ERATO Sagawa Information-to-Energy Interconversion Project

Quasiprobability thermodynamic uncertainty relation

[arXiv:2508.14354]

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Second law of thermodynamics

- Second law of thermodynamics

$$\text{EPR} \quad \dot{\Sigma} = \frac{dS}{dt} - \sum_{\nu} \beta_\nu \dot{Q}_\nu \geq 0$$

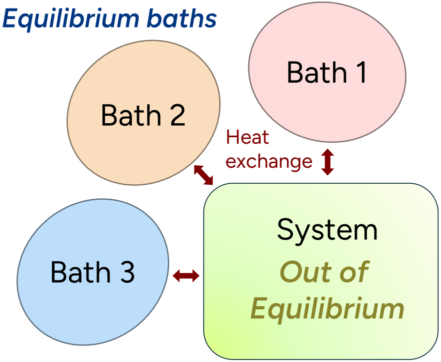
S : system's entropy,

β_ν : bath ν's inverse temperature,

$\dot{Q}_\nu$: heat flux from bath ν $(k_{\mathrm{B}} = 1)$

- Entropy production is a fundamental cost;

  it's in a trade-off relationship with various costs



Thermodynamic uncertainty relation

- TUR : trade-off with presicion Barato & Seifert, PRL **114**, 15810 (2015)

$$\dot{\Sigma} \geq \frac{2J_X^2}{S_X} \geq 0$$

  J_X/S_X : magnitude/fluctuation of current of a "quantity" X

- "Fluctuation" is *dynamical*

- If we focus on internal observables = Hermitian operators X
  - Single time statistics is insufficient (e.g., $\mathrm{Var}(X) = \langle X^2 \rangle - \langle X \rangle^2$)
  - In general, $[X(t), X(t')] \neq 0$; thus, not simultaneously measurable generally

    Previous extensions have given up this "generality"
  - Quasiprobabilities can characterize fluctuations generally!

Preliminaries

- Consider open quantum systems described by the GKSL master equation

$$\frac{d\rho}{dt} = \mathcal{L}(\rho) = -i[H, \rho] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right)$$

H : Hamiltonian, L_k : jump operators ($\hbar = 1$)

- Local detailed balance: $L_{-k} = e^{-s_k/2} L_k^\dagger$ (s_k : entropy flux into a bath)

1. Each jump is reversible
2. The ratio is associated with entropy increase s_k in a bath

- Entropy production rate (EPR)

$$\dot{\Sigma}(\rho) = \frac{d}{dt} S(\rho) + \sum_k s_k \operatorname{tr}(L_k^\dagger L_k \rho) \geq 0$$

$S(\rho) = -\operatorname{tr}(\rho \ln \rho)$: von Neumann entropy

Attempts of quantum TUR

- Two-point measurement based approach

  e.g., Sacchi, PRE **103**, 102111 (2019), Timpanaro et al., PRL **123**, 090604 (2019)

  - Initial state = incoherent (e.g., Gibbs state)
  - Later influence of initial coherence is lost

- Unraveling based approach

  e.g., Vu & Saito, PRL **128**, 140602 (2022), Vu, PRX Quantum **6**, 010343 (2025)

  - Consider quantities associated with jumps (rather than the state itself)
  - Unlike classical systems, jumps cannot characterize internal observables' (= Hermitian operators') change

Main result

- For any observable X and state ρ, KY and Hamazaki, arXiv:2508.14354

$$\dot{\Sigma}(\rho) \geq \frac{2|J_X^{\mathrm{d}}(\rho)|^2}{m_X(\rho)}$$

- $J_X^{\mathrm{d}}(\rho) = \mathrm{tr}\left(\rho \mathcal{D}^\dagger(X)\right)$: Dissipative current
  - $\mathcal{D}^\dagger(X) = \sum_k (L_k^\dagger X L_k - \frac{1}{2}\{L_k^\dagger L_k, X\})$
  - $\frac{d}{dt}\langle X\rangle_{\rho(t)} = J_X^{\mathrm{H}}(\rho(t)) + J_X^{\mathrm{d}}(\rho(t))$ with $J_X^{\mathrm{H}}(\rho) = i\,\mathrm{tr}(\rho[H,X])$
- $m_X(\rho) = \lim_{\Delta t\to 0} \frac{\langle(\Delta X)^2\rangle}{\Delta t}$: Short-time fluctuation w.r.t. the *TMH quasiprobability*

TMH Quasiprobability

- Terletsky–Margenau–Hill (TMH) quasiprobability

  Terletsky, J. Exp. Theor. Phys. **7**, 1290 (1937), Margenau and Hill, PTP **26**, 722 (1961)

$$P_{ji}^{\mathrm{TMH}} = \frac{1}{2}\,\mathrm{tr}\left(\{e^{\mathcal{L}^\dagger \Delta t}(\Pi_j), \Pi_i\}\rho(t)\right)$$

  for $X = \sum_i x_i \Pi_i$ ($\Pi_i = |i\rangle\langle i|$)

  cf. $\frac{d\rho}{dt} = \mathcal{L}(\rho) = -i[H,\rho] + \sum_k (L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\})$

- $\langle(\Delta X)^n\rangle := \sum_{i,j}(x_j - x_i)^n P_{ji}^{\mathrm{TMH}}$

  cf. $\dot{\Sigma}(\rho) \geq \dfrac{2|J_X^{\mathrm{d}}(\rho)|^2}{m_X(\rho)}$ and $m_X(\rho) = \lim_{\Delta t \to 0} \dfrac{\langle(\Delta X)^2\rangle}{\Delta t}$

Quasiprobability as an extension of classical probability

- Classical random variable $\hat{x}$ taking x_i at probability p_i
- Markov jump process with transition rate R_{ji};

$$P(j, t + \Delta t; i, t) = P_{ji} = [e^{R\Delta t}]_{ji} p_i(t)$$

with $R_{ii} = -\sum_{j(\neq i)} R_{ji} =: -\lambda_i$ (escape rate); $\langle (\Delta \hat{x})^n \rangle = \sum_{i,j} (x_j - x_i)^n P_{ji}$

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- Naive quantum extension

$$P_{ji}^{\mathrm{tpm}} = \langle j | e^{\mathcal{L}\Delta t}(\Pi_i) | j \rangle p_i(t) \quad \text{with} \quad p_i(t) = \mathrm{tr}(\rho(t)\Pi_i)$$

$$= \mathrm{tr}\left(e^{\mathcal{L}^\dagger \Delta t}(\Pi_j)\underline{\Pi_i \rho(t) \Pi_i}\right) \quad \text{coherence is broken}$$

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- TMH quasiprobability $P_{ji}^{\mathrm{TMH}} = \underline{\mathrm{Re}}\, \mathrm{tr}\left(e^{\mathcal{L}^\dagger \Delta t}(\Pi_j)\underline{\Pi_i \rho(t)}\right)$

Properties of quasiprobability

- $\sum_j P_{ji}^{\mathrm{TMH}} = p_i(t)$ and $\sum_i P_{ji}^{\mathrm{TMH}} = p_j(t+\Delta t) \to$ Joint probability
- P_{ji}^{TMH} can be negative $\to$ *Quasi*probability

  Negativity is associated with "contextuality" (a kind of non-classicality)
- If $[X, e^{\mathcal{L}^\dagger \Delta t}(X)] = 0$ or $[X, \rho(t)] = 0$, then $P_{ji}^{\mathrm{TMH}} = P_{ji}^{\mathrm{tpm}} \geq 0$

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👍 TUR $\dot{\Sigma}(\rho) \geq \dfrac{2|J_X^{\mathrm{d}}(\rho)|^2}{m_X(\rho)}$ with $m_X(\rho) = \lim_{\Delta t \to 0} \dfrac{\langle (\Delta X)^2 \rangle}{\Delta t} = \lim_{\Delta t \to 0} \dfrac{\mathrm{Var}(\Delta X)}{\Delta t}$

$\mathrm{Var}(\Delta X) = \langle (\Delta X)^2 \rangle - \langle \Delta X \rangle^2$

- Benefits of interpretation?

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- Tajima & Funo, PRL **127**, 190604 (2021)

  Dissipationless current $\frac{d}{dt}\langle H\rangle = O(N)$ while $\dot{\Sigma} = O(1)$

  $\uparrow$ Qunatum coherence is crucial

  $\uparrow\uparrow$ Negativity also matters (and provides stricter conditions)

Dissipationless current

- Quasiprobability TUR is rewritten as

$$Q_X(\rho) := \frac{|J_X^{\mathrm{d}}(\rho)|^2}{\dot{\Sigma}(\rho)} \le \frac{1}{2} m_X(\rho)$$

Intuitively, $J_X^{\mathrm{d}}(\rho) = O(N) \implies \dot{\Sigma}(\rho) = O(N)$; In fact, $Q_X(\rho) \le O(N)$ if classical

- For an N-fold degenerate two-level Hamiltonian H, there exists a ρ such that

$$\frac{d}{dt}\langle H\rangle_\rho = J_H^{\mathrm{d}}(\rho) = O(N) \quad \text{while} \quad \dot{\Sigma}(\rho) = O(1)$$

Tajima & Funo, PRL **127**, 190604 (2021)

- Dissipationless current requires $m_H(\rho) = O(N^2)$
- What is necessary for P_{ji}^{TMH} to get $m_X(\rho) > O(N)$ (*anomalous scaling*)?

Necessary conditions for dissipationless current

- $X = \sum_s x_s \sum_{j=1}^{N} \Pi_{s,j}$ with $\Pi_{s,j} = |s, j\rangle\langle s, j|$
- $m_X(\rho)$ cannot exhibit anomalous scaling unless *every* eigenbasis satisfies at least one of the following two conditions: KY and Hamazaki, arXiv:2508.14354

  (Q1): $\exists s, s'$ s.t. $\lim_{N\to\infty} \frac{\mathcal{J}_{s's}(\rho)}{N} = -\infty$ (negativity)

  (Q2): $\lim_{N\to\infty} \frac{\bar{\lambda}(\rho)}{N} = \infty$ (non-classical enhancement of "average escape rate")

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  (Q2): $\displaystyle\lim_{N\to\infty} \frac{\bar{\lambda}(\rho)}{N} = \infty$ (non-classical enhancement of "average escape rate")
- If $\rho = \sum_{s,j} p_{sj}\Pi_{s,j}$, with $R_{(s'j')(sj)} = \sum_k |\langle s', j'|L_k|s,j\rangle|^2$, $\lambda_{sj} = \sum_{s',j'(\neq s,j)} R_{(s'j')(sj)}$

$$\mathcal{J}_{s's}(\rho) = \begin{cases} \sum_{j,j'} R_{(s'j')(sj)} p_{sj} & (s \neq s') \\ \sum_{j,j'(\neq j)} R_{(sj')(sj)} p_{sj} & (s = s') \end{cases}, \quad \bar{\lambda}(\rho) = \sum_{s,j} \lambda_{sj} p_{sj}$$

Escape rate cannot grow faster than the number of evacuation targets increases

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  (Q2): $\lim_{N\to\infty} \frac{\bar{\lambda}(\rho)}{N} = \infty$ (non-classical enhancement of "average escape rate")
- Else $T_{s'j'sj}(\rho) := \frac{1}{2}\mathrm{tr}\left(\{\mathcal{L}^\dagger \Pi_{s',j'}, \Pi_{s,j}\}\rho\right) = \lim_{\Delta t\to 0} P^{\mathrm{TMH}}_{(s'j'),(sj)}/\Delta t$ and

$$\mathcal{J}_{s's}(\rho) = \begin{cases} \sum_{j,j'} T_{s'j'sj}(\rho) & (s \neq s') \\ \sum_{j,j'(\neq j)} T_{sj'sj}(\rho) & (s = s') \end{cases}, \quad \bar{\lambda}(\rho) = -\sum_{s,j} T_{sjsj}(\rho)$$

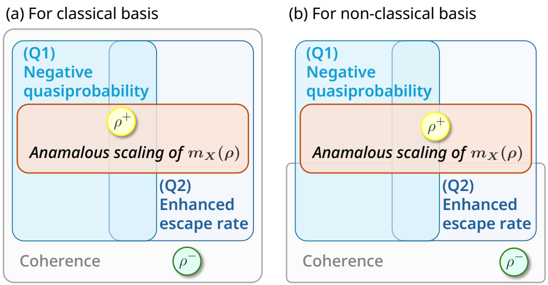
Relation to coherence

- For a given eigenbasis $\{|s,j\rangle\}$, $C_{l_1}(\rho) := \sum_{(s,j)\neq(s',j')} |\langle s,j|\rho|s',j'\rangle|$
- If $C_{l_1}(\rho) = O(1)$ regarding a classical basis, then (Q1) and (Q2) are violated
  - We regard a basis as classical if, as the number of states increases,

    (1) each jump rate $|\langle s',j'|L_k|s,j\rangle|$ does not increase,

    (2) the number of connections between every single pair of states does not increase

    $\#\{k \mid |\langle s',j'|L_k|s,j\rangle| > 0\} = O(1)$



Summary

- Derived a new quantum TUR via the Terletsky–Margenau–Hill quasiprobability
- Quasiprobabilities enable us to characterize dynamical fluctuations of observables in quantum dynamics
- We can connect two-types of non-classicality: quasiprobability's anomalous behavior and dissipationless current

Refs.

- KY and Hamazaki, arXiv:2508.14354 : quasiprobability TUR
- KY, Maekawa, Nagayama, and Ito, Phys. Rev. Res. 7, 013244 (2025) : technical basis

Outline of derivation

- Rewrite quantum thermodynamics KY et al., PRR **7**, 013244 (2025)

  e.g., Current $\mathbb{J}(\rho)$, Thermodynamic force $\mathbb{F}(\rho)$, Gradient $\nabla_{\mathbb{L}}$

- TUR-like inequality arises from a geometric representation of the EPR

$$|J_X^{\mathrm{d}}(\rho)|^2 \leq \dot{\Sigma}(\rho)\|\nabla_{\mathbb{L}}X\|_{\Gamma\otimes\rho}^2 \quad \text{(CS inequality)}$$

- Connect the fluctuation-like quantity $\|\nabla_{\mathbb{L}}X\|_{\Gamma\otimes\rho}^2$ to the quasiprobability

  KY and Hamazaki, arXiv:2508.14354