

Quantum geometric tensor determines  
the i.i.d. conversion rate  
in the resource theory of asymmetry  
for any compact Lie group

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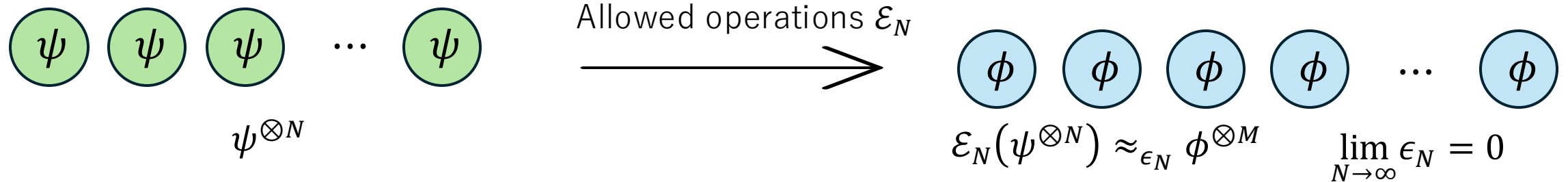
[[KY](#), Y. Mitsuhashi, T. Shitara, and H. Tajima arxiv:2411.04766]

# Introduction & Overview

- IID state conversion theory
  - a central question in resource theory
  - provides foundations for manipulating and quantifying resources
- The resource theory of asymmetry (RTA)
  - resource theory for quantifying symmetry breaking, including coherence
- We establish the IID pure-state conversion theory in RTA for any compact Lie group
- Application: The resource requirement for asymptotic state conversion in quantum thermodynamics

# IID conversion theory

A key problem in resource theories: asymptotic IID conversion theory



Conversion rate :  $R(\psi \rightarrow \phi) := \sup\{\text{conversion rate achievable with allowed operations}\}$

Why important? Because it addresses central questions:

- How to transform resource states?
- How to quantify resource?

Cf.) Entanglement theory: IID conversion rate via LOCCs  $R(\psi \rightarrow \phi) = \frac{S_{EE}(\psi)}{S_{EE}(\phi)}$

This fact establishes entanglement entropy as a standard measure of entanglement.

**Establishing the IID theory in RTA identifies a standard measure of symmetry breaking.**

# What is asymmetry (symmetry breaking)?

Energy coherence: superposition between energy eigenstates with different energy levels.

$$\text{Ex.) coherence bit } |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad H = E|1\rangle\langle 1|$$

More abstractly, we say that a state has energy coherence if it is *not* invariant under time-translation

$$\rho \text{ is asymmetric under time-translation iff } \exists t \quad \mathcal{U}_t(\rho) \neq \rho \quad \mathcal{U}_t(\rho) := e^{-iHt}\rho e^{iHt}$$

In general, under a symmetry described by a group  $G$ ,

$$\begin{aligned} \rho \text{ is asymmetric iff } & \exists g \in G \quad \mathcal{U}_g(\rho) \neq \rho & \mathcal{U}_g(\rho) := U(g)\rho U(g)^\dagger \\ \rho \text{ is symmetric iff } & \forall g \in G \quad \mathcal{U}_g(\rho) = \rho & U : \text{projective unitary rep.} \end{aligned}$$

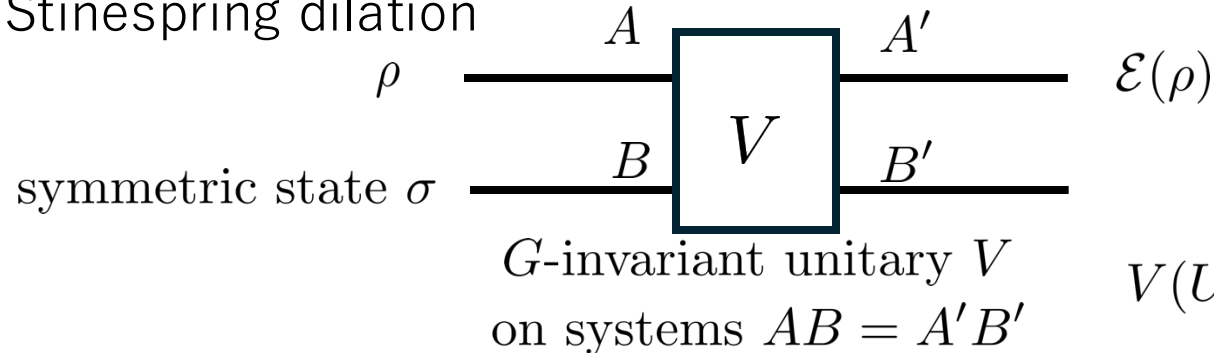
Resource Theory of Asymmetry (RTA) is a framework quantifying asymmetry(, including coherence).

# Free states and free operations in RTA

RTA is defined by

- Free states: symmetric states  $\rho$  s.t.  $\forall g \in G \quad \mathcal{U}_g(\rho) = \rho$
- Free operations:  $G$ -covariant operations  $\mathcal{E}$  is  $G$ -covariant  $\stackrel{\text{def.}}{\iff} \forall g \in G, \quad \mathcal{E} \circ \mathcal{U}_g = \mathcal{U}'_g \circ \mathcal{E}$

covariant Stinespring dilation



$$V(U_A(g) \otimes U_B(g)) = (U_{A'}(g) \otimes U_{B'}(g))V$$

This condition implies  $V(X_{A,\mu} + X_{B,\mu}) = (X_{A',\mu} + X_{B',\mu})V$   
 : conservation law

$\{X_\mu\}_{\mu=1}^{\dim G}$  : generators of infinitesimal transf.

- For example,
- time-translation symmetry:  $X = H$ , i.e., the Hamiltonian
  - $SO(3)$ -symmetry:  $X_i = L_i$ , i.e., angular momentum operators

# Main result: IID conv. theory in RTA

Quantum geometric tensor (QGT):

a matrix characterizing infinitesimal changes of a quantum state under parameter variations

[J. P. Provost and G. Vallee, Commun. Math. Phys. 6, 289 (1980)] [M. V. Berry, in *Geometric Phases in Physics*, 5 (1989)]

For pure states  $\mathcal{U}_g(\psi)$  parametrized by  $g \in G$ , the QGT  $\mathcal{Q}^\psi$  is a Hermitian matrix of  $\dim G \times \dim G$ ,

$$\mathcal{Q}_{\mu\nu}^\psi = \langle \psi | X_\mu X_\nu | \psi \rangle - \langle \psi | X_\mu | \psi \rangle \langle \psi | X_\nu | \psi \rangle \quad \{X_\mu\}_{\mu=1}^{\dim G} : \text{generators}$$

Theorem: conversion rate for pure states  $\psi$  and  $\phi$

In RTA for any connected compact Lie group  $G$ , if  $\text{Sym}_G(\psi) \subset \text{Sym}_G(\phi)$ ,

$\text{Sym}_G(\rho) := \{g \in G \mid \mathcal{U}_g(\rho) = \rho\}$  ( $\approx$  periods)

$$R(\psi \rightarrow \phi) = \sup\{r \geq 0 \mid \mathcal{Q}^\psi \geq r \mathcal{Q}^\phi\}$$

and otherwise  $R(\psi \rightarrow \phi) = 0$ .

[In our paper, formula is extended to any compact Lie group]

(cf.) In entanglement theory,  $R(\psi \rightarrow \phi) = \frac{S_{\text{EE}}(\psi)}{S_{\text{EE}}(\phi)} = \sup\{r \geq 0 \mid S_{\text{EE}}(\psi) \geq r S_{\text{EE}}(\phi)\}$

**In RTA, the QGT plays the same role as entanglement entropy does in entanglement theory.**

# Comparison with prior research

Theorem: conversion rate for pure states  $\psi$  and  $\phi$

In RTA for any connected compact Lie group  $G$ , if  $\text{Sym}_G(\psi) \subset \text{Sym}_G(\phi)$ ,

$$R(\psi \rightarrow \phi) = \sup\{r \geq 0 \mid Q^\psi \geq rQ^\phi\}$$

and otherwise  $R(\psi \rightarrow \phi) = 0$ .

$\text{Sym}_G(\rho) := \{g \in G \mid \mathcal{U}_g(\rho) = \rho\}$  ( $\approx$  periods)

[In our paper, formula is extended to any compact Lie group]

Prior studies: only few examples

- U(1): pure states w/ the same periods  
[Gour-Spekkens (2008)] [Marvian (2022)]
- SU(2): specific sets of pure states  
[Gour-Spekkens (2008)]
- SO(3): specific sets of pure states  
[Yang-Chiribella-Hu (2017)]
- Marvian-Spekkens conjecture:  
condition for reversible conversion  
[Marvian-Spekkens (2014)]

} Separate analyses  
of  
specific examples

} Unresolved for  
over a decade

Our result: general theory

- **Any** compact Lie group
- **Any** pure states
- Both **reversible and irreversible** conversion
  - MS conjecture is proven as a corollary

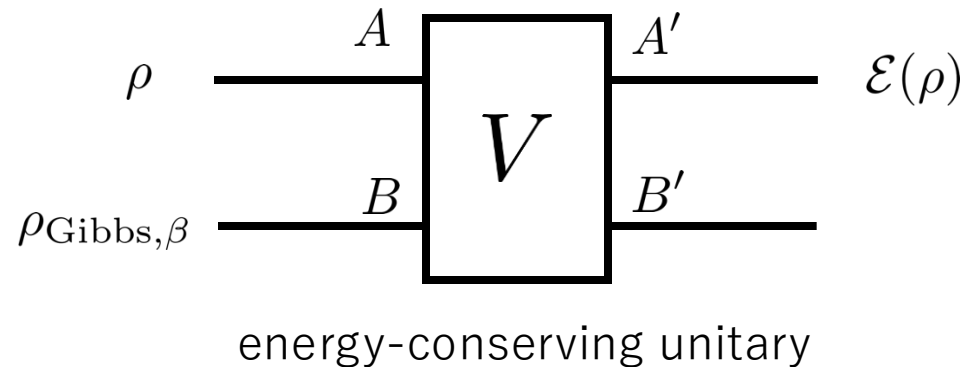
[KY, Y. Mitsuhashi, T. Shitara, and H. Tajima arxiv:2411.04766]

Our formula significantly extends and unifies previous work and resolves a long-standing conjecture.

# Application to quantum thermodynamics

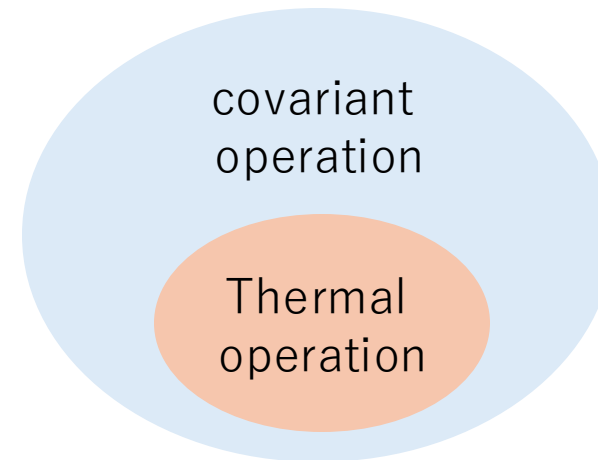
For simplicity, we will focus on time-translation symmetry.

Thermal operation: interaction w/ thermal bath



$\rho_{\text{Gibbs},\beta}$  is a symmetric state

$\Rightarrow$  any thermal operation is a covariant operation



- Thermal operations cannot generate energy coherence.
- An incoherent state cannot be converted to a coherent states via thermal operations.
- How much coherence is required to enable such conversions?

# Coherence requirement for reversibility

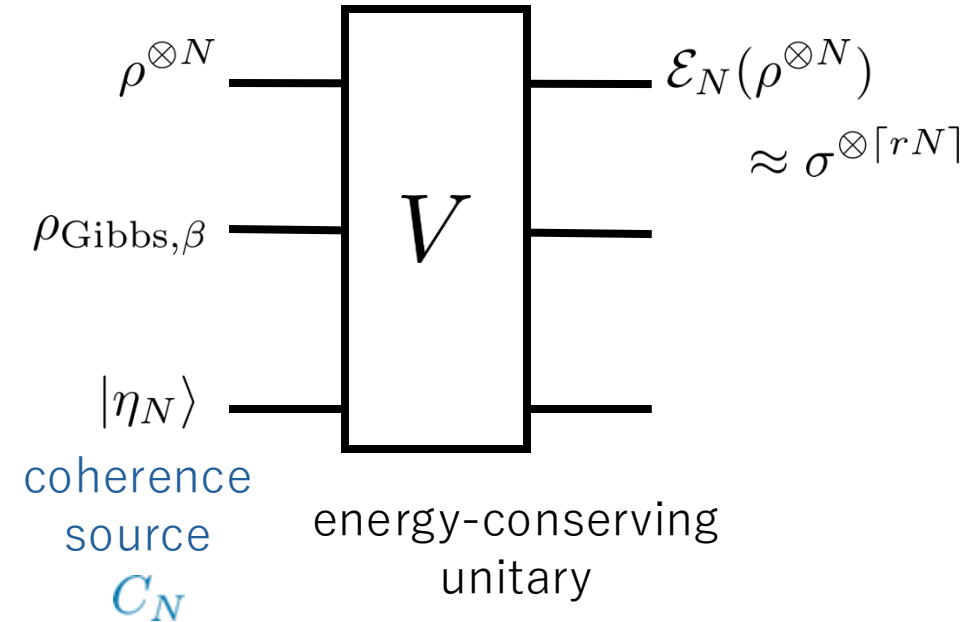
A generalized scenario: thermal bath + coherence source

“The amount of required coherence” is quantified by the energy range  $\|H_{C_N}\|_\infty$  of the coherence source system  $C_N$ .

Reversible conv. can be achieved with  $C_N$  s.t.  $\lim_{N \rightarrow \infty} \frac{\|H_{C_N}\|_\infty}{N} = 0$

[P. Faist, T. Sagawa, K. Kato, H. Nagaoka, and F. G. Brandão (2019)]

[T. Sagawa, P. Faist, K. Kato, K. Matsumoto, H. Nagaoka, and F. G. Brandão (2021)]



$\|H_{C_N}\|_\infty$  does not depend on the state  $|\eta_N\rangle$ , which therefore cannot be detailed information of coherence.

**Can we obtain more refined information by adopting the QGT as a measure of coherence?**

# Macroscopic coherence required for reversibility

A generalized scenario: thermal bath + coherence source

QGT for time-translation symmetry = energy variance

$$\text{Var}(\psi, H) := \langle \psi | H^2 | \psi \rangle - \langle \psi | H | \psi \rangle^2$$

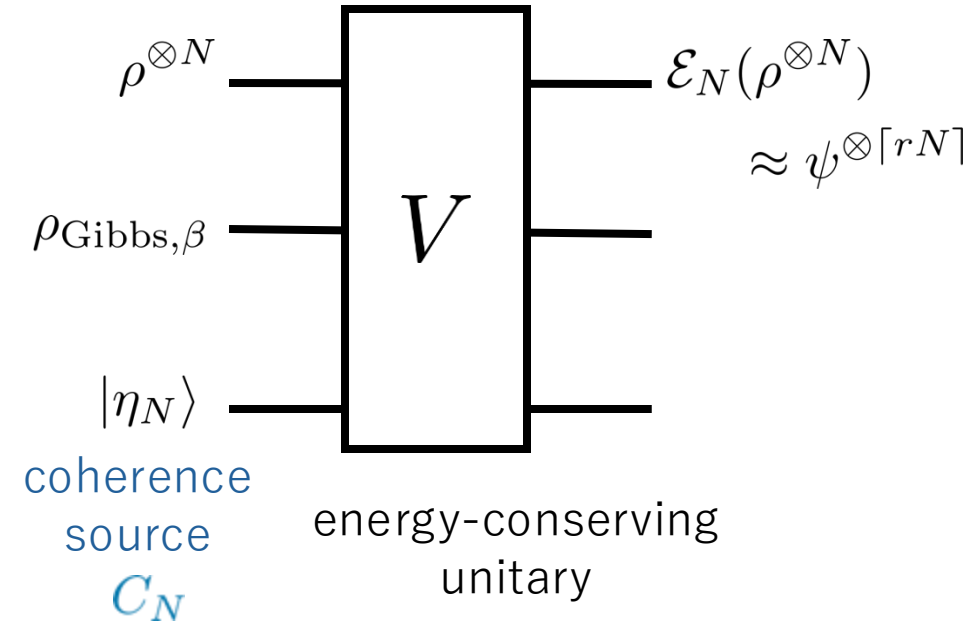
- Consider
- $\rho$  is full-rank
  - $\psi$  is a pure state

If  $\rho$  is convertible to  $\psi$  with a rate  $r > 0$ , then

$$\lim_{N \rightarrow \infty} \frac{\text{Var}(\eta_N, H_{C_N})}{N} \geq r \text{Var}(\psi, H)$$

In particular, when we prepare coherence bits as a coherence source, we need  $O(N)$  such bits.

**In quantum thermodynamics,  
there are infinitely many cases in which macroscopic coherence is required for state conversion.**



# Summary

- We have established the IID conversion theory in the RTA for any continuous symmetry described by a compact Lie group
- The QGT is the standard measure of asymmetry (i.e., the symmetry breaking of a state).
- Quantifying the coherence (i.e., time-translation asymmetry) with the QGT, we showed that a macroscopic amount of coherence is required for asymptotic conversion in quantum thermodynamics
- Takeaway: Let's quantify symmetry breaking with the QGT!