Duality between dissipation-coherence trade-off and thermodynamic speed limit for noisy oscillations

Based on [R. N. & Sosuke Ito, arXiv:2509.06421]

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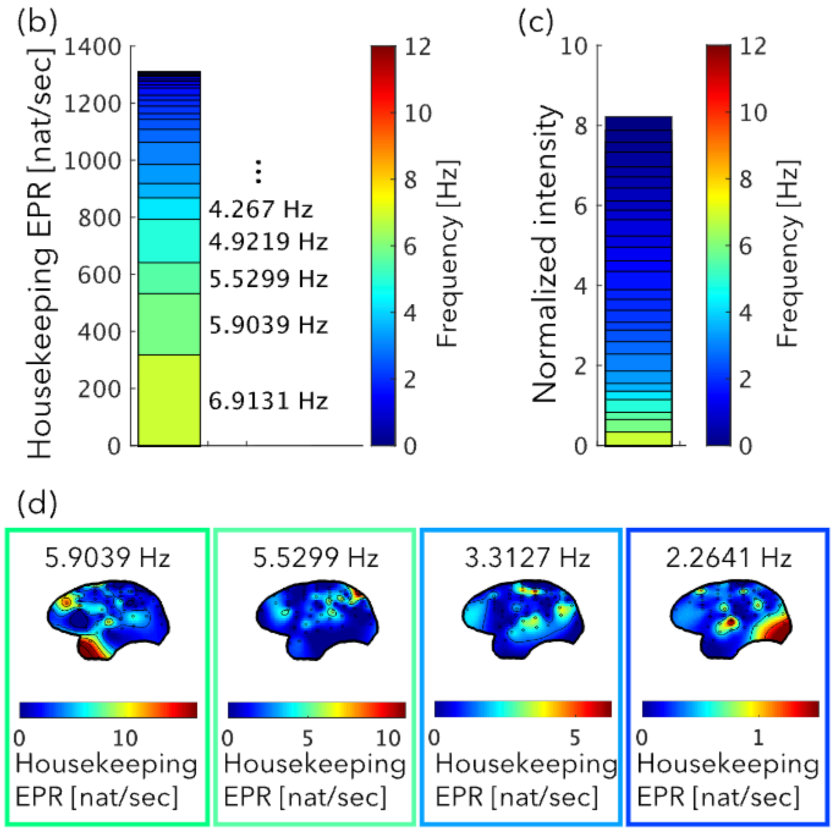
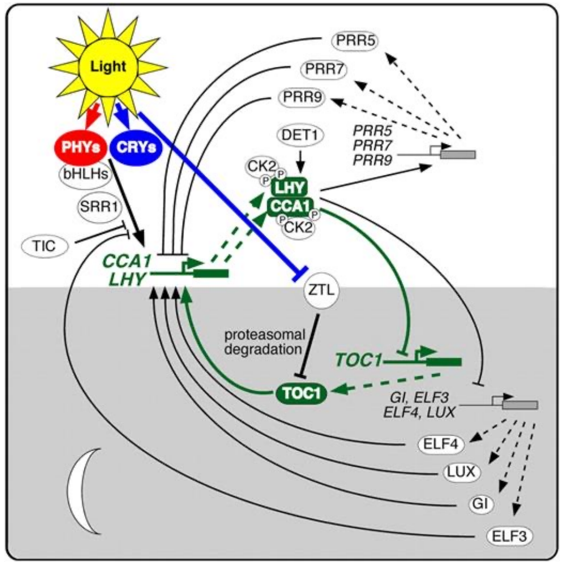
Introduction

Noisy oscillations are ubiquitous in biological phenomena

Brain & Consciousness



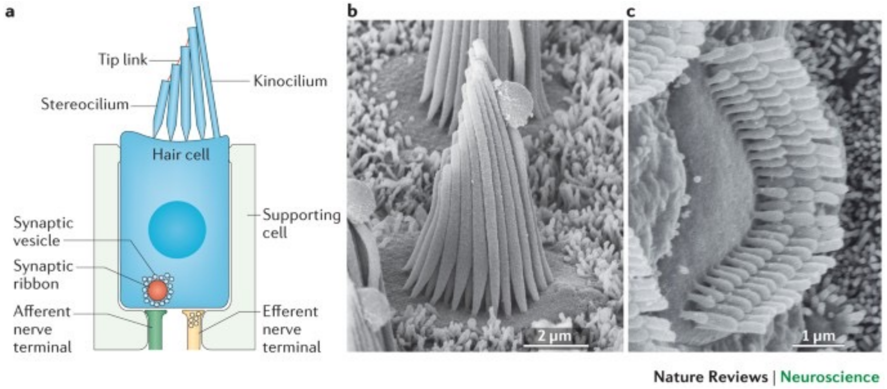
[D. Sekizawa, S. Ito, & M. Oizumi, PRX 2024]



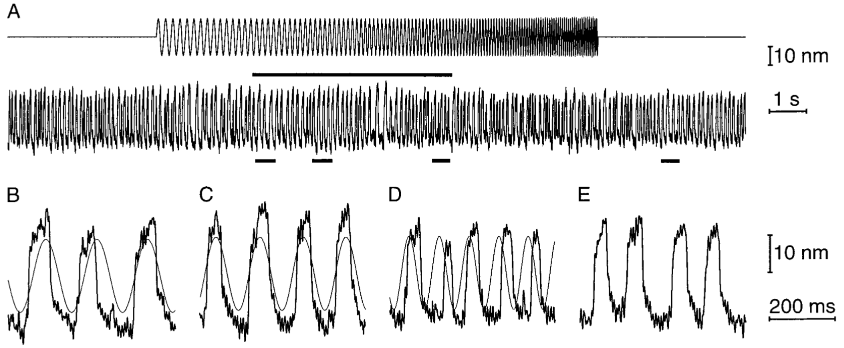
[C. R. McClung, Plant Cell 2006]

Circadian Rhythms

Hair-Cell Bundle



[A. J. Hudspeth, Nat. Rev. Neurosci. 2014]



[P. Martin & A. J. Hudspeth, PNAS 1999]

Introduction

Phase correlation (coherence) of the oscillation is essential for maintaining vital functions → measured with #coherent oscillations

$$\mathcal{N} := \tau_{\mathrm{c}} \times \tau_{\mathrm{p}}^{-1}$$

- τ_{c} : Correlation time

  $=$ characterizing the decay of the autocorrelation function $\langle a(\boldsymbol{x}_t) a(\boldsymbol{x}_0)\rangle_{\mathrm{st}} \sim \mathrm{e}^{-t/\tau_{\mathrm{c}}}$
- τ_{p} : Period of the oscillation (τ_{p}^{-1} is the frequency)
- $\mathcal{N}$: Number of oscillations that occur before the autocorrelations break down

Noise breaks coherence!!

Introduction

We need to pay thermodynamic cost to maintain the coherence

→ quantified by Dissipation-coherence trade-off (DCT)

$$\Sigma_{\tau_\mathrm{p}} \geq 4\pi^2\mathcal{N}$$

- Σ_{τ_p} : Entropy Production (EP) required for one oscillatory period
- Conjectured in Markov jump processes [L. Oberreiter, U. Seifert, & A. C. Barato, PRE 2022]
- Proved for stochastic limit cycles in the weak-noise regime under one of the following conditions:

  ① the diffusion coefficient matrix is proportional to the identity matrix

  ② near the critical point of the Hopf bifurcation [D. Santolin & G. Falasco, PRL 2025]

Questions

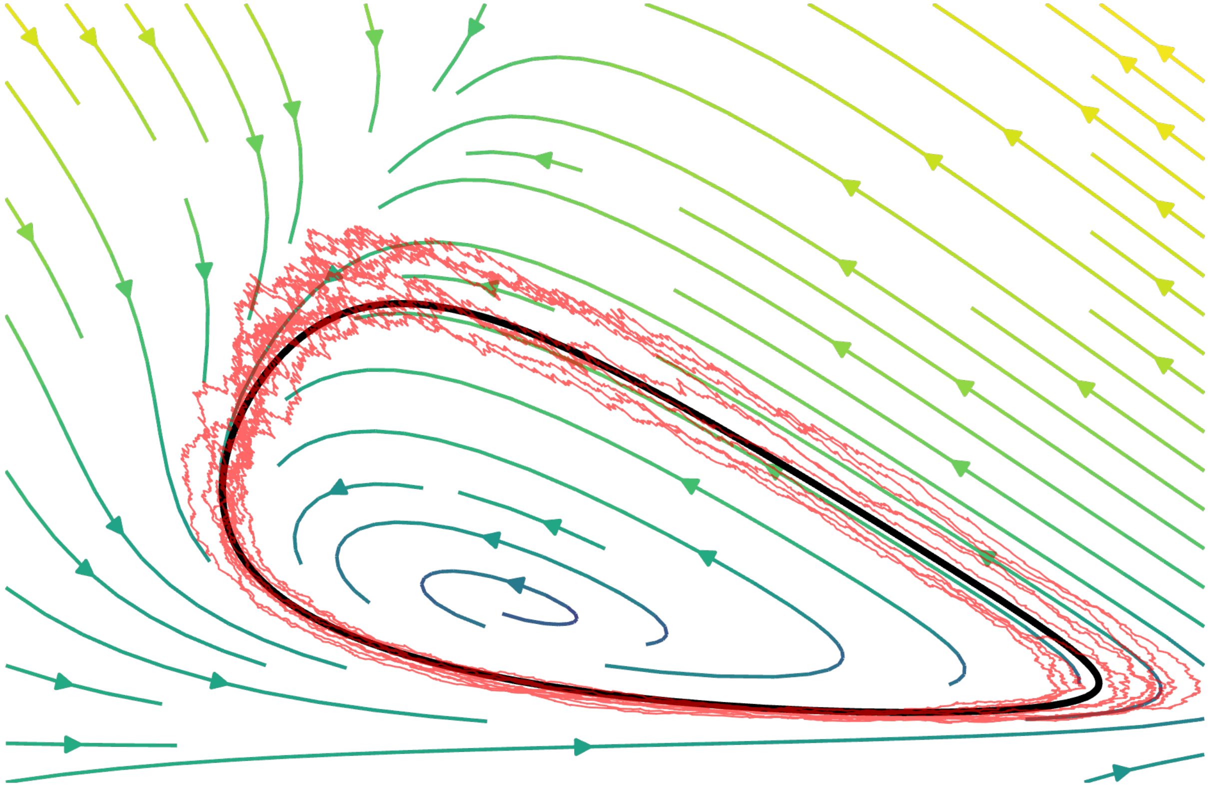
- **Can we derive the DCT for general stochastic limit cycles in the weak-noise regime?**
  - Conditions in the proof are too strong to apply stochastic chemical systems
  - Numerically, the DCT holds for the systems without the conditions

- **Is there any connection between the DCT and the well-known trade-offs?**
  - Thermodynamic uncertainty relation (TUR) [A. C. Barato & U. Seifert, PRL 2015]
  - Thermodynamic speed limit (TSL) [N. Shiraishi, K. Funo, & K. Saito, PRL 2018]

Setup: Stochastic limit cycle

Langevin eq.

$$\dot{\boldsymbol{x}}_t = \boldsymbol{F}(\boldsymbol{x}_t) + \sqrt{2\epsilon}\,\mathsf{G}(\boldsymbol{x}_t) \bullet \boldsymbol{\xi}_t$$



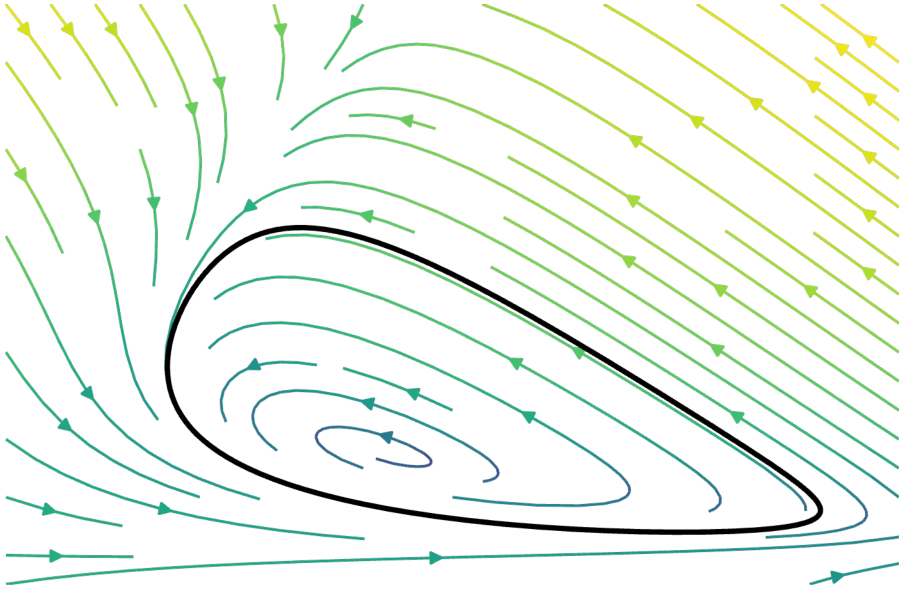
Setup: Stochastic limit cycle

Langevin eq. $$\dot{\boldsymbol{x}}_t = \boxed{\boldsymbol{F}(\boldsymbol{x}_t)} + \sqrt{2\epsilon}\mathsf{G}(\boldsymbol{x}_t) \bullet \boldsymbol{\xi}_t$$

First term: time evolution without noise

- Assumption: deterministic time evolution $\dot{x}_t = \boldsymbol{F}(x_t)$ has a stable limit cycle
- Period of the limit cycle: τ_p
- Limit cycle: $\{\boldsymbol{x}_t^{\mathrm{LC}}\}_{t\in\mathbb{R}}$

$$\dot{\boldsymbol{x}}_t^{\mathrm{LC}} = \boldsymbol{F}(\boldsymbol{x}_t^{\mathrm{LC}}),\ \boldsymbol{x}_{t+\tau_\mathrm{p}}^{\mathrm{LC}} = \boldsymbol{x}_t^{\mathrm{LC}}$$



Setup: Stochastic limit cycle

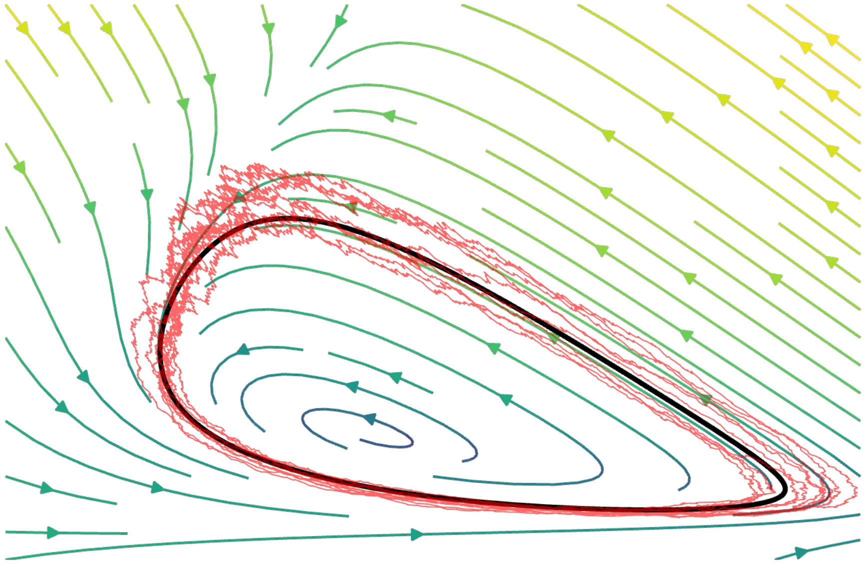
Langevin eq.

$$\dot{\boldsymbol{x}}_t = \boldsymbol{F}(\boldsymbol{x}_t) + \boxed{\sqrt{2\epsilon}\mathsf{G}(\boldsymbol{x}_t) \bullet \boldsymbol{\xi}_t}$$

Second term: effect of noise

- $\epsilon > 0$: Noise intensity
- $\mathsf{G}(\boldsymbol{x})$: Direction of noise (use Ito product)
- $\boldsymbol{\xi}_t$: White Gaussian noise

$$(\langle(\boldsymbol{\xi}_t)_\rho\rangle = 0,\ \langle(\boldsymbol{\xi}_t)_\rho(\boldsymbol{\xi}_{t'})_{\rho'}\rangle = \delta_{\rho\rho'}\delta(t-t'))$$



Setup: Stochastic limit cycle

Langevin eq.

$$\dot{\boldsymbol{x}}_t = \boldsymbol{F}(\boldsymbol{x}_t) + \sqrt{2\epsilon}\,\mathsf{G}(\boldsymbol{x}_t) \bullet \boldsymbol{\xi}_t$$

Fokker–Planck eq.

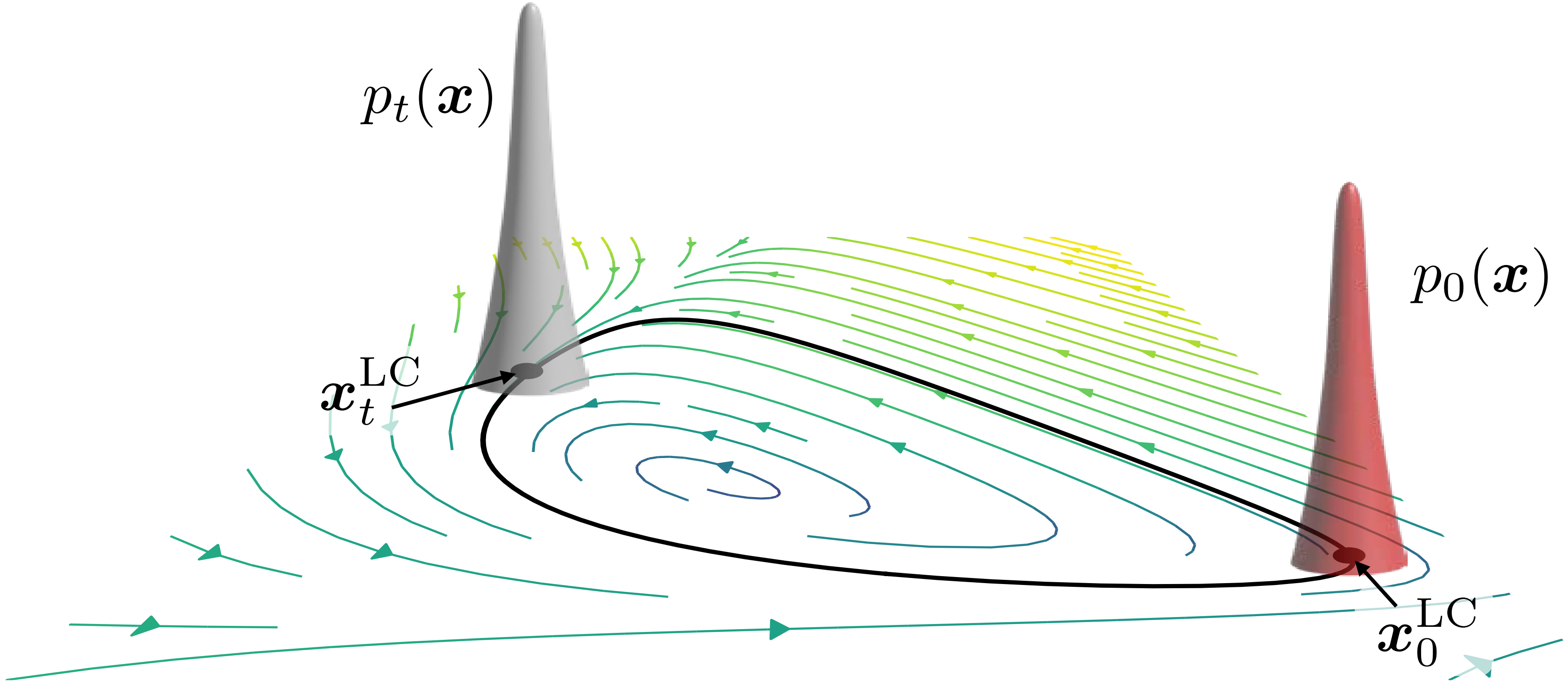
$$\partial_t p_t(\boldsymbol{x}) = -\boldsymbol{\nabla} \cdot \left(p_t(\boldsymbol{x})\boldsymbol{\nu}_t(\boldsymbol{x})\right)$$

- **Mean local velocity:** $\boldsymbol{\nu}_t(\boldsymbol{x}) := \boldsymbol{F}(\boldsymbol{x}) - \epsilon\boldsymbol{\nabla} \cdot \mathsf{D}(\boldsymbol{x}) - \epsilon\mathsf{D}(\boldsymbol{x})\boldsymbol{\nabla} \ln p_t(\boldsymbol{x})$
- **Diffusion coefficient matrix (assume positive-definiteness):** $\mathsf{D}(\boldsymbol{x}) := \mathsf{G}(\boldsymbol{x})\mathsf{G}(\boldsymbol{x})^\top$

Setup: Weak-noise regime

Fokker–Planck eq. $$\partial_t p_t(\boldsymbol{x}) = -\boldsymbol{\nabla} \cdot (p_t(\boldsymbol{x})\boldsymbol{\nu}_t(\boldsymbol{x}))$$

- **Consider $\epsilon \to 0$ & assume that $p_0(x)$ concentrates on x_0^{LC}**
  → $p_t(\boldsymbol{x})$ $(t \in [0, \tau_{\mathrm{p}}])$ **also concentrates on $\boldsymbol{x}_t^{\mathrm{LC}}$**



Setup: Weak-noise regime

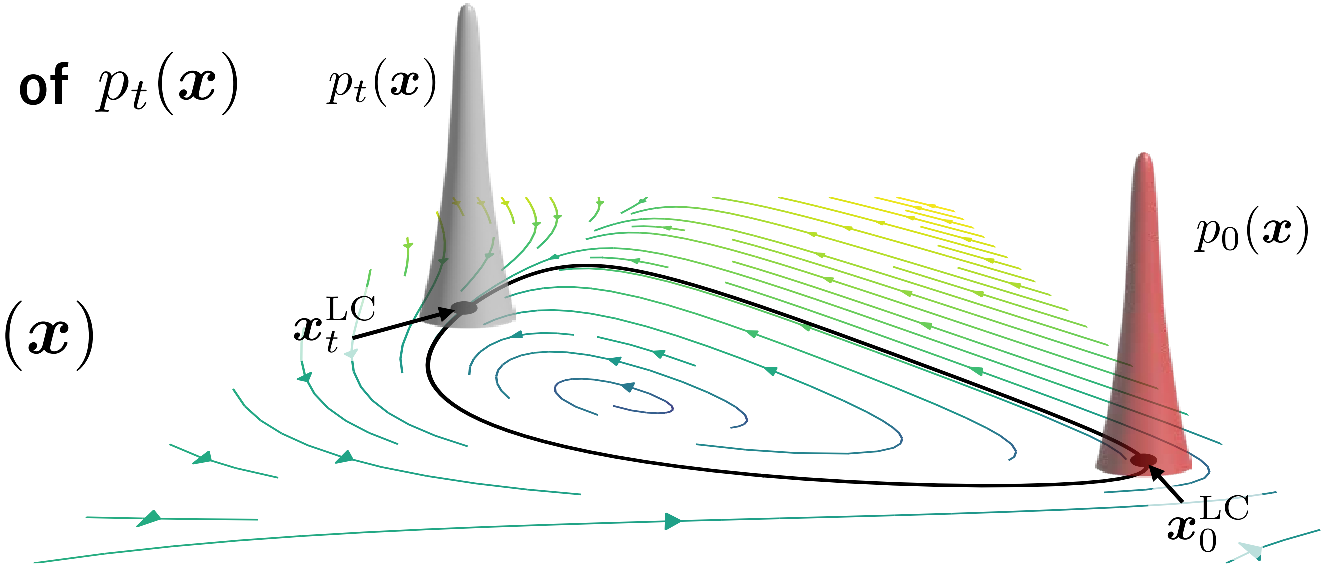
EP required for one period of the oscillation ($\epsilon \to 0$)

$$\Sigma_{\tau_\mathrm{p}} = \frac{1}{\epsilon} \int_0^{\tau_\mathrm{p}} dt\, \boldsymbol{F}(\boldsymbol{x}_t^{\mathrm{LC}})^\top \mathsf{D}(\boldsymbol{x}_t^{\mathrm{LC}})^{-1} \boldsymbol{F}(\boldsymbol{x}_t^{\mathrm{LC}})$$

- EP $=$ entropy change in heat bath $+$ change in system's Shannon entropy
- Regarded as a thermodynamic cost
- Derivation ⋯ Use the large deviation form of $p_t(\boldsymbol{x})$

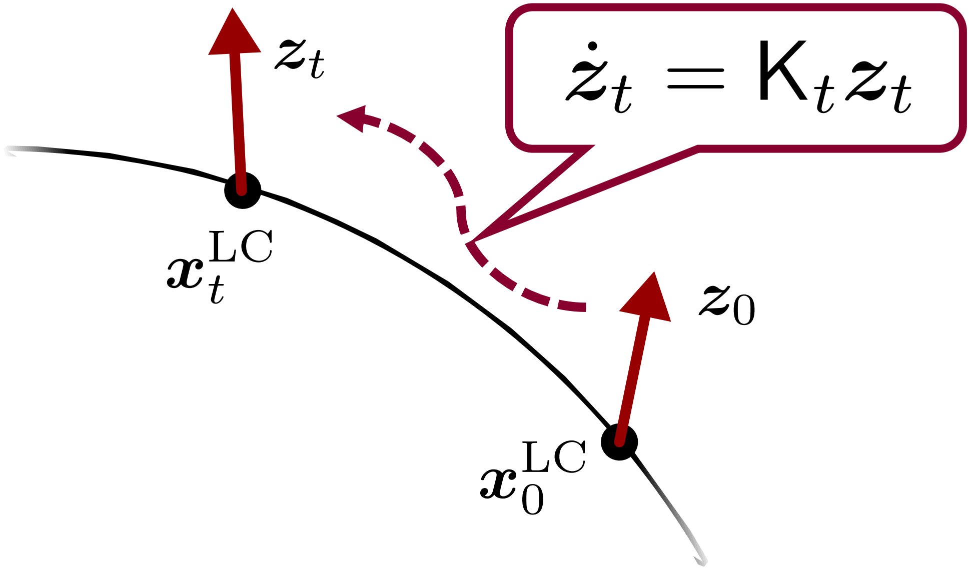
  & Laplace approximation in the EP rate

$$\dot{\Sigma}_t = \frac{1}{\epsilon} \int d\boldsymbol{x}\, p_t(\boldsymbol{x}) \boldsymbol{\nu}_t(\boldsymbol{x})^\top \mathsf{D}(\boldsymbol{x})^{-1} \boldsymbol{\nu}_t(\boldsymbol{x})$$



Setup: Stability of limit cycle

Deterministic time evolution of small deviation from the limit cycle

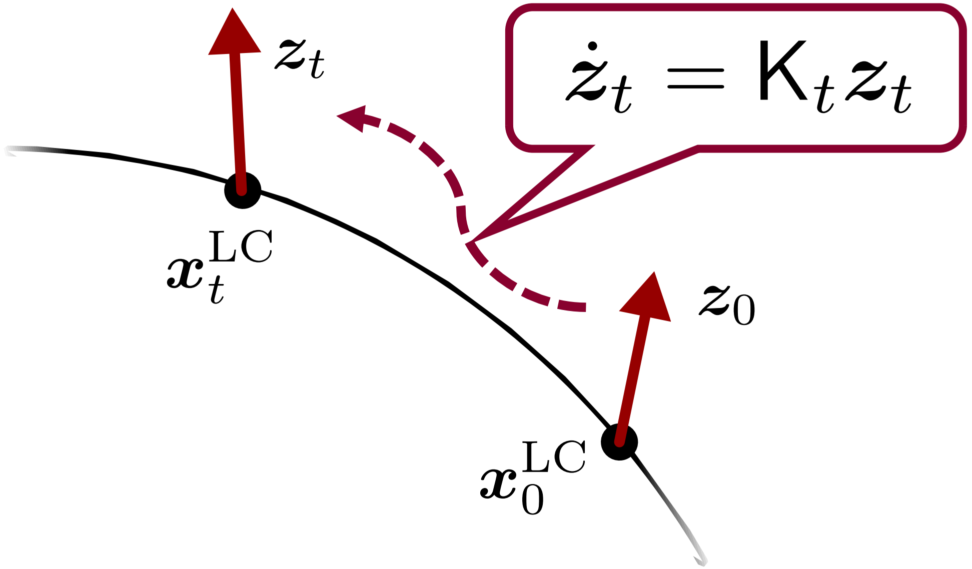


- $[\mathsf{K}_t]_{ij} := \partial_{x_i} F_j(\boldsymbol{x}_t^{\mathrm{LC}})$ **: Jacobian on the limit cycle**
- **Solved by the fundamental matrix as** $\boldsymbol{z}_t = \Phi_t \boldsymbol{z}_0$

  Φ_t **: solution of** $\Phi_0 = \mathsf{I},\ \dot{\Phi}_t = \mathsf{K}_t \Phi_t$

Setup: Stability of limit cycle

Deterministic time evolution of small deviation from the limit cycle



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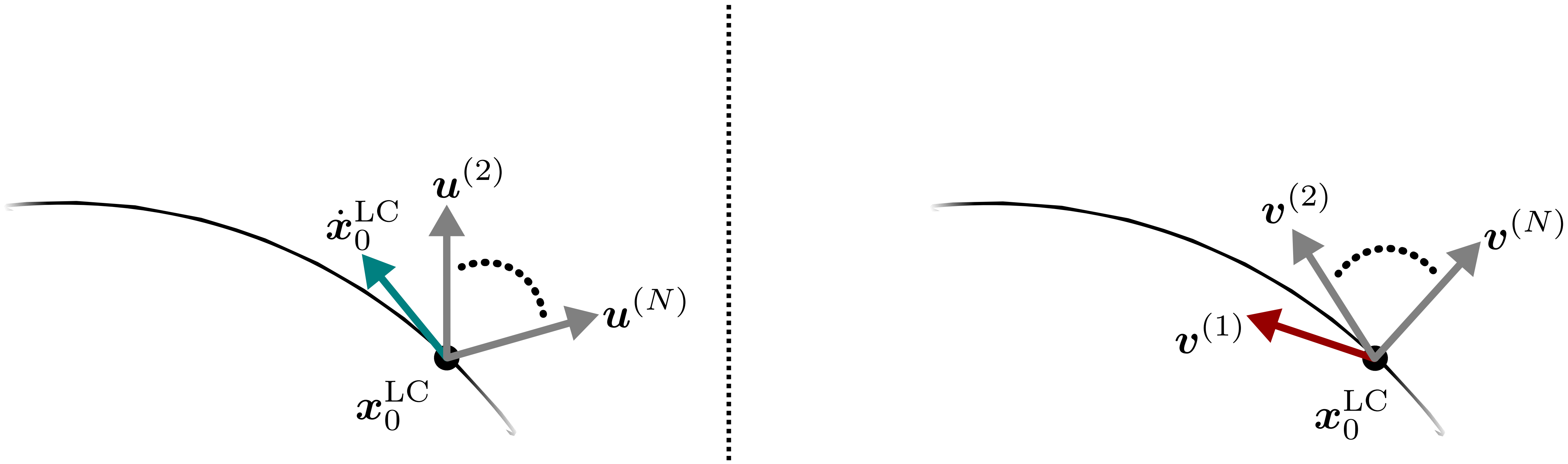
Stability is characterized by the monodromy matrix $\Phi_{\tau_{\mathrm{p}}}$

Spectral decomp. $\Phi_{\tau_{\mathrm{p}}} = \dot{\boldsymbol{x}}_0^{\mathrm{LC}} \boldsymbol{v}^{(1)\top} + \sum_{i=2}^{N} \lambda_i \boldsymbol{u}^{(i)} \boldsymbol{v}^{(i)\top}\ \ (|\lambda_i| < 1)$

Deviations perpendicular to the limit cycle decay after one period

Setup: Stability of limit cycle

$\Phi_{\tau_\mathrm{p}} = \dot{\boldsymbol{x}}_0^{\mathrm{LC}} \boldsymbol{v}^{(1)\top} + \sum_{i=2}^{N} \lambda_i \boldsymbol{u}^{(i)} \boldsymbol{v}^{(i)\top}$ **provides dual bases**



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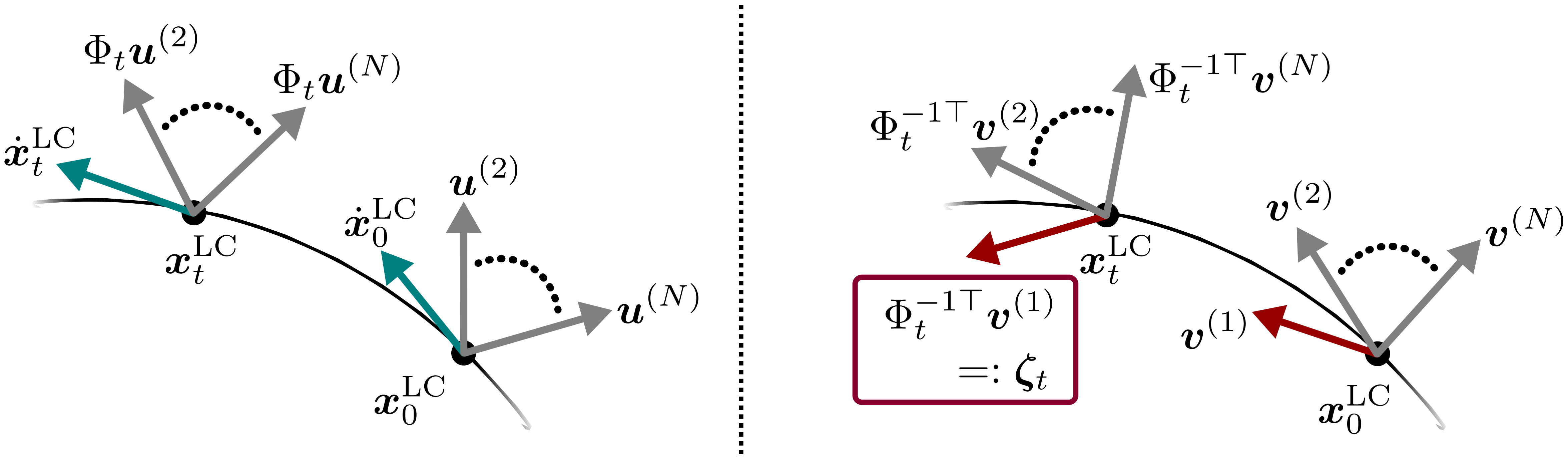
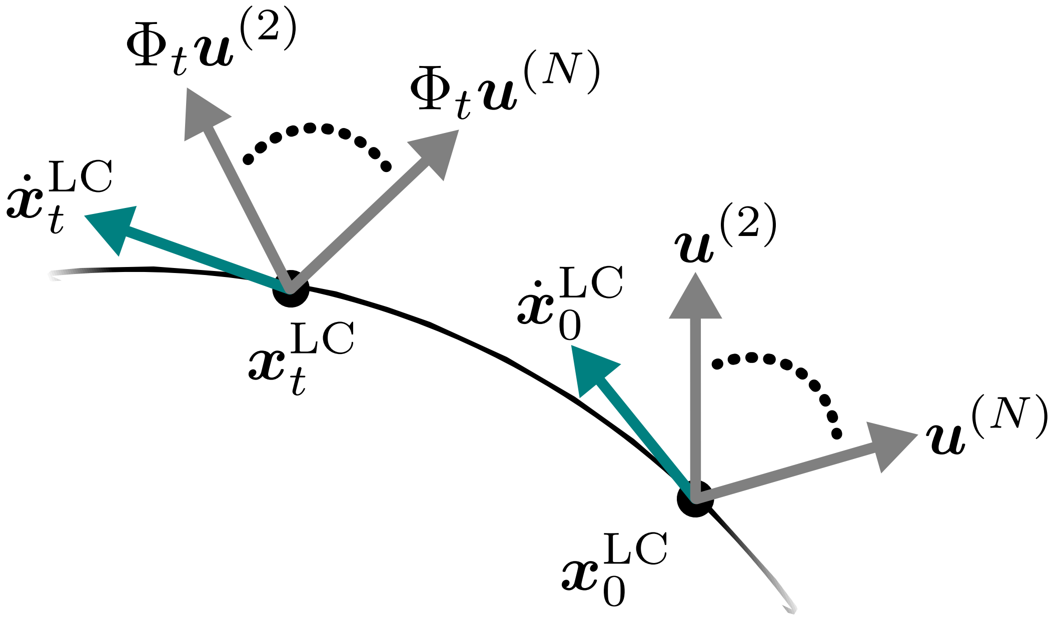
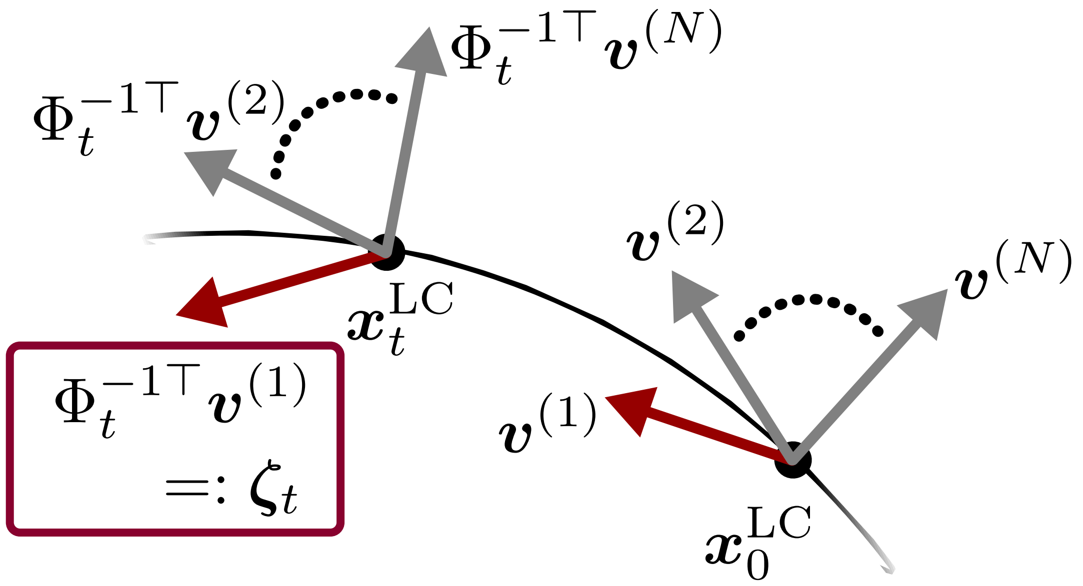


- Chain rule in $\ddot{\boldsymbol{x}}_t^{\mathrm{LC}} = d_t[\boldsymbol{F}(\boldsymbol{x}_t^{\mathrm{LC}})] \rightarrow \ddot{\boldsymbol{x}}_t^{\mathrm{LC}} = \mathsf{K}_t \dot{\boldsymbol{x}}_t^{\mathrm{LC}} \rightarrow \dot{\boldsymbol{x}}_t^{\mathrm{LC}} = \Phi_t \dot{\boldsymbol{x}}_0^{\mathrm{LC}}$
- Biorthogonality b/w right & left eigenvectors → Biorthogonality b/w $\{\Phi_t \boldsymbol{u}^{(n)}\}$ & $\{\Phi_t^{-1\top} \boldsymbol{v}^{(n)}\}$
- $\zeta_t := \Phi_t^{-1\top} \boldsymbol{v}^{(1)}$ is the dual of $\Phi_t \dot{\boldsymbol{x}}_0^{\mathrm{LC}} = \dot{\boldsymbol{x}}_t^{\mathrm{LC}}$

Setup: Stability of limit cycle

$\Phi_{\tau_{\mathrm{p}}} = \dot{\boldsymbol{x}}_0^{\mathrm{LC}} \boldsymbol{v}^{(1)\top} + \sum_{i=2}^{N} \lambda_i \boldsymbol{u}^{(i)} \boldsymbol{v}^{(i)\top}$ provides dual bases





$\Phi_t^{-1\top} \boldsymbol{v}^{(1)} =: \zeta_t$

Correlation time

$$\tau_{\mathrm{c}} = \frac{\tau_{\mathrm{p}}^3}{4\pi^2\epsilon} \left[\int_0^{\tau_{\mathrm{p}}} dt\, \boldsymbol{\zeta}_t^\top \mathsf{D}(\boldsymbol{x}_t^{\mathrm{LC}}) \boldsymbol{\zeta}_t \right]^{-1}$$

[P. Gaspard, J. Stat. Phys. 2002]

Stability determines the time until correlation breaks down

Method: Short time TUR

$$\dot{\Sigma}_t \geq \frac{2\mathbb{E}[J_t^{\boldsymbol{\omega}}]^2}{\Delta t \mathrm{Var}[J_t^{\boldsymbol{\omega}}]}$$

[S. Otsubo, S. Ito, A. Dechant, & T. Sagawa, PRE 2020]

- **Current of observable $\omega(x)$ in the short-time limit $\Delta t \to 0$: $J_t^{\boldsymbol{\omega}} \Delta t = \boldsymbol{\omega}(\boldsymbol{x}_t) \circ d\boldsymbol{x}_t$**

* Use the Stratonovich product

Method: Short time TUR

$$\dot{\Sigma}_t \geq \frac{2\mathbb{E}[J_t^{\boldsymbol{\omega}}]^2}{\Delta t \, \mathrm{Var}[J_t^{\boldsymbol{\omega}}]}$$

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* Use the Stratonovich product

Use weak-noise regime & Perform time integration

$$\Sigma_{\tau_{\mathrm{p}}} \geq \frac{\left[\int_0^{\tau_{\mathrm{p}}} dt \, \boldsymbol{\omega}(\boldsymbol{x}_t^{\mathrm{LC}})^\top \boldsymbol{F}(\boldsymbol{x}_t^{\mathrm{LC}})\right]^2}{\epsilon \int_0^{\tau_{\mathrm{p}}} dt \, \boldsymbol{\omega}(\boldsymbol{x}_t^{\mathrm{LC}})^\top \mathsf{D}(\boldsymbol{x}_t^{\mathrm{LC}}) \boldsymbol{\omega}(\boldsymbol{x}_t^{\mathrm{LC}})}$$

Result I: Thermodynamic speed limit

Using $\boldsymbol{F}(\boldsymbol{x}_t^{\mathrm{LC}})$ as $\omega(\boldsymbol{x}_t^{\mathrm{LC}})$ ⋯

$$\Sigma_{\tau_{\mathrm{p}}} \geq \frac{l_{\mathrm{LC}}^2}{\tau_{\mathrm{p}} D_{\mathrm{LC}}}$$

- **Euclidean length of the limit cycle:** $l_{\mathrm{LC}} := \int_0^{\tau_{\mathrm{p}}} dt\, \|\dot{\boldsymbol{x}}_t^{\mathrm{LC}}\| = \int_0^{\tau_{\mathrm{p}}} dt\, \|\boldsymbol{F}(\boldsymbol{x}_t^{\mathrm{LC}})\|$
- **Diffusion intensity along the limit cycle:** $D_{\mathrm{LC}} := \dfrac{\epsilon \int_0^{\tau_{\mathrm{p}}} dt\, \dot{\boldsymbol{x}}_t^{\mathrm{LC}\top} \mathsf{D}(\boldsymbol{x}_t^{\mathrm{LC}}) \dot{\boldsymbol{x}}_t^{\mathrm{LC}}}{\int_0^{\tau_{\mathrm{p}}} dt\, \|\dot{\boldsymbol{x}}_t^{\mathrm{LC}}\|^2}$

Large-amplitude or high-frequency oscillations → More dissipation

Result II: Dissipation-coherence trade-off

Using ζ_t as $\omega(\boldsymbol{x}_t^{\mathrm{LC}})$ ⋯

$$\Sigma_{\tau_{\mathrm{p}}} \geq 4\pi^2 \mathcal{N}$$

Derivation: use $\zeta_t^{\top} \dot{\boldsymbol{x}}_t^{\mathrm{LC}} = 1$ and the closed form of $\mathcal{N} = \tau_{\mathrm{c}} \times \tau_{\mathrm{p}}^{-1}$ in the time-integrated TUR

- Applicable even when $\mathsf{D}(\boldsymbol{x})$ is not proportional to the identity matrix
- Applicable far from the critical point of the Hopf bifurcation
- ζ_t is the dual of $\dot{\boldsymbol{x}}_t^{\mathrm{LC}} = \boldsymbol{F}(\boldsymbol{x}_t^{\mathrm{LC}})$ → DCT is the dual of the TSL in terms of TUR's observable

Maintaining coherence for a longer time → More dissipation

Example: Noisy Rössler model

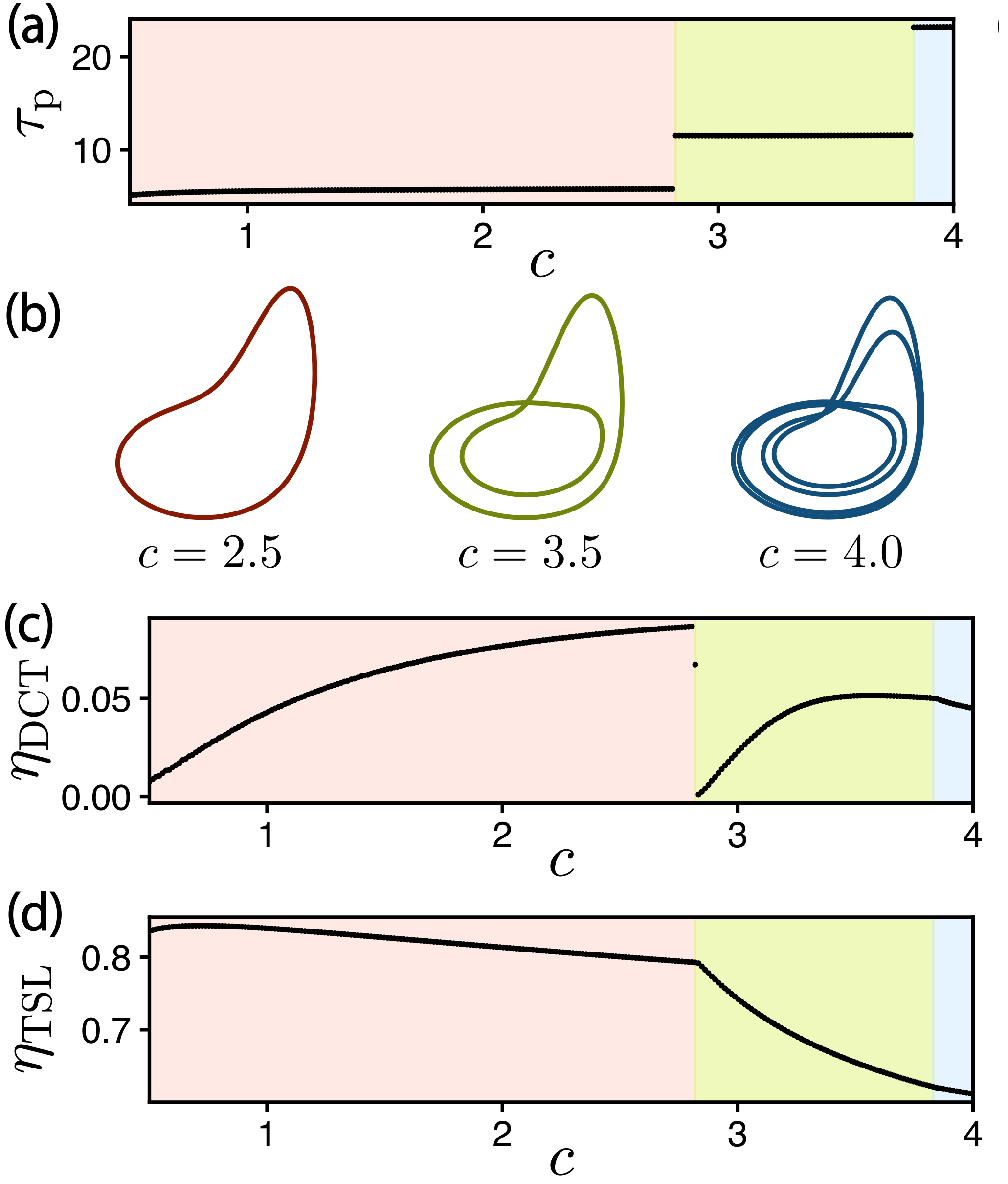
$$\boldsymbol{F}(\boldsymbol{x}) := \begin{pmatrix} -x_2 - x_3 \\ x_1 + 0.2x_2 \\ 0.2 + x_1 x_3 - c x_3 \end{pmatrix}, \quad \mathsf{D} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$c \in [0.5, 40]$ **: Period-doubling bifurcation**

Efficiencies: $\eta_{\mathrm{DCT}} := \dfrac{4\pi^2 \mathcal{N}}{\Sigma_{\tau_{\mathrm{p}}}}, \ \eta_{\mathrm{TSL}} := \dfrac{l_{\mathrm{LC}}^2}{\tau_{\mathrm{p}} D_{\mathrm{LC}} \Sigma_{\tau_{\mathrm{p}}}}$

- **Before the first bifurcation**
  η_{DCT} **increases, while** η_{TSL} **decreases**
- **At the first bifurcation**
  η_{DCT} **changes discontinuously,**
  while η_{TSL} **changes continuously**

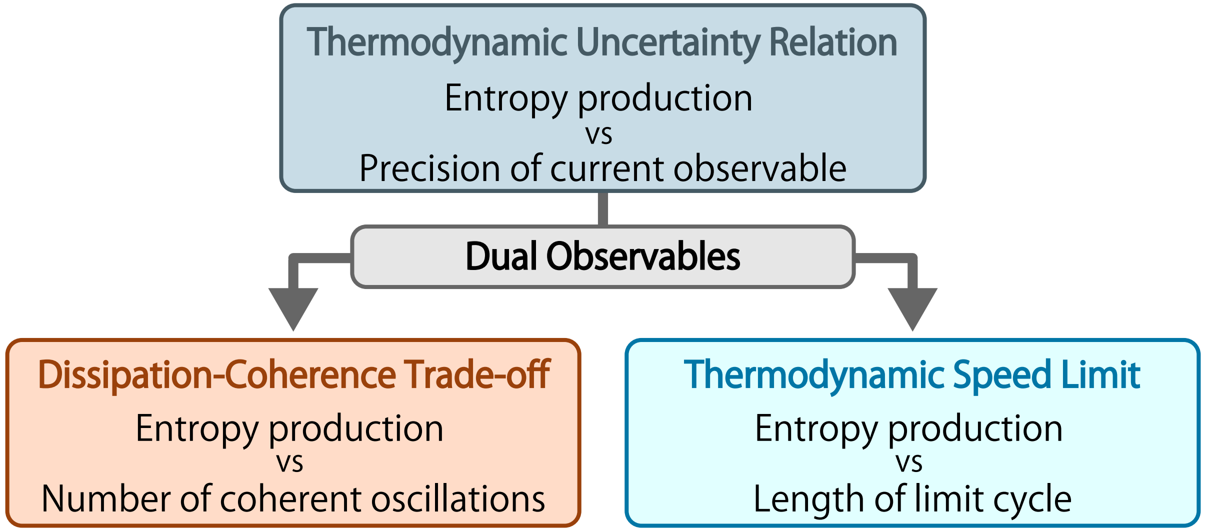
Different trade-off → Different behavior



Summary

- We derived the DCT in the weak-noise limit without further assumptions
- We also derived a new TSL as the dual of the DCT based on TUR

- The bounds are also generalized to chemical systems with positive semi-definite diffusion coefficient matrices
- Another proof with finite-time TUR & steady-state EP [A. Kolchinsky, arXiv:2510.14101]



Thermodynamic Uncertainty Relation
Entropy production
vs
Precision of current observable

Dual Observables

Dissipation-Coherence Trade-off
Entropy production
vs
Number of coherent oscillations

Thermodynamic Speed Limit
Entropy production
vs
Length of limit cycle

Appendix

Application to chemical systems

Chemical Langevin eq. $\dot{\boldsymbol{x}}_t = \mathsf{S}\boldsymbol{j}(\boldsymbol{x}_t) + \sqrt{2\epsilon}\mathsf{S}\mathsf{A}^{\frac{1}{2}}(\boldsymbol{x}_t) \bullet \boldsymbol{\xi}_t$

- $(\boldsymbol{x}_t)_\alpha$: Concentration of species α at time t
- $(\mathsf{S})_{\alpha\rho}$: Stoichiometric matrix (Net increase in #molecules of species α through reaction ρ)
- $(\boldsymbol{j})_\rho$: Net current of reaction ρ (Decomposed into fw/rev fluxes as $(\boldsymbol{j})_\rho = j_\rho^+ - j_\rho^-$)
- ϵ^{-1} : System size (weak-noise regime = macroscopic limit)
- $(\mathsf{A})_{\rho\rho'} := \delta_{\rho\rho'}(j_\rho^+ + j_\rho^-)/2$: Dynamical activity matrix
- $\mathsf{D} = \mathsf{S}\mathsf{A}\mathsf{S}^\top$: Scaled diffusion coefficient matrix

Some eigenvalues of the scaled diffusion coefficient matrix can be 0 due to the linear conservation laws in $\ker \mathsf{S}^\top$

→ Impossible to define D^{-1} and the EP

Application to chemical systems

Some eigenvalues of the scaled diffusion coefficient matrix can be 0 due to the linear conservation laws in $\ker \mathsf{S}^\top$

→ Impossible to define D^{-1} and the EP

Reduction of degrees of freedom

- $N_L :=$ **#Independent conservation laws** $= \dim(\ker \mathsf{S}^\top)$
- **The conservation laws determine the time evolution of N_L species using other $N - N_L$**
- **Removing these species →** D **becomes positive-definite**
- **EP for reduced Langevin eq. is a lower bound of the EP for corresponding chemical master eq.**
- **Trade-offs are consequences of the TUR for deterministic chemical systems**

[K. Yoshimura & S. Ito, PRL 2021]

Application to chemical systems

$$A \rightleftharpoons X_1 + X_3,\;\; X_1 + X_3 \rightleftharpoons X_2$$

$$2X_1 + X_2 \rightleftharpoons 3X_1 + X_3$$

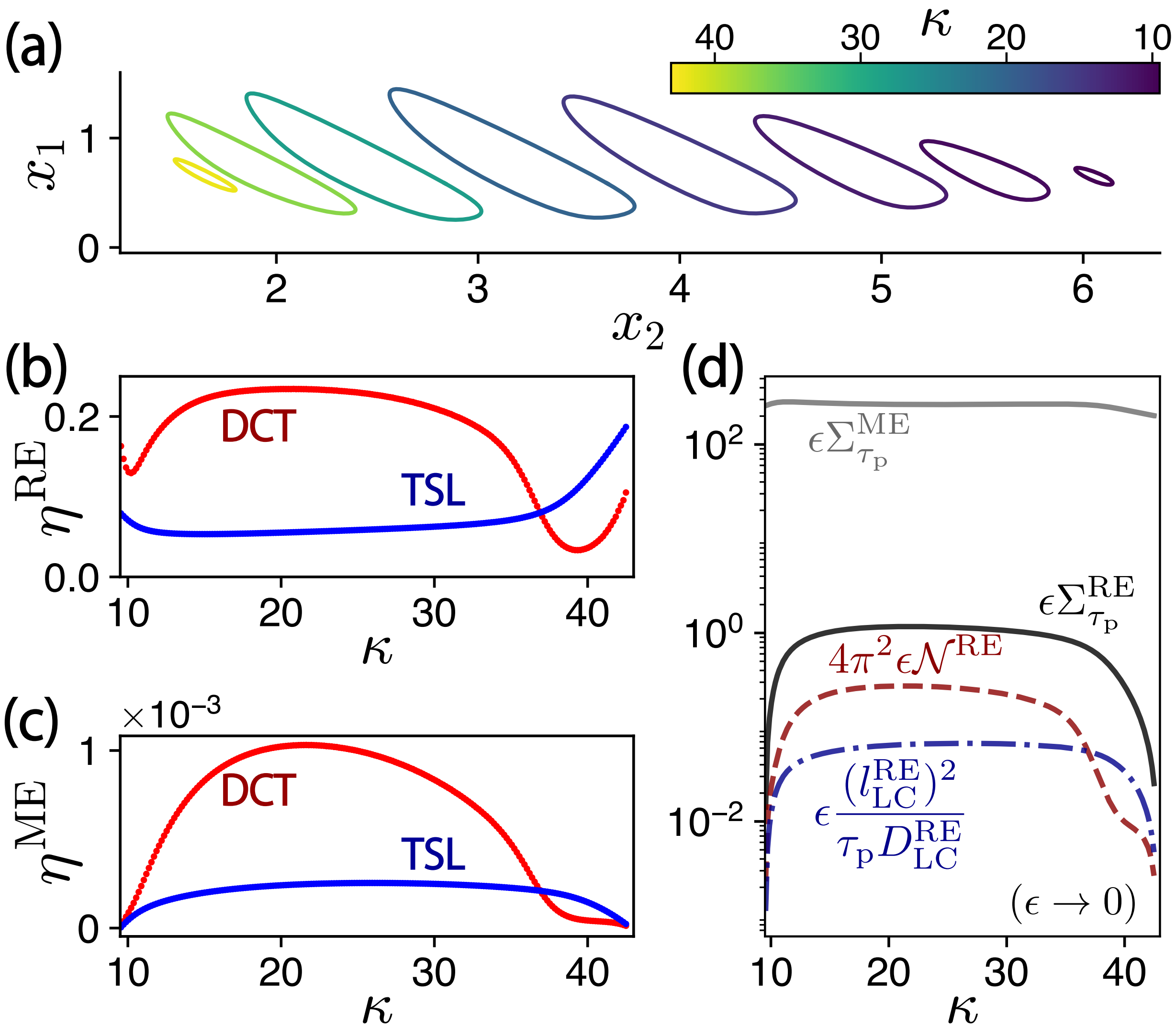
$$\mathsf{S} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix},\quad \boldsymbol{j}(\boldsymbol{x}) = \begin{pmatrix} 1 - x_1 x_3 \\ 30 x_1 x_3 - x_2 \\ \kappa x_1^2 x_2 - x_1^3 x_3 \end{pmatrix}$$

$x_1 - x_3$ **is conserved → Reduce** x_3

The EP for chemical master eq.

$$\Sigma^{\mathrm{ME}}_{\tau_{\mathrm{p}}} = \frac{1}{\epsilon}\int_0^{\tau_{\mathrm{p}}} dt \sum_{\rho=1}^{M} j_\rho(\boldsymbol{x}^{\mathrm{LC}}_t) \ln \frac{j^+_\rho(\boldsymbol{x}^{\mathrm{LC}}_t)}{j^-_\rho(\boldsymbol{x}^{\mathrm{LC}}_t)}$$

[D. Santolin & G. Falasco, PRL 2025]



Sketch of the derivation of correlation time

- Conventional derivations: Gaussian approximation of the rate function + Hamilton–Jacobi
- Without Large deviation = Gaussian approximation is enough [Y. –C. Cheng & H. Qian, J. Stat. Phys. 2021]

STEP1: Definition of correlation function

$$\langle a(\boldsymbol{x}_t)a(\boldsymbol{x}_0)\rangle_{\mathrm{st}} = \int d\boldsymbol{x} \int d\boldsymbol{x}'\, a(\boldsymbol{x})a(\boldsymbol{x}')p_t(\boldsymbol{x}|\boldsymbol{x}')p^{\mathrm{st}}(\boldsymbol{x}')$$

Steady-state distribution in the weak-noise limit

$$p^{\mathrm{st}}(\boldsymbol{x}) = \frac{1}{\tau_{\mathrm{p}}}\int_0^{\tau_{\mathrm{p}}} dt\, \delta(\boldsymbol{x}-\boldsymbol{x}_t^{\mathrm{LC}})$$

$$\langle a(\boldsymbol{x}_t)a(\boldsymbol{x}_0)\rangle_{\mathrm{st}} = \frac{1}{\tau_{\mathrm{p}}}\int d\boldsymbol{x} \int_0^{\tau_{\mathrm{p}}} dt'\, a(\boldsymbol{x})a(\boldsymbol{x}_{t'}^{\mathrm{LC}})p_t(\boldsymbol{x}|\boldsymbol{x}_{t'}^{\mathrm{LC}})$$

Sketch of the derivation of correlation time

STEP2: Gaussian approximation of the propagator

$$\langle a(\boldsymbol{x}_t)a(\boldsymbol{x}_0)\rangle_{\mathrm{st}} = \frac{1}{\tau_{\mathrm{p}}}\int d\boldsymbol{x}\int_0^{\tau_{\mathrm{p}}} dt'\, a(\boldsymbol{x})a(\boldsymbol{x}^{\mathrm{LC}}_{t'})\underline{p_t(\boldsymbol{x}|\boldsymbol{x}^{\mathrm{LC}}_{t'})}$$

- Starting from $\boldsymbol{x}^{\mathrm{LC}}_{t'}$, the state at time t is in the neighborhood of $\boldsymbol{x}^{\mathrm{LC}}_{t+t'}$
- Linearized time evolution of $\boldsymbol{z}_{t|t'} := \boldsymbol{x}_t - \boldsymbol{x}^{\mathrm{LC}}_{t'+t}$: $\dot{\boldsymbol{z}}_{t|t'} = \mathsf{K}_{t'+t}\boldsymbol{z}_{t|t'} + \sqrt{2\epsilon}\,\mathsf{G}(\boldsymbol{x}^{\mathrm{LC}}_{t'+t})\cdot\boldsymbol{\xi}_t$
- Covariance matrix $\mathsf{V}_{t|t'} := \langle \boldsymbol{z}_{t|t'}\boldsymbol{z}^{\top}_{t|t'}\rangle$ (Mean is always zero/Covariance is initially zero)
- Time evolution $\dot{\mathsf{V}}_{t|t'} = \mathsf{K}_{t'+t}\mathsf{V}_{t|t'} + \mathsf{V}_{t|t'}\mathsf{K}^{\top}_{t'+t} + 2\epsilon\mathsf{D}(\boldsymbol{x}^{\mathrm{LC}}_{t'+t})$

  ← Solved by $\mathsf{V}_{t|t'} = 2\epsilon\Phi_{t'+t}\left\{\int_0^t ds\,\Phi^{-1}_{t'+s}\mathsf{D}(\boldsymbol{x}^{\mathrm{LC}}_{t'+s})\Phi^{-1\top}_{t'+s}\right\}\Phi^{\top}_{t'+t}$

- $p_t(\boldsymbol{x}|\boldsymbol{x}^{\mathrm{LC}}_{t'})$ is approximated by the Gaussian $\mathcal{G}[\boldsymbol{x}^{\mathrm{LC}}_{t+t'};\mathsf{V}_{t|t'}]$

Sketch of the derivation of correlation time

STEP3: Long time approximation

$$\mathsf{V}_{t|t'} = 2\epsilon\Phi_{t'+t}\left\{\int_0^t ds\,\Phi_{t'+s}^{-1}\mathsf{D}(\boldsymbol{x}_{t'+s}^{\mathrm{LC}})\Phi_{t'+s}^{-1\top}\right\}\Phi_{t'+t}^{\top}$$

- **The monodromy matrix satisfies** $\Phi_{t+\tau_{\mathrm{p}}} = \Phi_t\Phi_{\tau_{\mathrm{p}}}$
- **Large** t **& stability of the limit cycle** $\rightarrow$ $\Phi_{t'+t} \simeq \dot{\boldsymbol{x}}_{t'+t}^{\mathrm{LC}}\boldsymbol{v}^{(1)\top}$

- **Periodicity leads to** $\mathsf{V}_{t|t'} \simeq \dfrac{2\epsilon t}{\tau_{\mathrm{p}}}\dot{\boldsymbol{x}}_{t'+t}^{\mathrm{LC}}\left\{\int_0^{\tau_{\mathrm{p}}} ds\,\boldsymbol{\zeta}_s^{\top}\mathsf{D}(\boldsymbol{x}_s^{\mathrm{LC}})\boldsymbol{\zeta}_s\right\}\dot{\boldsymbol{x}}_{t'+t}^{\mathrm{LC}\top}$

Sketch of the derivation of correlation time

STEP4: Gaussian integral

$$\begin{aligned}\langle a(\boldsymbol{x}_t)a(\boldsymbol{x}_0)\rangle_{\mathrm{st}} &\simeq \frac{1}{\tau_{\mathrm{p}}}\int d\boldsymbol{x}\int_0^{\tau_{\mathrm{p}}} dt'\, a(\boldsymbol{x})a(\boldsymbol{x}_{t'}^{\mathrm{LC}})\mathcal{G}_{\boldsymbol{x}}[\boldsymbol{x}_{t'+t}^{\mathrm{LC}}; \mathsf{V}_{t|t'}] \\ &= \frac{1}{\tau_{\mathrm{p}}}\int d\boldsymbol{z}\int_0^{\tau_{\mathrm{p}}} dt'\, a(\boldsymbol{x}_{t'+t}^{\mathrm{LC}}+\boldsymbol{z})a(\boldsymbol{x}_{t'}^{\mathrm{LC}})\mathcal{G}_{\boldsymbol{z}}[\boldsymbol{0}; \mathsf{V}_{t|t'}]\end{aligned}$$

- Long time approximation → Gaussian dist. is delta function-like for $\boldsymbol{z}_\perp$

  (the components perpendicular to the limit cycle)

- For the component parallel to the limit cycle, we can introduce $z = (\boldsymbol{z}-\boldsymbol{z}_\perp)\|\dot{\boldsymbol{x}}_{t'+t}^{\mathrm{LC}}\|^{-1}$

  & rewrite $a(\boldsymbol{x}_{t'+t}^{\mathrm{LC}}+\boldsymbol{z}) \Rightarrow a(\boldsymbol{x}_{t'+t+z}^{\mathrm{LC}})$

- Gaussian integral for z → The closed form of correlation time