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Stochastic thermodynamics for non-Markov processes

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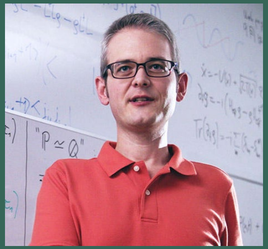
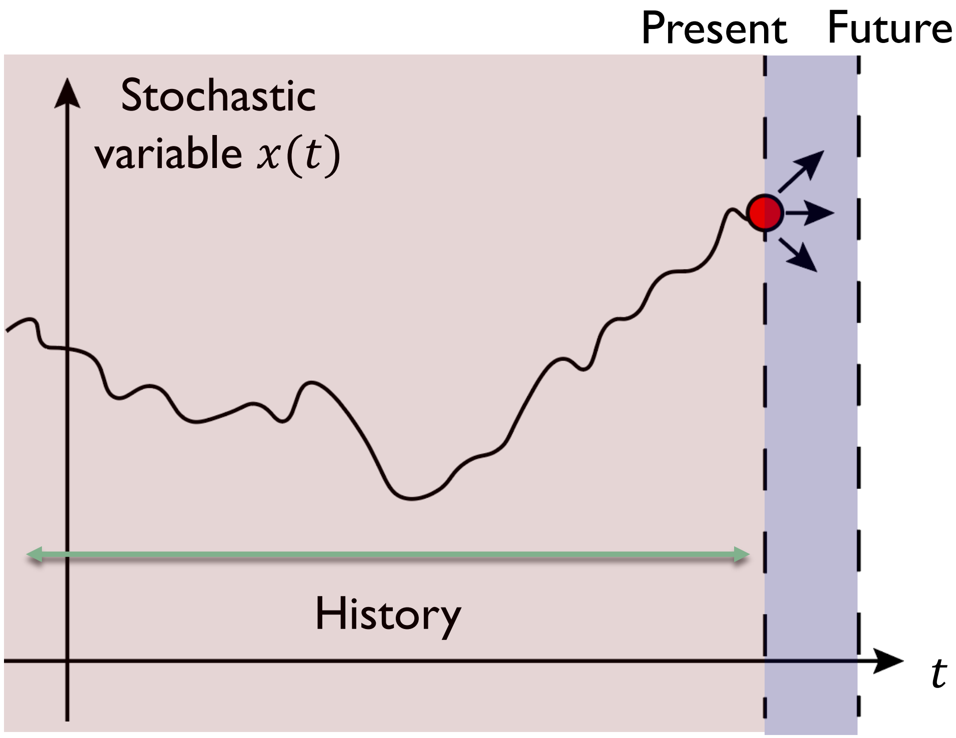
Collaborators



Andreas Dechant
Kyoto Univ.

- KK and D. Sornette, Phys. Rev. Res. **6**, 023270 (2024).
- KK and A. Dechant, arXiv:2506.04726 (2025).

Background:
Markov vs. non-Markov processes



Let me review the standard solutions for Markov processes

- Markov = requiring only the current info $x(t)$ for 1-step future
- Non-Markov = all the history $\{x(s)\}_{s \le t}$ is required for prediction
- NOTE: various tools are mainly prepared only for Markov processes

Solution: SDE ⇔ master equation

White Gaussian

White compound Poisson

Stochastic differential equation (SDE)

$$\frac{d\hat{v}}{dt} = -a(\hat{v}) + b(\hat{v}) \cdot \hat{\xi}^{\mathrm{G}} + \hat{\xi}^{\mathrm{CP}}_{\lambda(y|\hat{v})}$$

- 1-variable SDE = Gaussian + compound Poisson noise

⇕

Master equation (time-evolution of the probability density function (PDF))

$$\frac{\partial P(v,t)}{\partial t} = \left[\frac{\partial}{\partial v}a(v) + \frac{1}{2}\frac{\partial^2}{\partial v^2}b^2(v)\right]P(v,t) + \int_{-\infty}^{\infty} dy[\lambda(y|v-y)P(v-y,t) - \lambda(y|v)P(v,t)]$$

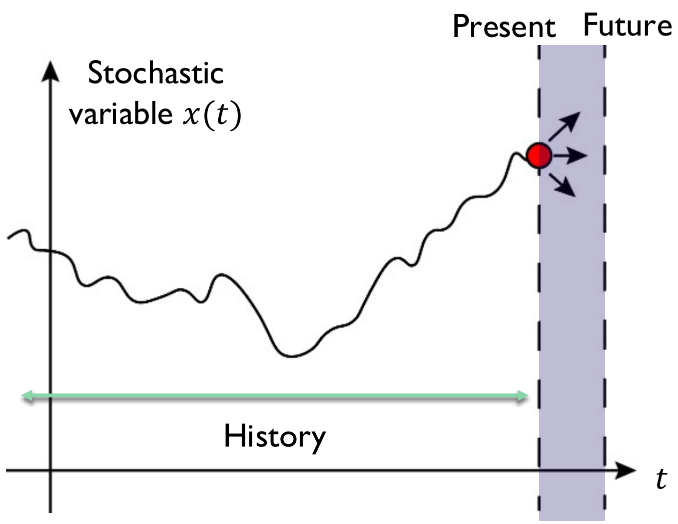
- Always linear regarding $P(v,t)$
- Within linear algebra in a wide sense

⇒ Standard solution: SDE⇔master equation
Master equation is solved within linear algebra

Tools are established for Markovian process
⇒ Stochastic thermo is established for Markovian systems

$$\frac{dv}{dt} = -U'(x) - \gamma v + \sqrt{2\gamma T}\xi^G \quad \Rightarrow \quad \frac{\partial P_t}{\partial t} = L P_t$$

Master Eq. or
Fokker-Planck Eq.



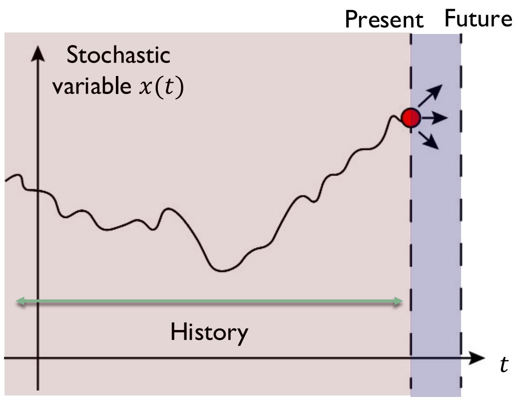
- Various tools available for Markovian processes
- Various thermodynamic inequalities
  - 2^{nd} law
  - TUR
  - …

Assuming Markovian, various thermodynamic bounds are revealed

Long-term motivation:
stochastic thermodynamics for non-Markovian processes

$$\frac{dv}{dt} = [\text{history dependent noise}] \Rightarrow \frac{\partial P_t}{\partial t} = LP_t$$

Master Eq.??
Fokker-Planck Eq.??



Various bounds??:

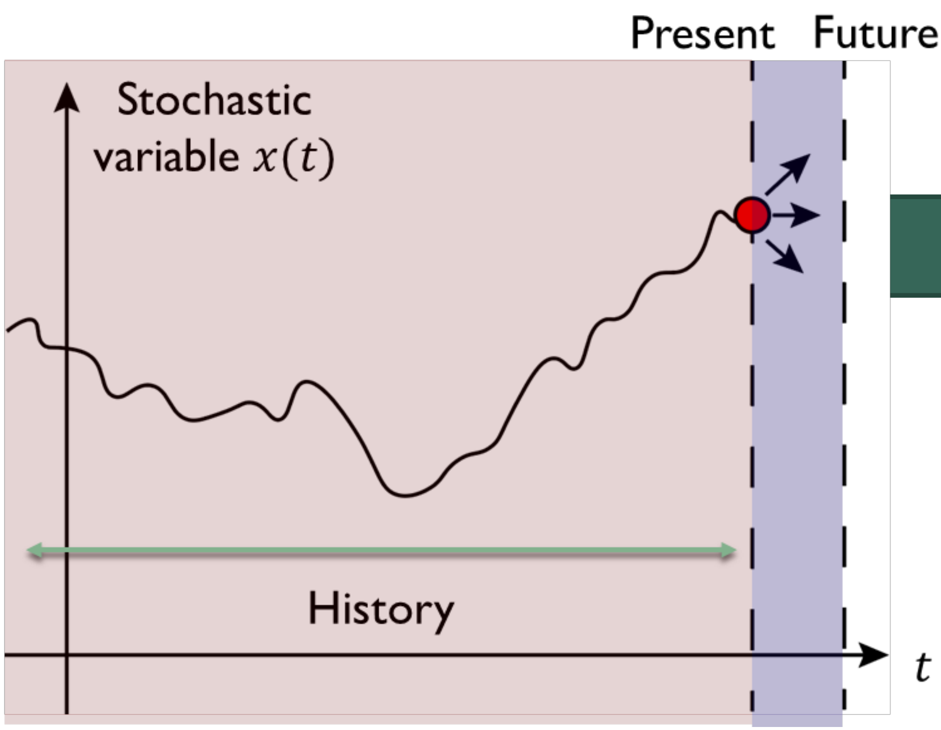
- 2^{nd} law
- TUR
- …

- Not fully formulated yet regarding non-Markovian systems…
- Elusive yet regarding the various thermodynamic bounds

Revelation of thermodynamic bounds for general non-Markovian systems

My research question:

Q: systematic solutions to non-Markovian systems?



Questions

Q: stochastic thermodynamics for non-Markov processes?
Q: master equations for non-Markov processes?

- Master Eq. available for *only specific class* of systems (e.g., renewal process)
- *Unavailable* for general non-Markovian dynamics…

Goal

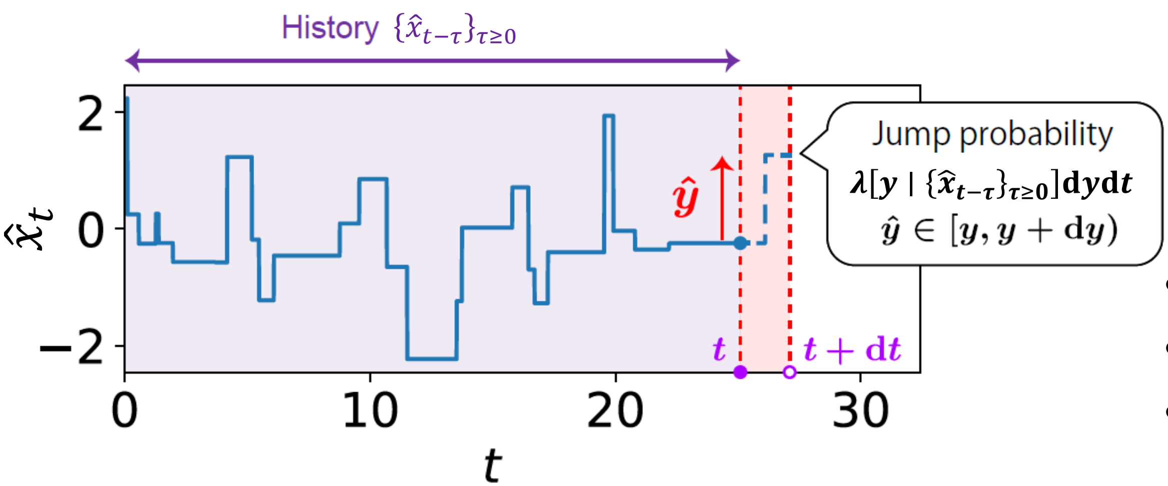
Derivation of master Eqs. and stochastic thermos for *general* non-Markov dynamics by developing *Markovian embedding*

Main results: Fourier embedding and stochastic thermodynamics

[1] K. Kanazawa and A. Dechant, arXiv:2506.04726
[2] A. Dechant and K. Kanazawa, in preparation (2025)

Model: non-Markov jump model for position $\hat{x}_t$

-- embedding based on the Fourier transform --



$$\frac{\mathrm{d}\hat{x}_t}{\mathrm{d}t} = \hat{\xi}^{\mathrm{CP}}_{\lambda_{a}[y|\{x_{t-\tau}\}_{\tau\geq 0}]}$$

- Intensity: $\lambda_a[y \mid \{\hat{x}_{t-\tau}\}_{\tau\geq 0}]$
- Control parameter: a
- $\hat{x}_t$: even-parity variable

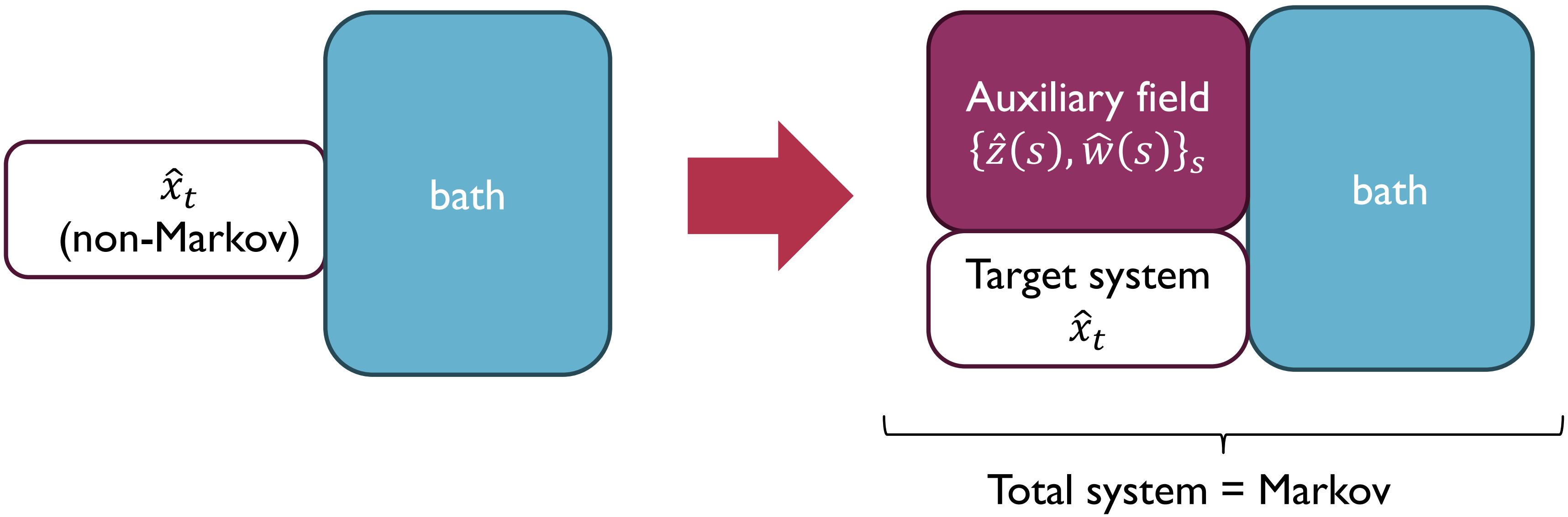
Fourier-transform of historical velocity

$$\hat{v}_t := \frac{d\hat{x}_t}{dt}$$

$$\hat{w}_t(s) := \int_0^\infty \hat{v}_{t-\tau}\cos(s\tau)\,\mathrm{d}\tau, \quad \hat{z}_t(s) := \int_0^\infty \hat{v}_{t-\tau}\sin(s\tau)\,\mathrm{d}\tau, \quad \hat{v}_{t-\tau} := \frac{\mathrm{d}\hat{x}_{t-\tau}}{\mathrm{d}t}$$

$$\hat{\Gamma}_t := (\hat{x}_t, \{\hat{z}_t(s)\}_{s>0}, \{\hat{w}_t(s)\}_{s>0})$$

Schematic of the embedding



- Total system is Markov after embedding
- Target system is non-Markov, after tracing out
- The embedding-variable set is not unique;
  there are many candidates, such as Laplace and Fourier embedding
  - ✓ Laplace embedding: K. Kanazawa and D. Sornette, Phys. Rev. Res. **6**, 023270 (2024).

Fourier-type embedding deduces the Markov SPDE
– equivalent to Markov field theory –

Embedding

$$\hat{w}_t(s) := \int_0^\infty \hat{v}_{t-\tau}\cos(s\tau)\,\mathrm{d}\tau, \quad \hat{z}_t(s) := \int_0^\infty \hat{v}_{t-\tau}\sin(s\tau)\,\mathrm{d}\tau, \quad \hat{v}_{t-\tau} := \frac{\mathrm{d}\hat{x}_{t-\tau}}{\mathrm{d}t}$$

$$\hat{\Gamma}_t := (\hat{x}_t, \{\hat{z}_t(s)\}_{s>0}, \{\hat{w}_t(s)\}_{s>0})$$

SPDE

Target system

Fields dynamics

$$\frac{\mathrm{d}\hat{x}_t}{\mathrm{d}t} = \hat{\xi}^{\mathrm{CP}}_{\lambda_{\boldsymbol{a}}[y|\hat{\Gamma}_t]}, \quad \frac{\partial \hat{z}_t(s)}{\partial t} = s\hat{w}_t(s), \quad \frac{\partial \hat{w}_t(s)}{\partial t} = -s\hat{z}_t(s) + \hat{\xi}^{\mathrm{CP}}_{\lambda_{\boldsymbol{a}}[y|\hat{\Gamma}_t]}$$

- Auxiliary variables = harmonic oscillators + jump noise

The necessary and sufficient condition can be met
for the time-reversal symmetry in the field master equation

Field master equation by the Fourier embedding

SPDE

$$\frac{\mathrm{d}\hat{x}_t}{\mathrm{d}t} = \hat{\xi}^{\mathrm{CP}}_{\lambda_{\boldsymbol{a}}[y|\hat{\Gamma}_t]}, \quad \frac{\partial \hat{z}_t(s)}{\partial t} = s\hat{w}_t(s), \quad \frac{\partial \hat{w}_t(s)}{\partial t} = -s\hat{z}_t(s) + \hat{\xi}^{\mathrm{CP}}_{\lambda_{\boldsymbol{a}}[y|\hat{\Gamma}_t]},$$

Master Eq.

$$\frac{\partial P_t[\Gamma]}{\partial t} = \left(\mathcal{L}_{\mathrm{A}} + \mathcal{L}_{\mathrm{J}}\right) P_t[\Gamma],$$

Advective

Jump

$$\mathcal{L}_{\mathrm{A}} P_t[\Gamma] := \int_0^\infty \mathrm{d}s \left\{ s z(s) \frac{\delta P_t[\Gamma]}{\delta w(s)} - s w(s) \frac{\delta P_t[\Gamma]}{\delta z(s)} \right\},$$

$$\mathcal{L}_{\mathrm{J}} P_t[\Gamma] := \int_{-\infty}^\infty \mathrm{d}y \left\{ \lambda_{\boldsymbol{a}}[y \mid \Gamma - \Delta\Gamma_y] P_t[\Gamma - \Delta\Gamma_y] - \lambda_{\boldsymbol{a}}[y \mid \Gamma] P_t[\Gamma] \right\}$$

- Master equation for an arbitrary non-Markov jumps
- Advantage: *time-reversal symmetry* can be formulated

Necessary and sufficient condition of time-reversal symmetry 1 (regarding the total-system intensity)

- Let us define the *total-system intensity*

$$\Lambda_{\boldsymbol{a}}[\Gamma \mid \Gamma'] := \lambda_{\boldsymbol{a}}[x - x' \mid \Gamma']\delta[z - z']\delta[w - w' - s(x - x')]$$

- The necessary and sufficient condition of the time-reversal symmetry:

$$\Lambda_{\boldsymbol{a}}\left[\Gamma \mid \Gamma'\right] P_{\mathrm{can};\boldsymbol{a}}[\Gamma'] = \Lambda_{\boldsymbol{a}}\left[\Gamma'^{*} \mid \Gamma^{*}\right] P_{\mathrm{can};\boldsymbol{a}}[\Gamma]$$

where * signifies the time reversal of the state variables.

$$\boxed{\frac{1}{\beta}\ln\frac{\Lambda_{\boldsymbol{a}}\left[\Gamma^{*} \mid \Gamma'^{*}\right]}{\Lambda_{\boldsymbol{a}}\left[\Gamma' \mid \Gamma\right]} = E_{\mathrm{tot}}[\Gamma', \boldsymbol{a}] - E_{\mathrm{tot}}[\Gamma, \boldsymbol{a}]}$$

Canonical dist. assumption

$$E_{\mathrm{tot}}[\Gamma, \boldsymbol{a}] = E_{\mathrm{tot}}[\Gamma^{*}, \boldsymbol{a}]$$

$$P_{\mathrm{can};\boldsymbol{a}}[\Gamma] = e^{\beta(F(\boldsymbol{a}) - E_{\mathrm{tot}}[\Gamma, \boldsymbol{a}])}, \quad e^{-\beta F(\boldsymbol{a})} := \int \mathrm{d}\Gamma e^{-\beta E_{\mathrm{tot}}[\Gamma, \boldsymbol{a}]}$$

Assumption I:
Necessary and sufficient condition of time-reversal symmetry 2 (regarding the *target-system intensity*)

- Assume that the total intensity satisfies the time-reversal symmetry:

$$\Lambda_{\boldsymbol{a}}\left[\Gamma \mid \Gamma'\right] P_{\mathrm{can};\boldsymbol{a}}[\Gamma'] = \Lambda_{\boldsymbol{a}}\left[\Gamma'^{*} \mid \Gamma^{*}\right] P_{\mathrm{can};\boldsymbol{a}}[\Gamma]$$

- The jump intensity only for the target system (*the target-system intensity*) satisfies the following condition (which we call ***Assumption I***)

$$\boxed{\frac{1}{\beta}\ln\frac{\lambda_{\boldsymbol{a}}\left[x - x' \mid \Gamma'^{*}\right]}{\lambda_{\boldsymbol{a}}\left[x' - x \mid \Gamma\right]} = E_{\mathrm{tot}}[\Gamma', \boldsymbol{a}] - E_{\mathrm{tot}}[\Gamma, \boldsymbol{a}]}$$

- Remark: this condition is irrelevant to the selection of the embedding
- Time-reversal symmetry *only for the target system* is formulated, which is *independent of the Fourier embedding*

For the original variable $\hat{X}_t := (\{\hat{x}_{t-\tau}\}_{\tau\geq 0})$

$$\frac{1}{\beta}\ln\frac{\lambda_{\boldsymbol{a}}\left[x - x' \mid X'^{*}\right]}{\lambda_{\boldsymbol{a}}\left[x' - x \mid X\right]} = E_{\mathrm{tot}}[X', \boldsymbol{a}] - E_{\mathrm{tot}}[X, \boldsymbol{a}]$$

Assumption 2:
invariance of the total intensity under time reversal

- Let us define the total intensity: $\Lambda_{\mathrm{tot};\boldsymbol{a}}[\Gamma] := \int \Lambda_{\boldsymbol{a}}[\Gamma' \mid \Gamma]\, \mathrm{d}\Gamma'$
- Total intensity is assumed to be invariant by time reversal which we call ***Assumption 2***

$$\Lambda_{\mathrm{tot};\boldsymbol{a}}[\Gamma] = \Lambda_{\mathrm{tot};\boldsymbol{a}}[\Gamma^*]$$

- Standard assumption even for Markovian stochastic thermodynamics
- Under Assumptions 1 and 2: the general solution is obtained for the stationary master equation (i.e., restriction of the canonical dist.)

$$(\mathcal{L}_{\mathrm{A}} + \mathcal{L}_{\mathrm{J}}) P_{\mathrm{can}}[\Gamma] = 0$$

General solution of the steady-state field master equation (i.e., the restriction of the canonical dist.)

- The canonical dist. must satisfies the following master equation

$$\left(\mathcal{L}_{\mathrm{A}}+\mathcal{L}_{\mathrm{J}}\right)P_{\mathrm{can};\boldsymbol{a}}[\Gamma]=0$$

- While this is a functional-derivative equation, its general (complete) solution can be obtained by the method of characteristics

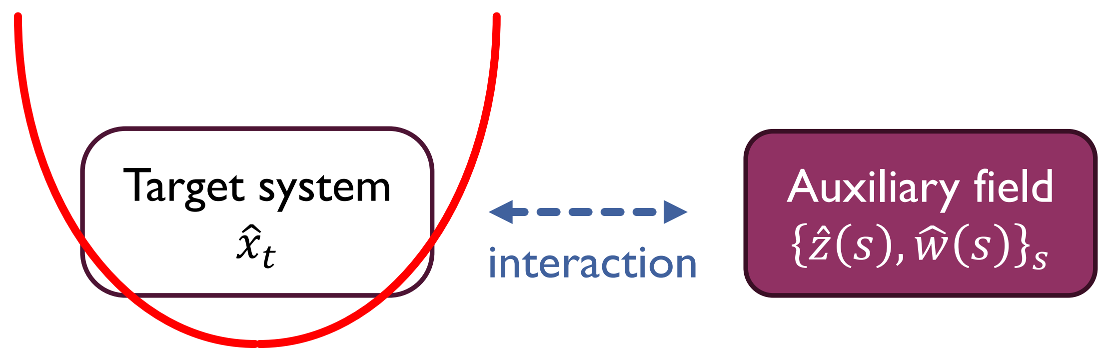
Necessary condition
as the solution of the master equation

$$P_{\mathrm{can};\boldsymbol{a}}[\Gamma]\propto e^{-E_{\mathrm{tot}}[\Gamma,\boldsymbol{a}]}$$

$$E_{\mathrm{tot}}[\Gamma,\boldsymbol{a}]=E_{\mathrm{tot}}\left[x,\{\varepsilon_s\}_s,\boldsymbol{a}\right],\quad \varepsilon_s(z(s),w(s)):=\frac{1}{2}\left(s^2z^2(s)+w^2(s)\right)$$

Assumption 3: Controllable energy is associated only with the target system

Controlled by
the external parameter a



- Let us focus on a subclass
  where the total energy = controllable energy + non-controllable energy

$$E_{\rm tot}[x, \{\varepsilon_s\}_s, \boldsymbol{a}] := \boxed{E_{\rm ctrl}(x, \boldsymbol{a})} + E_{\rm nctrl}[x, \{\varepsilon_s\}_s]$$

Controllable part depends only on x

- Sufficient condition that the second law does not depend on the selection of the embedding variable
  (i.e., the "gauge invariance" of the second law)

Simple example: quadratic non-controllable energy

$$P_{\mathrm{can};\boldsymbol{a}}[\Gamma] = \mathbf{1}_{S_{\mathrm{C}}}[\Gamma] e^{(F(\boldsymbol{a}) - E_{\mathrm{tot}}[\Gamma, \boldsymbol{a}])}, \quad E_{\mathrm{tot}}[\Gamma, \boldsymbol{a}] = E_{\mathrm{nctrl}}(x, \boldsymbol{a}) + \int_0^\infty \mathrm{d}s M(s) \left[w^2(s) + z^2(s) \right]$$

$\mathbf{1}_{S_{\mathrm{C}}}[\Gamma]$: the indicator function corresponding to the Fourier embedding
(Remark: conditional ergodicity, as will be explained later…)

- Example: Quadratic energy regarding the field variable
  - $E_{\mathrm{cntrl}}(x, a)$: the controllable part of energy by the external parameter
  - : positive weight function, controlling the contribution of the wavenumber s
- We can construct specific models which has this stationary PDF of the field master equation (see the next slides)

1st law of stochastic thermodynamics (under *Assumptions 1 and 2*)

Stochastic heat and work

$$\mathrm{d}\hat{Q} := \begin{cases} E_{\mathrm{tot}}[\hat{\Gamma}', \boldsymbol{a}] - E_{\mathrm{tot}}[\hat{\Gamma}, \boldsymbol{a}] = \dfrac{1}{\beta} \ln \dfrac{\Lambda_{\boldsymbol{a}}[\hat{\Gamma}^* \mid \hat{\Gamma}'^*]}{\Lambda_{\boldsymbol{a}}[\hat{\Gamma}' \mid \hat{\Gamma}]} & (\mathrm{prob.}{=}\Lambda_{\boldsymbol{a}}[\hat{\Gamma}' \mid \hat{\Gamma}]\, \mathrm{d}\Gamma'\, \mathrm{d}t) \\ 0 & (\mathrm{prob.}{=}1 - \Lambda_{\boldsymbol{a}}[\hat{\Gamma}' \mid \hat{\Gamma}]\, \mathrm{d}\Gamma'\, \mathrm{d}t) \end{cases}$$

$$\mathrm{d}\hat{W} := \mathrm{d}\boldsymbol{a} \frac{\partial}{\partial \boldsymbol{a}} E_{\mathrm{tot}}[\Gamma, \boldsymbol{a}]$$

1st law (energy conservation)

$$\mathrm{d}E_{\mathrm{tot}}[\hat{\Gamma}, \boldsymbol{a}] = \mathrm{d}\hat{W} + \mathrm{d}\hat{Q}$$

2^{nd} law for the entropy production (under *Assumptions 1 and 2*)

Entropy production

$$\dot{\sigma}_{\rm sys} := -\frac{\mathrm{d}}{\mathrm{d}t}\int P_t[\Gamma]\ln P_t[\Gamma]\,\mathrm{d}\Gamma$$

$$\dot{\sigma}_{\rm bath} := -\beta\left\langle \frac{\mathrm{d}\hat{Q}}{\mathrm{d}t}\right\rangle = \int \Lambda[\Gamma' \mid \Gamma]P_t[\Gamma]\ln\frac{\Lambda[\Gamma' \mid \Gamma]}{\Lambda[\Gamma^* \mid \Gamma'^*]}\,\mathrm{d}\Gamma\,\mathrm{d}\Gamma'$$

$$\Downarrow$$

2^{nd} law

$$\dot{\sigma}_{\rm tot} := \dot{\sigma}_{\rm sys} + \dot{\sigma}_{\rm bath} = \int \mathrm{d}\Gamma\,\mathrm{d}\Gamma'\Lambda_{\boldsymbol{a}}[\Gamma' \mid \Gamma]P_t[\Gamma]\ln\frac{\Lambda_{\boldsymbol{a}}[\Gamma' \mid \Gamma]P_t[\Gamma]}{\Lambda_{\boldsymbol{a}}[\Gamma^* \mid \Gamma'^*]P_t[\Gamma']} \geq 0$$

"Gauge-invariance" of the 2nd law for the irreversible work (under *Assumptions 1, 2, and 3*)

- Free energy:

$$F(\boldsymbol{a}) := -\beta^{-1} \ln \mathcal{Z}(\boldsymbol{a}), \quad \mathcal{Z}(\boldsymbol{a}) := \int \exp\left(-\beta E_{\mathrm{ctrl}}(x, \boldsymbol{a}) - \beta E_{\mathrm{nctrl}}[x, \varepsilon]\right) \mathrm{d}\Gamma.$$

- Free energy difference *does not depend* on the embedding variables:

$\boxed{\Delta F = \textit{irrelevant} \text{ to the embedding variables selection}}$

- Quasi-static work formula is closed only with $E_{sys}[\hat{x}, \boldsymbol{a}]$:

$$\langle \Delta \hat{W} \rangle := \int_{\boldsymbol{a}_{\mathrm{ini}}}^{\boldsymbol{a}_{\mathrm{fin}}} \mathrm{d}\boldsymbol{a} \left\langle \frac{\partial}{\partial \boldsymbol{a}} E_{\mathrm{sys}}(\hat{x}, \boldsymbol{a}) \right\rangle_{\mathrm{can}} = F(\boldsymbol{a}_{\mathrm{fin}}) - F(\boldsymbol{a}_{\mathrm{ini}})$$

- "Gauge-invariant" 2nd law for the work: irrelevant to embedding variables

$$\boxed{\langle \Delta \hat{W} \rangle \geq \Delta F}$$

Quadratic non-controllable energy (QnCE) model

$$\lambda_{\boldsymbol{a}}[y \mid \Gamma] = \lambda_0 \rho_{\boldsymbol{a}}(y \mid x) \exp\left[-\frac{\beta}{2}\left\{E_{\mathrm{ctrl}}(x', \boldsymbol{a}) - E_{\mathrm{ctrl}}(x, \boldsymbol{a}) + \int_0^\infty \mathrm{d}s M(s) \left(\frac{y^2}{2} + y w(s)\right)\right\}\right],$$

$$\rho_{\boldsymbol{a}}(y \mid x) = \rho_{\boldsymbol{a}^*}(\epsilon y \mid \epsilon x), \quad \rho_{\boldsymbol{a}}(y \mid x) = \rho_{\boldsymbol{a}^*}(-\epsilon y \mid \epsilon x'), \quad y := x' - x.$$

- For the above intensity functional, we have the canonical PDF:

$$P_{\mathrm{can};\boldsymbol{a}}[\Gamma] = \mathbf{1}_{S_{\mathrm{C}}}[\Gamma] e^{(F(\boldsymbol{a}) - E_{\mathrm{tot}}[\Gamma, \boldsymbol{a}])}, \quad E_{\mathrm{tot}}[\Gamma, \boldsymbol{a}] = E_{\mathrm{sys}}(x, \boldsymbol{a}) + \int_0^\infty \mathrm{d}s M(s)\left[w^2(s) + z^2(s)\right]$$

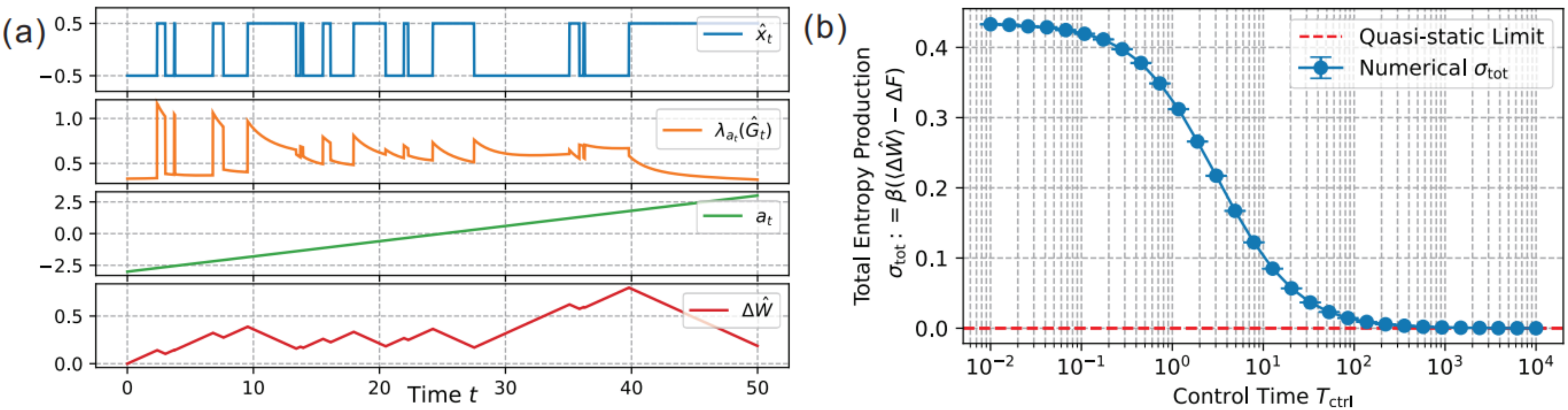
- Stochastic work and heat:

$$\mathrm{d}Q := E_{\mathrm{tot}}[\Gamma', \boldsymbol{a}] - E_{\mathrm{tot}}[\Gamma, \boldsymbol{a}] = E_{\mathrm{ctrl}}(x', \boldsymbol{a}) - E_{\mathrm{ctrl}}(x, \boldsymbol{a}) + \int_0^\infty \mathrm{d}s M(s) \left(\frac{y^2}{2} + y w(s)\right)$$

- Assumptions 1-3 are satisfied.
- Second law holds:

$$\sigma_{\mathrm{tot}} = \beta\left(\langle \Delta W \rangle - \Delta F\right) \geq 0.$$

Example 1: two-level system



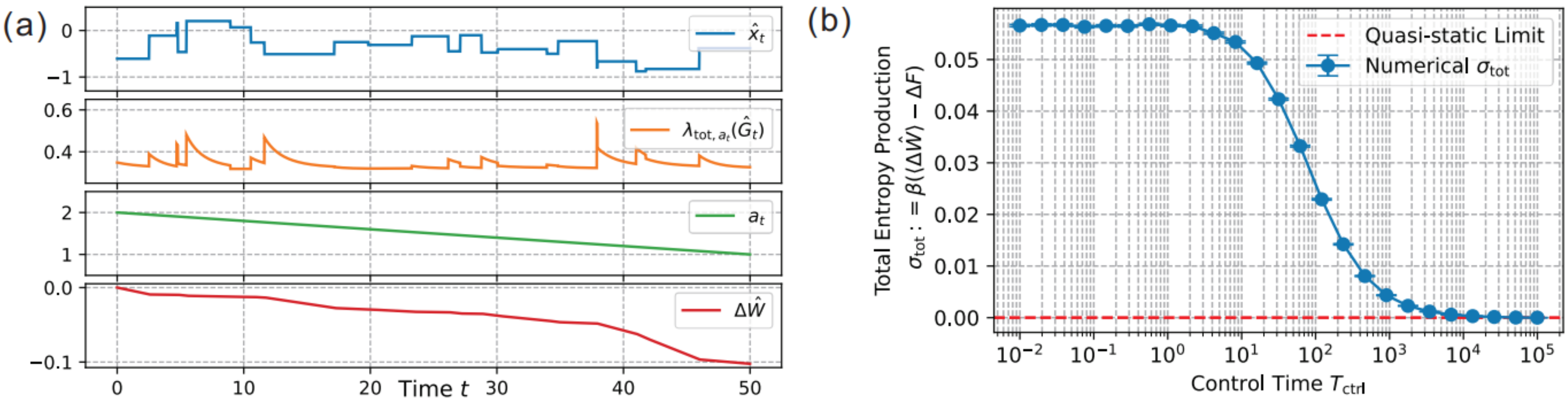
$$E_{\mathrm{ctrl}}(\hat{x}, a_t) = -a_t \hat{x}, \quad E_{\mathrm{nctrl}}[\hat{z}, \hat{w}] = \frac{1}{2}\int_0^\infty M(s)\left[\hat{w}^2(s) + \hat{z}^2(s)\right] \mathrm{d}s.$$

$$\lambda_{\boldsymbol{a}}[y \mid \{x_{t-\tau}\}_{\tau \geq 0}] = \tilde{\lambda}_0 \exp\left[-\frac{y\beta}{2}\left\{-a + \int_0^\infty \phi(\tau)\hat{v}_{t-\tau}\,\mathrm{d}\tau\right\}\right], \quad \phi(\tau) := \int_0^\infty M(s)\cos(s\tau)\,\mathrm{d}s.$$

Memory function

The 2nd law holds! $\sigma_{\mathrm{tot}} = \beta\left(\langle \Delta W \rangle - \Delta F\right) \geq 0$

Example 2: non-Markov random walks



$$E_{\mathrm{ctrl}}(\hat{x}, a) = \frac{a}{2}\hat{x}^2, \quad E_{\mathrm{nctrl}}[\hat{z}, \hat{w}] = \frac{1}{2}\int_0^\infty M(s)\left[\hat{w}^2(s) + \hat{z}^2(s)\right] \mathrm{d}s.$$

Harmonic potential

$$\lambda_a[y \mid \{x_{t-\tau}\}_{\tau \geq 0}] = \lambda_0 \rho(y) \exp\left[-\frac{\beta}{2}\left\{E_{\mathrm{ctrl}}(x+y, a) - E_{\mathrm{ctrl}}(x, a) + \frac{y^2}{2}\phi(0) + y\int_0^\infty \phi(\tau) v_{t-\tau}\, \mathrm{d}\tau\right\}\right]$$

Memory function

The 2nd law holds! $\sigma_{\mathrm{tot}} = \beta\left(\langle \Delta W \rangle - \Delta F\right) \geq 0$

Discussion 1: entropy production

Another approach: Laplace embedding

K. Kanazawa and D. Sornette, Phys. Rev. Lett. **125**, 138301 (2020)
K. Kanazawa and D. Sornette, Phys. Rev. Lett. **127**, 188301 (2021)
K. Kanazawa and D. Sornette, Phys. Rev. Res. **6**, 023270 (2024)

Laplace transform of the historical velocity: $\hat{Z}_t(s) := \int_0^\infty \hat{v}_{t-\tau} e^{-s\tau}\, \mathrm{d}\tau$

Markov SPDE:

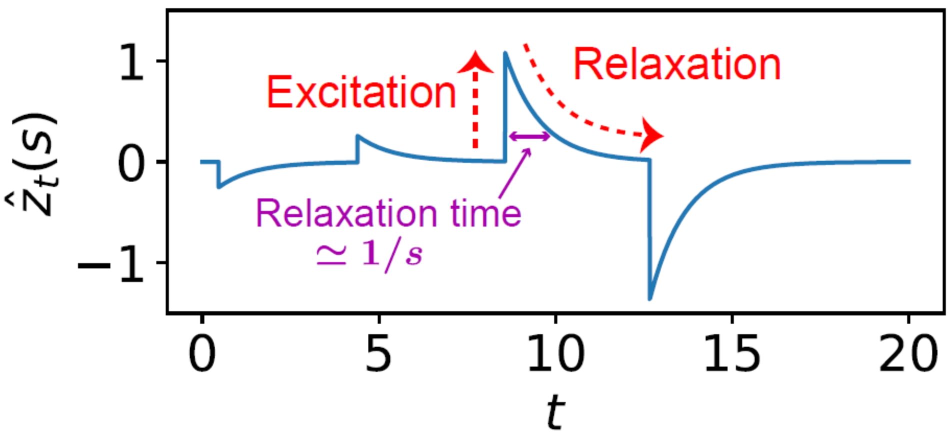
$$\frac{\mathrm{d}\hat{x}_t}{\mathrm{d}t} = \hat{\xi}^{\mathrm{CP}}_{\lambda_{\boldsymbol{a}}[y|\hat{G}_t]}, \quad \frac{\mathrm{d}\hat{Z}_t(s)}{\mathrm{d}t} = -s\hat{Z}_t(s) + \hat{\xi}^{\mathrm{CP}}_{\lambda_{\boldsymbol{a}}[y|\hat{G}_t]}, \quad \hat{G}_t := \left(\hat{x}_t, \{\hat{Z}_t(s)\}_s\right)$$

Field master equation:

$$\begin{aligned}\frac{\partial P_t[G]}{\partial t} &= \int \mathrm{d}s \frac{\delta}{\delta Z(s)}\left(sZ(s)P_t[G]\right) + \int \mathrm{d}y \left\{\lambda[y \mid G - \Delta G_y]P_t[G - \Delta G_y] - \lambda[y \mid G]P_t[G]\right\} \\ &= \int \mathrm{d}s \frac{\delta}{\delta Z(s)}\left(sZ(s)P_t[G]\right) + \int \mathrm{d}G' \left\{\tilde{\Lambda}[G \mid G']P_t[G'] - \tilde{\Lambda}[G' \mid G]P_t[G]\right\}\end{aligned}$$

Q: Does the Laplace embedding work for stochastic thermodynamics?
A: No…

Problem of the Laplace-type embedding: time-reversal symmetry *cannot* be formulated in principle



Poisson jump

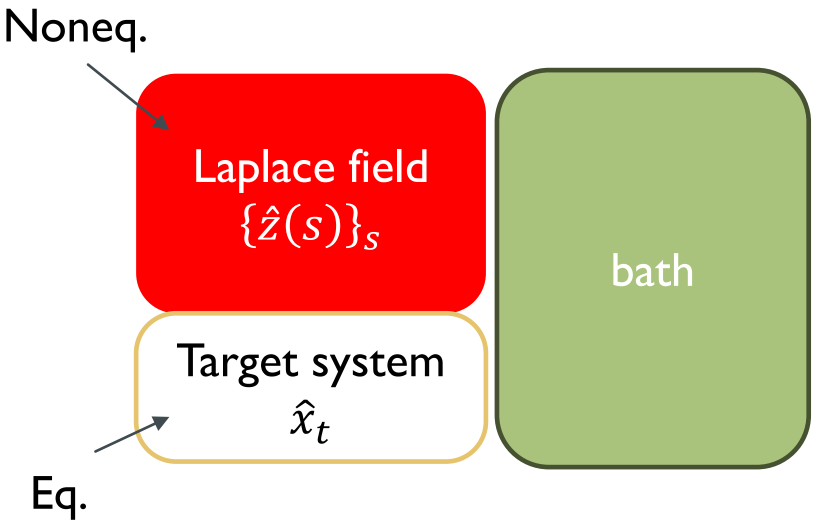
$$\frac{\mathrm{d}\hat{Z}_t(s)}{\mathrm{d}t} = -s\hat{Z}_t(s) + \hat{\xi}^{\mathrm{CP}}_{\lambda_a[y|\hat{G}_t]}$$

1st order SPDE

- 1st-order differential Eq. + jump
  - ✓ Trivially, irreversible relaxation occurs…
  - ✓ In principle, no reverse paths are realized (exception = Gaussian limit)
- Due to the 1st-order differential Eq., the time-reversal symmetry *never* exists

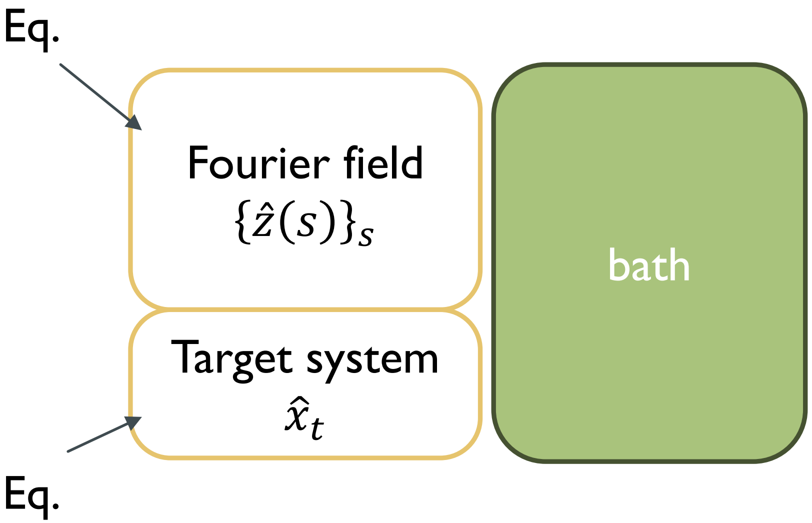
Discussion: what is the essential difference between the Laplace and Fourier embedding?

Laplace embedding



- Total intensity does not satisfy the time-reversal symmetry, even if the target system is in equilibrium
- Laplace field is always nonequilibrium

Fourier embedding



- Total intensity satisfies the time-reversal symmetry, if the target system is in equilibrium
- Fourier field can be equilibrium

Decomposition of the Laplace entropy based on the house-keeping entropy and excess entropy

Laplace embedding

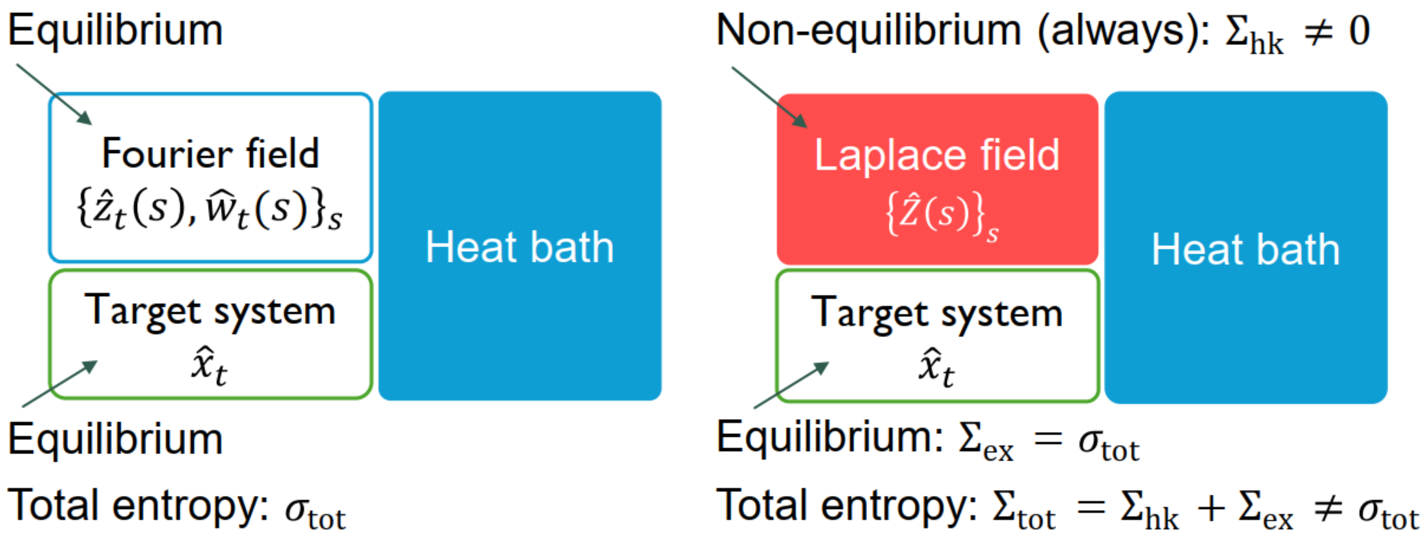
$$\hat{G}_t := \left(\hat{x}_t, \{\hat{Z}_t(s)\}_s\right) \qquad \frac{\mathrm{d}\hat{x}_t}{\mathrm{d}t} = \hat{\xi}^{\mathrm{CP}}_{\lambda[y|\hat{G}_t]}, \quad \frac{\mathrm{d}\hat{Z}_t(s)}{\mathrm{d}t} = -s\hat{Z}_t(s) + \hat{\xi}^{\mathrm{CP}}_{\lambda_{\boldsymbol{a}}[y|\hat{G}_t]}$$

$$\dot{\Sigma}_{\mathrm{tot}} = \dot{\Sigma}_{\mathrm{hk}} + \dot{\Sigma}_{\mathrm{ex}}$$

$$\begin{cases} \dot{\Sigma}_{\mathrm{hk}} \neq 0 \\ \dot{\Sigma}_{\mathrm{ex}} := \displaystyle\int \lambda[y \mid \Gamma] P_t[\Gamma] \ln \frac{\lambda[y \mid \Gamma] P_t[\Gamma]}{\lambda[-y \mid \Gamma'^*] P_t[\Gamma']} \, \mathrm{d}y \, \mathrm{d}\Gamma = \dot{\sigma}_{\mathrm{tot}} \end{cases}$$

Identical to Fourier's entropy!

- The total system by the Laplace embedding is always nonequilibrium $\dot{\Sigma}_{\mathrm{tot}} \neq 0$, even if the target system is equilibrium (i.e., even if *Assumption 1* is satisfied).
- However, if we look at only the excess entropy (experimentally relevant), there is no difference between the Laplace and Fourier embedding
- Thermodynamic laws for the *target system* hold even for Laplace embedding, if *Assumptions 1-3* are satisfied, while the *total system* is nonequilibrium.



(a) Fourier embedding (b) Laplace embedding

$$\dot{\Sigma}_{\mathrm{tot}} = \dot{\Sigma}_{\mathrm{hk}} + \dot{\Sigma}_{\mathrm{ex}}$$

$$\begin{cases} \dot{\Sigma}_{\mathrm{hk}} \neq 0 \\ \dot{\Sigma}_{\mathrm{ex}} := \int \lambda[y \mid \Gamma] P_t[\Gamma] \ln \dfrac{\lambda[y \mid \Gamma] P_t[\Gamma]}{\lambda[-y \mid \Gamma'^*] P_t[\Gamma']} \mathrm{d}y \, \mathrm{d}\Gamma = \dot{\sigma}_{\mathrm{tot}} \end{cases}$$

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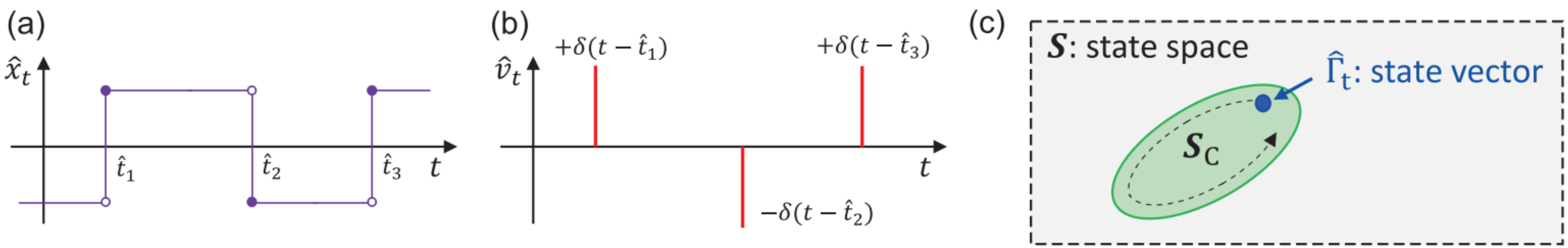
Discussion 2: conditional ergodicity

Conditional ergodicity of the field master equation

Q: Is the field master equation ergodic (irreducible)?: does it have the unique solution?
A: No. It has the unique solution only if its initial condition is within an appropriate class. (i.e., conditionally ergodic)



- We consider a very large solution space, which cannot be explored in principle
  1. Suppose a two-level system $\hat{x}_t = \pm 1/2$ (Fig. a)
  2. The velocity can explore only limited space… $\hat{v}_t = \sum_{k=1}^{\infty}(-1)^{k-1}\delta(t-\hat{t}_k)$
- To understand this point, let us consider Markov models with finite states…

Markov jump process with finite states (a reducible case)

$$\frac{\partial P_i(t)}{\partial t} = \sum_j H_{ij} P_i(t)$$

p_i: probability, $\sum_i p_i = 1$

H_{ij}: probability matrix

- If the matrix H_{ij} is irreducible, the master equation has a unique stationary solution:

$$\lim_{t\to\infty} P_i(t) = \pi_i$$

- If the matrix H_{ij} is reducible, the stationary solution of the master equation is given by a superposition of stationary PDF for closed communicating classes:

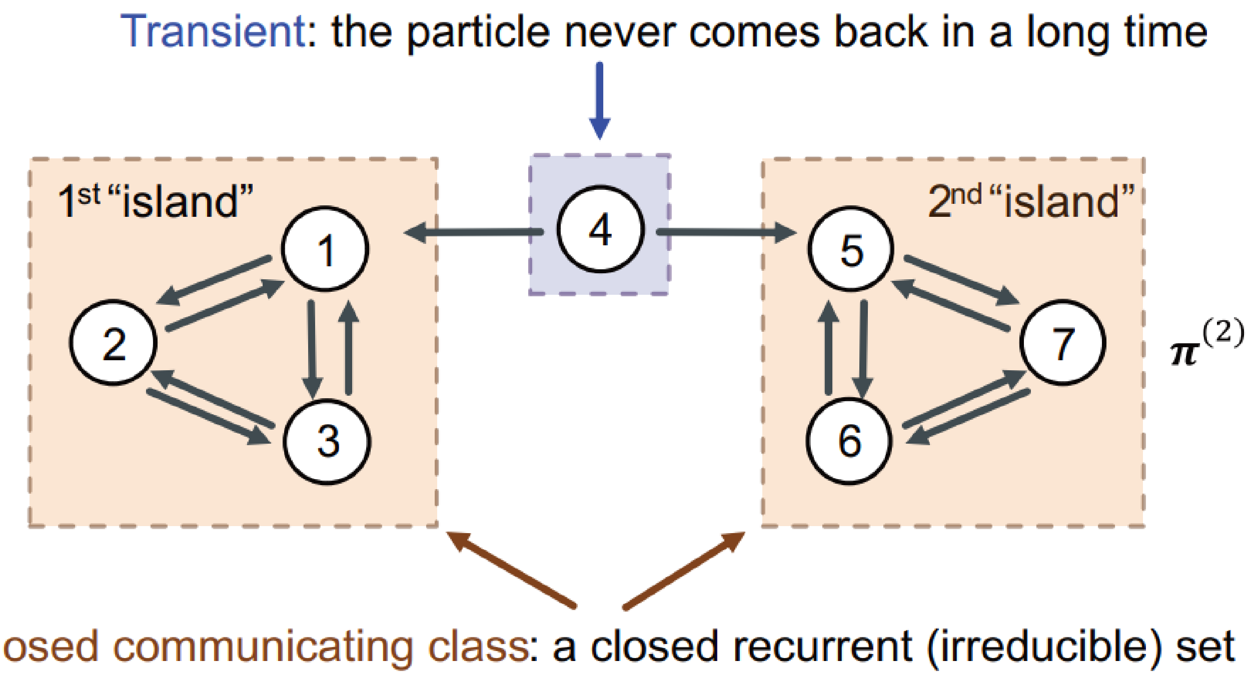
$$\lim_{t\to\infty} P_i(t) = \sum_k c^{(k)} \pi_i^{(k)}, \qquad c^{(k)} \geq 0$$

- In other words, the stationary solution is unique only when we restrict the communicating class for the initial condition (i.e., conditionally ergodic).

Theorem: all states are classified into either a recurrent communicating class or transient set

Example 1 (a 7-level system with reducible probability matrix):

$$\boldsymbol{\pi}^{(1)} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



$$\boldsymbol{\pi}^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

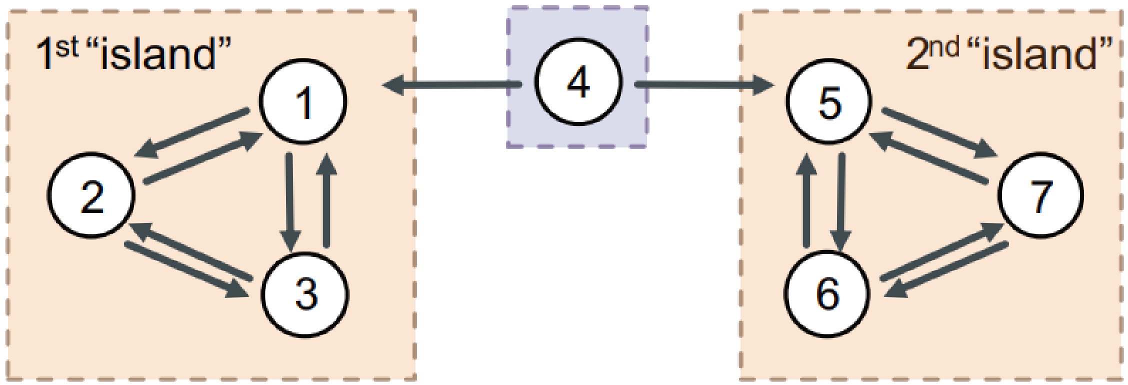
- Theorem for finite-state Markov: all states = either *recurrent* or *transient*
- Stationary PDF:

$$\lim_{t\to\infty} P_i(t) = \sum_k c^{(k)} \pi_i^{(k)}, \qquad c^{(k)} \geq 0$$

corresponding to the i-th "island"

Example 1 (a 7-level system with reducible probability matrix): Constructing the stationary PDF corresponding to the initial condition

$$\boldsymbol{\pi}^{(1)} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



$$\boldsymbol{\pi}^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

- Suppose you find a positive stationary PDF $\mathcal{P}_i$: $\displaystyle\sum_j H_{ij}\mathcal{P}_j = 0, \qquad \mathcal{P}_i > 0$
- Suppose that your initial condition is in the 1st "island".
- The corresponding stationary PDF is unique (i.e., conditionally ergodic):

$$\lim_{t\to\infty} P_i(t) \propto \mathcal{P}_i \mathbf{1}_i^{(1)} \geq 0, \qquad \mathbf{1}_i^{(1)} := \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Projection to the 1st "island"
by multiplying the indicator func.

Another example:
Markov jump processes with discrete jumps

$$\frac{\partial P_t(x)}{\partial t} = \int_{-\infty}^{\infty} \mathrm{d}y[\lambda(y \mid x-y)P_t(x-y) - \lambda(y \mid x)P_t(x)]$$

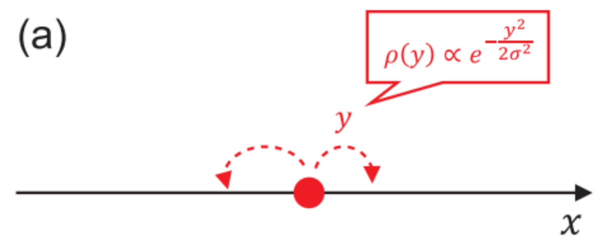
$$\lambda(y \mid x) = \rho(y)\exp^{-\frac{\beta}{2}[E(x+y)-E(x)]}, \quad E(x) = \frac{x^2}{2}$$

$$\Rightarrow \quad P_{\mathrm{can}}(x) \propto e^{-\beta E(x)}$$

- Markov jump processes with time-reversal symmetry
- Canonical PDF exists as one of the stationary PDFs.
- Q. Is it the unique stationary PDF?
  - ✓ A. Yes, if $\rho(y)$ is continuous and decent (irreducible);

    $$\rho(y) \propto e^{-y^2/2\sigma^2} \implies P_{\mathrm{can}}(x) \propto e^{-E(x)} \quad \text{(unique)}$$
  - ✓ A. But, sometimes, no… (reducible)

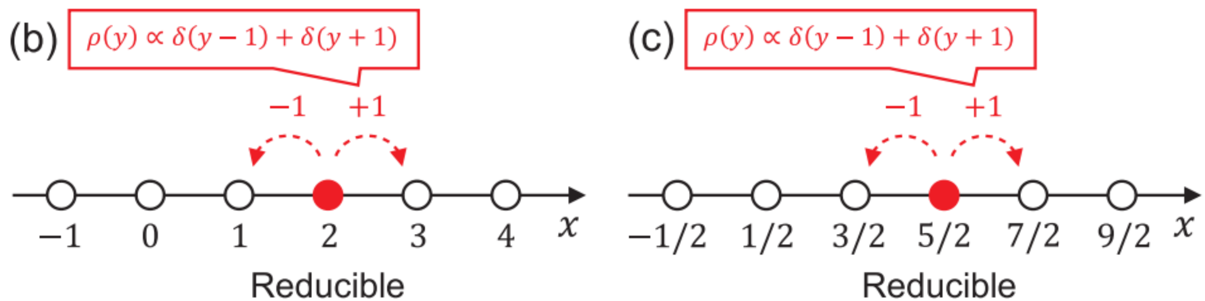
(a)



Irreducible

Another example:
Markov jump processes with discrete jumps

$$\rho(y) = \frac{1}{2}[\delta(y-1) + \delta(y+1)]$$



Case 1: if $x_0 \in \mathbb{Z} := \{0, \pm 1, \dots\}$ (integers)

- Defined only at integers

$$P_{\mathrm{can}}(x) \propto \sum_{k=-\infty}^{\infty} e^{-\beta E(k)} \delta(x-k)$$

$$= \mathbf{1}_{\boldsymbol{Z}}(x) e^{-\beta E(x)}$$

$$\mathbf{1}_{\boldsymbol{Z}}(x) := \sum_{k=-\infty}^{\infty} \delta(x-k)$$

Case 2: if $x_0 \in \mathbb{Z}_{1/2} := \{\pm 1/2, \pm 3/2, \dots\}$ (half integers)

- Defined only at half integers

$$P_{\mathrm{can}}(x) \propto \sum_{k=-\infty}^{\infty} e^{-\beta E(k)} \delta(x-k+1/2)$$

$$= \mathbf{1}_{\boldsymbol{Z}_{1/2}}(x) e^{-\beta E(x)}$$

$$\mathbf{1}_{\boldsymbol{Z}_{1/2}}(x) := \sum_{k=-\infty}^{\infty} \delta(x-k+1/2)$$

As a unified expression, we obtain the formula with the indicator function corresponding to the initial condition $P_{\mathrm{can}}(x) \propto \mathbf{1}_C(x) e^{-E(x)}$ (conditionally ergodic)

Going back to the conditional ergodicity of the fME

$$P_{\mathrm{can};\boldsymbol{a}}[\Gamma] = \mathbf{1}_{S_{\mathrm{C}}}[\Gamma] e^{(F(\boldsymbol{a}) - E_{\mathrm{tot}}[\Gamma, \boldsymbol{a}])}$$

$\mathbf{1}_{S_{\mathrm{C}}}[\Gamma]$: the indicator function corresponding to the Fourier embedding

- Technical note: the field master equation is conditionally ergodic
  1. Reducible: several stationary PDFs because of the too large embedded space
  2. Embedding requires a special class of initial conditions
     - ✓ Kramers-Kronig relation btw Fourier cosine $\{w(s)\}_s$ and sine transforms $\{z(s)\}_s$

$$\hat{w}_t(s) := \int_0^\infty \hat{v}_{t-\tau} \cos(s\tau)\, \mathrm{d}\tau, \quad \hat{z}_t(s) := \int_0^\infty \hat{v}_{t-\tau} \sin(s\tau)\, \mathrm{d}\tau$$

$$\Rightarrow \quad \hat{z}_0(s) = -\mathcal{H}\left[\hat{w}_0(s)\right], \quad \mathcal{H}[f] := \frac{2s}{\pi} \int_0^\infty \frac{f(u)}{u^2 - s^2}\, \mathrm{d}u$$

     - ✓ Velocity is composed of delta functions; not an arbitrary function
       - ✓ E.g., two-level system: $\hat{v}_t = +\delta(t - t_1) - \delta(t - t_2) + \cdots$
         Plus and minus come alternatively…
  3. Indicator function multiplied to extract the consistent solution
     (= *conditional ergodicity*, only within the recurrent communicating class)

Summary

Summary: Fourier-type embedding + field master equations $\longrightarrow$ the time-reversal symmetry can be satisfied!

KK and A. Dechant, arXiv:2506.04726

$$\{\hat{x}_{t-\tau}\}_{\tau \geq 0}$$

$$\frac{\mathrm{d}\hat{v}_t}{\mathrm{d}t} = \hat{\xi}^{\mathrm{CP}}_{\lambda(y|\{\hat{v}_\tau\}_{\tau \leq t})}$$

$$\hat{\Gamma} := (\hat{x}_t, \{\hat{z}_t(s)\}_{s>0}, \{\hat{w}_t(s)\}_{s>0})$$

$$\frac{\mathrm{d}\hat{x}_t}{\mathrm{d}t} = \hat{\xi}^{\mathrm{CP}}_{\lambda_{\boldsymbol{a}}[y|\hat{\Gamma}_t]}, \quad \frac{\partial \hat{z}_t(s)}{\partial t} = \hat{w}_t(s)$$

$$\frac{\partial \hat{w}_t(s)}{\partial t} = -s^2 \hat{z}_t(s) + s\hat{\xi}^{\mathrm{CP}}_{\lambda_{\boldsymbol{a}}[y|\hat{\Gamma}_t]}$$

- Fourier embedding
  - ✓ Time-reversal symmetry for the total system can be formulated
- Stochastic thermodynamics formulated for general non-Markov jumps
  - ✓ "Gauge-invariance" of the second law:
    embedding selection is *irrelevant* to the thermodynamic laws for irreversible work