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# Promoting Fluctuation Theorems into Covariant Forms

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Ji-Hui Pei, Jin-Fu Chen, H. T. Quan, Phys. Rev. Lett. **134**, 237102 (2025) [Editors’ Suggestions]

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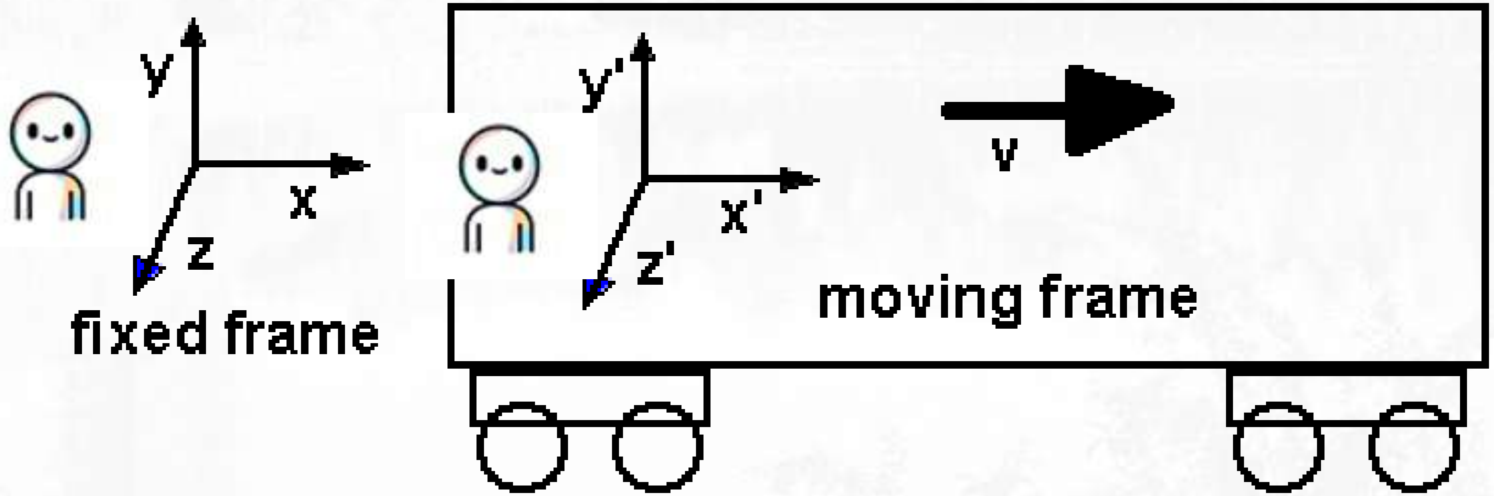
# Background: Covariance of physical laws

All inertial frames of reference are equivalent, and there is no special inertial frame! -----Albert Einstein

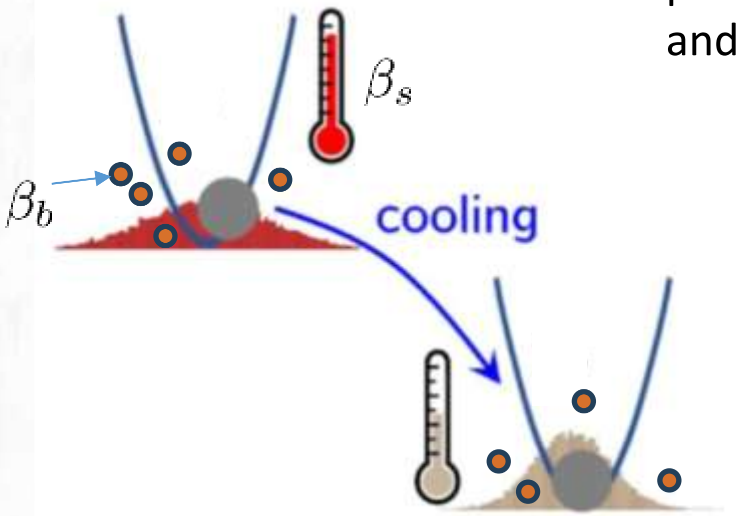
- Newton's equation is covariant under Galileo transformation (1687)
- Maxwell's equations are covariant under Lorentz transformation (1904)



# Fluctuation Theorems

generalization of the second law

- (Noncovariant) heat exchange fluctuation theorem




probability ratio between forward and backward trajectories

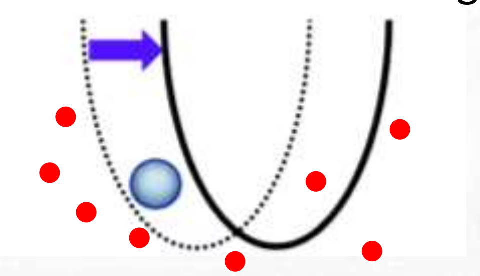
$$\frac{\Pr(\omega)}{\Pr(\tilde{\omega})} = e^{-(\beta_b - \beta_s)Q[\omega]}$$

$$(\beta_b - \beta_s)\langle Q \rangle \geq 0$$

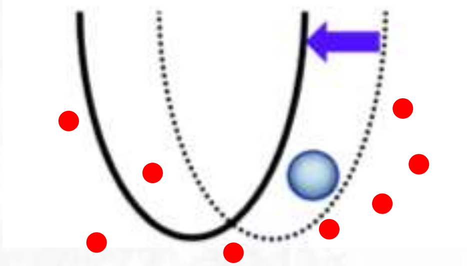
C. Jarzynski and D. K. Wójcik, Phys. Rev. Lett. 92.230602 (2004).

- (Noncovariant) work fluctuation theorem

forward driving



backward driving



$$\frac{\Pr(\omega)}{\tilde{\Pr}(\tilde{\omega})} = e^{\beta(W[\omega] - \Delta F)}$$

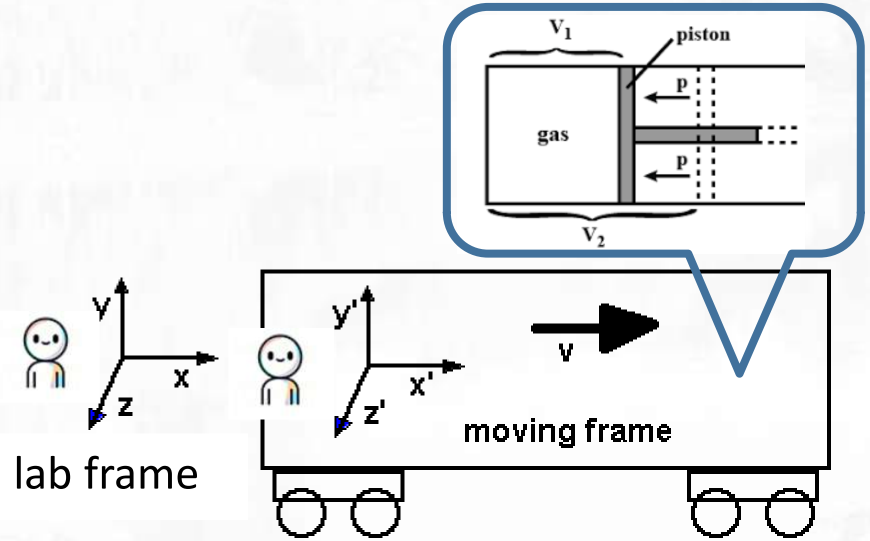
$$\langle W \rangle - \Delta F \geq 0$$

G. E. Crooks, J. Stat. Phys. 90, 1481 (1998).

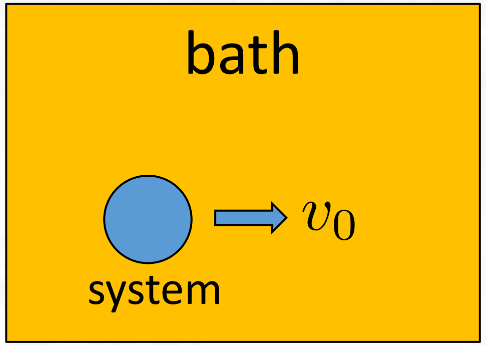
# Fluctuation theorems are not covariant

An implicit requirement of fluctuation theorems (and 2nd law): system and bath are at rest relative to the observer

- NOT compatible with the principle of covariance
- Invalid when system and bath are in relative motion




No Fluctuation theorem unless in comoving frame




No Fluctuation theorem in any frame

# Our aim

(for classical, relativistic systems)

■ Promoting fluctuation theorems into covariant forms

--- FTs valid for moving systems and baths

# Van Kampen formulation of relativistic thermodynamics

A covariant formulation of equilibrium thermodynamics in special relativity

inverse temperature 4-vector $\beta_\mu = \beta u_\mu$

(rest) inverse temperature

4-velocity $u_\mu = \eta_{\mu\nu} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$

$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$

work
heat

energy + momentum →

work 4-vector $W^\mu$
heat 4-vector $Q^\mu$

ensemble average level

concrete expression based on quasi-static volume work

Covariant equilibrium distribution $\rho_{\mathrm{eq}} = \exp(-\beta_\mu P^\mu)/Z$

Einstein summation rule

Equilibrium thermodynamic relation $\Delta P^\mu = W^\mu + Q^\mu$ $\Delta S = -\beta_\mu Q^\mu$

first law of thermodynamics

Clausius relation

Remark: so far only works for equilibrium thermodynamics

N.G. van Kampen, Phys. Rev. 173, 295 (1968)

# Our aim

- Promoting fluctuation theorems into covariant forms

FTs valid for moving systems and baths (relative to the observer)

- Bridging the relativistic thermodynamics and stochastic thermodynamics

Extend relativistic thermodynamics into nonequilibrium process

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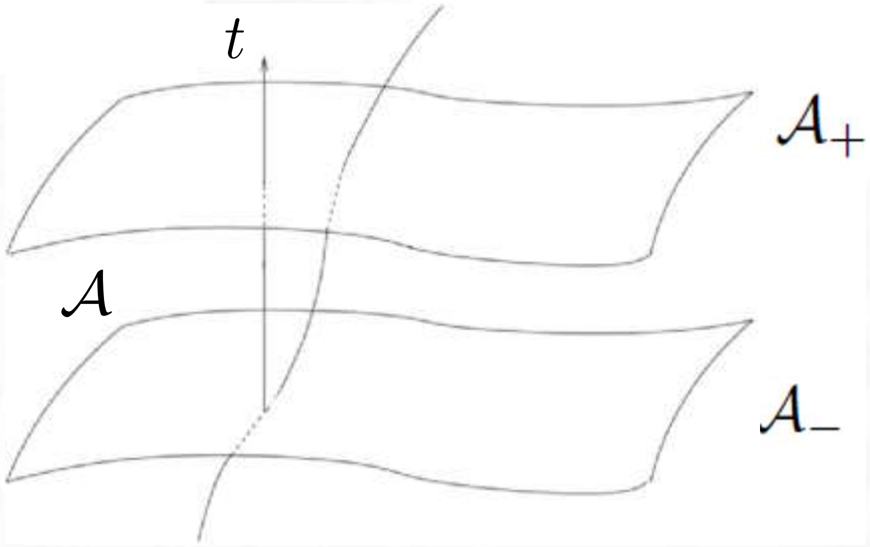
- Relativistic thermodynamic process
- Covariant Fluctuation theorems
- Summary

# Finite-time Thermodynamic process

"at the same time" → spacelike hypersurface




final hypersurface

initial hypersurface

space-time diagram

initial state $\rho\Big|_{\mathcal{A}_-}$

External driving field

$$h(x) \quad x \in \mathcal{A}$$

4-momentum change associated with a random trajectory

$$\Delta P^\mu = P^\mu\Big|_{\mathcal{A}_+} - P^\mu\Big|_{\mathcal{A}_-}$$

# Stochastic thermodynamic quantities

Action $I_{\text{tot}} = I[h(x)] + I_{\text{B}} + I_{\text{I}}$

System    Bath    Interaction

driving field

$h(x)$

**Particle systems**

$$I[z, h(x)] = \int \mathrm{d}s L$$

trajectory work 4-vector

$$W^{\mu} = -\int_{\mathcal{A}_-}^{\mathcal{A}_+} \mathrm{d}s \frac{\partial L}{\partial h} \frac{\partial h}{\partial x_{\mu}}$$

associated with external driving

trajectory heat 4-vector

$$Q^{\mu} = \Delta P^{\mu} - W^{\mu}$$

**Field systems**

$$I[\phi(x), h(x)] = \int \mathrm{d}^4 x \mathcal{L}$$

trajectory work 4-vector

$$W^{\mu} = -\int_{\mathcal{A}} \mathrm{d}^4 x \frac{\partial \mathcal{L}}{\partial h} \frac{\partial h}{\partial x_{\mu}}$$

trajectory heat 4-vector

$$Q^{\mu} = \Delta P^{\mu} - W^{\mu}$$

# Stochastic thermodynamic quantities

$W^{\mu}$ $Q^{\mu}$ 4-vectors under Lorentz transformation

$\mu = 0$ energy component $\mu = 1, 2, 3$ momentum components

$W^{\mu}$ 4- momentum change due to external driving

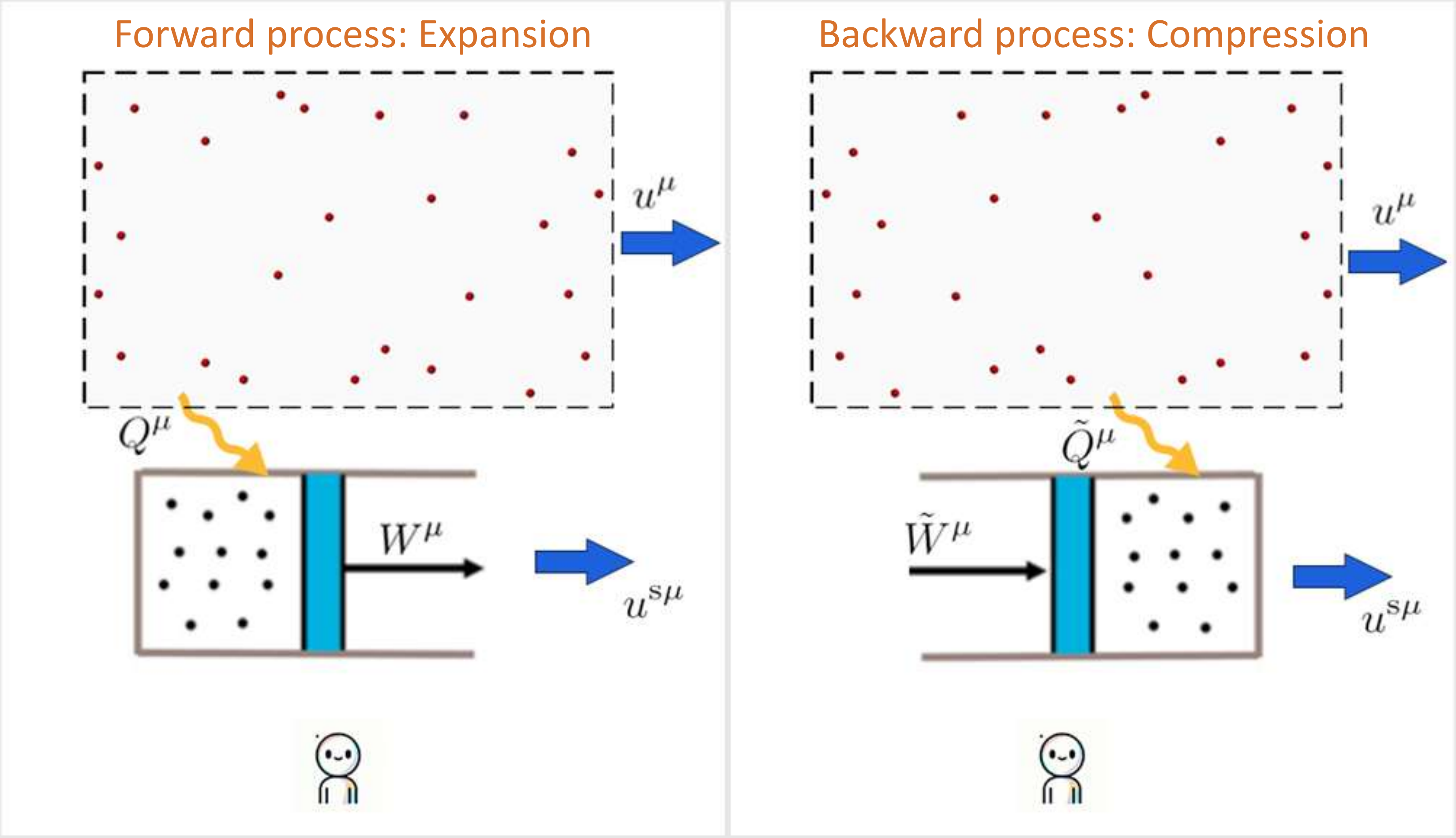
$Q^{\mu}$ 4- momentum change due to interaction with the bath

First law at the random trajectory level $\Delta P^{\mu} = Q^{\mu} + W^{\mu}$

generalization of van Kampen's definition to nonequilibrium and stochastic processes

# Backward process

Backward process: spacetime-reversal process




Driving field reversed

$$h(x) \to \tilde{h}(x) = h(-x)$$

Velocity does not change

$$u^\mu \to u^\mu$$

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# Covariant fluctuation theorems for work

- Covariant FT of work for a driving process

$$\frac{\mathrm{Pr}(\omega)}{\tilde{\mathrm{Pr}}(\tilde{\omega})} = \mathrm{e}^{\beta_\mu W^\mu[\omega] - \beta\Delta F}$$

(Lorentz covariant)

initial state of forward process: $\rho_{eq}(\beta_\mu, h_{ini})$
initial state of backward process: $\rho_{eq}(\beta_\mu, h_{fin})$
4-inverse temperature $\beta_\mu$ are the same as heat bath

- Integral FT (covariant Jarzynski's equality) and the covariant second law

$$\langle \exp(-\beta_\mu W^\mu) \rangle = \exp(-\beta \Delta F) \qquad \beta_\mu \langle W^\mu \rangle \geq \beta \Delta F$$

In the rest frame, it recovers the traditional FT

$$\beta_\mu W^\mu = \beta W^0_{\mathrm{rest}}$$

→ work (energy component) measured in the rest frame of the bath

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# Covariant fluctuation theorems for work

In a general inertial frame, bath 4-velocity $u_\mu = (\gamma, -\gamma\vec{v})$ $\qquad \gamma = (1-\vec{v}^2)^{-1/2}$

$$\beta_\mu \langle W^\mu \rangle - \beta \Delta F \geq 0$$

$$\beta\gamma \left( \langle W^0 \rangle - \sum_{i=1}^{3} v^i \langle W^i \rangle \right) - \beta \Delta F \geq 0$$

no upper bound for energy component only $\langle W^0 \rangle$

The momentum of the moving bath is a resource to extract work

# Covariant fluctuation theorems for heat

- Covariant heat exchange FT in pure relaxation process

$$\frac{\Pr(\omega)}{\Pr(\tilde{\omega})} = \mathrm{e}^{(\beta^{\mathrm{s}}_{\mu} - \beta_{\mu}) Q^{\mu}[\omega]}$$

initial state is in equilibrium $\rho_{eq}(\beta_{\mu}^{s})$
forward process and backward process are the same

system initial 4-inverse temperature $\beta^{\mathrm{s}}_{\mu} = \beta^{\mathrm{s}} u^{\mathrm{s}}_{\mu}$

bath 4-inverse temperature $\beta_{\mu} = \beta u_{\mu}$

- Integral fluctuation theorem

$$\left\langle \exp(-\beta^{\mathrm{s}}_{\mu} + \beta_{\mu}) Q^{\mu} \right\rangle = 1$$

- Covariant second law at ensemble average level

$$(\beta^{\mathrm{s}}_{\mu} - \beta_{\mu}) \langle Q^{\mu} \rangle \geq 0$$

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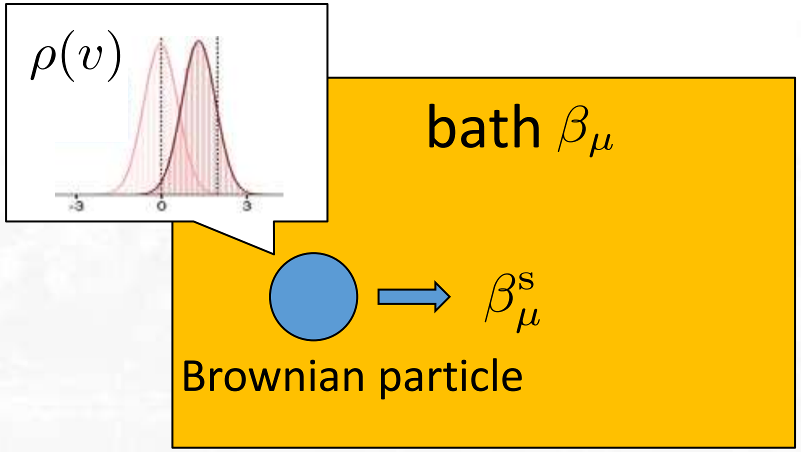
# Systems in relative motion

$\beta_\mu \nparallel \beta^{\mathrm{s}}_\mu \Rightarrow u_\mu \neq u^{\mathrm{s}}_\mu$ system and bath are in relative motion

$$\frac{\Pr(\omega)}{\Pr(\tilde{\omega})} = \mathrm{e}^{-(\beta_\mu - \beta^{\mathrm{s}}_\mu) Q^\mu [\omega]}$$




No rest reference frame

irreversibility characterized by $(\beta^{\mathrm{s}}_\mu - \beta_\mu) Q^\mu$

momentum components must be considered, regardless of the frame

# Entropy production

- Total Entropy production (k=1)

$$\Sigma = \Delta S - \beta_\mu Q^\mu$$

$$\Delta S = \ln \rho_{\mathcal{A}_-} - \ln \rho_{\mathcal{A}_+}$$

inverse temperature 4-vector of bath

- Entropy production fluctuation theorem:

$$\frac{\Pr(\omega)}{\tilde{\Pr}(\tilde{\omega})} = \exp(\Sigma[\omega]).$$

arbitrary initial state of forward process, initial state of the backward process is the final state of initial process

(Lorentz covariant entropy production fluctuation theorem)

# Nonrelativistic limit

In nonrelativistic limit $v \ll c$

3-velocity of the bath $\vec{v}$

$$\beta_\mu = \left(\frac{1}{T}, -\frac{\vec{v}}{T}\right)$$

$W^0_{\text{rest}}$

work fluctuation theorem (Galileo covariant)

$$\frac{\Pr(\omega)}{\tilde{\Pr}(\tilde{\omega})} = \exp\left[\beta\left(W^0 - \sum_{i=1}^{3} v^i W^i\right) - \beta \Delta F\right]$$

heat fluctuation theorem (Galileo covariant)

$$\frac{\Pr(\omega)}{\tilde{\Pr}(\tilde{\omega})} = \exp\left[-(\beta - \beta^{\mathrm{s}})Q^0 + \sum_{i=1}^{3}(\beta v^i - \beta^{\mathrm{s}} v^{\mathrm{s}i})Q^i\right]$$

Momentum components play a role for nonrelativistic moving objects

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# Summary

- Generalize work and heat 4-vectors to relativistic nonequilibrium process

- Promote fluctuation theorems into covariant forms

- Valid for arbitrary uniformly moving system or bath

$$\frac{\Pr(\omega)}{\tilde{\Pr}(\tilde{\omega})} = \mathrm{e}^{\beta_\mu W^\mu[\omega] - \beta \Delta F} \qquad \frac{\Pr(\omega)}{\Pr(\tilde{\omega})} = \mathrm{e}^{-(\beta_\mu - \beta^{\mathrm{s}}_\mu) Q^\mu[\omega]}$$

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# Thank you for your attention

# Example: field system

Stochastic Klein-Gordon field (inertial model A)

$$\partial^\mu \partial_\mu \phi + \kappa^\mu \partial_\mu \phi + m^2 \phi = h(x) + \sqrt{2\kappa/\beta}\,\xi(x)$$

$\kappa^\mu = \kappa u^\mu$

friction coefficient

external driving

$\beta$ rest inverse temperature

Lagrangian density $\mathcal{L} = \int \mathrm{d}^4 x \frac{1}{2}[\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2]$

$$W^\nu = \int_{\mathcal{A}} \mathrm{d}^4 x (-\phi \partial^\nu h) \qquad Q^\nu = \int_{\mathcal{A}} \mathrm{d}^4 x \partial^\nu \phi (\partial^\mu \partial_\mu \phi + m^2 \phi + V' - h)$$

# Example: work statistics

Consider a driving process: in the infinite past and infinite future $h(x) = 0$

work is performed during $-\infty < t < +\infty$, finally dissipated into the heat bath

Joint distribution of 4-work is Gaussian, with

$$\langle W^\mu \rangle = -\int \mathrm{d}^4x\mathrm{d}^4y\partial^\mu h(x)\Delta(x-y)(\partial^\nu\partial_\nu - \kappa^\nu\partial_\nu + m^2)h(y)$$

$$\langle W^\mu W^\nu \rangle - \langle W^\mu \rangle \langle W^\nu \rangle = \int \mathrm{d}^4x\mathrm{d}^4y\frac{2\kappa}{\beta}\partial^\mu h(x)\Delta(x-y)\partial^\nu h(y)$$

$$\Delta(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}p_\mu x^\mu}}{(-p^\mu p_\mu + m^2)^2 + (\kappa_\mu p^\mu)^2}$$

The work distribution satisfies the covariant FT

$$\left\langle \exp(-\int \mathrm{d}^4x\beta_\nu W^\nu(x)) \right\rangle = 1$$

# Example: particle system

Relativistic Ornstein-Uhlenbeck process under electromagnetic field

$$\frac{\mathrm{d}p^i}{\mathrm{d}t} = f^i - \kappa \frac{p^i}{p^0} + \sqrt{2\kappa/\beta}\xi^i$$

$$\vec{f} = q\vec{E} + q\vec{v} \times \vec{B}$$

$\kappa$ friction coefficient

$\beta$ rest inverse temperature of bath

detailed balance condition satisfied

$$\mathrm{d}Q^\mu = \mathrm{d}p^\mu - F^{\mu\nu}\mathrm{d}x_\nu \qquad \mathrm{d}W^\mu = \partial^\mu A^\nu \mathrm{d}x_\nu$$

4-heat can also be expressed via the friction and noise force

$$Q^0 = \int_{t_\mathrm{i}}^{t_\mathrm{f}} (-\kappa v^i + \sqrt{2\kappa/\beta}\xi^i) v^i \mathrm{d}t \qquad Q^i = \int_{t_\mathrm{i}}^{t_\mathrm{f}} (-\kappa v^i + \sqrt{2\kappa/\beta}\xi^i) \mathrm{d}t$$

# Example: heat statistics

For a massless particle, relaxation process in (1+1)d can be solved exactly

transition probability from momentum $p$ to $q$

$$P_t(p|q) = \frac{1}{\sqrt{4\pi\kappa t/\beta}} \exp[-\frac{(p-q)^2}{4\kappa t/\beta} - \frac{\beta}{2}(|p|-|q|) - \frac{\beta\kappa}{4}]$$
$$+ \frac{\beta}{4}\exp(-\beta|p|)\,\mathrm{erfc}[\frac{1}{\sqrt{4\kappa/\beta}}(\frac{|p|+|q|}{\sqrt{t}} - \kappa\sqrt{t})]$$

joint distribution of heat 2-vector

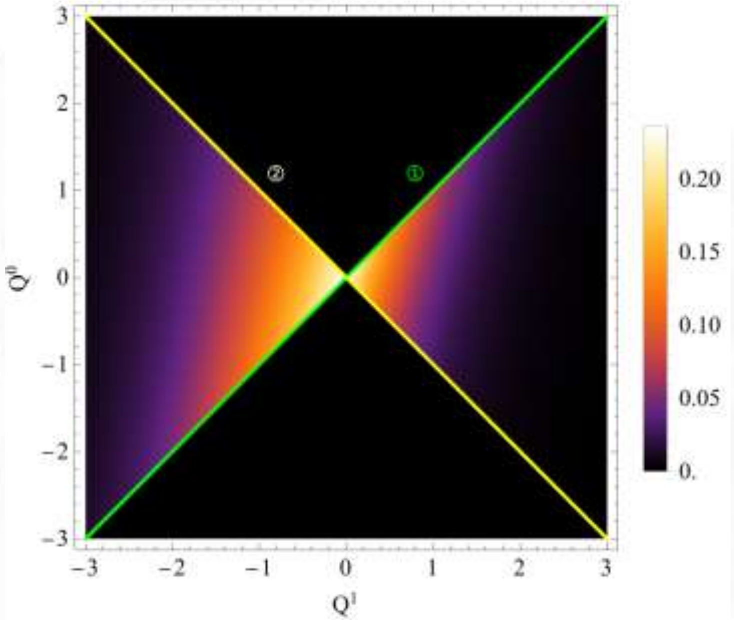
$$P_t(Q^0, Q^1) = \int \mathrm{d}p\mathrm{d}q\delta(Q^0 - (|p|-|q|))\delta(Q^1 - (p-q))P_t(p|q)\rho_0(q)$$

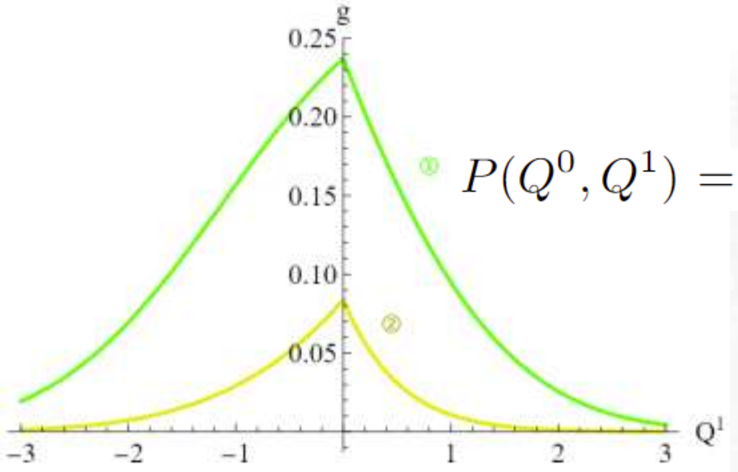
# Example: heat statistics

- Joint 2-heat distribution

$$\beta^s_{\text{ini}} = (5/4, 3/4) \quad \beta^b = (1, 0) \qquad (\beta^s)^2 = (\beta^b)^2$$

same initial temperature as heat bath, moving rightwards




$$P(Q^0, Q^1) = \delta(Q^0 \pm Q^1) g(Q^1)$$

on lines $Q^1 = \pm Q^2$, degenerated to 1-d distribution

- satisfies the covariant heat exchange fluctuation theorem