

Thermodynamics based on optimal transport

**Sosuke Ito, Universal Biology Institute, the University of Tokyo
Kyoto workshop on quantum thermodynamics and stochastic thermodynamics 2025**

2025.12.9



Collaborators

Thermodynamics based on optimal transport

Review: SI, *Information Geometry* 7, Suppl 1, 441-483 (2024).

M. Nakazato and SI, *Phys. Rev. Res.* 3, 043093 (2021).

A. Dechant, S-I Sasa and SI, *Phys. Rev. Res.* 4, L012034 (2022).

A. Dechant, S-I Sasa and SI, *Phys. Rev. E.* 106, 024125 (2022).

K. Yoshimura, A. Kolchinsky, A. Dechant and SI, *Phys. Rev. Res.* 5, 013017 (2023).

Y. Fujimoto and SI, *Phys. Rev. Res.* 6, 013023 (2024).

K. Yoshimura and SI, *Phys. Rev. Res.* 6, L022057 (2024).

D. Sekizawa, SI, M. Oizumi, *Phys. Rev. X* 14, 041003 (2024).

A. Kolchinsky, A. Dechant, K. Yoshimura and SI, arXiv:2206.14599. arXiv:2412.08432.

R. Nagayama, K. Yoshimura, A. Kolchinsky and SI, *Phys. Rev. Res.* 7, 033011 (2025).

K. Yoshimura, Y. Maekawa, R. Nagayama and SI, *Phys. Rev. Res.* 7, 013244 (2025).

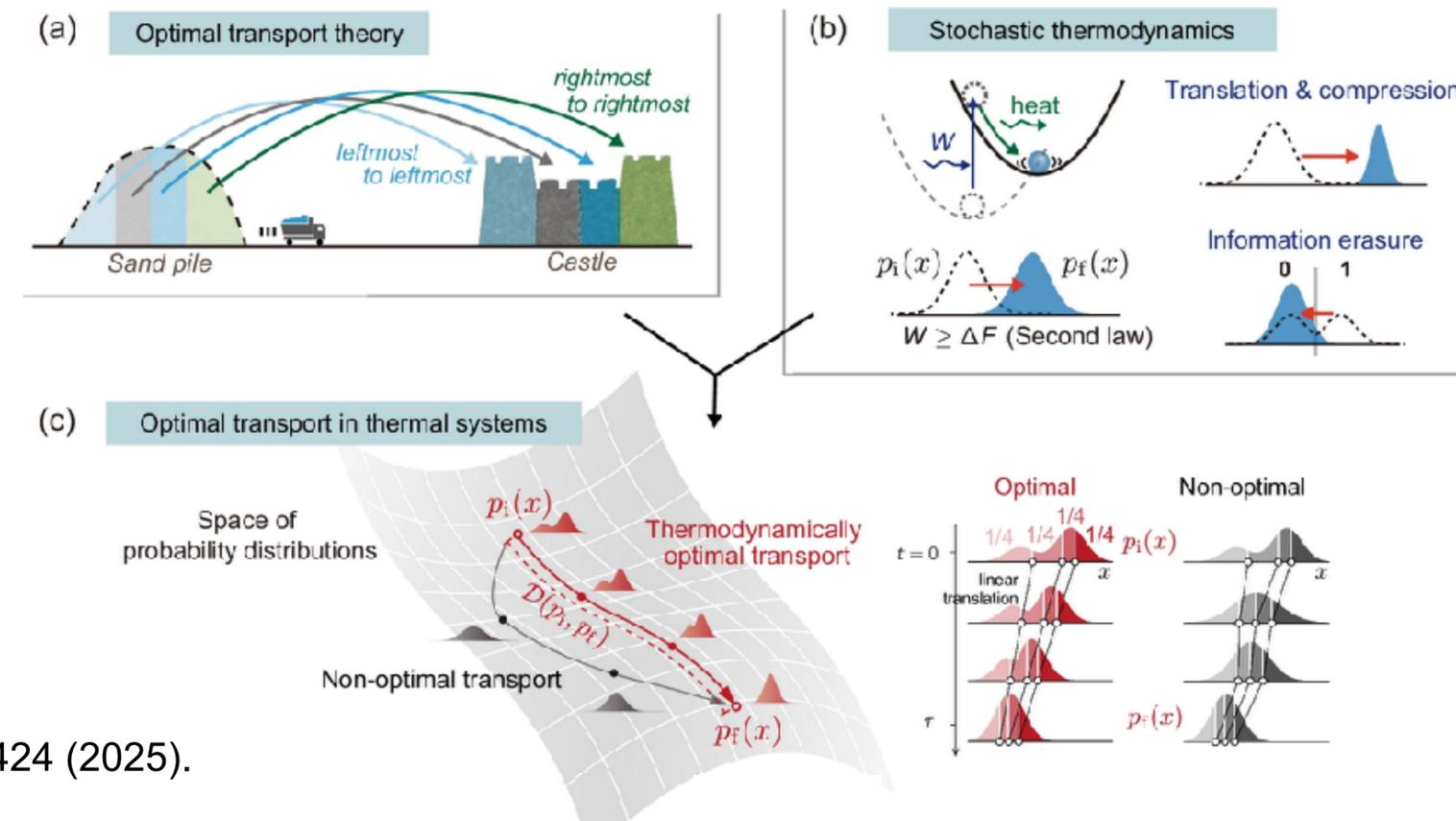
K. Ikeda, T. Uda, D. Okanohara and SI, *Phys. Rev. X*, 15, 031031 (2025).

R. Nagayama, K. Yoshimura and SI, *Phys. Rev. Res.* 7, 013307 (2025).

S. Oikawa, Y. Nakayama, SI, T. Sagawa and S. Toyabe, *Nature Communications* 16, 10424 (2025).

Y. Maekawa, K. Yoshimura, R. Nagayama and SI, arXiv: 2509.21985 (2025).

D. Sekizawa, SI, M. Oizumi, arXiv:2510.21340 (2025).



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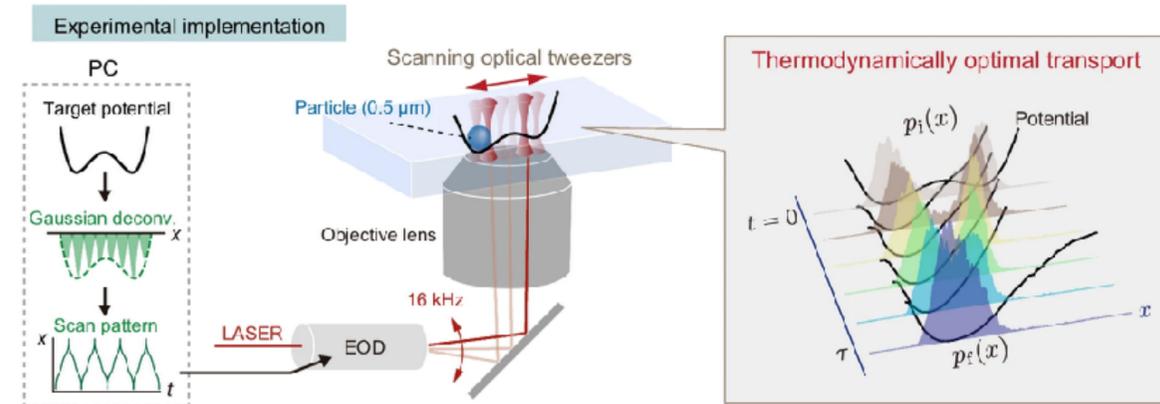
Artemy Kolchinsky (UTokyo⇒Ponpeu Fabra U), **Ryuna Nagayama (UTokyo)**, **Yoh Maekawa (UTokyo)**, Kotaro Ikeda (UTokyo), Tomoya Uda (UTokyo)

Andreas Dechant (KyotoU), Shin-ichi Sasa (KyotoU), Daisuke Okanohara (Preferred Networks), Daiki Sekizawa (UTokyo), Masafumi Oizumi (UTokyo), Shingo Oikawa (Tohoku U), Yohei Nakayama (Tohoku U), Takahiro Sagawa (UTokyo), Shoichi Toyabe (Tohoku U)

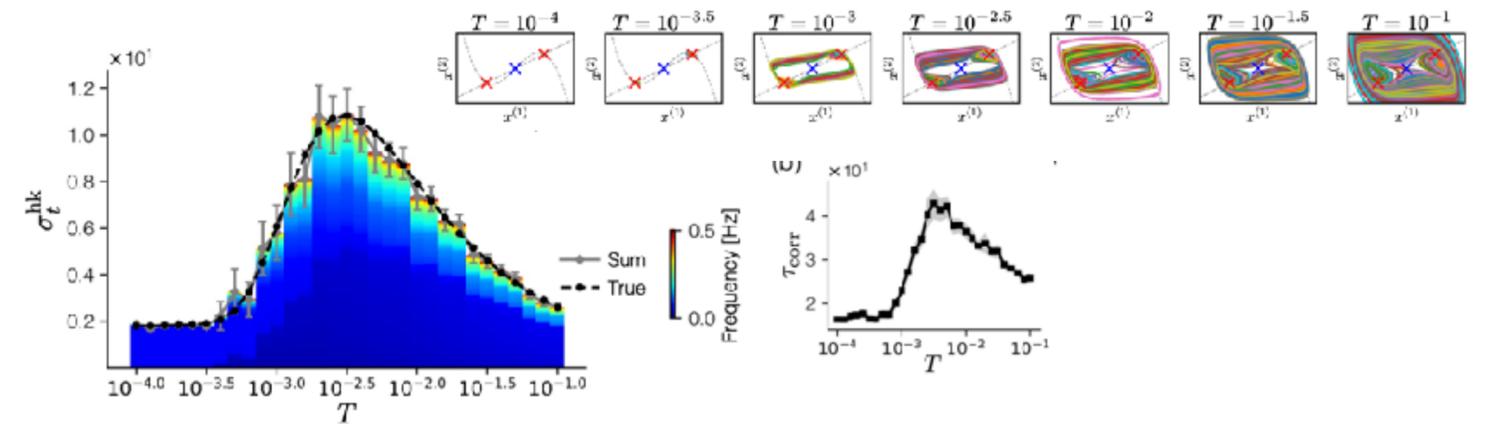
Contents

Introduction: **The thermodynamic framework based on optimal transport**
 see SI, *Information Geometry* 7, Suppl 1, 441-483 (2024). ...etc.

Experiment achieving minimal dissipation:
 S. Oikawa, Y. Nakayama, SI, T. Sagawa and
 S. Toyabe, *Nature Communications* 16, 10424 (2025).



Koopman mode decomposition of dissipation:
 D. Sekizawa, SI, M. Oizumi, arXiv:2510.21340 (2025),



Quantum generalization of the thermodynamic framework:
 K. Yoshimura, Y. Maekawa, R. Nagayama and SI. *Phys. Rev. Res.* 7, 013244 (2025).

	Classical	Quantum
Gradient	∇	∇_L
Continuity	$dx/dt = \nabla^T J(x)$	$\mathcal{D}(\rho) = \nabla_L^T J(\rho)$
Current	$J_e = J_e^+ - J_e^-$	$J(\rho) = [L, \Gamma \otimes \rho]$
Force	$F_e = \ln(J_e^+ / J_e^-)$	$\mathbb{F}(\rho) = [L, \ln(\Gamma \otimes \rho)]$
Conservative	$F = -\nabla\phi$	$\mathbb{F}(\rho) = -\nabla_L\phi$
Detailed balance	$J(x^{eq}) = 0$	$J(\rho^{eq}) = 0$
EPR	$J^T F$	$\text{tr}(J(\rho)^T \mathbb{F}(\rho))$

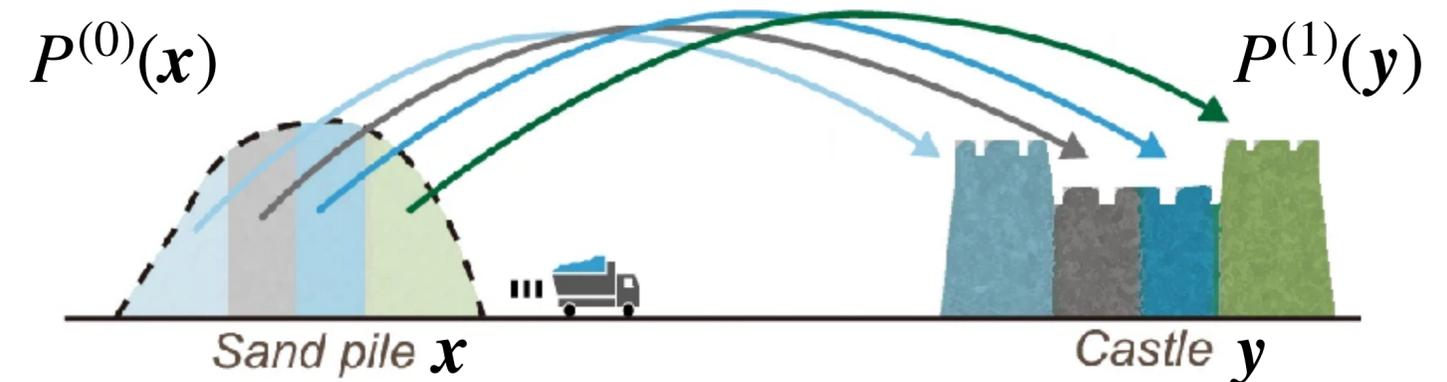
Introduction: Optimal transport problem

Optimal transport problem
[2-Wasserstein distance]

Villani, C. (2009). *Optimal transport: old and new* (Vol. 338, p. 23). Berlin: Springer.

$$\mathcal{W}_2(P^{(0)}, P^{(1)}) = \sqrt{\inf_{\pi} \int dx \int dy \pi(x, y) \|x - y\|^2}$$

$$\text{s.t. } \pi(x, y) \geq 0, \quad \int dy \pi(x, y) = P^{(0)}(x), \quad \int dx \pi(x, y) = P^{(1)}(y)$$



Axioms of a metric

① $\mathcal{W}_2(P^{(0)}, P^{(1)}) \geq 0, \mathcal{W}_2(P^{(0)}, P^{(1)}) = 0 \Leftrightarrow P^{(0)} = P^{(1)}$

② $\mathcal{W}(P^{(0)}, P^{(1)}) = \mathcal{W}(P^{(1)}, P^{(0)})$ ③ $\mathcal{W}(P^{(0)}, P^{(2)}) \leq \mathcal{W}(P^{(0)}, P^{(1)}) + \mathcal{W}(P^{(1)}, P^{(2)})$

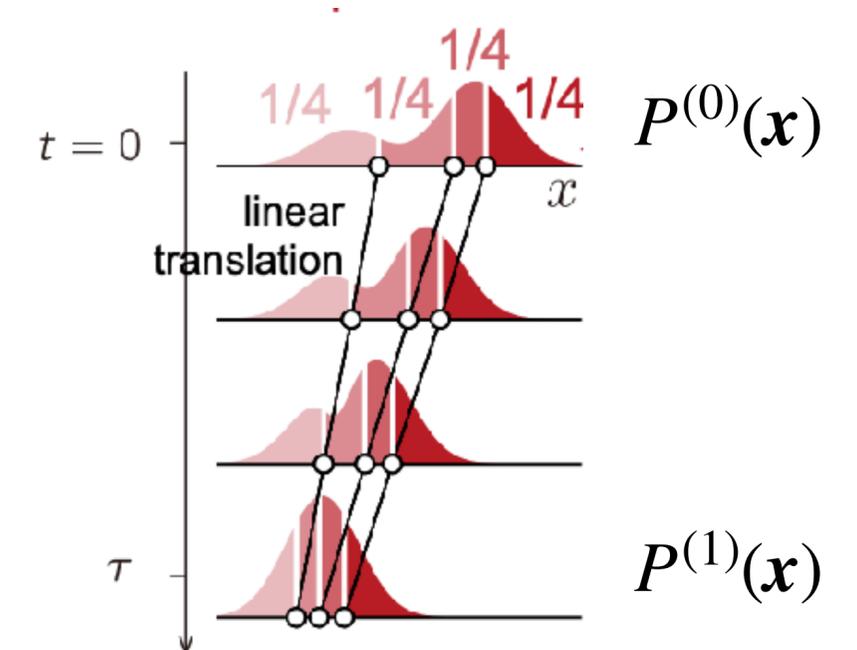
Introduction: Benamou-Brenier formula

Benamou-Brenier formula

J-D. Benamou & Y. Brenier. *Numerische Mathematik* 84, 375-393 (2000).

$$\mathcal{W}_2(P^{(0)}, P^{(1)}) = \sqrt{\inf_{(P'_t, \mathbf{u}'_t)_{0 \leq t \leq \tau}} \tau \int_0^\tau dt \int d\mathbf{x} \|\mathbf{u}'_t(\mathbf{x})\|^2 P'_t(\mathbf{x})}$$

$$\text{s. t. } \partial_t P'_t(\mathbf{x}) = -\nabla \cdot (\mathbf{u}'_t(\mathbf{x}) P'_t(\mathbf{x})), \quad P'_0(\mathbf{x}) = P^{(0)}(\mathbf{x}), \quad P'_\tau(\mathbf{x}) = P^{(1)}(\mathbf{x})$$



Optimal solution

$$\mathbf{u}_t^{\text{opt}}(\mathbf{x}) = \nabla \phi_t^{\text{opt}}(\mathbf{x}), \quad \partial_t \phi_t^{\text{opt}}(\mathbf{x}) = -\frac{1}{2} \|\nabla \phi_t^{\text{opt}}(\mathbf{x})\|^2$$

$$\partial_t P_t^{\text{opt}}(\mathbf{x}) = -\nabla \cdot (\mathbf{u}_t^{\text{opt}}(\mathbf{x}) P_t^{\text{opt}}(\mathbf{x})), \quad P_0^{\text{opt}}(\mathbf{x}) = P^{(0)}(\mathbf{x}), \quad P_\tau^{\text{opt}}(\mathbf{x}) = P^{(1)}(\mathbf{x})$$

Introduction:

Stochastic thermodynamics and optimal transport

Fokker-Planck equation $(k_B = 1)$

$$\partial_t P_t(\mathbf{x}) = -\nabla \cdot (\boldsymbol{\nu}_t(\mathbf{x}) P_t(\mathbf{x}))$$

$$\boldsymbol{\nu}_t(\mathbf{x}) = \mu \mathbf{F}_t(\mathbf{x}) - \mu T \nabla \ln P_t(\mathbf{x})$$

Entropy production rate

$$\dot{\Sigma}_t := \frac{1}{\mu T} \int d\mathbf{x} P_t(\mathbf{x}) \|\boldsymbol{\nu}_t(\mathbf{x})\|^2$$

Minimal dissipation (speed limit)

E. Aurell, et al. *Journal of statistical physics* 147, 487-505 (2012).

M. Nakazato and SI. *Phys. Rev. Res.* 3, 043093 (2021).

$$\left(\frac{w_d}{T} := \right) \int_0^\tau dt \dot{\Sigma}_t \geq \frac{[\mathcal{W}_2(P_0, P_\tau)]^2}{\mu T \tau} \quad \therefore \text{Benamou-Brenier formula}$$

w_d : Dissipated work

($w_d = W - \Delta F$ if the initial state and the final state are equilibrium states.)

Optimal protocol to achieve minimal dissipation:

1. Conservative force: $F_t(\mathbf{x}) = -\nabla U_t(\mathbf{x})$

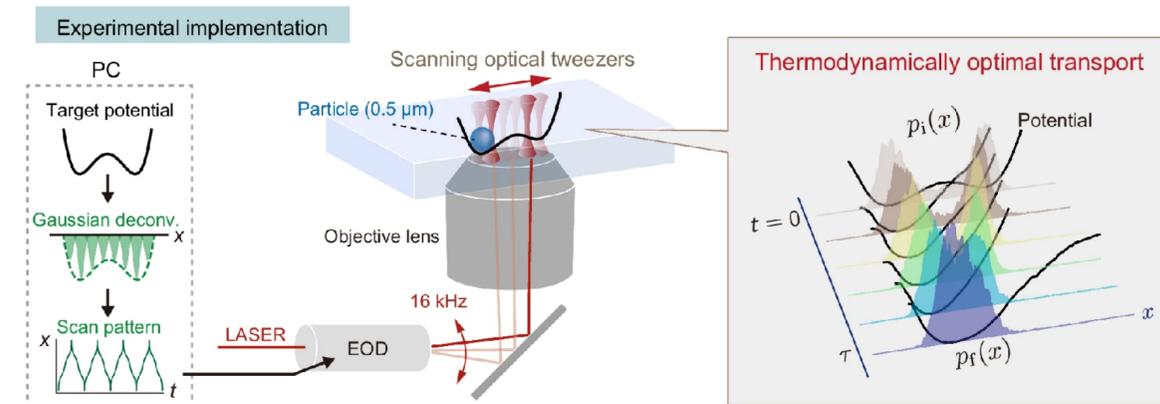
2. Geodesic: $\lim_{\Delta t \rightarrow 0} \frac{\mathcal{W}_2(P_t, P_{t+\Delta t})}{\Delta t} = \frac{\mathcal{W}_2(P_0, P_\tau)}{\tau} = \text{const.}$

Contents

The thermodynamic framework: Review) SI, *Information Geometry* 7, Suppl 1, 441-483 (2024)...etc.

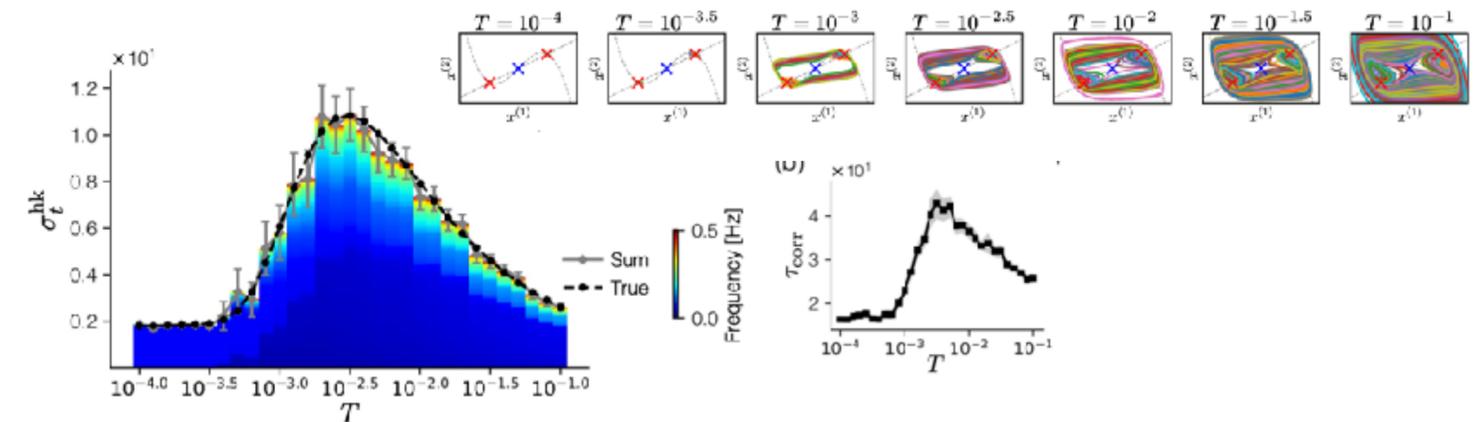
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Koopman mode decomposition of dissipation:

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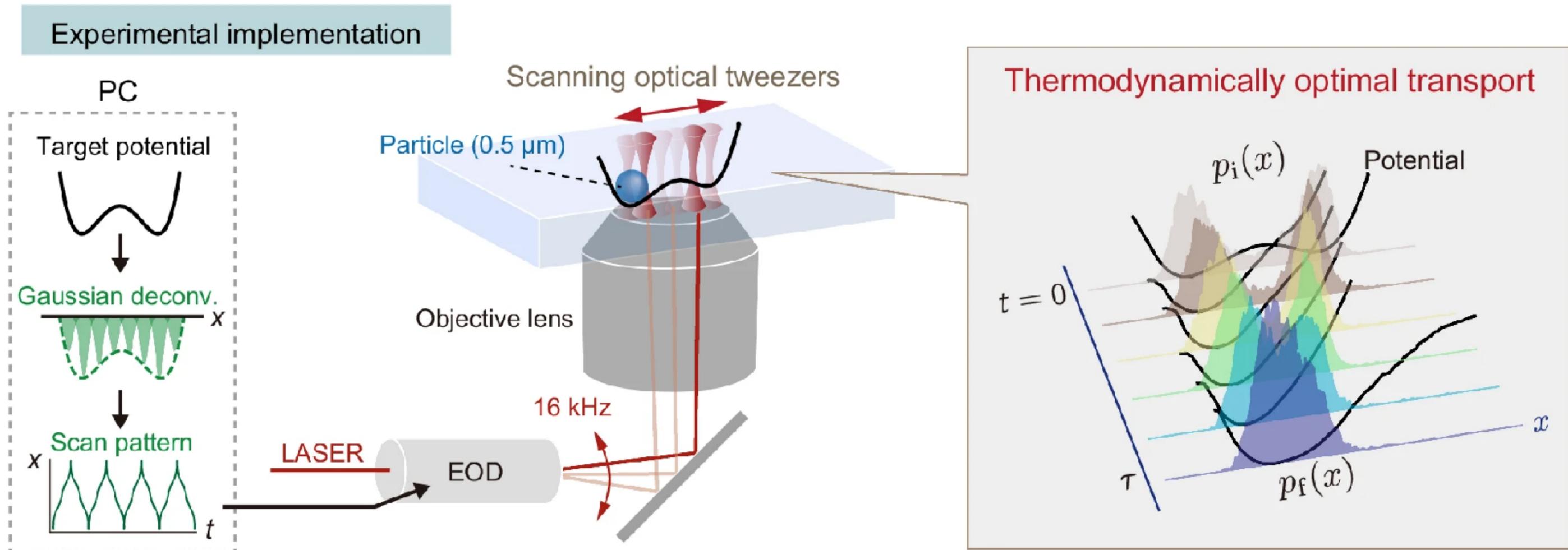


Quantum generalization of the thermodynamic framework:

K. Yoshimura, Y. Maekawa, R. Nagayama and SI. *Phys. Rev. Res.* 7, 013244 (2025).

	Classical	Quantum
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EPR	$J^T F$	$\text{tr}(J(\rho)^T \mathbb{F}(\rho))$

Experimental implementation



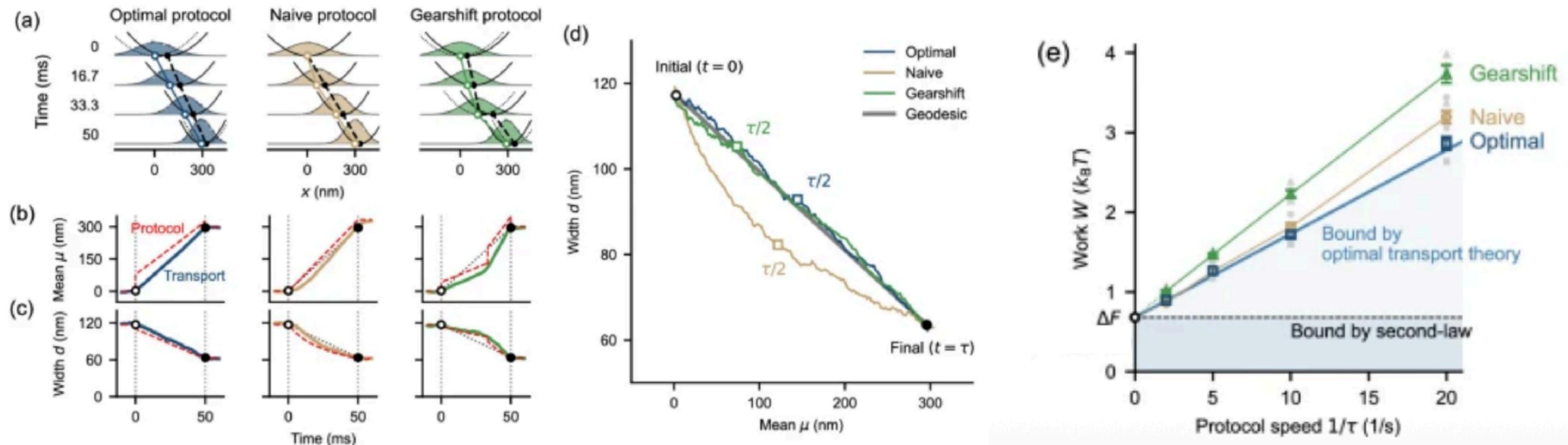
Optimal transport protocol vs other protocols for 1d Gaussian distribution

$P_t(\mathbf{x}) \sim \mathcal{N}(\mu_t, d_t^2)$: 1d Gaussian distribution with μ_t mean and d_t^2 variance

1. Conservative force: $F_t(\mathbf{x}) = -\nabla U_t(\mathbf{x})$. \rightarrow OK. (The 1d force is conservative.)

Optimal protocol is discussed in M. Nakazato and SI. Phys. Rev. Res. 3, 043093 (2021).

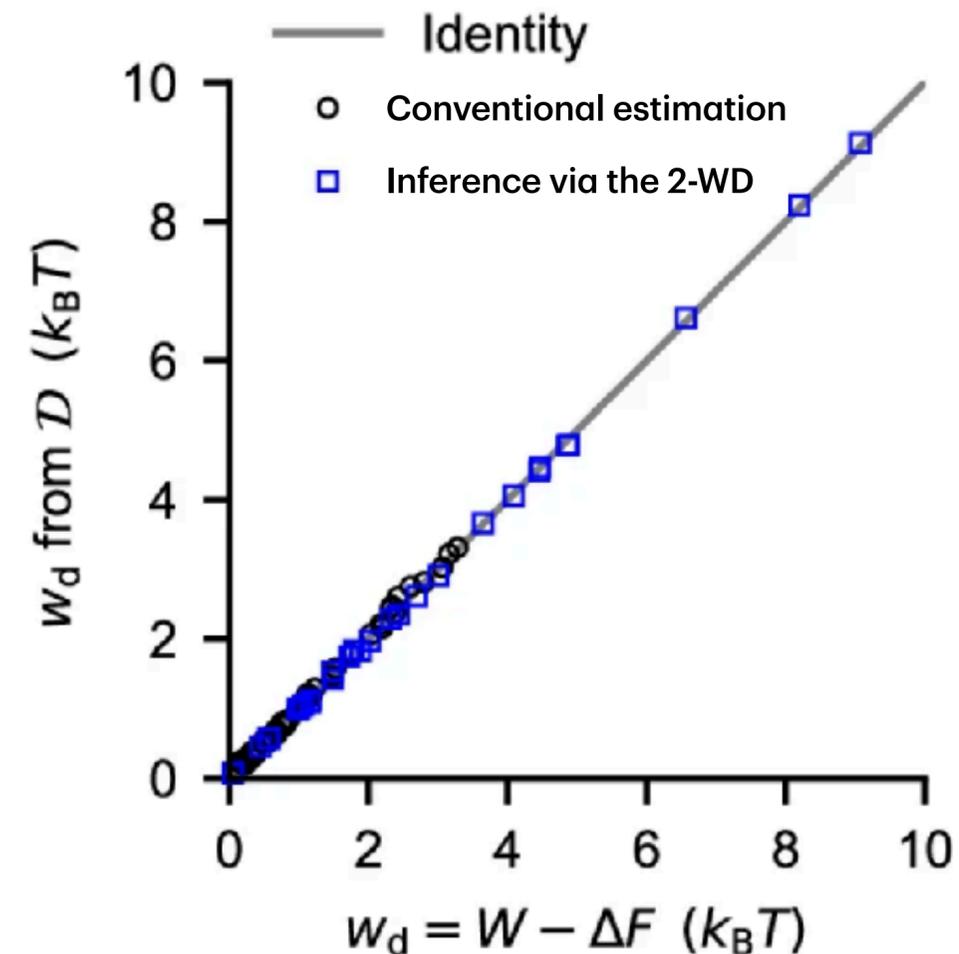
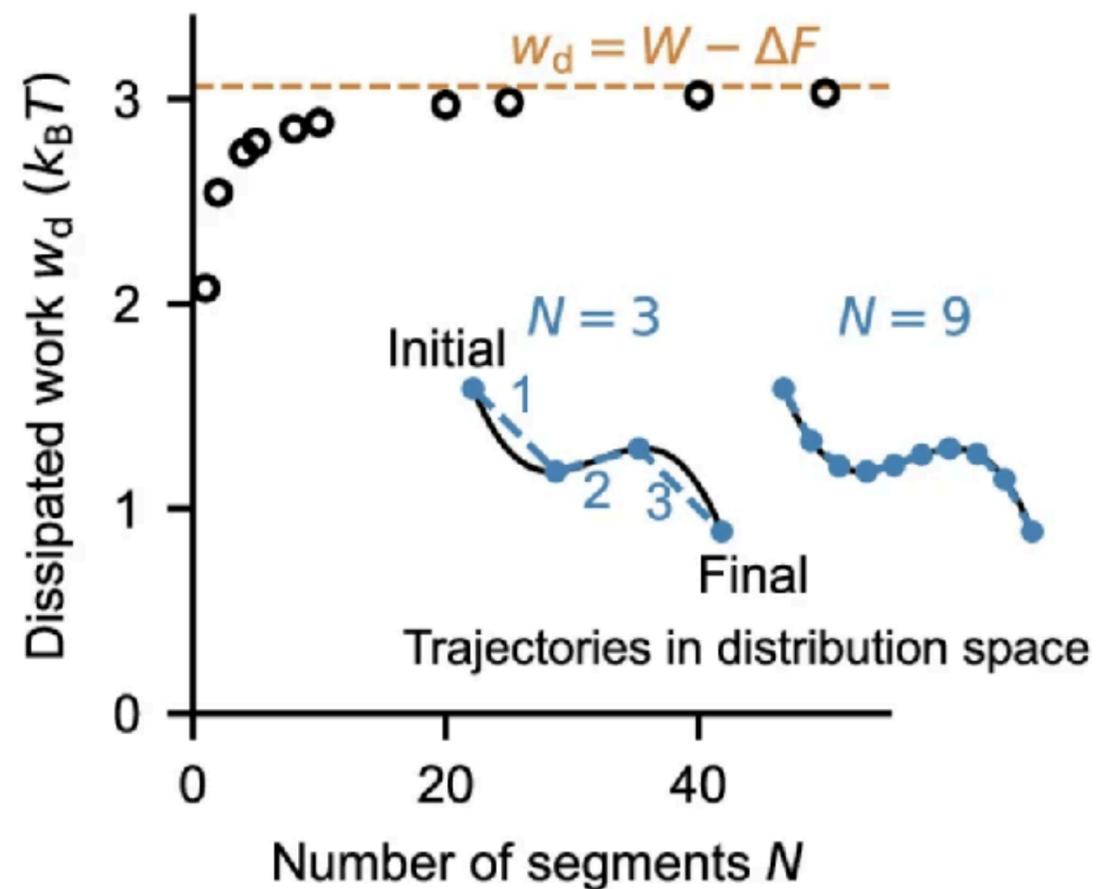
2. Geodesic: $\lim_{\Delta t \rightarrow 0} \frac{\mathcal{W}_2(P_t, P_{t+\Delta t})}{\Delta t} = \frac{\mathcal{W}_2(P_0, P_\tau)}{\tau} = \text{const.} \rightarrow \lim_{\Delta t \rightarrow 0} \frac{\mathcal{W}_2(P_t, P_{t+\Delta t})}{\Delta t} = \sqrt{(\partial_t \mu_t)^2 + (\partial_t d_t)^2}, \mu_t = \mu_0 + \frac{t}{\tau}(\mu_\tau - \mu_0), d_t = d_0 + \frac{t}{\tau}(d_\tau - d_0)$



Inferring the entropy production via the 2-Wasserstein distance

Without a nonconservative force: $w_d = T \int_0^\tau dt \dot{\Sigma}_t \simeq T \sum_{i=0}^{N=\lceil \tau/\Delta t \rceil} \frac{[\mathcal{W}_2(P_{i\Delta t}, P_{(i+1)\Delta t})]^2}{\Delta t}$ ($N \rightarrow \infty, \Delta t \rightarrow 0$)

M. Nakazato and SI. Phys. Rev. Res. 3, 043093 (2021).



Finite-time bit erasure

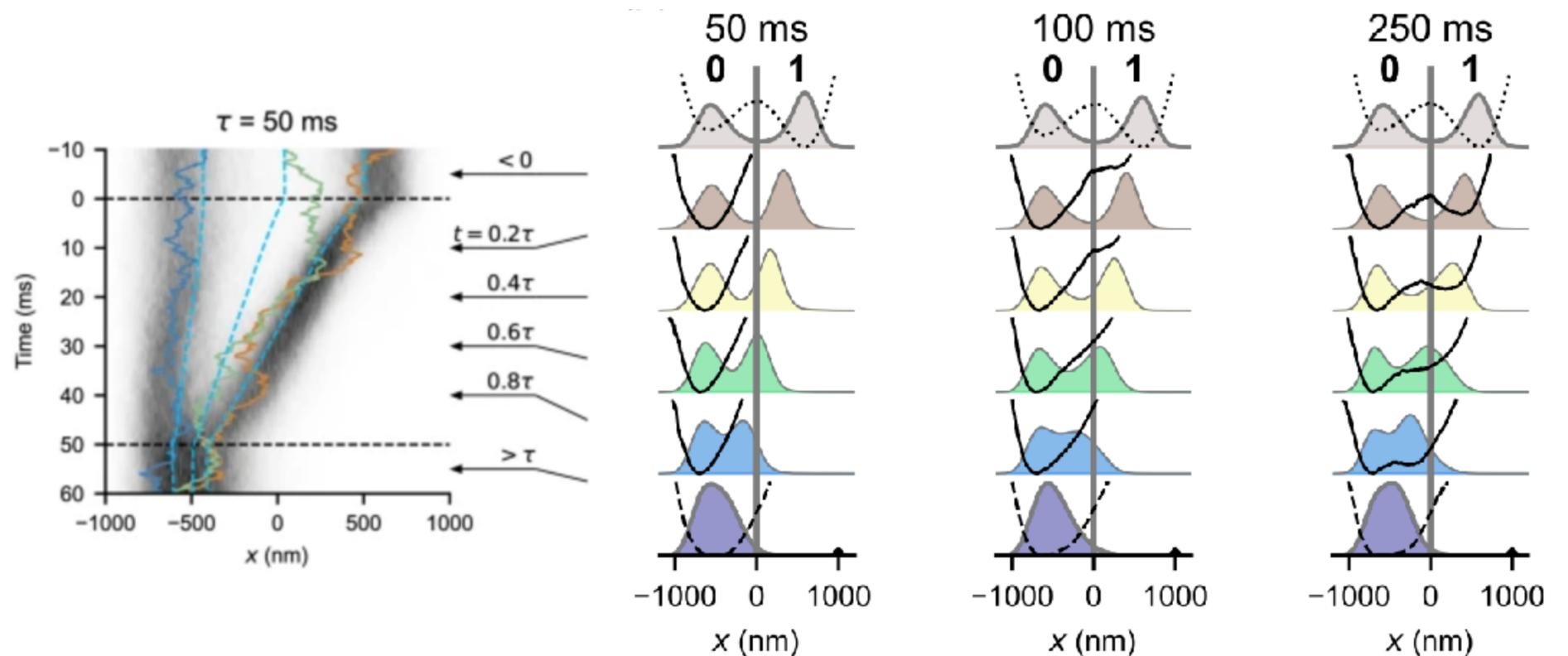
Assumption: Initial state and final state are equilibrium.

$$\int_0^\tau dt \dot{\Sigma}_t = \frac{1}{T}(W - \Delta F) \quad \Delta F \simeq \ln 2$$

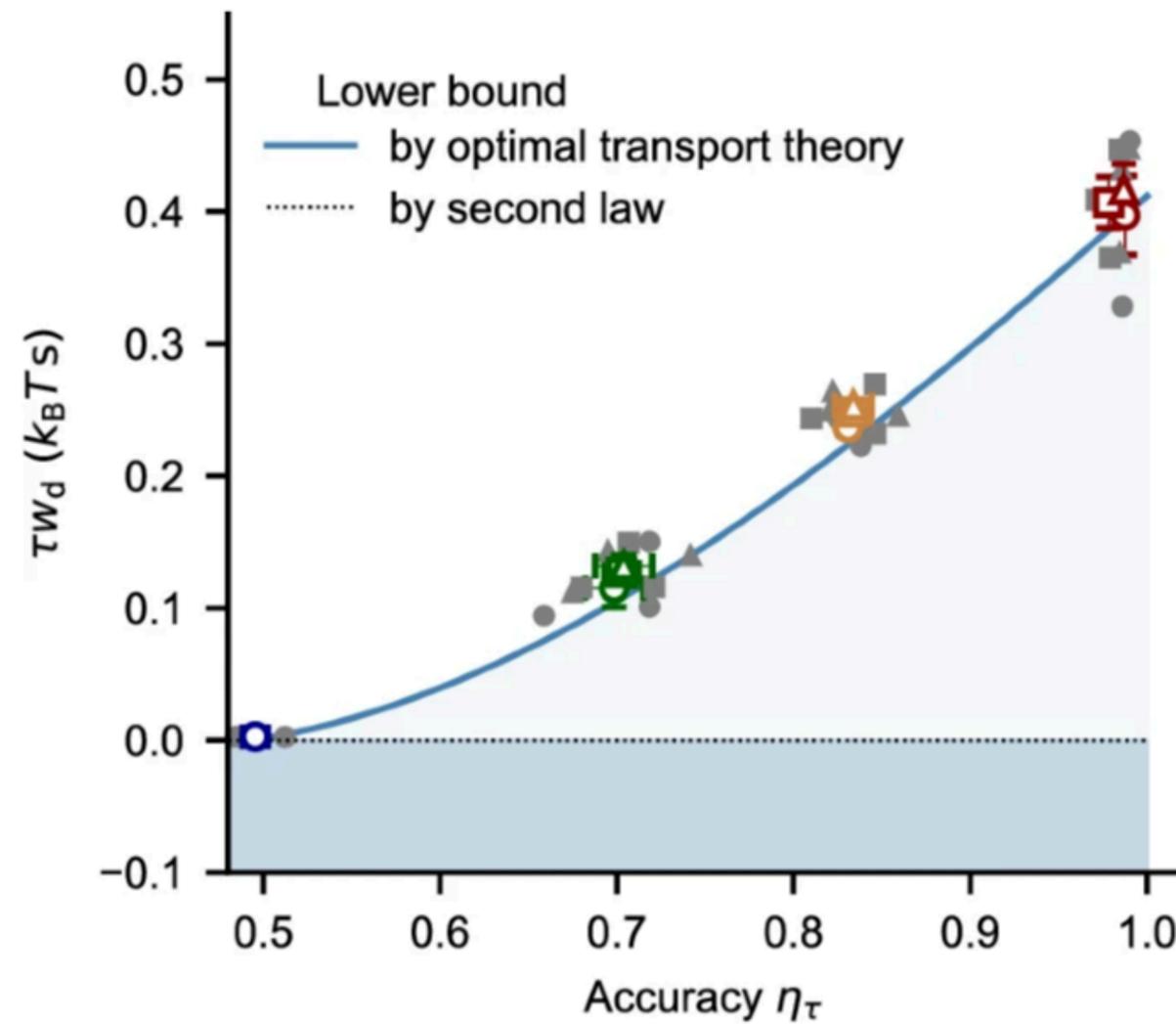
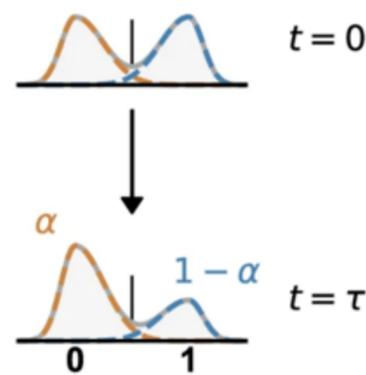
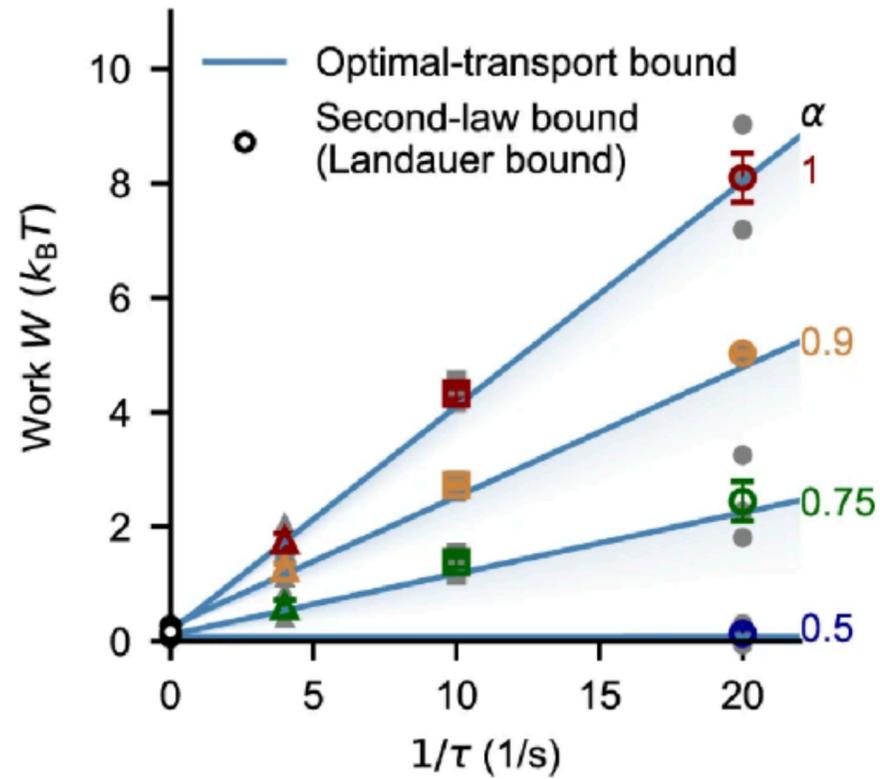
(Bit erasure)

Finite-time bit erasure

$$W \geq T \ln 2 + \frac{[\mathcal{W}_2(P_0, P_\tau)]^2}{\mu\tau}$$



Accuracy of bit erasure vs dissipated work

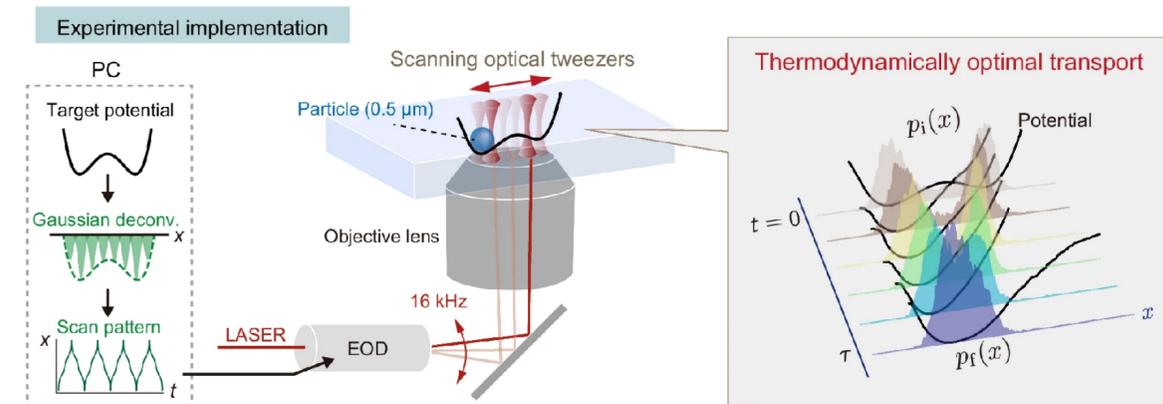


$$\eta_0 = \int_{-\infty}^0 dx P_0(x) = 0.5 \quad \rightarrow \quad \eta_\tau = \int_{-\infty}^0 dx P_\tau(x)$$

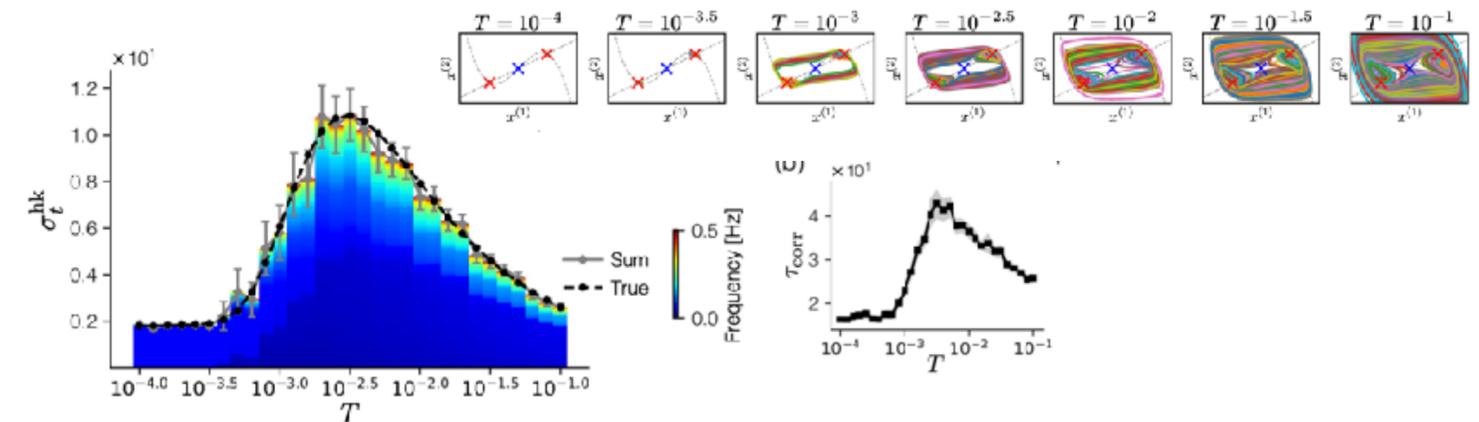
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Detailed balance	$J(x^{eq}) = 0$	$J(\rho^{eq}) = 0$
EPR	$J^T F$	$\text{tr}(J(\rho)^T \mathbb{F}(\rho))$

Introduction: Minimal entropy production rate

$$\dot{\Sigma}_t := \frac{1}{\mu T} \int d\mathbf{x} P_t(\mathbf{x}) \|\boldsymbol{\nu}_t(\mathbf{x})\|^2 \geq \frac{1}{\mu T} \left[\lim_{\Delta t \rightarrow 0} \frac{\mathcal{W}_2(P_t, P_{t+\Delta t})}{\Delta t} \right]^2$$

M. Nakazato and SI. Phys. Rev. Res. 3, 043093 (2021).

Equality condition: $F_t(\mathbf{x}) = -\nabla U_t(\mathbf{x})$

Excess entropy production rate

$$\dot{\Sigma}_t^{\text{ex}} = \frac{1}{\mu T} \left[\lim_{\Delta t \rightarrow 0} \frac{\mathcal{W}_2(P_t, P_{t+\Delta t})}{\Delta t} \right]^2 (\geq 0)$$

C. Maes and K. Netočný, *Journal of Statistical Physics*, 154, 188-203 (2014).

A. Dechant, S-I Sasa and SI. Phys. Rev. Res. 4, L012034 (2022).

A. Dechant, S-I Sasa and SI, Phys. Rev. E. 106, 024125 (2022).

Housekeeping entropy production rate

$$\dot{\Sigma}_t^{\text{hk}} := \dot{\Sigma}_t - \dot{\Sigma}_t^{\text{ex}} (\geq 0)$$

In the steady state, $\dot{\Sigma}_t^{\text{ex}} \Big|_{\text{st}} = 0$ $\dot{\Sigma}_t^{\text{hk}} \Big|_{\text{st}} = \dot{\Sigma}_t \Big|_{\text{st}}$

Conservative force, $\dot{\Sigma}_t^{\text{ex}} = \dot{\Sigma}_t$ $\dot{\Sigma}_t^{\text{hk}} = 0$

Introduction: Geometric decomposition

A. Dechant, S-I Sasa and SI, Phys. Rev. Res. 4, L012034 (2022).
A. Dechant, S-I Sasa and SI, Phys. Rev. E. 106, 024125 (2022).

Fokker-Planck equation

$$\partial_t P_t(\mathbf{x}) = -\nabla \cdot \mathbf{j}(\mathbf{x}) = -\nabla \cdot (\mathbf{L}_t(\mathbf{x}) \mathbf{f}_t(\mathbf{x}))$$

$$\mathbf{f}_t(\mathbf{x}) = \frac{\mathbf{F}_t(\mathbf{x})}{T} - \nabla \ln P_t(\mathbf{x}) : \text{Thermodynamic driving force}$$

$$\mathbf{L}_t(\mathbf{x}) = \mathbf{D}_t P_t(\mathbf{x}) : \text{Onsager matrix}$$

Entropy production rate

$$\dot{\Sigma}_t = \int d\mathbf{x} [\mathbf{f}_t(\mathbf{x})]^\top \mathbf{L}_t(\mathbf{x}) \mathbf{f}_t(\mathbf{x}) =: \langle \mathbf{f}_t, \mathbf{f}_t \rangle_{\mathbf{L}_t}$$

Potential $\phi_t^{\text{opt}}(\mathbf{x})$ (corresponding to optimal solution of Benamou-Brenier formula)

$$\text{A solution of } -\nabla \cdot (\mathbf{L}_t(\mathbf{x}) \mathbf{f}_t(\mathbf{x})) = -\nabla \cdot (\mathbf{L}_t(\mathbf{x}) \nabla \phi_t^{\text{opt}}(\mathbf{x}))$$

Excess entropy production rate

$$\dot{\Sigma}_t^{\text{ex}} = \langle \nabla \phi_t^{\text{opt}}, \nabla \phi_t^{\text{opt}} \rangle_{\mathbf{L}_t}$$

Housekeeping entropy production rate

$$\dot{\Sigma}_t^{\text{hk}} := \dot{\Sigma}_t - \dot{\Sigma}_t^{\text{ex}} = \langle \mathbf{f}_t - \nabla \phi_t^{\text{opt}}, \mathbf{f}_t - \nabla \phi_t^{\text{opt}} \rangle_{\mathbf{L}_t}$$

Orthogonality

$$\langle \nabla \phi_t^{\text{opt}}, \mathbf{f}_t - \nabla \phi_t^{\text{opt}} \rangle_{\mathbf{L}_t} = 0$$

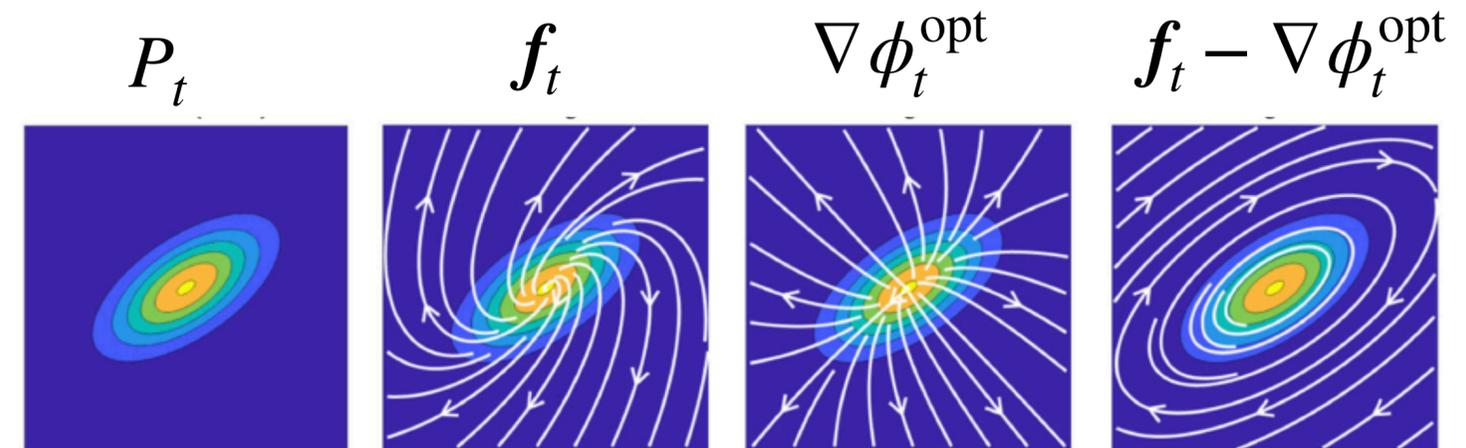
Housekeeping entropy production rate and oscillation

Excess entropy production rate Housekeeping entropy production rate

$$\dot{\Sigma}_t^{\text{ex}} = \langle \nabla \phi_t^{\text{opt}}, \nabla \phi_t^{\text{opt}} \rangle_{L_t} \quad \dot{\Sigma}_t^{\text{hk}} := \dot{\Sigma}_t - \dot{\Sigma}_t^{\text{ex}} = \langle \mathbf{f}_t - \nabla \phi_t^{\text{opt}}, \mathbf{f}_t - \nabla \phi_t^{\text{opt}} \rangle_{L_t}$$

$$\nabla \cdot (\mathbf{L}_t(\mathbf{x})[\mathbf{f}_t(\mathbf{x}) - \nabla \phi_t^{\text{opt}}(\mathbf{x})]) = 0$$

Oscillatory contributions



(Figure from) D. Sekizawa, SI, M. Oizumi, Phys. Rev. X 14, 041003 (2024).

Q: Could we discuss the oscillatory contributions to the housekeeping entropy production rate (or the steady-state entropy production rate)?

Oscillatory dynamics based on the housekeeping velocity field and its Koopman mode

Oscillatory dynamics (deterministic)

$$\frac{d}{ds}\mathbf{x}(s) = \boldsymbol{\nu}_t^{\text{hk}}(\mathbf{x}(s))$$

Housekeeping velocity field: $\boldsymbol{\nu}_t^{\text{hk}}(\mathbf{x}) := D_t[f_t(\mathbf{x}) - \nabla\phi_t^{\text{opt}}(\mathbf{x})]$

Koopman operator \mathcal{K}

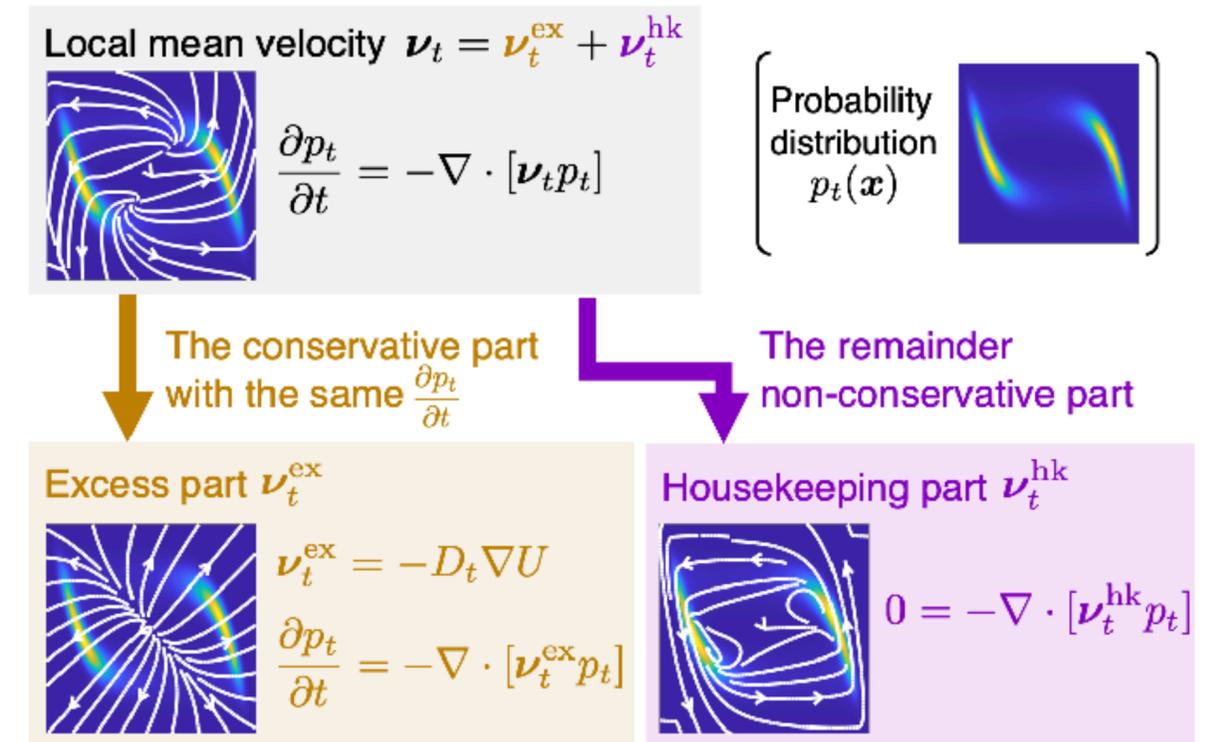
For any $g(\mathbf{x}(s))$,

$$\frac{d}{ds}g(\mathbf{x}(s)) = \mathcal{K}g(\mathbf{x}(s)) := \nabla g(\mathbf{x}(s)) \cdot \boldsymbol{\nu}_t^{\text{hk}}(\mathbf{x}(s))$$

Koopman eigenmode $\psi_i(\mathbf{x}(s))$ [Oscillatory mode]

$$\frac{d}{ds}\psi_i(\mathbf{x}(s)) = \mathcal{K}\psi_i(\mathbf{x}(s)) = \lambda_i\psi_i(\mathbf{x}(s))$$

$$\chi_i := \frac{\text{Im}[\lambda_i]}{2\pi} \quad \text{:frequency}$$



Koopman mode decomposition of the housekeeping entropy production rate

An expression of identity function

$$\text{Id}(\mathbf{x}) = \mathbf{x} = \sum_i \nu_i \psi_i(\mathbf{x}) \quad \rightarrow \quad \nu_t^{\text{hk}}(\mathbf{x}) = \mathcal{K} \text{Id}(\mathbf{x}) = \sum_i \lambda_i \nu_i \psi_i(\mathbf{x})$$

Koopman mode decomposition of the housekeeping entropy production rate

$$\dot{\Sigma}_t^{\text{hk}} = \langle D_t \nu_t^{\text{hk}}, D_t \nu_t^{\text{hk}} \rangle_{L_t} = (2\pi)^2 \sum_i \underbrace{(\chi_i)^2}_{\text{Frequency}} \underbrace{\langle D_t \overline{\nu_i \psi_i}, D_t \nu_i \psi_i \rangle_{L_t}}_{\text{Intensity}}$$

Complex conjugate
↓

∴ Skew-symmetric property $\int d\mathbf{x} g_2(\mathbf{x}) P_t(\mathbf{x}) [\mathcal{K} g_1(\mathbf{x})] = - \int d\mathbf{x} [\mathcal{K} g_2(\mathbf{x})] P_t(\mathbf{x}) g_1(\mathbf{x})$

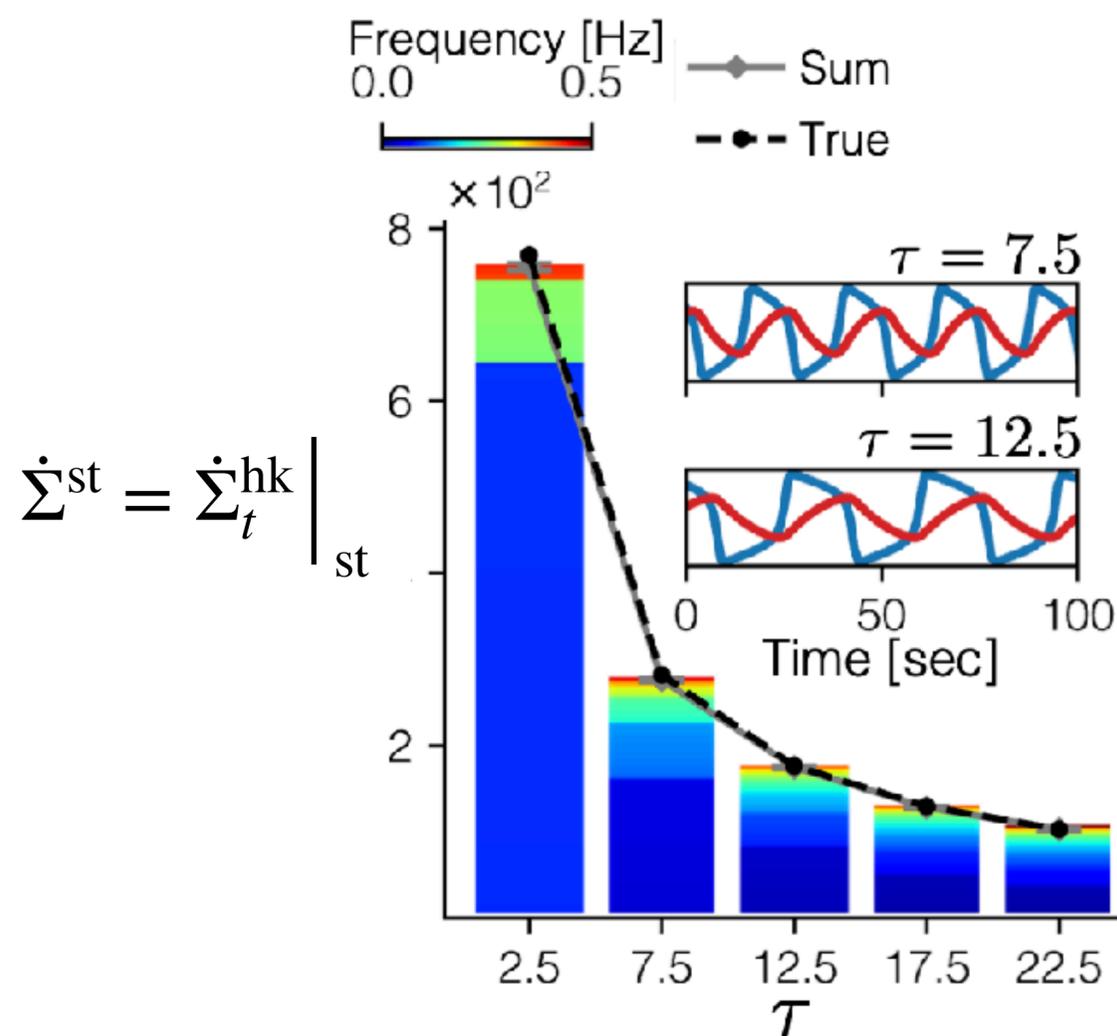
$$\rightarrow (\lambda_i + \bar{\lambda}_i) \int d\mathbf{x} \bar{\nu}_i(\mathbf{x}) P_t(\mathbf{x}) \nu_i(\mathbf{x}) = 0$$

Example: Noisy FitzHugh-Nagumo model

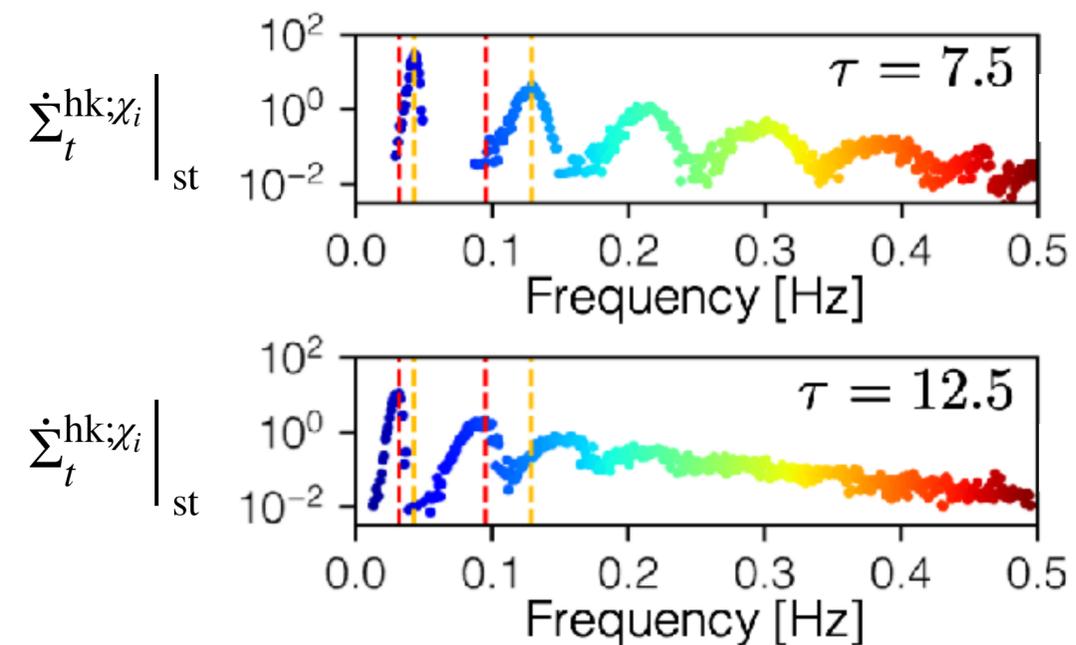
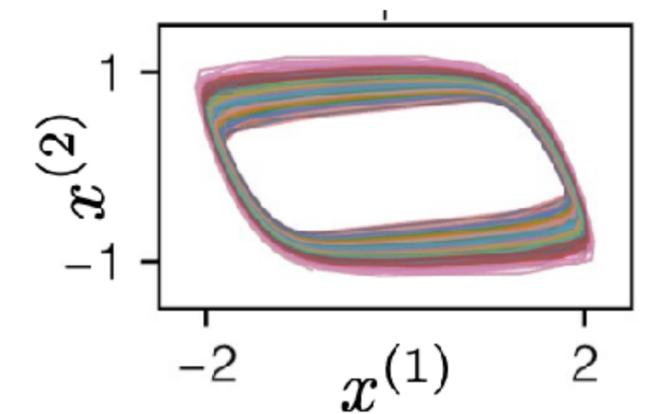
$$\mathbf{x} = \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix} \quad F_t(\mathbf{x}) = \begin{pmatrix} x^{(1)} - [x^{(1)}]^3/3 - x^{(2)} - I \\ (x^{(1)} + a - bx^{(2)})/\tau \end{pmatrix} \quad D_t = T\mathbf{I}$$

Langevin description

$$[\dot{\mathbf{x}}(t) = F_t(\mathbf{x}(t)) + \sqrt{2T}\xi(t)]$$



Oscillatory dynamics in the steady state
[limit cycle] $a = 0, b = 0.5, I = 0, T = 10^{-3}$

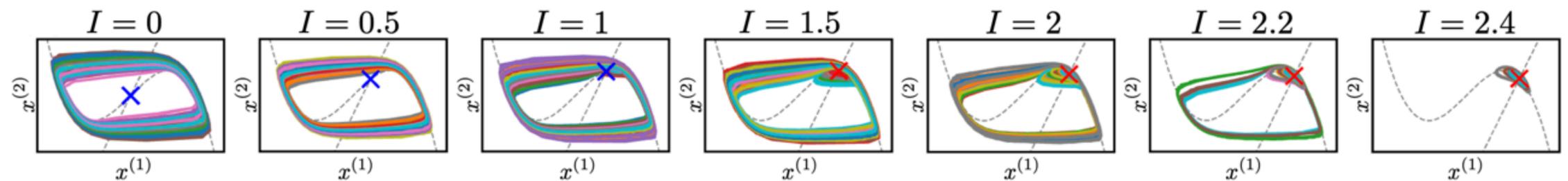


$$\dot{\Sigma}_t^{\text{hk};\chi_i} := (2\pi)^2 (\chi_i)^2 \langle D_t \bar{\mathbf{v}}_i \bar{\psi}_i, D_t \mathbf{v}_i \psi_i \rangle_{L_t}$$

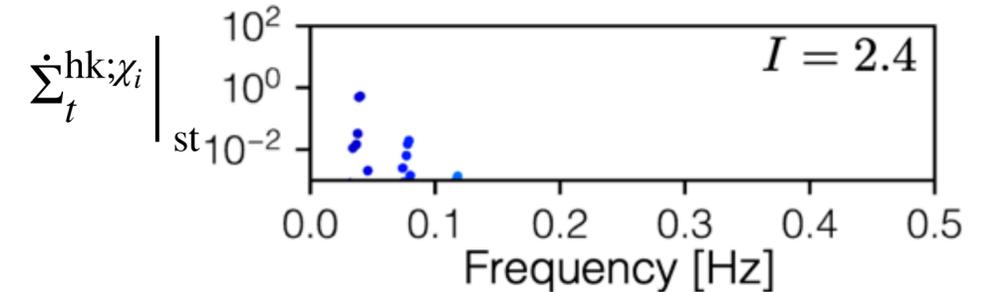
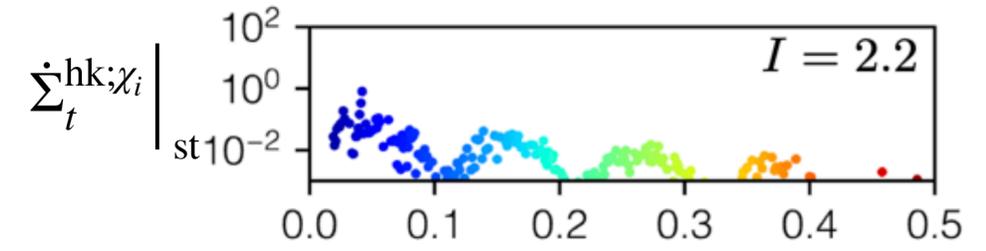
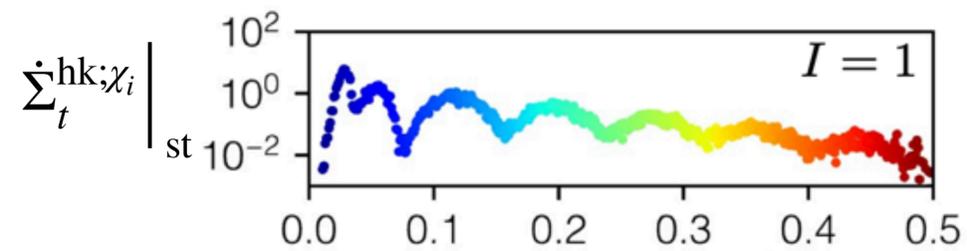
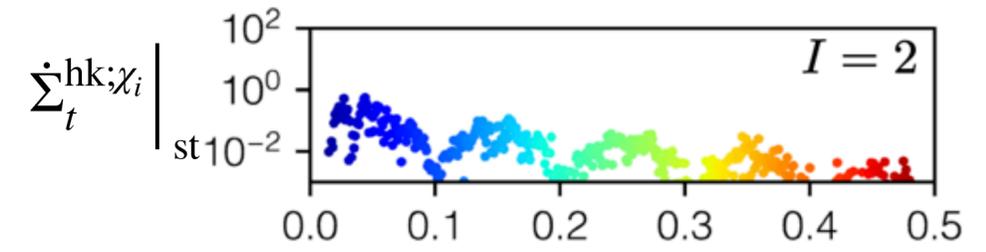
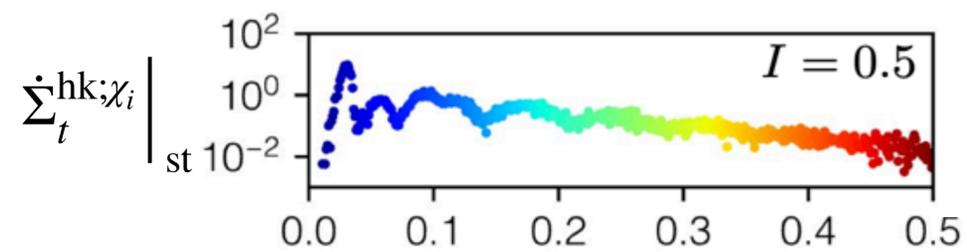
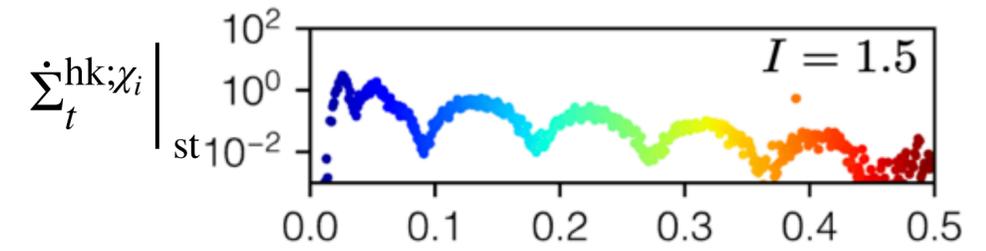
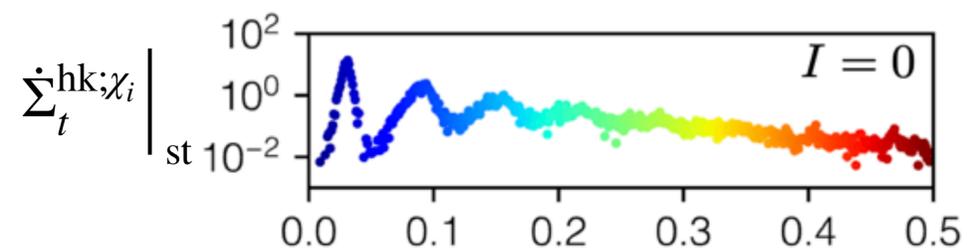
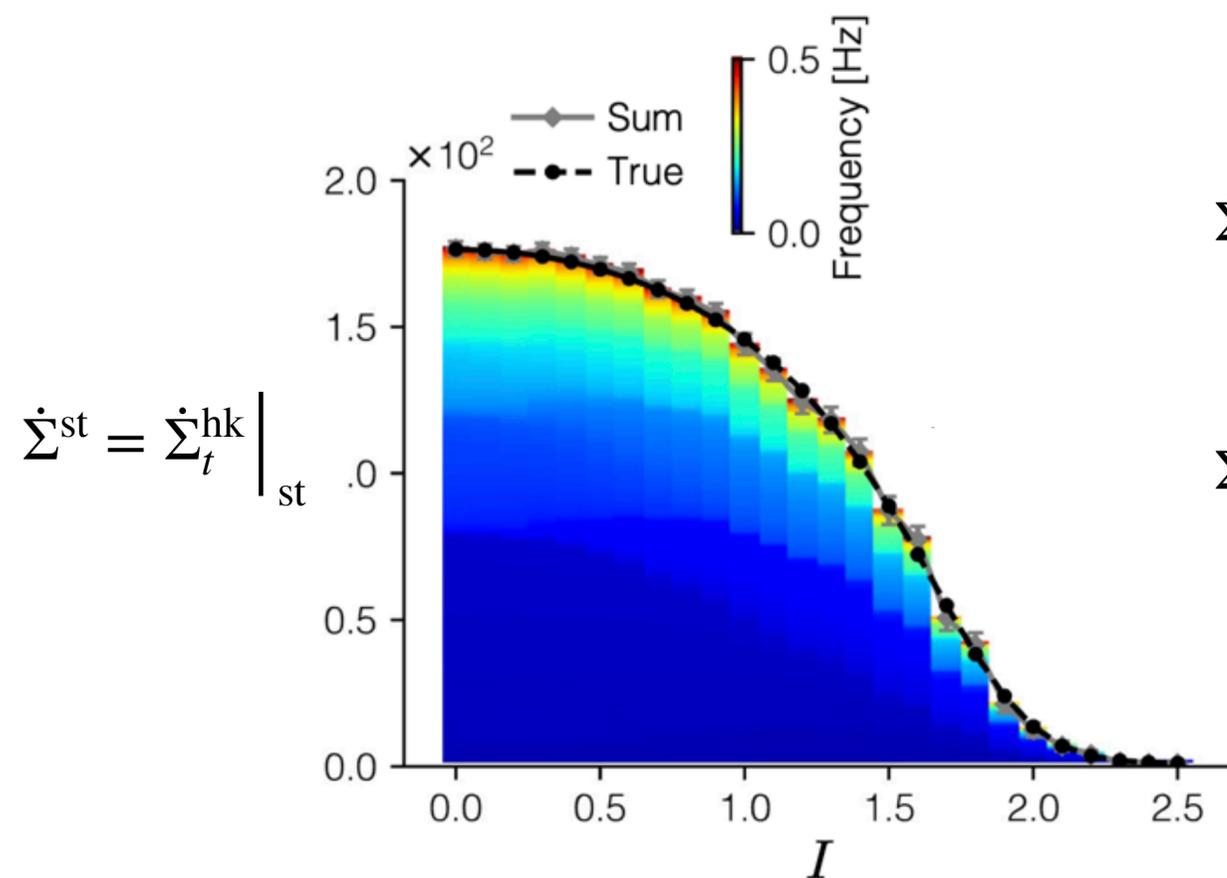
Example: Noisy FitzHugh-Nagumo model

[Hopf bifurcation]

Oscillatory dynamics
in the steady state
[Hopf bifurcation]



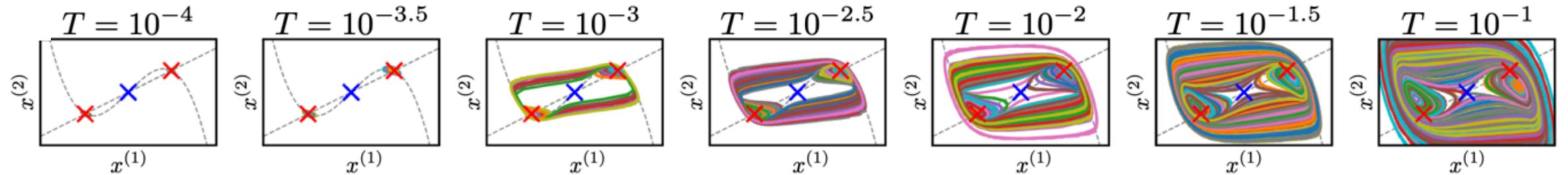
$$a = 0, b = 0.5, \tau = 12.5, T = 10^{-3}$$



Example: Noisy FitzHugh-Nagumo model

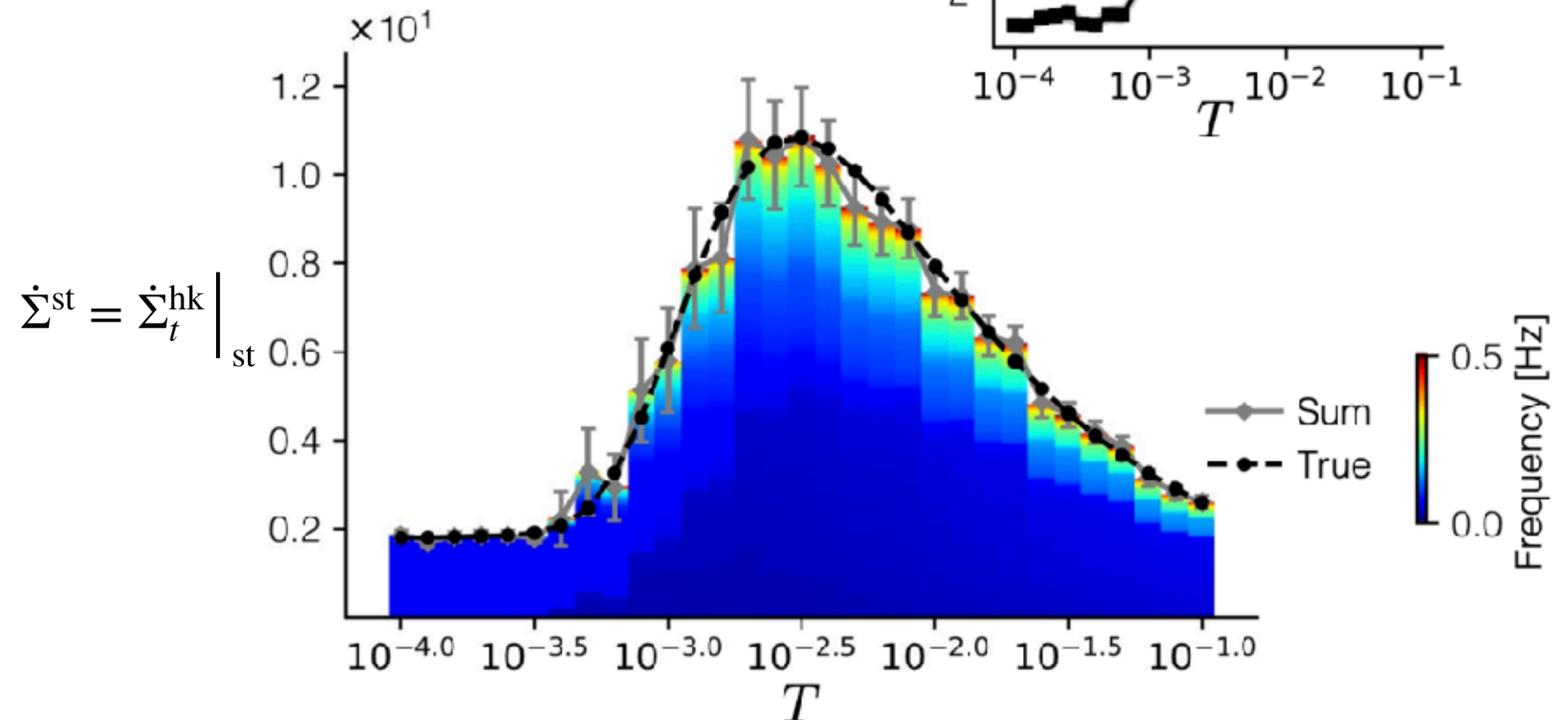
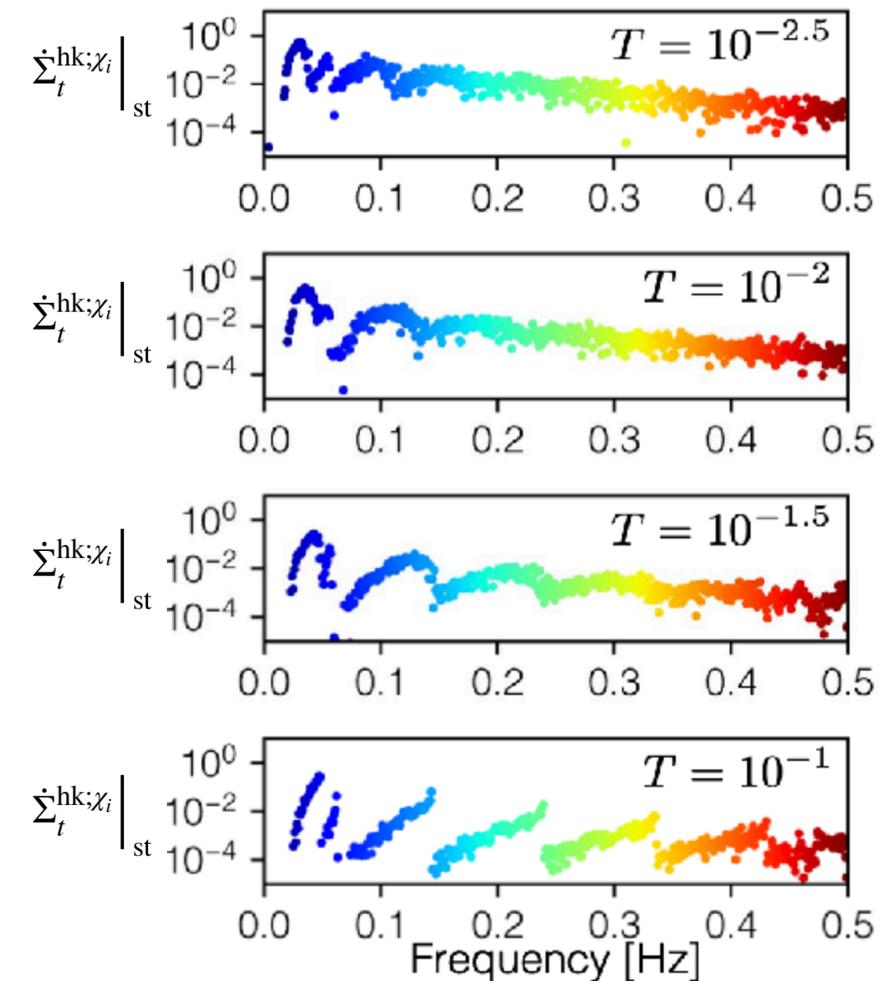
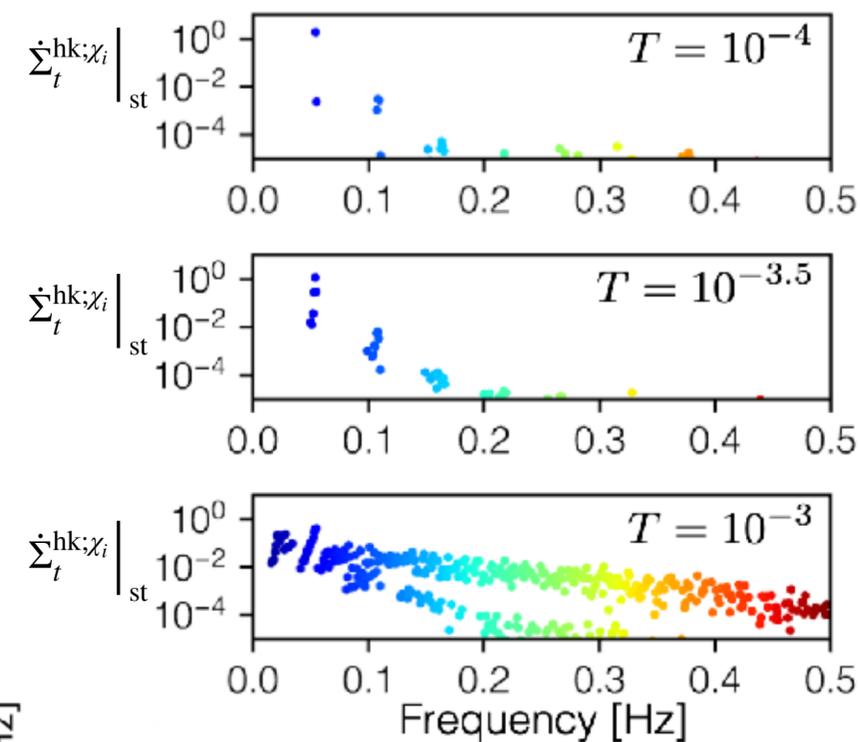
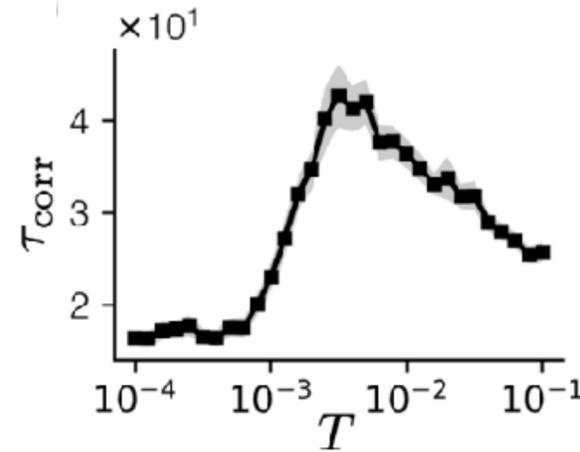
[Stochastic resonance]

Oscillatory dynamics
in the steady state
[Stochastic resonance]



$a = 0, b = 2, I = 0, \tau = 12.5$

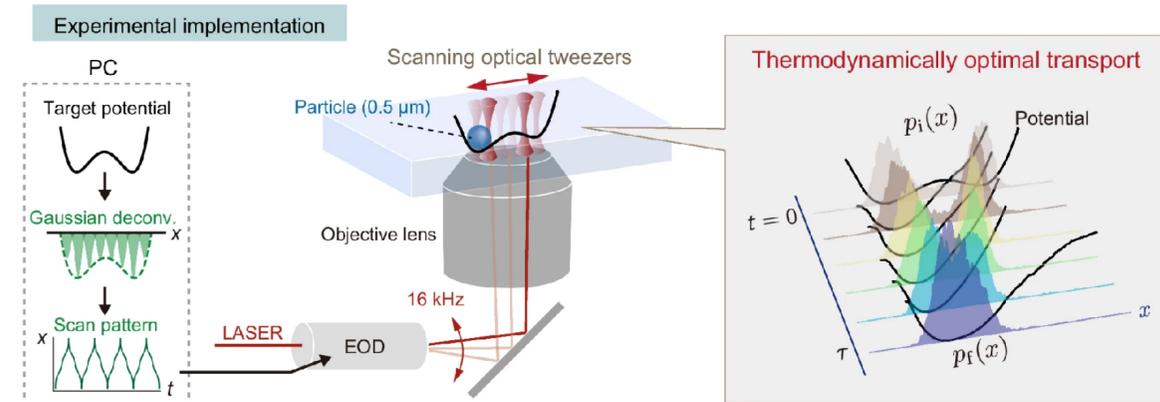
$$\tau_{\text{corr}} := \int_0^{\infty} ds \frac{\text{Cov}[x^{(1)}(t)x^{(1)}(t+s)]}{\text{Var}[x^{(1)}(t)]}$$



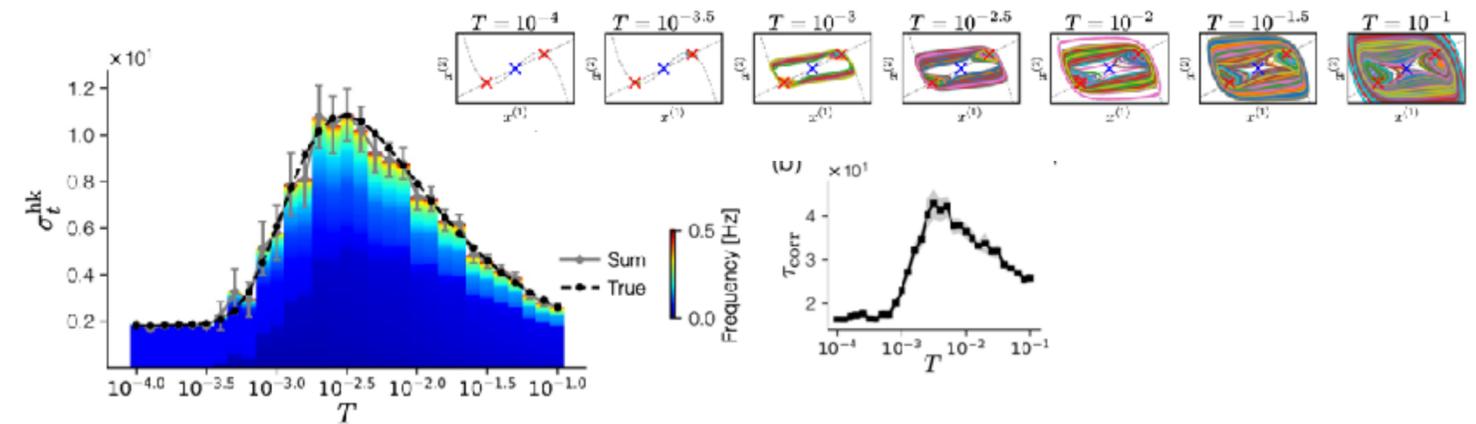
Contents

Introduction: **The thermodynamic framework based on optimal transport**
 see SI, *Information Geometry* 7, Suppl 1, 441-483 (2024). ...etc.

Experiment achieving minimal dissipation:
 S. Oikawa, Y. Nakayama, SI, T. Sagawa and
 S. Toyabe, *Nature Communications* 16, 10424 (2025).



Koopman mode decomposition of dissipation:
 D. Sekizawa, SI, M. Oizumi, arXiv:2510.21340 (2025),



Quantum generalization of the thermodynamic framework:
 K. Yoshimura, Y. Maekawa, R. Nagayama and SI. *Phys. Rev. Res.* 7, 013244 (2025).

	Classical	Quantum
Gradient	∇	∇_L
Continuity	$dx/dt = \nabla^T J(x)$	$\mathcal{D}(\rho) = \nabla_L^T J(\rho)$
Current	$J_e = J_e^+ - J_e^-$	$J(\rho) = [L, \Gamma \otimes \rho]$
Force	$F_e = \ln(J_e^+ / J_e^-)$	$\mathbb{F}(\rho) = [L, \ln(\Gamma \otimes \rho)]$
Conservative	$F = -\nabla\phi$	$\mathbb{F}(\rho) = -\nabla_L\phi$
Detailed balance	$J(x^{eq}) = 0$	$J(\rho^{eq}) = 0$
EPR	$J^T F$	$\text{tr}(J(\rho)^T \mathbb{F}(\rho))$

Can we generalize the thermodynamic framework based on optimal transport for quantum systems?

Open quantum dynamics [GKSL equation]

$$\partial_t \rho = -i[H, \rho] + D(\rho), \quad D(\rho) = \sum_k \gamma_k \left(L_k \rho L_k^\dagger - \frac{1}{2} \{L_k^\dagger L_k, \rho\} \right)$$

Q. Can we generalize the thermodynamic framework for quantum systems?

We can discuss the geometric decomposition of the entropy production rate into the excess entropy production rate and the housekeeping entropy production rate.

K. Yoshimura, Y. Maekawa, R. Nagayama and SI. Phys. Rev. Res. 7, 013244 (2025).

An analogy between the continuity equation and the GKSL equation

GKSL equation

$$\partial_t \rho = -i[H, \rho] + D(\rho),$$

$$D(\rho) = \sum_k \left[\gamma_k \left(L_k \rho L_k^\dagger - \frac{1}{2} \{L_k^\dagger L_k, \rho\} \right) + \gamma_{-k} \left(L_{-k} \rho L_{-k}^\dagger - \frac{1}{2} \{L_{-k}^\dagger L_{-k}, \rho\} \right) \right] \quad L_k = L_{-k}^\dagger$$

“Continuity equation”

$$D(\rho) = \nabla_{\mathbb{L}}^* \mathbb{J}$$

$$\text{cf.}) \partial_t P_t(\mathbf{x}) = -\nabla \cdot \mathbf{j}(\mathbf{x})$$

$$\mathbb{J}(\rho) := \bigoplus_k \begin{pmatrix} 0 & J_{-k} \\ J_k & 0 \end{pmatrix}$$

$$\nabla_{\mathbb{L}}^* \mathbb{J} := \sum_k ([J_k, L_{-k}] + [J_{-k}, L_k])$$

$$\mathbb{L} := \bigoplus_k \begin{pmatrix} 0 & L_{-k} \\ L_k & 0 \end{pmatrix}$$

$$J_k(\rho) := \frac{1}{2} (\gamma_k L_k \rho - \gamma_{-k} \rho L_k)$$

$$\nabla_{\mathbb{L}}^* \mathbb{J} := \text{tr}_K[\mathbb{J}, \mathbb{L}]$$

Thermodynamic driving force and Onsager operator

Thermodynamic driving force

$$\mathbb{F} := \bigoplus_k \begin{pmatrix} 0 & F_{-k} \\ F_k & 0 \end{pmatrix} \quad F_k(\rho) := \ln \left(\frac{\gamma_k}{\gamma_{-k}} \right) L_k + [L_k, \ln \rho]$$

Onsager operator $\mathcal{L}_{\Gamma \otimes \rho}$

$$\mathcal{L}_{\Gamma \otimes \rho}(\mathbb{A}) = \int_0^1 ds (\Gamma \otimes \rho)^s \mathbb{A} (\Gamma \otimes \rho)^{1-s} \quad \Gamma := \bigoplus_k \begin{pmatrix} \gamma_k/2 & 0 \\ 0 & \gamma_{-k}/2 \end{pmatrix} \quad \Gamma \otimes \rho := \bigoplus_k \begin{pmatrix} \gamma_k \rho/2 & 0 \\ 0 & \gamma_{-k} \rho/2 \end{pmatrix}$$

$$\mathcal{L}_{\Gamma \otimes \rho}(\mathbb{F}) = \mathbb{J}$$

$$\text{cf.) } \mathbf{L}_t(\mathbf{x}) \mathbf{f}_t(\mathbf{x}) = \mathbf{j}(\mathbf{x})$$

Entropy production rate

Entropy production rate ($k_B = 1$)

$$\dot{\Sigma} := -\operatorname{tr}[D(\rho)\ln\rho] + \sum_k \ln\left(\frac{\gamma_k}{\gamma_{-k}}\right) [\gamma_k \operatorname{tr}(L_k^\dagger L_k \rho) - \gamma_{-k} \operatorname{tr}(L_{-k}^\dagger L_{-k} \rho)]$$

$$= \operatorname{tr}[[\mathbb{F}]^\dagger \mathbb{J}] = \operatorname{tr}[[\mathbb{F}]^\dagger \mathcal{L}_{\Gamma \otimes \rho}(\mathbb{F})] =: \langle \mathbb{F}, \mathbb{F} \rangle_{\mathcal{L}_{\Gamma \otimes \rho}}$$

$$\text{cf.) } \dot{\Sigma}_t = \int dx f(\mathbf{x}) \cdot \mathbf{j}_t(\mathbf{x}) = \int dx f(\mathbf{x}) \cdot \mathbb{L}_t(\mathbf{x}) f_t(\mathbf{x}) = \langle f_t, f_t \rangle_{\mathbb{L}_t}$$

Gradient operator

“(Negative) divergence” operator

$$\mathbb{Y} := \bigoplus_k \begin{pmatrix} 0 & Y_{-k} \\ Y_k & 0 \end{pmatrix} \quad \nabla_{\perp}^* \mathbb{Y} := \sum_k ([Y_k, L_{-k}] + [Y_{-k}, L_k])$$

“Gradient” operator

$$\nabla_{\perp} X := \bigoplus_k \begin{pmatrix} 0 & [X, L_{-k}] \\ [X, L_k] & 0 \end{pmatrix}$$

Adjointness

$$\text{tr}[(\mathbb{Y})^\dagger \nabla_{\perp} X] = \text{tr}[(\nabla_{\perp}^* \mathbb{Y})^\dagger X]$$

$$\text{cf.) } \int d\mathbf{x} \mathbf{Y}(\mathbf{x}) \cdot [\nabla X(\mathbf{x})] = \int d\mathbf{x} [-\nabla \cdot \mathbf{Y}(\mathbf{x})] X(\mathbf{x})$$

Geometric decomposition of the entropy production rate

Potential ϕ^{opt} cf.) $(\partial_t P_t(\mathbf{x}) = -) - \nabla \cdot (\mathbf{L}_t(\mathbf{x}) \mathbf{f}_t(\mathbf{x})) = - \nabla \cdot (\mathbf{L}_t(\mathbf{x}) \nabla \phi_t^{\text{opt}}(\mathbf{x}))$

A solution of $(D(\rho) = -) \nabla_{\perp}^* \mathcal{L}_{\Gamma \otimes \rho}(\mathbb{F}) = \nabla_{\perp}^* \mathcal{L}_{\Gamma \otimes \rho}(\nabla_{\perp} \phi^{\text{opt}})$

Orthogonality $\langle \nabla_{\perp} \phi^{\text{opt}}, \mathbb{F} - \nabla_{\perp} \phi^{\text{opt}} \rangle_{\mathcal{L}_{\Gamma \otimes \rho}} = 0$ cf.) $\langle \nabla \phi_t^{\text{opt}}, \mathbf{f}_t - \nabla \phi_t^{\text{opt}} \rangle_{\mathbf{L}_t} = 0$

Geometric decomposition of the entropy production rate

$$\dot{\Sigma} := \langle \mathbb{F}, \mathbb{F} \rangle_{\mathcal{L}_{\Gamma \otimes \rho}} = \langle \nabla_{\perp} \phi^{\text{opt}}, \nabla_{\perp} \phi^{\text{opt}} \rangle_{\mathcal{L}_{\Gamma \otimes \rho}} + \langle \mathbb{F} - \nabla_{\perp} \phi^{\text{opt}}, \mathbb{F} - \nabla_{\perp} \phi^{\text{opt}} \rangle_{\mathcal{L}_{\Gamma \otimes \rho}} = \dot{\Sigma}^{\text{ex}} + \dot{\Sigma}^{\text{hk}}$$

$$\dot{\Sigma}^{\text{ex}} := \langle \nabla_{\perp} \phi^{\text{opt}}, \nabla_{\perp} \phi^{\text{opt}} \rangle_{\mathcal{L}_{\Gamma \otimes \rho}} (\geq 0), \quad \dot{\Sigma}^{\text{hk}} := \langle \mathbb{F} - \nabla_{\perp} \phi^{\text{opt}}, \mathbb{F} - \nabla_{\perp} \phi^{\text{opt}} \rangle_{\mathcal{L}_{\Gamma \otimes \rho}} (\geq 0)$$

$$\text{cf.) } \dot{\Sigma}_t^{\text{ex}} = \langle \nabla \phi_t^{\text{opt}}, \nabla \phi_t^{\text{opt}} \rangle_{\mathbf{L}_t}, \quad \dot{\Sigma}_t^{\text{hk}} := \langle \mathbf{f}_t - \nabla \phi_t^{\text{opt}}, \mathbf{f}_t - \nabla \phi_t^{\text{opt}} \rangle_{\mathbf{L}_t}$$

Excess entropy production rate and housekeeping entropy production rate in the GKSL equation

Property of the excess entropy production rate

$$\dot{\Sigma}^{\text{ex}} = \langle \nabla_{\perp} \phi^{\text{opt}}, \nabla_{\perp} \phi^{\text{opt}} \rangle_{\mathcal{L}_{\Gamma \otimes \rho}} = 0 \quad \text{if} \quad D(\rho) = 0 \quad [\partial_t \rho = -i[H, \rho] \neq 0]$$

Zero dissipator (\neq the steady state)

Property of the housekeeping entropy production rate

$$\dot{\Sigma}^{\text{hk}} = \langle \mathbb{F} - \nabla_{\perp} \phi^{\text{opt}}, \mathbb{F} - \nabla_{\perp} \phi^{\text{opt}} \rangle_{\mathcal{L}_{\Gamma \otimes \rho}} = 0 \quad \text{if} \quad \exists \phi, \quad \mathbb{F} = \nabla_{\perp} \phi$$

e.g.,) $\phi = -\ln \rho + \ln e^{-\beta H}$

Conservative driving force

Summary

For classical systems (overdamped Langevin systems),

The optimal-transport framework provides the formula of minimal dissipation.
The minimal dissipation is given by the 2-Wasserstein distance.
We experimentally verify the formula to achieve the optimal protocol.

**S. Oikawa, Y. Nakayama, SI, T. Sagawa and
S. Toyabe, Nature Communications 16, 10424 (2025).**

The optimal-transport framework provides the decomposition of the entropy production rate into conservative (excess) and nonconservative (housekeeping) contributions.
Especially, the housekeeping contribution can be decomposed into each oscillatory mode.
We analyze the nonlinear oscillatory phenomena (limit cycle, Hopf bifurcation and stochastic resonance) using this decomposition.

D. Sekizawa, SI, M. Oizumi, arXiv:2510.21340 (2025).

For quantum systems (the GKSL equation),

We decompose the entropy production rate into conservative (excess) and non-conservative (housekeeping) contributions, in parallel with the classical case.

K. Yoshimura, Y. Maekawa, R. Nagayama and SI. Phys. Rev. Res. 7, 013244 (2025).

(We also derived thermodynamic uncertainty relations, variational expressions...etc.)

Review: SI, Information Geometry 7, Suppl 1, 441-483 (2024).

Thank you for listening!