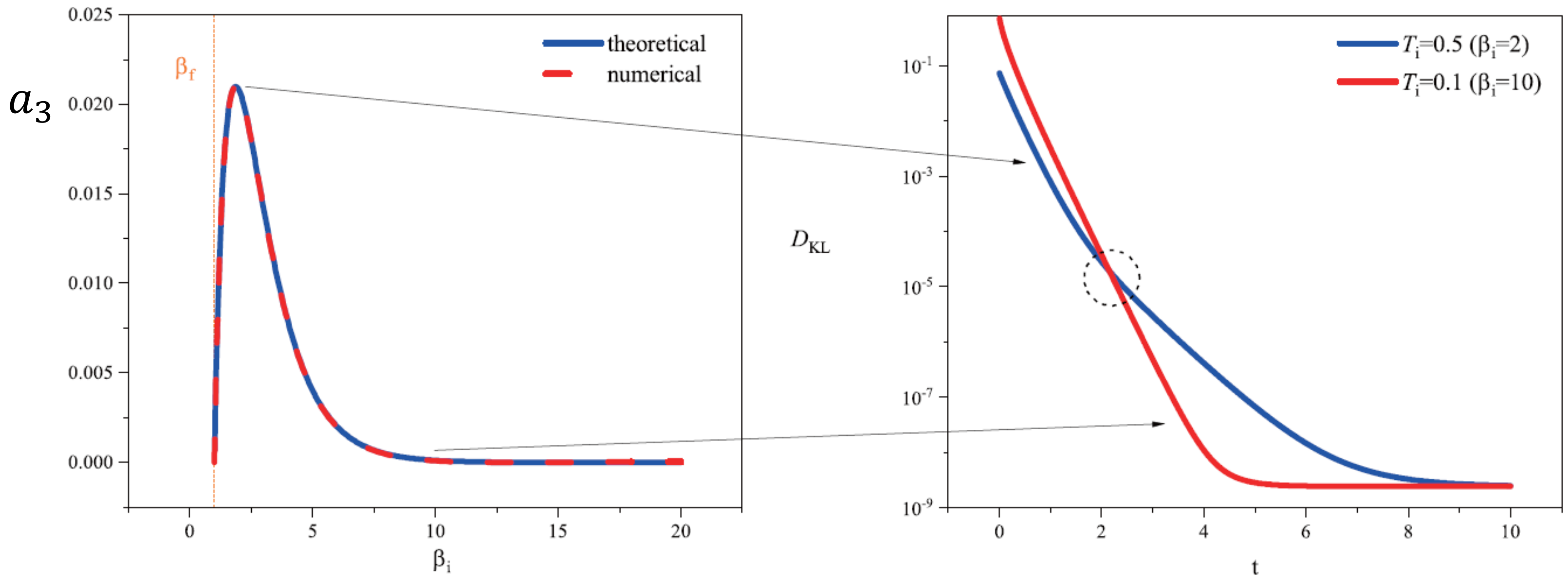
Results (1)

- We find there is the minimum for a_2 if $k_{\mathrm{I}} > 1$.



Symmetric case (Yue Liu)

- So far, we analyzed relaxation in an asymmetric potential.
- The symmetric potential also exhibits the Mpemba effect, although $a_2 = 0$ but a_3 has a maximum or a minimum.



Two-dimensional case

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- We can use a parallel argument in 2D problems.

$$U(r) = \begin{cases} U_{\rm in}(r) = \frac{a_{\rm in}}{2} r^2 + C_{\rm in}, & 0 \le r < r_-, \\ U_{\rm mid}(r) = \frac{a_{\rm mid}}{2} r^2 + b_{\rm mid} \ln r + C_{\rm mid}, & r_- \le r < r_+, \\ U_{\rm out}(r) = \frac{a_{\rm out}}{2} r^2 + b_{\rm out} \ln r + C_{\rm out}, & r \ge r_+, \end{cases}$$

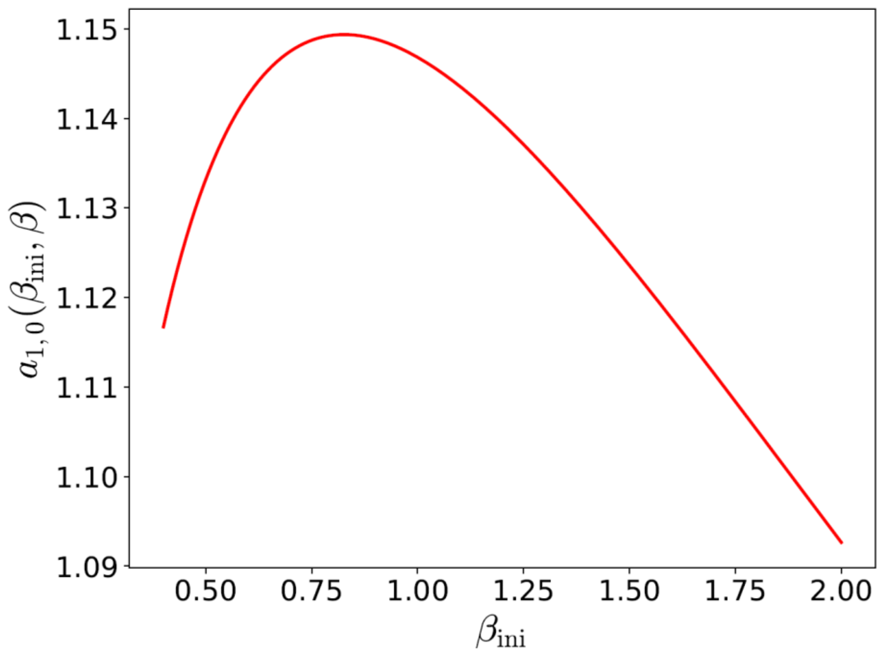
$$V_S(r) = \frac{1}{4T}\left(U'(r)\right)^2 + \frac{1}{2r}\frac{d}{dr}\left(rU'(r)\right).$$

$$V_S(r) = \begin{cases} a_{\rm in} + \dfrac{a_{\rm in} b_{\rm in}}{2T} + \dfrac{a_{\rm in}^2}{4T} r^2 + \dfrac{b_{\rm in}^2}{4T r^2}, & 0 \le r < r_-, \\ a_{\rm mid} + \dfrac{a_{\rm mid} b_{\rm mid}}{2T} + \dfrac{a_{\rm mid}^2}{4T} r^2 + \dfrac{b_{\rm mid}^2}{4T r^2}, & r_- \le r < r_+, \\ a_{\rm out} + \dfrac{a_{\rm out} b_{\rm out}}{2T} + \dfrac{a_{\rm out}^2}{4T} r^2 + \dfrac{b_{\rm out}^2}{4T r^2}, & r \ge r_+. \end{cases}$$

Eigenmode analysis

$$P(r,t) = \sum_m a_{m,0}\varphi_{m,0}(r)e^{-\beta U(r)/2-\lambda_{m,0}t} \approx a_{1,0}\varphi_{1,0}(r)e^{-\beta U(r)/2-\lambda_{1,0}t},$$

$$a_{m,n} = \frac{2\pi\delta_{n0}}{Z(\beta_{\rm ini})}\int_0^\infty dr r \exp\left[\left(\frac{\beta}{2}-\beta_{\rm ini}\right)U(r)\right]\varphi_{m,0}(r),$$



$T = 1$, $a_{\rm in} = 1.0$, $a_{\rm mid} = -0.5$, $a_{\rm out} = 0.8$, $\xi = 1.0$, and $\alpha = 3.0$

Summary

- We have solved two bistable potential models using Fokker-Planck equation with exact and simple analytic calculation.
  - 1D bistable case and 2D bistable case
  - Even relaxation in a symmetric potential exhibits the Mpemba effect.
  - The relaxation modes can be obtained exactly.
- Single well potential also exhibits the Mpemba effect.
- Effect of disorders in potentials?=> This will be important.
- So far, nobody has argued the quantum Mpemba effect starting from two equilibrium conditions.=> This will be an important problem.

Acknowledgment

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- Collaborators





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Thank you for your attention.