Anomalous Enhancement of Structural Stability due to Static Friction

20+10min
19pages

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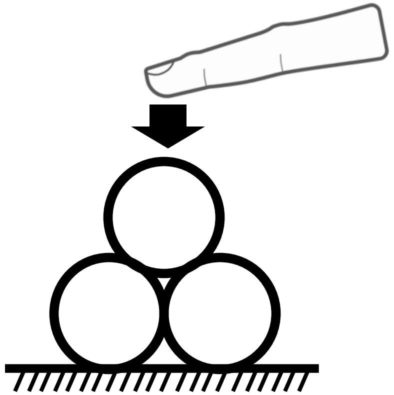
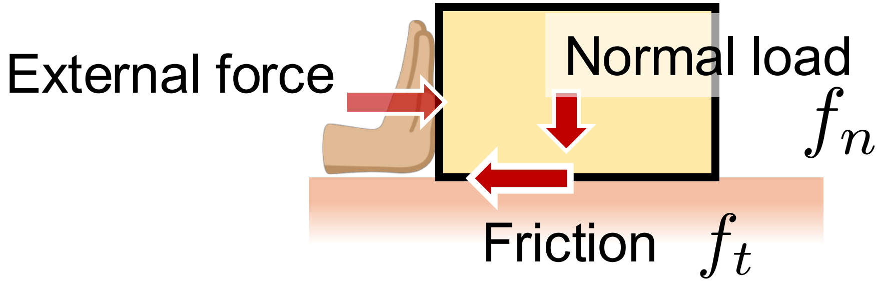


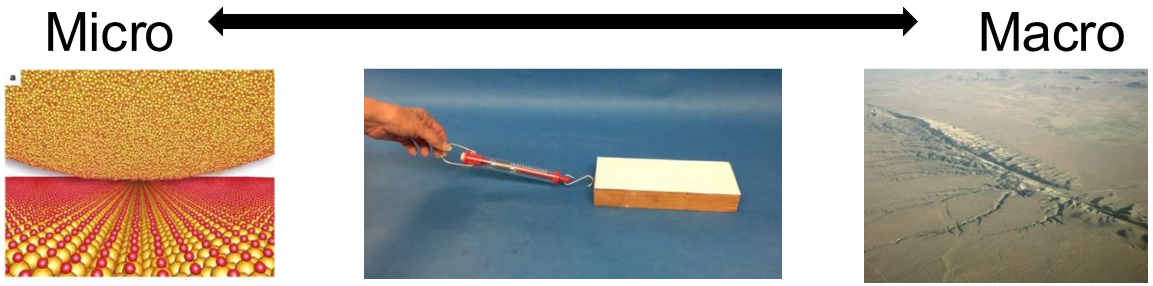
2025/07/09 Mini-workshop on “Jamming, rheology and granular matter”

Friction

What is Friction?

- A force that **resists external tangential force** at solid contact
- Observed over **a wide range of scales**





Amontons-Coulomb law (17-18th centuries)

- **Universal empirical laws** at the **macroscopic scale**
- The key point is that **tangential force can resist up to the following threshold**:

$$|f_t| \leq \mu f_n$$

Material-specific value: **Friction coefficient**

H. Matsukawa, The Physics of Friction, Iwanami Shoten (2012).

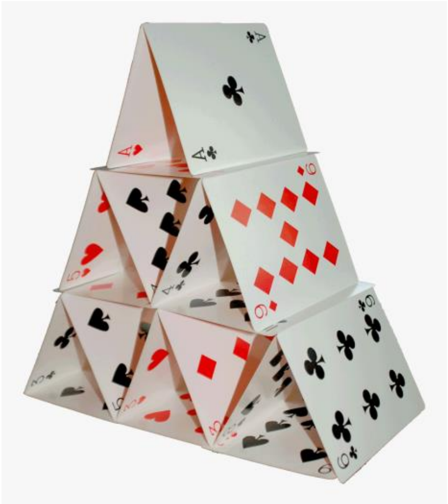
Friction-Stabilized Structure

■Examples of Friction-Stabilized Structure

- Let's consider **the static state** of a system **with friction**

Sandpile

House of cards



Arch bridge

The aqueduct in Maintenon, built in the 17th century

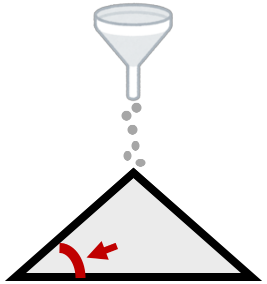
- As is well recognized, **friction and geometry** work together to suppress sliding and **enhance mechanical stability**

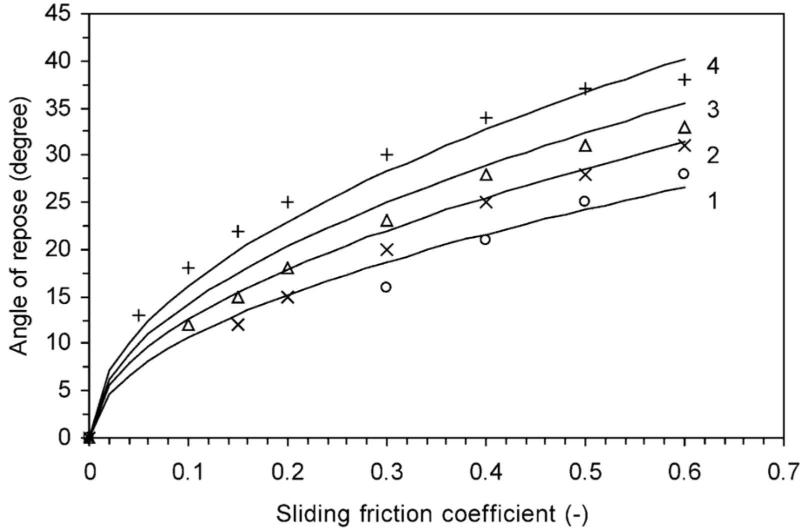
How do **these stability** depend on the **friction coefficient**?

Effects of Friction Coefficient

The effects of friction coefficient are usually gradual.

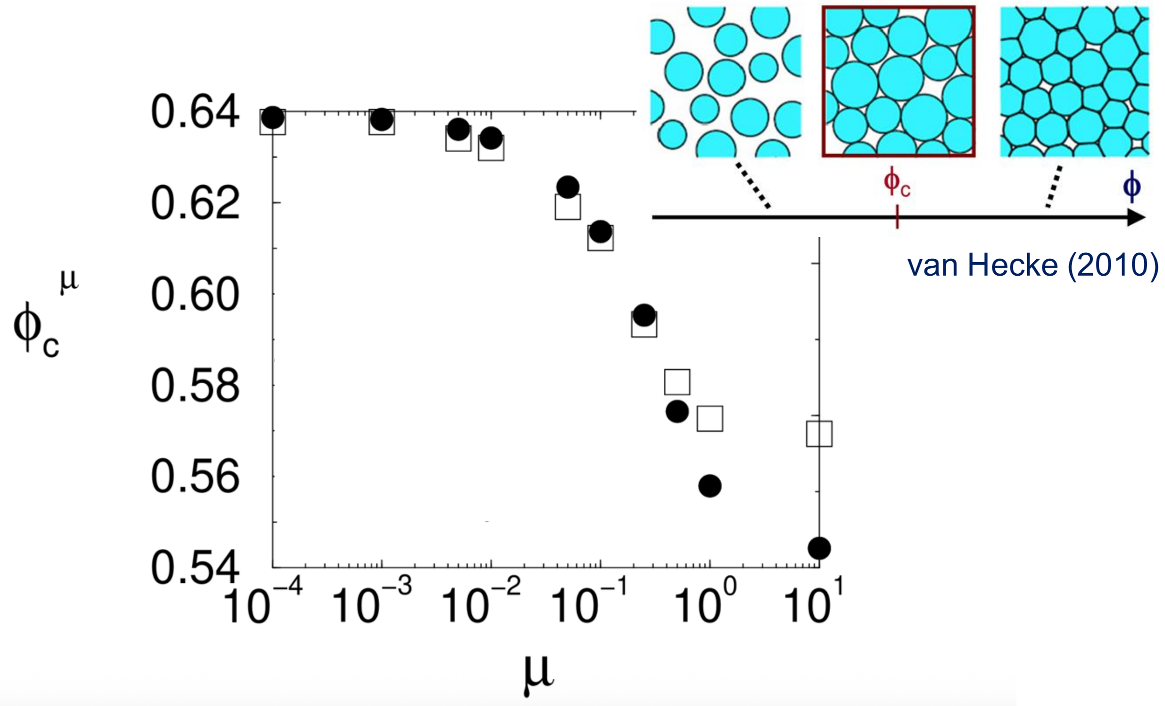
- **Angle of repose**





Y.C. Zhou et al., Powder Technology (2002).

- **Rigidity transition point**



van Hecke (2010)

L.E. Silbert, Soft Matter (2010).

✓ We focus on a phenomenon with **a singular dependence on the friction coefficient**.

Phenomena of Interest

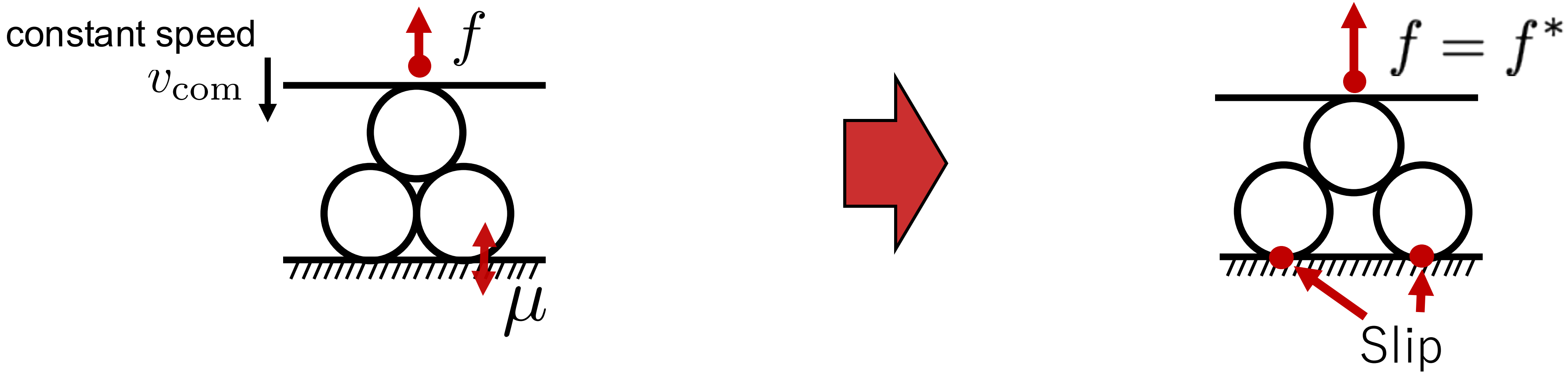
metal
floor

wood
floor

Credit: Tani Marie

Setup

① Compress quasi-statically ② Slip occurs when f = f*

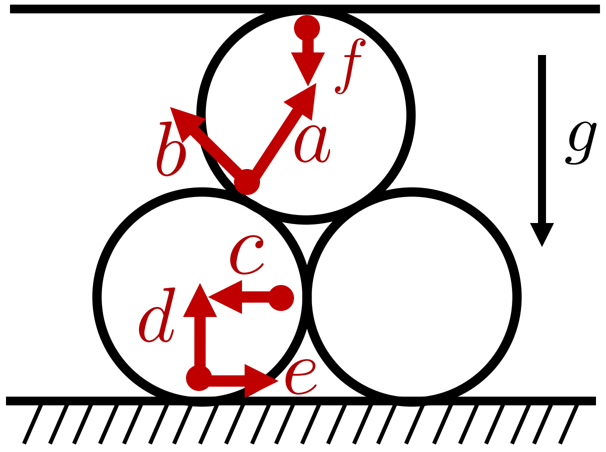


We define the threshold f* as the **yield force.**

✓ We investigate how **the yield force** depend on **the friction coefficient.**

Rigid-Body Case (1/2)

First, we consider the case of **rigid cylinders** (i.e., non-deformable).



m : mass of cylinder

μ : friction coefficient with the floor

- **Variables** (a, b, c, d, e, f) ← 6vars

- **Conditions**

  - Equilibrium of forces
  $$\begin{cases} \sqrt{3}a + b - f - mg = 0 \\ \sqrt{3}a + b - d + 2mg \\ -a + \sqrt{3}b + 2e = 0 \\ b - e = 0 \end{cases}$$

  - Slip condition (Coulomb law) $\boxed{e = \mu d}$

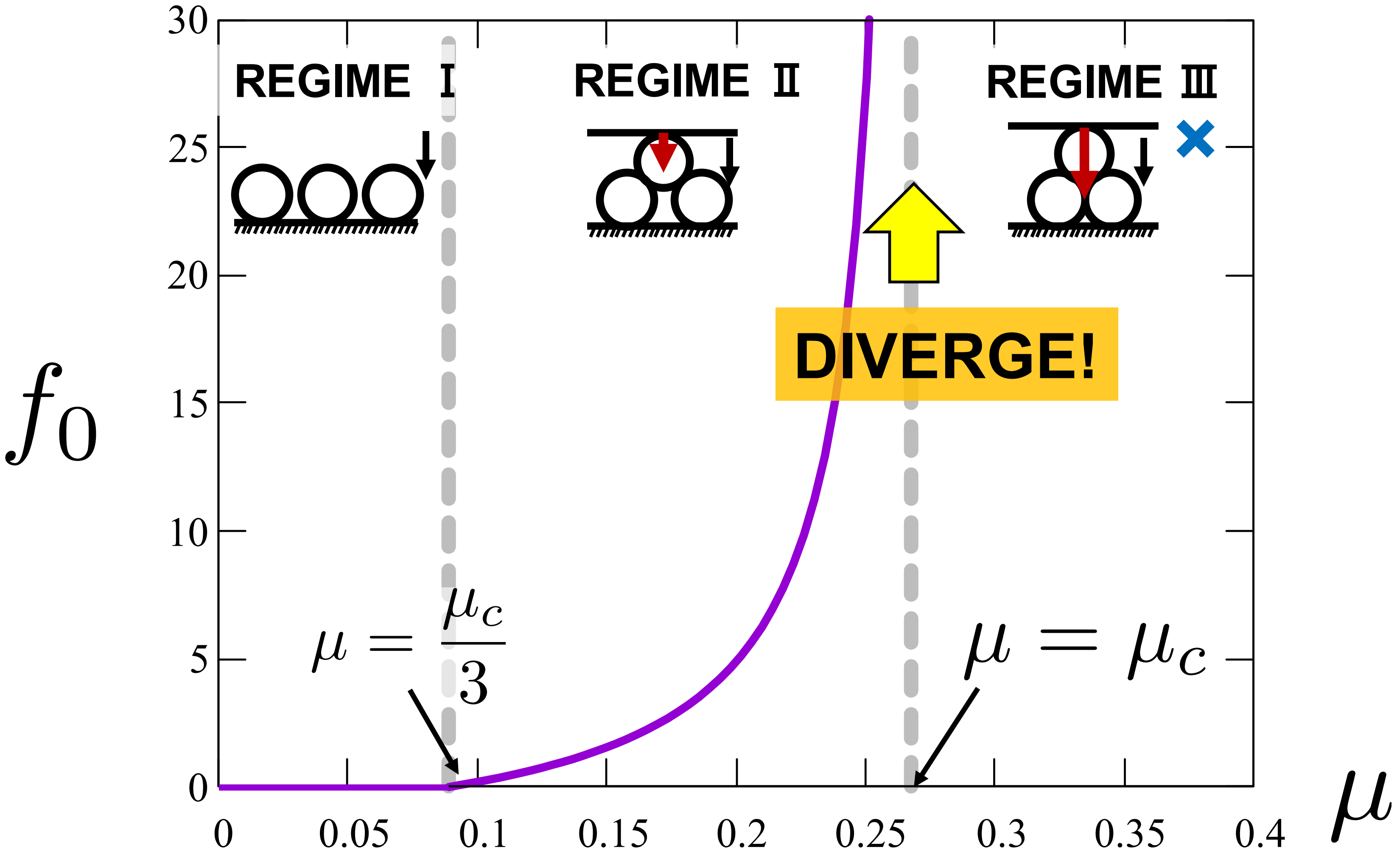
  - At the onset of sliding, we **assume** $\boxed{c = 0}$

6eqs.

Yield force
$$f_0 = \frac{3\mu_c - \mu}{\mu_c - \mu} mg$$

critical point
$$\mu_c = 2 - \sqrt{3}$$
$\simeq 0.268$

Rigid-Body Case (2/2)

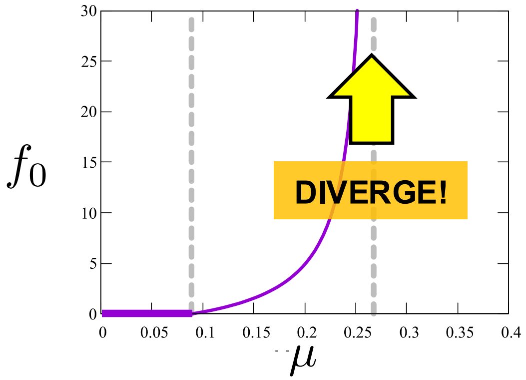


The yield force **diverges** at $\mu = \mu_c$

Question

■ Rigid-Body Case

- Yield force **diverges** at $\mu = \mu_c$
- However, a rigid-body is **an idealized limit**



■ Question

1. How is this phenomenon observed in realistic **elastic bodies**?
2. **Can the yield force be predicted** from the **material properties** (such as Young's modulus and Poisson's ratio)?

1. Model (DEM) & Simulation
2. Perturbation Analysis
3. Main Results

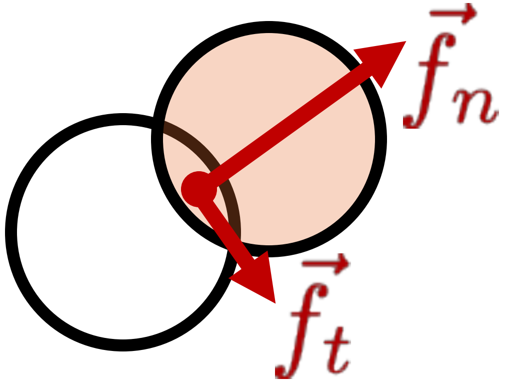
Outline

1. **Model (DEM) & Simulation**
2. Perturbation Analysis
3. Main Results

Model (DEM)

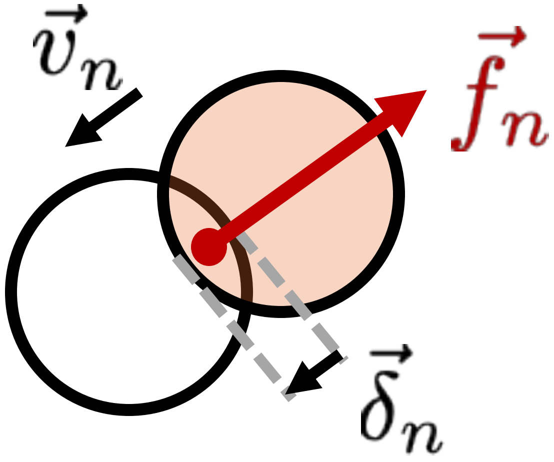
◆ DEM (Discrete Element Method)

P.A. Cundall and O.D.L. Strack, *Géotechnique* 29 (1979).



A contact force model based on Amontons-Coulomb law

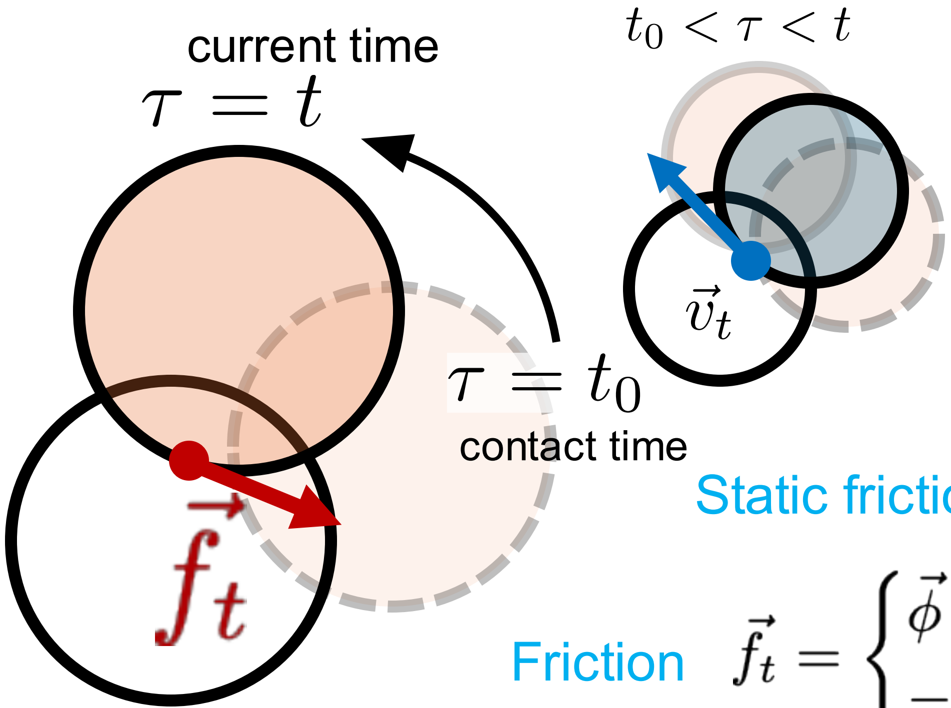
■ Normal component



$$\vec{f}_n = -k_n\vec{\delta}_n - \eta_n\vec{v}_n$$

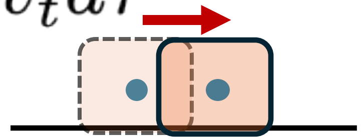
Elastic Damping

■ Tangential component



Tangential displacement

$$\vec{\delta}_t = \int_{t_0}^{t} \vec{v}_t d\tau$$



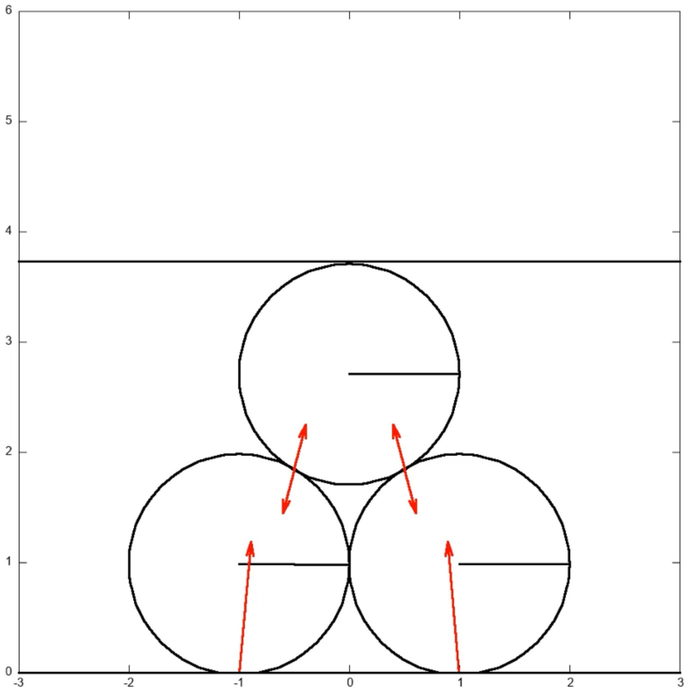
Elastic Damping

Static friction $\vec{\phi} = -k_t\vec{\delta}_t - \eta_t\vec{v}_t$

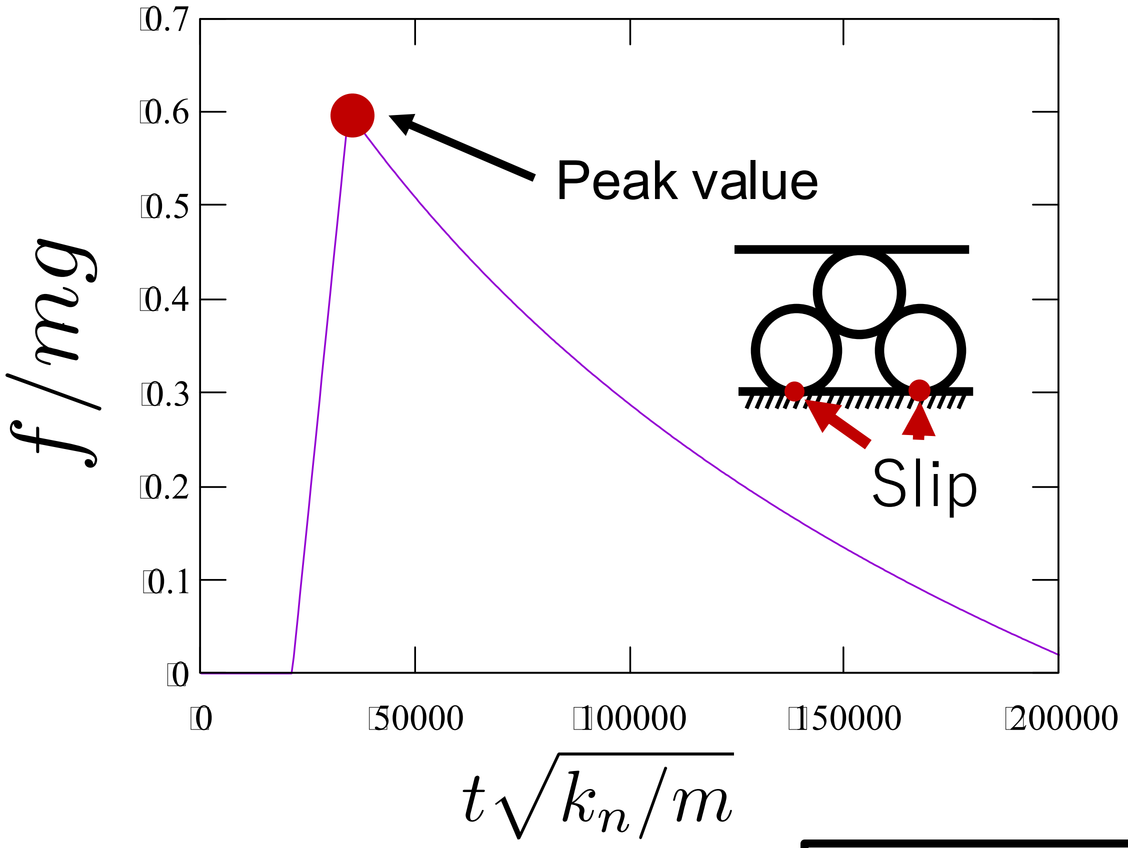
Friction $\vec{f}_t = \begin{cases} \vec{\phi} & \text{if } |\vec{\phi}| < \mu|\vec{f}_n|, \\ -\mu|\vec{f}_n|\frac{\vec{v}_t}{|\vec{v}_t|} & \text{otherwise.} \end{cases}$

Measurement of Yield Force in Simulation

We record the **normal force f(t)** during compression



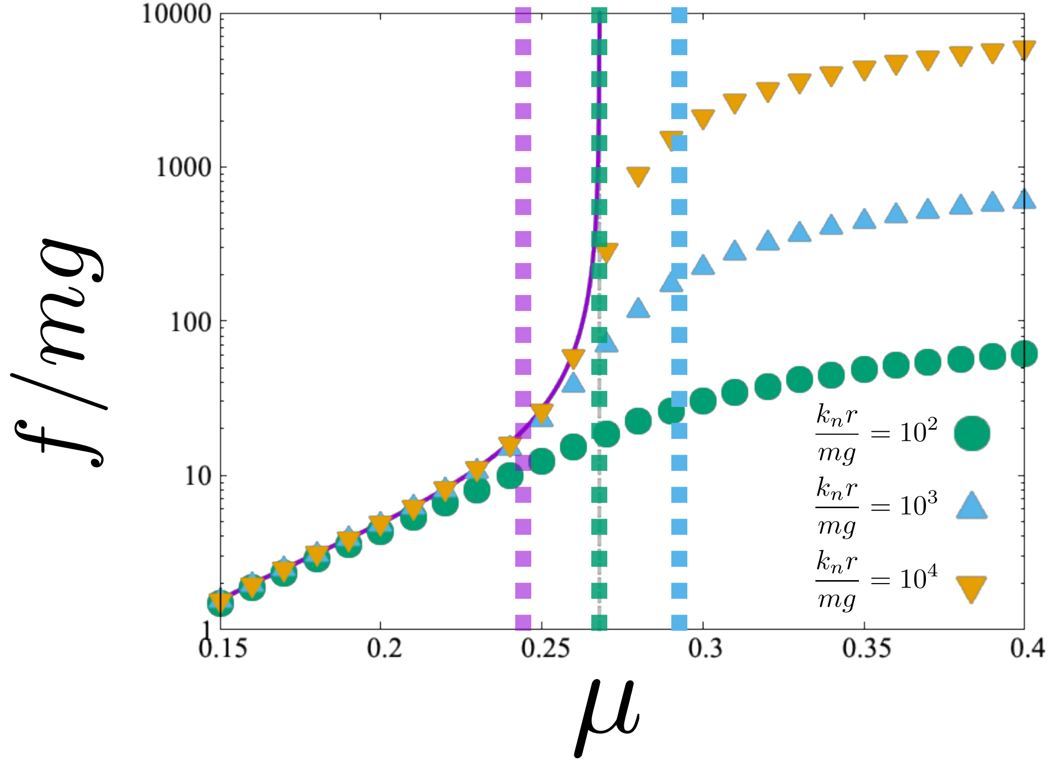
$$\frac{k_n r}{mg} = 10^2, k_t = 0.4k_n, \mu_d = 0.7$$

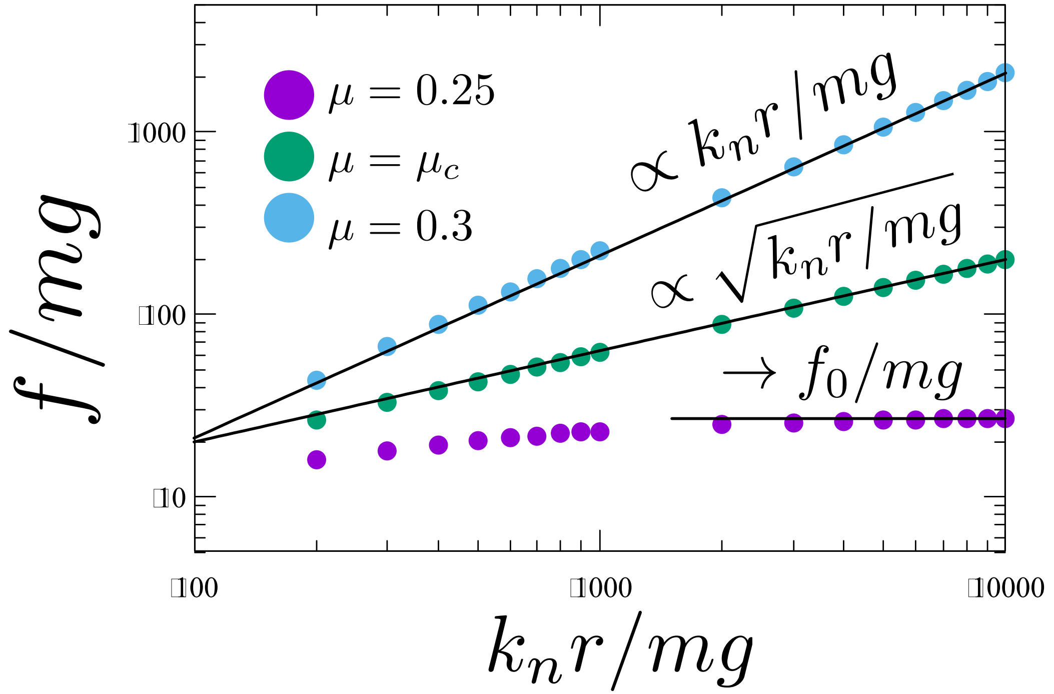


The yield force is defined as **the peak value of f(t)** →

$$\boxed{f\left(\mu, \frac{k_n r}{mg}\right)}$$

Numerical Results





- At $\mu = \mu_c$, the yield force exhibits an **anomalous increase** as $k_n r/mg \to \infty$.

- The following **scaling behavior** is observed.

$$f/mg \to \begin{cases} f_0/mg & \mu < \mu_c \\ O(\sqrt{k_n r/mg}) & \mu = \mu_c \qquad k_n r/mg \to \infty \\ O(k_n r/mg) & \mu > \mu_c \end{cases}$$

✓ Based on **DEM,** we derive the yield force and its scaling **using perturbation analysis.**

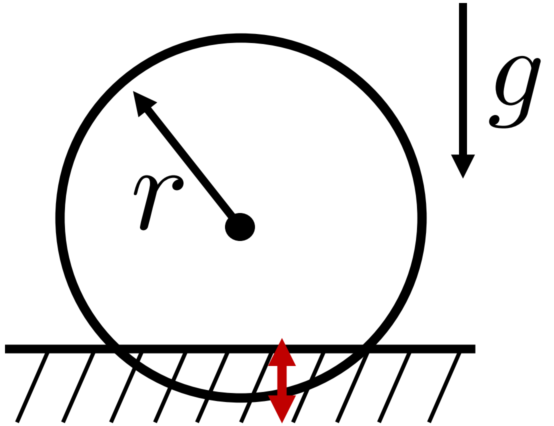
Outline

1. Model (DEM) & Simulation
2. **Perturbation Analysis**
3. Main Results

Perturbation and Small Parameter

Aim: To obtain **a perturbative expression** for the yield force **in the limit of large stiffness**

- Small parameter $$\boxed{\epsilon \equiv \frac{mg}{k_n r}}$$
  - $\epsilon = 0$ corresponds to a rigid-body limit.



displacement $\delta = \frac{mg}{k_n} \Rightarrow \epsilon = \frac{\delta}{r}$

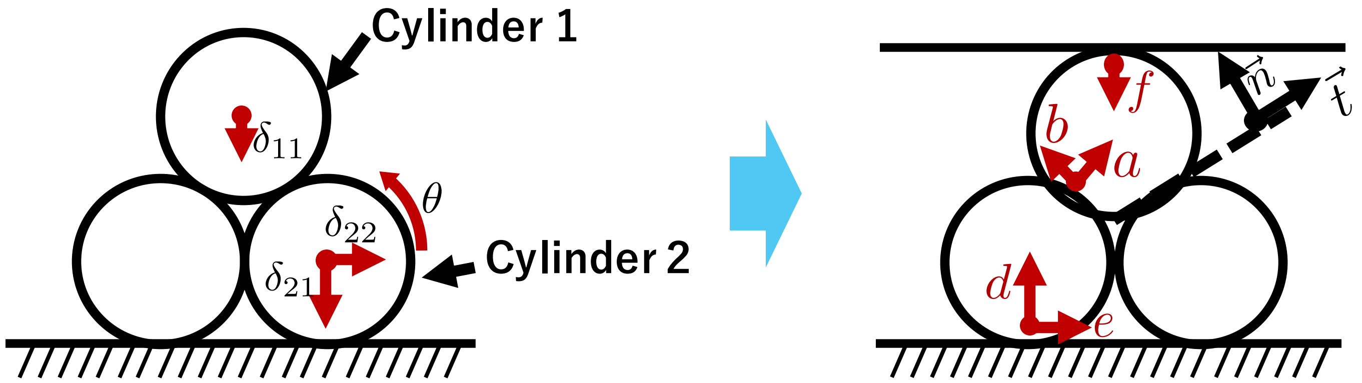
- We consider the following limit:

$$\boxed{\epsilon \to 0 \quad \text{with} \quad \kappa \equiv \frac{k_n}{k_t} \quad \text{(fixed)}}$$

Equation for the Yield force

Displacement and unit vector

- To describe the displacement, we define **four variables:** $(\delta_{11}, \delta_{21}, \delta_{22}, \theta)$
- To describe the geometric configuration, we define **unit vectors**: $\vec{n} = (n_x, n_y)$ $\vec{t} = (t_x, t_y)$



- Contact forces and unit vectors **can be described using displacements**:

$$a = a(\delta_{11}, \delta_{21}, \delta_{22}, \theta)$$

$$n_x = n_x(\delta_{11}, \delta_{21}, \delta_{22}, \theta)$$

etc.

Force balance and sliding condition

$$\begin{cases} f + 3mg = 2d \\ e = \mu d \end{cases} \quad \& \quad \begin{cases} b - e = 0 \\ an_x + b(t_x + 1) = 0 \\ 2an_y + 2bt_y - f - mg = 0 \end{cases}$$

5 Equations that determine $(\delta_{11}, \delta_{21}, \delta_{22}, \theta, f)$

Complex…

Equation to be Solved

- We introduce a **new variables**: $x = \delta_{11} - \delta_{21}, \quad y = \delta_{22}, \quad z = \delta_{21}$
- After all, previous equations leads to the following **two equations**:

Equation that determines f form z

$$\boxed{f + 3mg = 2k_n z}$$

Equation that determines z EQZ

$$\boxed{z = r\mathcal{D}(\mu, \alpha)\mathcal{E}(\mu, \alpha)} \quad \text{with } z \geq \frac{3}{2}\epsilon r$$

$\because f \geq 0$

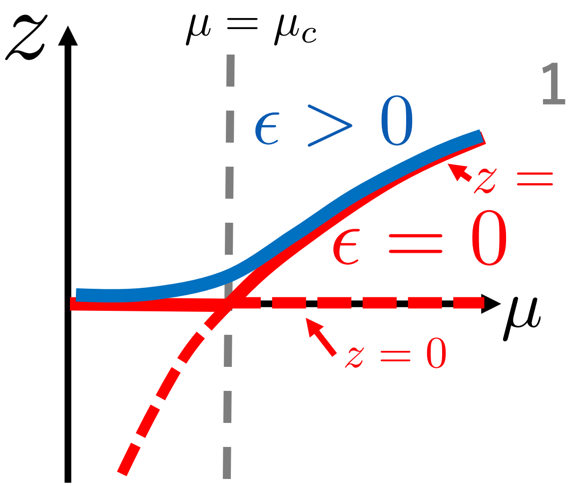
$$\alpha \equiv \frac{\epsilon r}{z}$$

$$\mathcal{D}(\mu, \alpha) \equiv \frac{\mu^2 + 2\sqrt{3}(1-\alpha)\mu - (1-\alpha)^2}{\mu\left((1-\alpha)^2 + \mu^2\right)}$$

$$\mathcal{E}(\mu, \alpha) \equiv \frac{(1-\alpha)^2}{(1-\alpha)^2 + \kappa\left((1-\alpha)^2 + \mu^2\right)}$$

All we have to do is to **solve EQZ** in the limit $\epsilon \to 0$ with κ fixed

Solution of EQZ

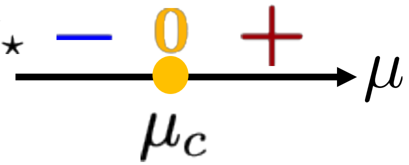


$\boxed{\text{EQZ}}$ $z = r\mathcal{D}(\mu,\alpha)\mathcal{E}(\mu,\alpha)$ with $z \geq \frac{3}{2}\epsilon r$

$\epsilon = 0$ ➡ solution $z = \begin{cases} 0 & \mu < \mu_c \\ z_\star & \mu \geq \mu_c \end{cases}$

$$z_\star \equiv r\frac{(\mu-\mu_c)(\mu+\mu_c^{-1})}{\mu(1+\mu^2)(1+\kappa(1+\mu^2))}$$

sign of $z_\star$



$\epsilon > 0$ ➡ not exactly solvable

Assume a form of solution

$$z = \begin{cases} \epsilon z_1^- + O(\epsilon^2) & \mu < \mu_c \\ z_\star + \epsilon z_1^+ + O(\epsilon^2) & \mu \geq \mu_c \end{cases} \xrightarrow{\text{EQZ}} z_1^- = \frac{\mu_c r}{\mu_c - \mu} \quad z_1^+ = \cdots$$

Naive perturbative expansion

The first-order correction term **diverges** as $\mu \to \mu_c$

The naive perturbation **breaks down**; **singular perturbation** is needed

Singular Perturbation

Idea We expand $z(z - z_\star)$ in powers of ϵ instead of z itself

$$\boxed{z = r\mathcal{D}(\mu, \alpha)\mathcal{E}(\mu, \alpha)} \Rightarrow \boxed{z(z - z_\star) = \epsilon r^2 \Phi(\mu) + O(\epsilon^2)}$$

EQZ

complicated function of μ

$f + 3mg = 2k_n z$

Singular perturbated solution

$$\boxed{\frac{f_\epsilon + 3mg}{mg} = \epsilon^{-1}\left(z_\star + \sqrt{z_\star^2 + 4\epsilon r^2 \Phi(\mu)}\right)} \quad \bigstar$$

Based on **Johnson's theory**: $\epsilon = \dfrac{\rho r g}{E^*}\left[\log\left(\dfrac{8E^*}{\rho r g}\right) - 1\right] \quad E^* = \dfrac{E}{2(1-\nu^2)}$ effective modulus

C. M. Pereira et al., Nonlin. Dyn. 63, 681 (2011).

- **E.g., wood cylinder** $E = 10\,\mathrm{GPa},\ \nu = 0.3,\ \rho = 0.5\,\mathrm{g/cm^3},\ r = w = 1\,\mathrm{cm},\ \kappa = 2.5$

$$\epsilon = 1.8 \times 10^{-7} \longrightarrow \begin{cases} f(\mu = 0.2) = 7.5 \times 10^{-2}\,\mathrm{N} \\ f(\mu = 0.4) = 5.3 \times 10^{4}\,\mathrm{N} \end{cases} \times 10^6 \text{ !!}$$

Outline

1. Model (DEM) & Simulation
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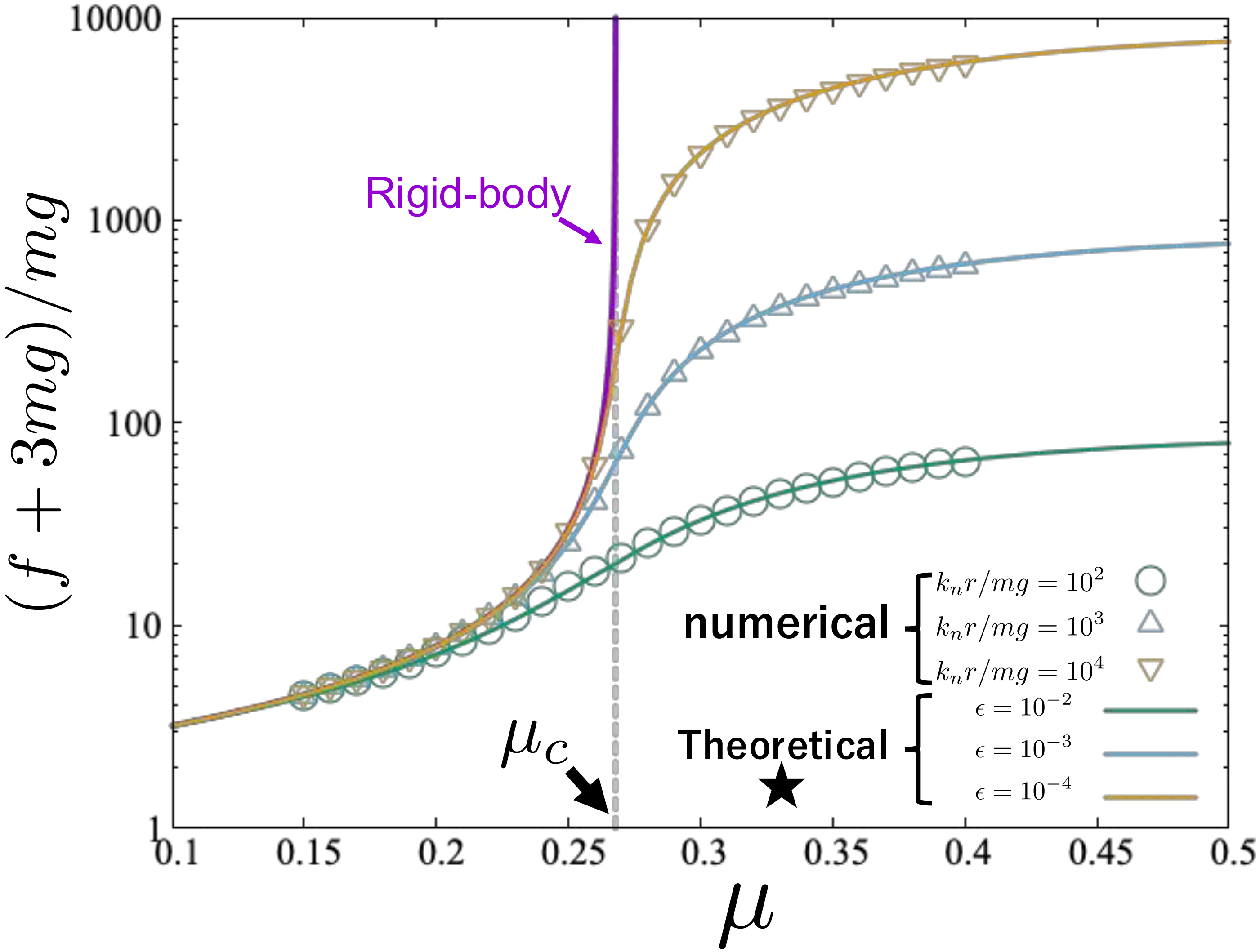
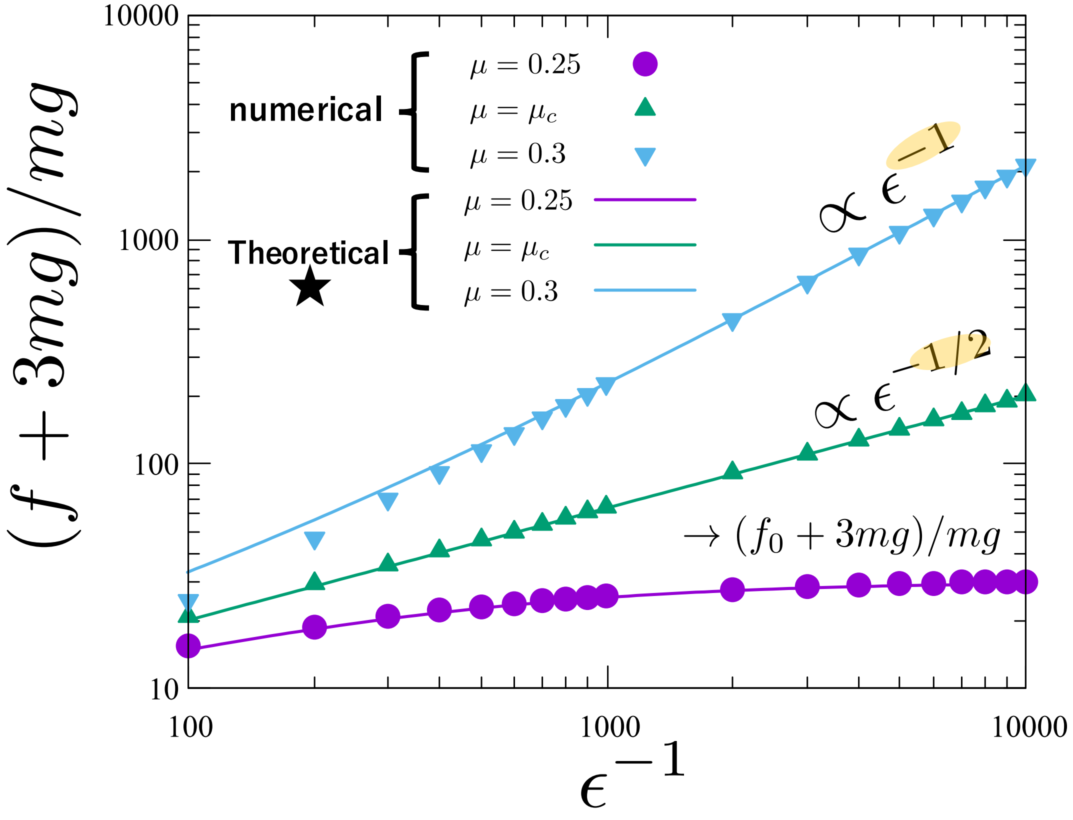
3. **Main Results**

Main result


Rigid-body
$(f + 3mg)/mg$
10000
1000
100
10
1
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
μ
μ_c
numerical
$k_n r/mg = 10^2$
$k_n r/mg = 10^3$
$k_n r/mg = 10^4$
Theoretical ★
$\epsilon = 10^{-2}$
$\epsilon = 10^{-3}$
$\epsilon = 10^{-4}$

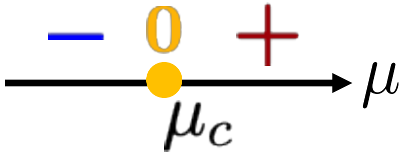
Scaling



$$\boxed{\frac{f_\epsilon + 3mg}{mg} = \epsilon^{-1}\left(z_\star + \sqrt{z_\star^2 + 4\epsilon r^2\Phi(\mu)}\right)} \quad ★$$

■ Derivation of the scaling

Remind sign of $z_\star$



- $\mu = \mu_c \Rightarrow (f_\epsilon + 3mg)/mg = \epsilon^{-1}\left(\cancel{z_\star} + \sqrt{\cancel{z_\star^2} + 4\epsilon r^2\Phi(\mu)}\right)$
  $= O(\epsilon^{-1/2})$
- $\mu > \mu_c \Rightarrow (f_\epsilon + 3mg)/mg = \epsilon^{-1}\left(z_\star + \sqrt{z_\star^2 + \cancel{4\epsilon r^2\Phi(\mu)}}\right)$
  $= O(\epsilon^{-1})$

We can derive scaling straightforwardly from ★

Summary and Outlook

We studied the yield force in a system of **three stacked cylinders**:

1. For **rigid-body case**, the yield force **diverges at μ_c**
2. For **elastic case**, the yield force shows **a singular increase near μ_c**
3. **A simple expression and the scaling behavior** can be derived using **singular perturbation analysis**

Are these results real?

⟶ Experiment