

ユニタリティと矛盾しない Sommerfeld 効果の計算

佐藤 亮介



K. Blum, R. Sato, T. R. Slatyer, 1603.01383, JCAP 06 (2016) 021

A. Parikh, R. Sato, T. R. Slatyer, 2410.18168

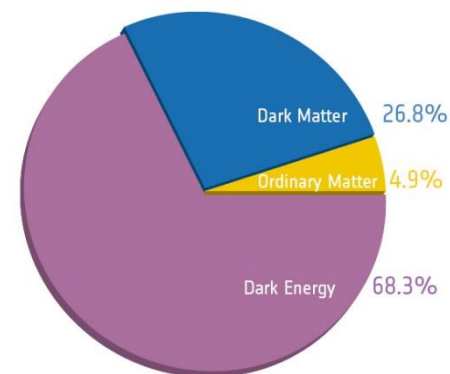
2025. 9. 5 @ 素粒子物理学の進展2025 (PPP2025)

Plan

1. Dark matter and Sommerfeld effect
2. Sommerfeld effect and unitarity

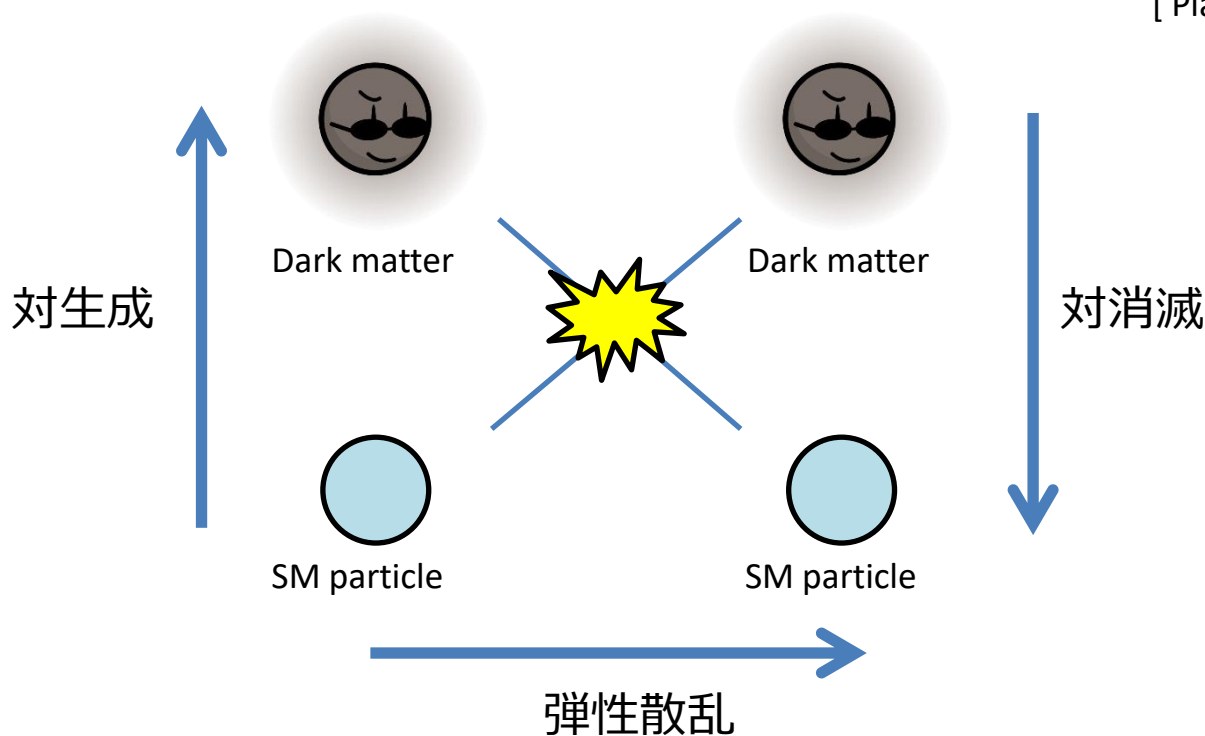
暗黒物質の対消滅

宇宙のエネルギーの27%は**暗黒物質**
その正体は未だ不明



[Planck collaboration]

暗黒物質が標準模型粒子と相互作用を持つと...



暗黒物質の対消滅

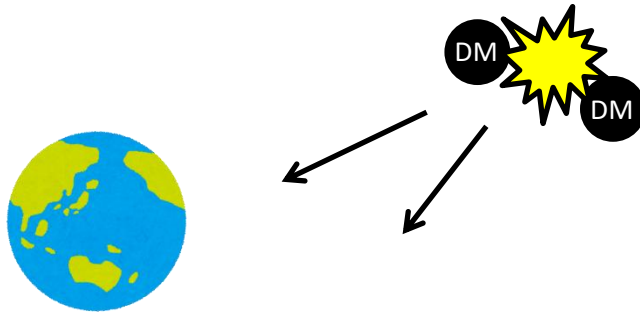
- Freeze-out 機構

現在の宇宙の残存量

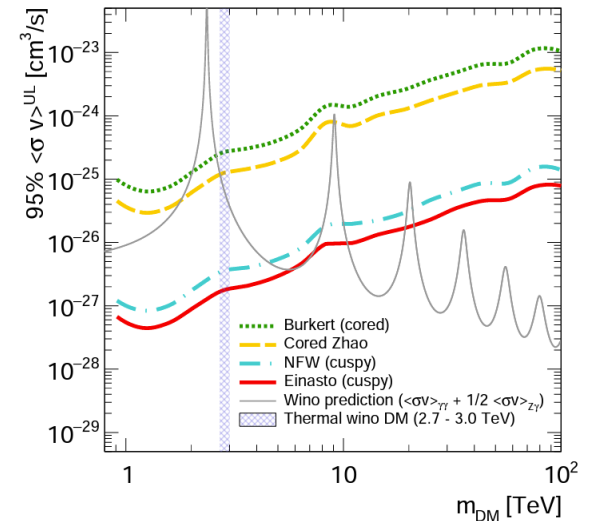
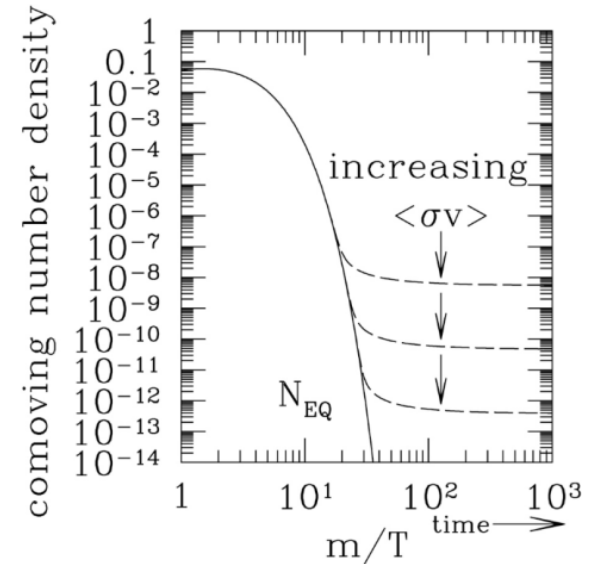
$$\rightarrow \Omega_{\text{DM}} h^2 \sim 0.1 \times \frac{3 \times 10^{-26} \text{ cm}^3/\text{s}}{\sigma v}$$

- 暗黒物質の間接探索

暗黒物質由来の高エネルギー宇宙線



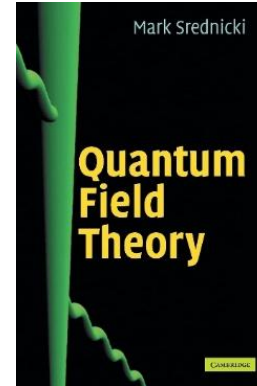
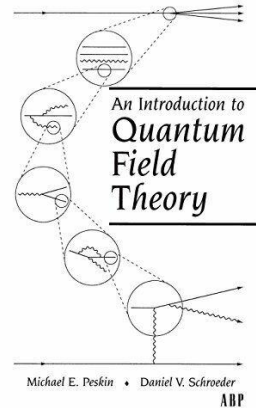
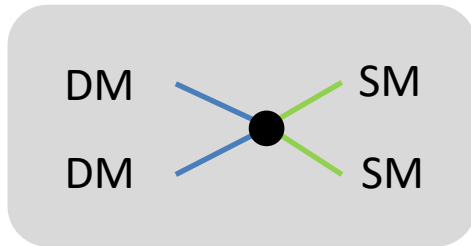
- ...



[MAGIC, 2212.10527]

σv の計算方法

結合定数が $O(1)$ 未満なら、基本的には摂動論で計算ができる



などなど

ヒッグスポータル暗黒物質の例

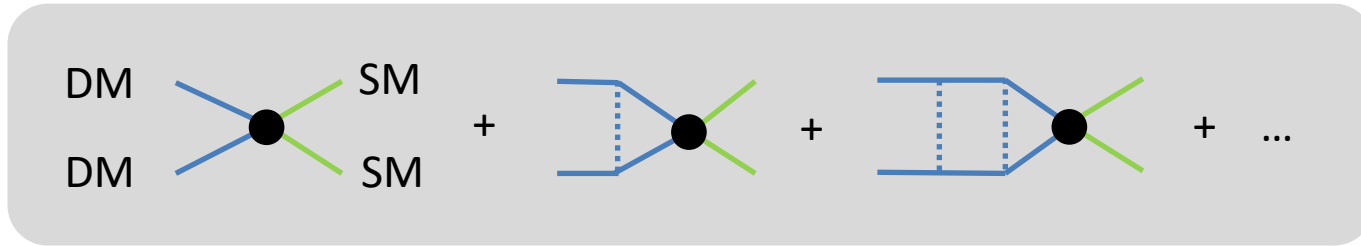
$$L \ni -\frac{\lambda}{4} \phi^2 h^2 \quad \Rightarrow \quad iM(\phi\phi \rightarrow hh) = -i\lambda \quad \Rightarrow \quad \sigma v_{rel} = \frac{\lambda^2}{64\pi^2 m_\phi^2} \sqrt{1 - \frac{m_h^2}{m_\phi^2}}$$

σv の計算方法

結合定数が0(1)未満でも、**非摂動的な効果**が効く場合がある！

暗黒物質が**軽いボゾン**と相互作用していると...

$$(m_{\text{boson}} \ll m_{\text{DM}})$$



$$M$$

$$\sim M \frac{\alpha}{v}$$

$$\sim M \left(\frac{\alpha}{v}\right)^2$$

- Wino / Higgsino dark matter
- ex) • SU(2) 5-plet dark matter
- ...

Sommerfeld効果
(しきい値共鳴効果)

[Sommerfeld (1931)]

[Hisano, Matsumoto, Nojiri (2003)]

計算の方針

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

See also [Agrawal, Parikh, Reece (2020)]

非相対論的な状況の対消滅に興味がある！

Schroedinger方程式が有効なはず。

$$E\psi = -\frac{1}{2\mu}\nabla^2\psi + V(x)\psi$$

- Freeze-out ($T \simeq m/20$)
- $v \simeq 10^{-3}c$ in galaxy

ポテンシャル項が有効理論的な記述を担う

$$V(r) = \begin{cases} V_{\text{short}}(r) & (r < a) & \text{complex} \\ & \text{短距離の効果} & \text{: 対消滅 (など)} \\ V_{\text{long}}(r) & (r \geq a) & \text{real} \\ & \text{長距離の効果} & \text{: 軽いボゾンの交換} \end{cases}$$

$$\text{ex) } V(r) \sim u \delta^3(x) + \frac{\alpha}{r} e^{-mr}$$

a は（とりあえず）手で決めた境界

Schroedinger方程式と対消滅

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

学部で習うような二体散乱問題

$$E\psi = -\frac{1}{2\mu}\nabla^2\psi + V(x)\psi \quad \text{with} \quad \psi \rightarrow e^{ipz} + f(\theta)\frac{e^{ikr}}{r}$$

確率の流速

$$\vec{j}(x) = \frac{1}{\mu}\text{Im}[\psi^*(x)\vec{\nabla}\psi(x)]$$



確率の“非保存”

$$\begin{aligned}\vec{\nabla} \cdot \vec{j}(x) &= 2\text{Im}V(x)|\psi(x)|^2 \\ &= 2\text{Im}V_{\text{short}}(x)|\psi(x)|^2\end{aligned}$$

Schroedinger方程式と対消滅

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

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断面積の定義

$$\sigma \times j_{in} = -\frac{dP}{dt}$$

入射波の流速

$$j_{in} = \frac{1}{\mu} \text{Im}[\psi_{in}^*(x)\vec{\nabla}\psi_{in}(x)] = \frac{p}{\mu} = v$$

対消滅のレート

$$-\frac{dP}{dt} = \int_{r < a} d^3x \, 2\text{Im}V_{\text{short}}(x)|\psi(x)|^2$$

Schroedinger方程式と対消滅

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

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対消滅断面積

$$\sigma v = \int_{r < a} d^3x \, 2\text{Im}V_{\text{short}}(x) |\psi(x)|^2$$

Schroedinger方程式と対消滅

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

$$\sigma v = \int_{r < a} d^3x \, 2\text{Im}V_{\text{short}}(x) |\psi(x)|^2 \quad \Rightarrow \quad \approx \int_{r < a} d^3x \, 2\text{Im}V_{\text{short}}(x) |\psi_{\text{long}}(x)|^2$$

$$\psi \approx \psi_{\text{long}} \quad \begin{array}{l} \text{(歪曲波ボルン近似)} \\ \text{(Distorted Wave Born Approximation)} \end{array}$$

σv が大きくなければ悪くないはず...
(あとで議論します)

$$\text{s. t.} \quad \left[-\frac{1}{2\mu} \nabla^2 + V_{\text{long}}(x) - E \right] \psi_{\text{long}} = 0$$

s-wave の場合

$$\sigma v \approx |\psi_{\text{long}}(0)|^2 \times \int_{r < a} d^3x \, 2\text{Im}V_{\text{short}}(x)$$

Enhancement factor

Schroedinger方程式と対消滅

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

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s-wave の場合

$$\sigma v \approx |\psi_{\text{long}}(0)|^2 \times (\sigma v)_0$$

[Hisano, Matsumoto, Nojiri (2002)]

[Arkani-hamed, Finkbeiner, Slatyer, Weiner (2008)]

etc

Enhancement factor

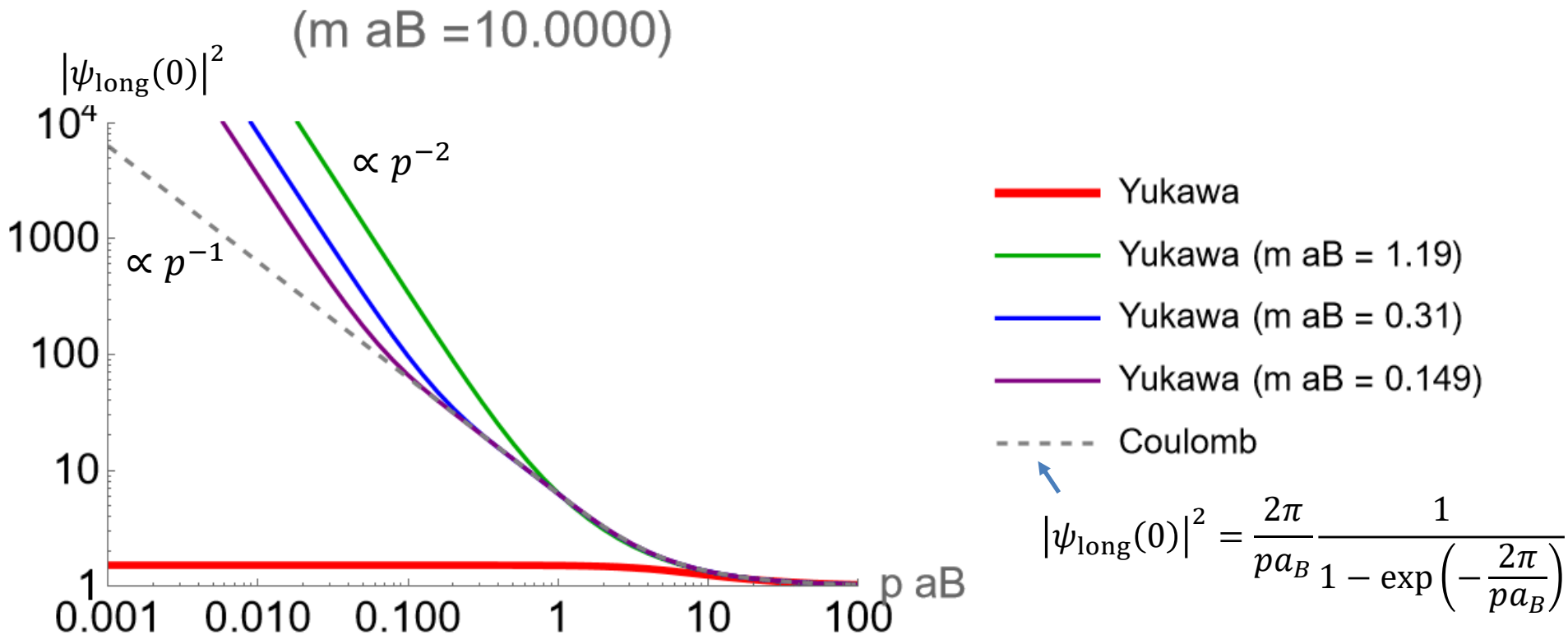
長距離力を考慮しない
対消滅断面積

Sommerfeld factor

$$V(r) = -\frac{\alpha}{r} \exp(-mr)$$

$$\text{ボア半径} : a_B \equiv \frac{1}{\alpha\mu}$$

- $m < \frac{1}{a_B}$ & $p < \frac{1}{a_B}$ のとき σv は大きい
- 特定の ma_B のとき特に σv は大きい

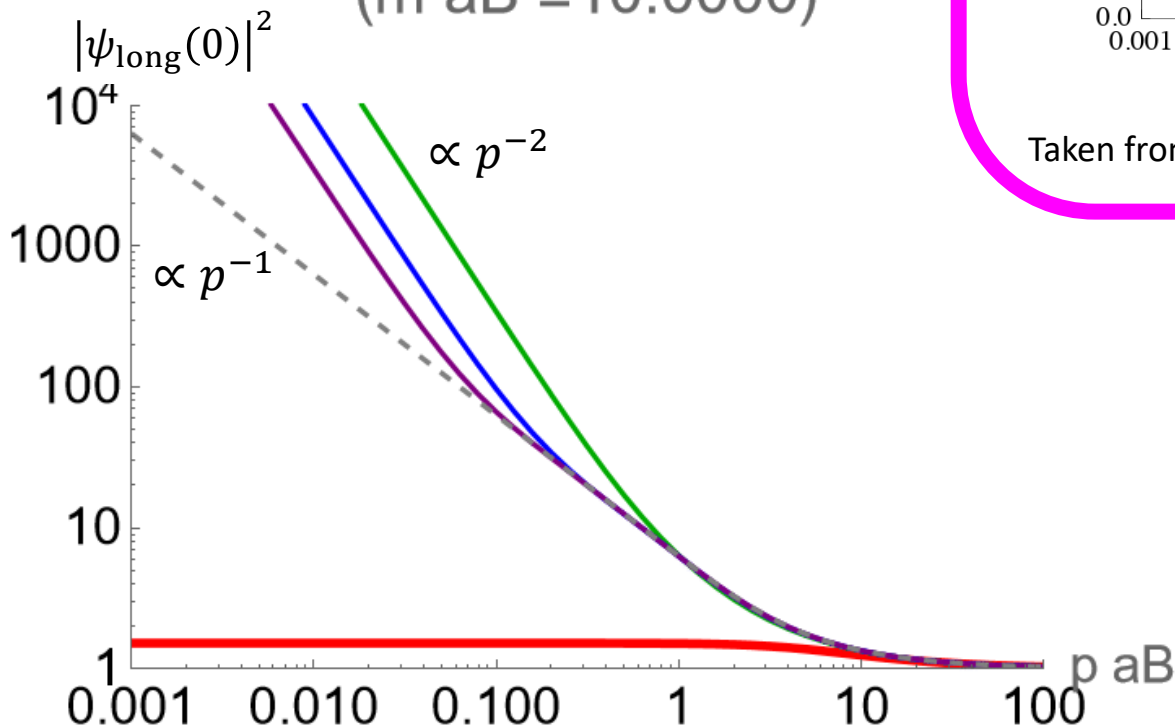


Sommerfeld factor

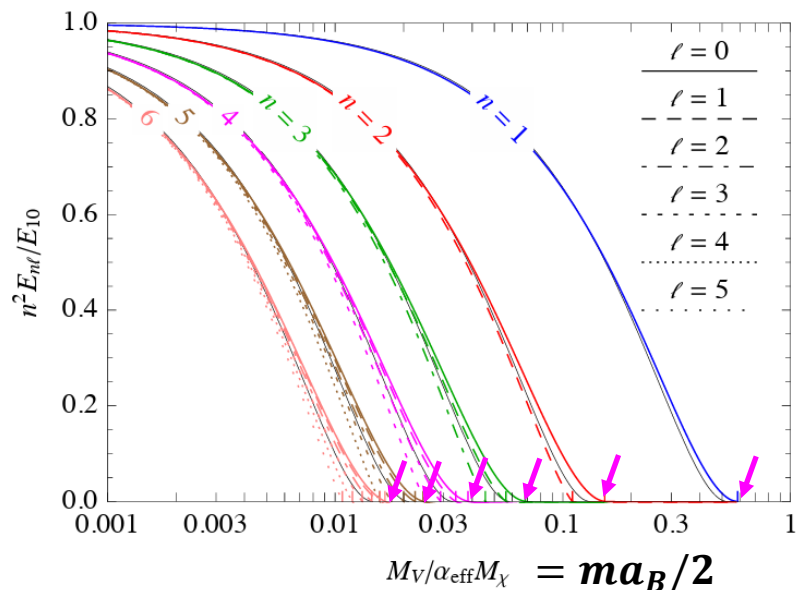
$$V(r) = -\frac{\alpha}{r} \exp(-mr) \quad \text{ボーア半}$$

- $m < \frac{1}{a_B}$ & $p <$
- 特定の ma_B のと

($m a_B = 10.0000$)



Binding energies in a Yukawa potential



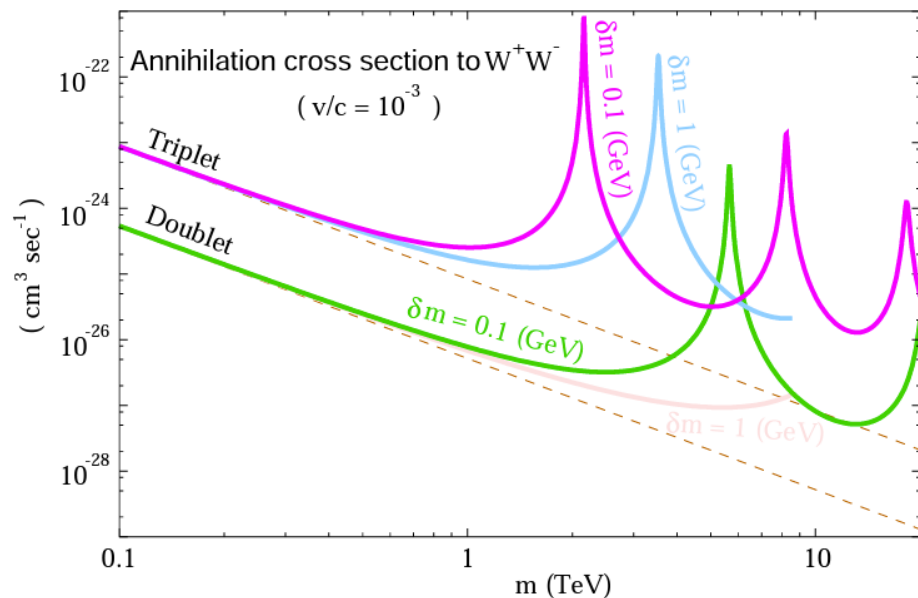
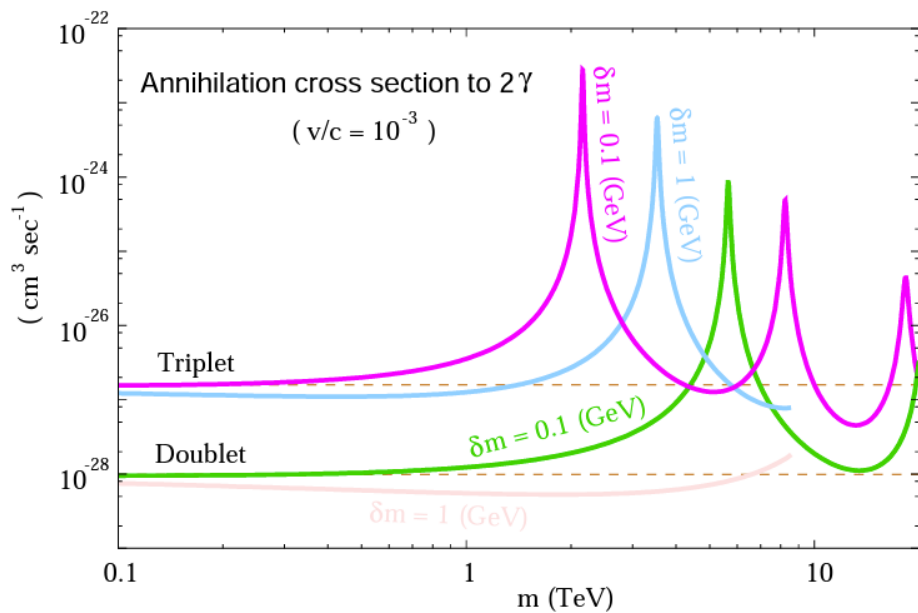
Taken from [Mitridate, Redi, Smirnov, Strumia (2017)]

- Yukawa ($m a_B = 1.19$)
- Yukawa ($m a_B = 0.31$)
- Yukawa ($m a_B = 0.149$)
- - - Coulomb

$$|\psi_{\text{long}}(0)|^2 = \frac{2\pi}{p a_B} \frac{1}{1 - \exp\left(-\frac{2\pi}{p a_B}\right)}$$

Wino / Higgsino 暗黒物質

破線： 普通の摂動計算 (w/o Sommerfeld effect)
実線： 非摂動的な計算 (w/ Sommerfeld effect)



[Hisano, Matsumoto, Nojiri (2003)]

大きくちがう！

Plan

1. Dark matter and Sommerfeld effect
2. Sommerfeld effect and unitarity

Unitarity bound

$$\sigma = \sigma_0 \times S(v)$$

LO cross section :

$$\sigma_0$$

Enhancement factor :

$$S(v) = |\psi(\mathbf{0})|^2$$

σ_0 と $S(v)$ は無関係



s-wave 断面積の上限 :

$$\sigma \leq \frac{\pi}{p^2}$$

[Griest, Kamionkowski (1992)]
[Landau-Lifshits's textbook]

こんなときに上限が気になる

1. $\sigma_0 v$ がそもそも大きい
2. zero energy resonance がある場合 ($S(v) \propto v^{-2} \rightarrow \sigma \propto v^{-3}$)
(for s-wave)

Unitarity bound

$$\sigma = \sigma_0 \times S(v)$$

LO cross
Enhanc

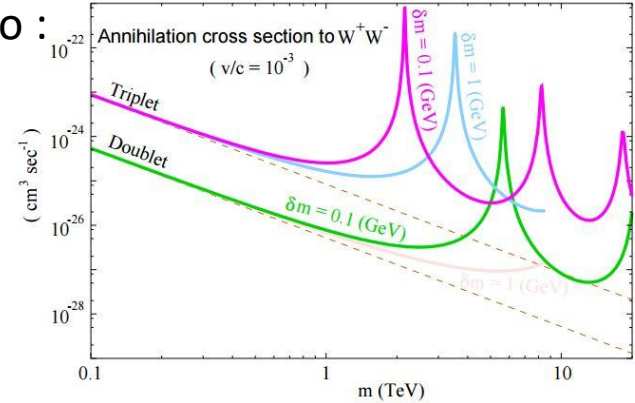
σ_0 と $S(v)$ (

s-wave 断面

こんなときに上限が気になる

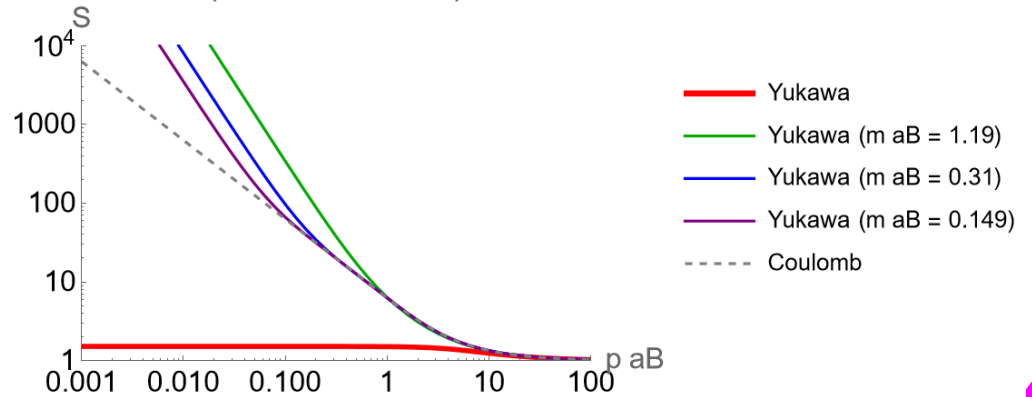
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(for s-wave)

Wino / Higgsino :



Yukawa potential :

(m aB = 10.0000)



Schroedinger方程式をちゃんと解かなくちゃ

何かいけないことをしただろうか... 公式の導出まで戻ってみる。

$$\sigma v = \int_{r < a} d^3x \, 2\text{Im}V_{\text{short}}(x) |\psi(x)|^2 \quad \Rightarrow \quad \approx \int_{r < a} d^3x \, 2\text{Im}V_{\text{short}}(x) |\psi_{\text{long}}(x)|^2$$

$$\psi \approx \psi_{\text{long}}$$

(歪曲波ボルン近似)
(Distorted Wave Born Approximation)

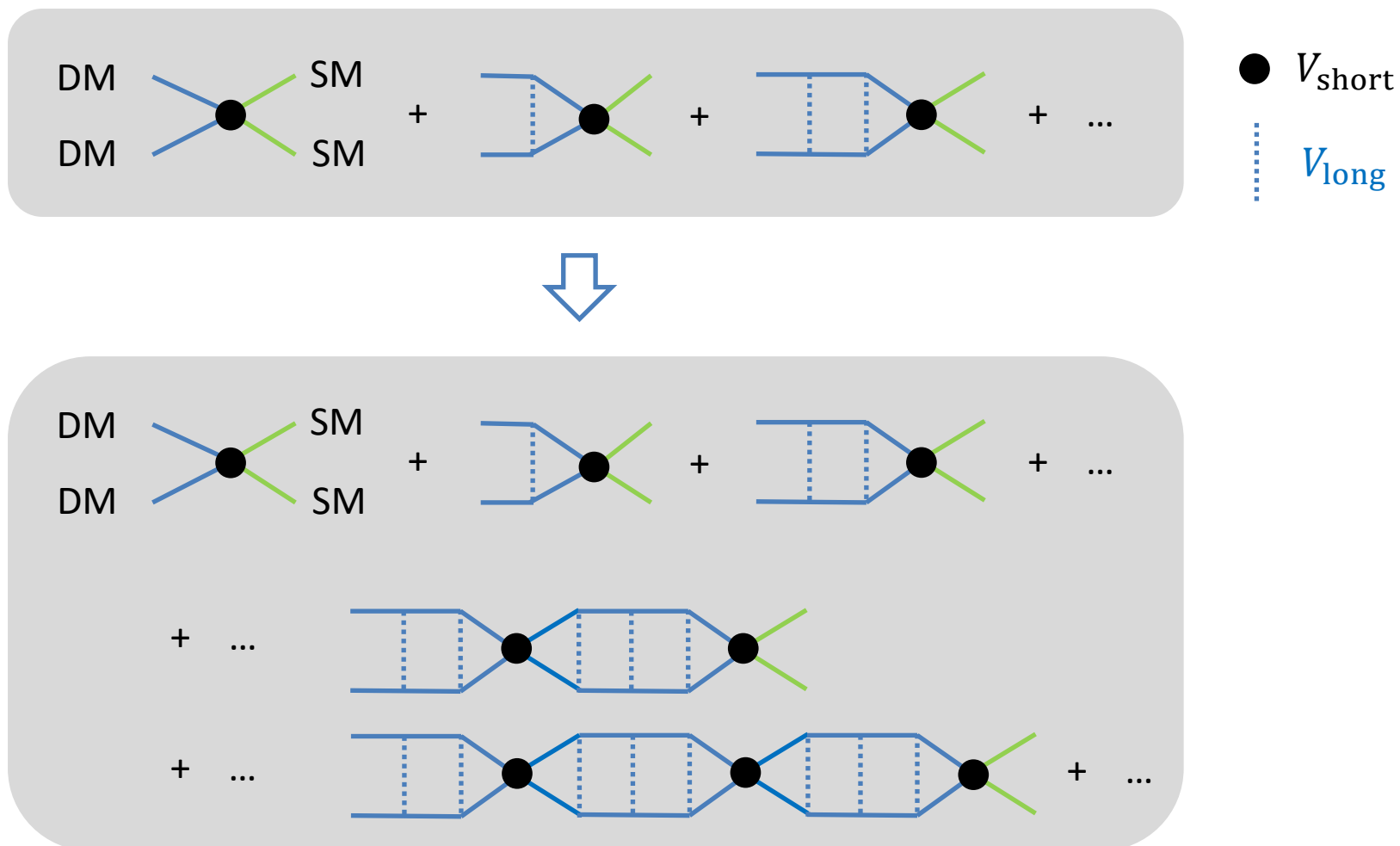
σv が大きくなければ悪くないはず...
(あとで議論します)

$$\text{s. t.} \quad \left[-\frac{1}{2\mu} \nabla^2 + V_{\text{long}}(x) - E \right] \psi_{\text{long}} = 0$$

σv が大きいつき、 ψ と ψ_{long} はだいぶ違うはず

Schroedinger方程式をちゃんと解かなくちゃ

ファインマンダイアグラムのな解釈：



S-matrixの計算

必要なのは、ちゃんと計算した波動関数：
$$\psi = \sum_{\ell} P_{\ell}(\cos \theta) \frac{(-1)^{\ell} \chi_{\ell}(r)}{pr}$$

Schroedinger eq.

$$\left[-\frac{1}{2\mu} \nabla^2 + V(r) - \frac{p^2}{2\mu} \right] \psi(r) = 0 \quad \Rightarrow \quad \left[-\frac{1}{2\mu} \frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{2\mu r^2} + V(r) - \frac{p^2}{2\mu} \right] \chi_{\ell}(r) = 0$$

$$V(r) = \begin{cases} V_{\text{short}}(r) & (r < a) & \text{complex} \\ V_{\text{long}}(r) & (r \geq a) & \text{real} \end{cases}$$

$r \rightarrow \infty$ における境界条件

$$\chi_{\ell}(r) \rightarrow \frac{S_{\ell} \exp(ipr) - \exp(-ipr)}{2i}$$

$r = a$ における境界条件 (V_{short} により決まる短距離の効果)

$\chi_{\ell}'(r)/\chi_{\ell}(r)$ at $r = a$ は p に依らないとする

Straightforward 計算

あとは、腕力でがんばる。

2.1. The full cross sections for elastic scattering and inclusive annihilation are

$$\sigma_{el} = \frac{2\pi}{3} (2\pi + 1) \left[\frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} + \frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} \exp(2\nu^2) \right]^{-1} \quad (2.20)$$

where ν is the reduced mass of the two-body system, ν^2 is the separation vector between the particles, $r = \nu$, and p is the momentum of other particle in the center-of-mass system. The asymptotic behavior of the wavefunction $\psi(r)$ for $r \rightarrow \infty$ is

$$\psi(r) \sim e^{i\pi} + i\theta(r) e^{-i\pi} \quad (2.22)$$

where $\theta(r)$ is the angle between the wave and ν in the direction of the initial plane wave. Let us take the following partial wave expansion:

$$\psi(r, \theta) = \sum_{\ell} (2\ell + 1) P_{\ell}(\cos\theta) \frac{u_{\ell}(r)}{r} \quad (2.23)$$

The radial wavefunction $u_{\ell}(r)$ satisfies the reduced Schrödinger equation:

$$\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - 2\mu(r) \right] u_{\ell}(r) = 0 \quad (2.24)$$

$\mu(r)$ can be expanded as $\mu(r) = \sum_{\ell} \mu_{\ell}(r) P_{\ell}(\cos\theta)$. We will frequently be useful to work with the full wavefunction:

$$\psi(r, \theta) = \sum_{\ell} a_{\ell}(r) P_{\ell}(\cos\theta) \quad (2.25)$$

where $a_{\ell}(r)$ and $a_{\ell}(\theta)$ are the standard Legendre function. These wavefunctions have the asymptotic behavior:

$$a_{\ell}(r) \sim \frac{e^{i\pi} + i\theta(r) e^{-i\pi}}{2} \quad (2.26)$$

At large r , this asymptotic behavior is:

$$a_{\ell}(r) \sim \cos(\ell - \pi/2), \quad a_{\ell}(\theta) \sim \cos(\ell - \pi/2) \quad (2.27)$$

Thus, we can read off the asymptotic behavior of $a_{\ell}(r)$ from the boundary condition of Eq. 2.26 as

$$\psi(r, \theta) \sim \sum_{\ell} (2\ell + 1) P_{\ell}(\cos\theta) \quad (2.28)$$

where $P_{\ell} = \sum_{\ell} (2\ell + 1) P_{\ell}(\cos\theta)$ and $S_1 = 1 + 2i\mu(r)$ is the S -matrix.

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The elastic cross section σ_{el} and the inclusive annihilation cross section σ_{in} are given by the asymptotic behavior of the potential $V(r)$, which we assume to be

$$V(r) = \frac{V_0}{r^2} \left[\frac{r^2 - r_0^2}{r^2} \right]^{-1} \quad (2.29)$$

where V_0 is the reduced mass of the two-body system, r_0 is the separation vector between the particles, $r = \nu$, and p is the momentum of other particle in the center-of-mass system. The asymptotic behavior of the wavefunction $\psi(r)$ for $r \rightarrow \infty$ is

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2.3. Scattering from the boundary condition

We are interested in the annihilation cross section when the language here defines the wavefunction from a finite size. We assume that the long-range force does not provide any annihilation effect directly, i.e. the corresponding potential is zero. This is useful to separate the short-range interactions which can include inelastic, absorptive channels and the long-range interactions which are well-described by a real potential at a short distance $r = a$.

$$V(r) = \begin{cases} V_0(r) & (r < a) \\ 0 & (r \geq a) \end{cases} \quad (2.10)$$

It is impossible to get the relation between the coefficient of r^2 and $r^{2\ell}$ only from the asymptotic behavior near the origin. This is because the sum of an infinite series of r^2 and $r^{2\ell}$ is another irregular solution, violating the same Schrödinger equation. The coefficient of $r^{2\ell}$ is determined by imposing the boundary condition at infinity (Eq. 2.10). For us, we just keep both the r^2 term and $r^{2\ell}$ term and write $C_{\ell} = a$ as

$$C_{\ell} = \frac{2\pi}{3} (2\ell + 1) \left[\frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} + \frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} \exp(2\nu^2) \right]^{-1} \quad (2.20)$$

Note that we also keep track of the term scaling as $|p|^{2\ell} \log(r)/r$, because it comes a regular solution which we will discuss later. Again, C_{ℓ} cannot be determined by the asymptotic behavior at the origin. On the other hand, σ_{el} can be determined by Eq. 2.10. There is a parameter which has been introduced to make the argument of the short-range $V_0(r)$ can be taken to be any value because the difference of r_0 can be absorbed by redefining C_{ℓ} . However, we will find it useful to take $r_0 = a$. We will discuss the behavior of C_{ℓ} for some examples of $V_0(r)$ in Sec. 2.6.

2.4. Momentum scaling of terms in $h(p)$

Now let us discuss the momentum dependence of the coefficient $h(p)$. We only keep the leading term and drop terms of $O(p^2/a^2)$ in Eq. 2.10. Thus, the boundary condition from the short-range physics can be expressed by two parameters: κ_0 and ζ_0 . By using Eq. 2.10-2.15, we obtain

$$h(p) = \kappa_0 + \zeta_0 p + i\sigma(p) \quad (2.31)$$

where κ_0 and ζ_0 are momentum-independent constants which are defined as

$$\kappa_0 = \frac{2\pi}{3} (2\ell + 1) \left[\frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} + \frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} \exp(2\nu^2) \right]^{-1} \quad (2.32)$$

$$\zeta_0 = \frac{2\pi}{3} (2\ell + 1) \left[\frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} + \frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} \exp(2\nu^2) \right]^{-1} \quad (2.33)$$

Instead of κ_0 and ζ_0 , we can use the two parameters k_0 and l_0 to parameterize the effects of the short-range physics. k_0 is $O(p^{2\ell})$ and the leading term in $h(p)$ in most of the cases. As we will see in explicit examples, $\sigma(p)$ can be large at small p if there exists a resonance or a bound state. l_0 can be determined by a reference solution given in Eq. A.1, and it is a polynomial of momentum p in general.

We can extract the momentum-dependent terms in $h(p)$ from the behavior of $C_{\ell}(r)$. Let us define $h(p)$ as

$$h(p) = \frac{2\pi}{3} (2\ell + 1) \left[\frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} + \frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} \exp(2\nu^2) \right]^{-1} \geq \kappa_0 + \zeta_0 p + i\sigma(p) \quad (2.34)$$

Here V_{long} is real and V_{short} has an imaginary part which provides an effective description of the annihilation of particles. We will primarily be interested in the case where the incoming momentum is much smaller than $1/a$, so we will generally have in mind the case where $1/a$ is parametrically smaller (or larger) than the mass of the annihilation particles, so the low-momentum approximation is automatically satisfied. As we will see below, it is also not necessary that the short-range interactions be fully captured by a non-relativistic potential V_{short} if instead using non-relativistic quantum mechanics, as long as we can calculate the S-matrix associated with the short-range interaction.

In order to describe the wavefunction $\psi(r)$, we introduce the function $F(r)$ and $G(r)$, which are solutions of the Schrödinger equation with the long-range force:

$$\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - 2\mu(r) \right] F(r) = 0 \quad (2.14)$$

$$\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - 2\mu(r) \right] G(r) = 0 \quad (2.15)$$

$F(r)$ is regular at the origin and $G(r)$ is irregular, and their asymptotic behavior at infinity is

$$F(r) \sim \cos(\ell - \pi/2) \quad (2.16)$$

where $\theta(r)$ is the standard phase shift induced by the long-range force. Since we assume V_{short} to be real, $F(r)$ and $G(r)$ have the asymptotic behavior

$$F(r) \sim C_0(r) \exp(i\pi) \quad (2.17)$$

where $C_0(r)$ is a function of r and ℓ . Determined by $V_{\text{long}}(r)$. Note that we use Eq. 2.17, replacing $F(r)$ and $G(r)$ in the limit $r \rightarrow \infty$, by using the fact that the Wronskian $F(r)G'(r) - F'(r)G(r)$ is independent of r , combined with the large r asymptotic given in Eq. 2.16.

We obtain $F(r) \sim \cos(\ell - \pi/2)$ and $G(r) \sim \exp(i\pi)$ for $r \rightarrow \infty$ if the long-range force is real, and hence $C_0(r) \sim 1$ in the case. Comparing Eq. 2.1 and Eq. 2.17, C_0 can be interpreted as the enhancement factor of the tensor scalar contribution to the origin, compared to the plane wave. As we will explicitly see later, C_0 is the conventional Sommerfeld factor.

The wavefunction which is consistent with the coefficient of r^2 in Eq. 2.10 given by

$$\psi(r) = \begin{cases} F(r) & (r < a) \\ G(r) & (r \geq a) \end{cases} \quad (2.18)$$

From this expression, we can read off the S-matrix as $S_1 = \exp(2i\theta(r) + i\pi)$. We do not specify an explicit form for $\sigma(p)$, and will only need to behave as $\sigma \rightarrow a$ to obtain the full S-matrix. Note that $h(p)$ is a complex parameter in general, and can be determined from the boundary condition at $r = a$

$$\tan \theta(p) = \frac{F(a) - F'(a)G(a)/G'(a)}{G(a) - G'(a)F(a)/F'(a)} \quad (2.19)$$

Note that ζ_0 is an integer $\sim \nu a^{2\ell}$. The difference between $h(p)$ and $h(p) + i\sigma(p)$ is calculated as

$$h(p) - [h(p) + i\sigma(p)] = \frac{2\pi}{3} (2\ell + 1) \left[\frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} + \frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} \exp(2\nu^2) \right]^{-1} \quad (2.20)$$

Since C_{ℓ} is determined by the long-range force, the RHS of the above equation is at most $O(\nu^{2\ell} a^{2\ell})$, where ν is the typical length scale of the long-range force (e.g. the Bohr radius). For example, this will be explicitly seen in Eq. 2.10 in the Coulomb potential case. On the other hand, σ is $O(\nu^{2\ell} a^{2\ell})$ and the difference between $h(p)$ and $h(p) + i\sigma(p)$ is negligible compared to σ . Thus, we can evaluate $h(p)$ by replacing $h(p) + i\sigma(p)$ in Eq. 2.10 with $h(p)$ as

$$h(p) = \kappa_0 + \zeta_0 p + i\sigma(p) \quad (2.30)$$

For practical purposes, it is useful to evaluate k_0 as

$$k_0 = h(p) + \Delta\nu(\nu, p, \nu) \quad (2.31)$$

where we define $\Delta\nu(\nu, p, \nu)$ as

$$\Delta\nu(\nu, p, \nu) = \Delta\nu(\nu, p, \nu) - \nu^2 \quad (2.32)$$

In Eq. 2.31, k_0 and $\Delta\nu(\nu, p, \nu)$ parameterize the effect of short-range physics and long-range physics, respectively. $\Delta\nu(\nu, p, \nu)$ has a dependence on ν and ν and is seen that the separation between long-range physics and short-range physics is not complete. However, $\Delta\nu(\nu, p, \nu)$ depends on ν via a term $[\nu^2 - \Delta\nu(\nu, p, \nu)] \exp(2\nu^2)$ and this term is always subdominant compared to k_0 in $h(p)$ terms. Therefore, we usually ignore the dependence in $\Delta\nu(\nu, p, \nu)$. Note that, in some special cases, $\sigma(p)$ does not depend on momentum. For example, this happens for the case with $\ell = 0$, or when the potential $V(r)$ has a $1/r$ behavior near the origin. In this case, the dependence completely disappears and the single parameter k_0 parameterizes the information of the short-range physics. However, for $\ell \geq 1$ and $V(r)$ containing a $1/r^2$ term, $\sigma(p)$ is general and depends on p . See the details in App. A.3.

The asymptotic behavior in Eq. 2.17, the long-range and short-range contributions to $h(p)$ can be separated into $h_0(p)$ (short-range effect) and $\Delta\nu(\nu, p, \nu)$ (long-range effect).

2.5. Applications for the S-matrix and cross sections

By using Eq. 2.17, S_1 can be written as

$$S_1 = \frac{h(p) + \Delta\nu(\nu, p, \nu) - i\sigma(p)}{h(p) + \Delta\nu(\nu, p, \nu) + i\sigma(p)} \exp(2\nu^2) \quad (2.33)$$

σ is related to $C_{\ell}(r) = \frac{2\pi}{3} (2\ell + 1) \left[\frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} + \frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} \exp(2\nu^2) \right]^{-1}$. The relation $\sigma = \nu^2 \ln |S_1|^{-1} + i\pi$ is a direct consequence of the unitarity of the S-matrix.

2.6. The full cross sections for elastic scattering and inclusive annihilation are

$$\sigma_{el} = \frac{2\pi}{3} (2\pi + 1) \left[\frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} + \frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} \exp(2\nu^2) \right]^{-1} \quad (2.20)$$

where ν is the reduced mass of the two-body system, ν^2 is the separation vector between the particles, $r = \nu$, and p is the momentum of other particle in the center-of-mass system. The asymptotic behavior of the wavefunction $\psi(r)$ for $r \rightarrow \infty$ is

$$\psi(r) \sim e^{i\pi} + i\theta(r) e^{-i\pi} \quad (2.22)$$

where $\theta(r)$ is the angle between the wave and ν in the direction of the initial plane wave. Let us take the following partial wave expansion:

$$\psi(r, \theta) = \sum_{\ell} (2\ell + 1) P_{\ell}(\cos\theta) \frac{u_{\ell}(r)}{r} \quad (2.23)$$

The radial wavefunction $u_{\ell}(r)$ satisfies the reduced Schrödinger equation:

$$\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - 2\mu(r) \right] u_{\ell}(r) = 0 \quad (2.24)$$

$\mu(r)$ can be expanded as $\mu(r) = \sum_{\ell} \mu_{\ell}(r) P_{\ell}(\cos\theta)$. We will frequently be useful to work with the full wavefunction:

$$\psi(r, \theta) = \sum_{\ell} a_{\ell}(r) P_{\ell}(\cos\theta) \quad (2.25)$$

where $a_{\ell}(r)$ and $a_{\ell}(\theta)$ are the standard Legendre function. These wavefunctions have the asymptotic behavior:

$$a_{\ell}(r) \sim \frac{e^{i\pi} + i\theta(r) e^{-i\pi}}{2} \quad (2.26)$$

At large r , this asymptotic behavior is:

$$a_{\ell}(r) \sim \cos(\ell - \pi/2), \quad a_{\ell}(\theta) \sim \cos(\ell - \pi/2) \quad (2.27)$$

Thus, we can read off the asymptotic behavior of $a_{\ell}(r)$ from the boundary condition of Eq. 2.26 as

$$\psi(r, \theta) \sim \sum_{\ell} (2\ell + 1) P_{\ell}(\cos\theta) \quad (2.28)$$

where $P_{\ell} = \sum_{\ell} (2\ell + 1) P_{\ell}(\cos\theta)$ and $S_1 = 1 + 2i\mu(r)$ is the S -matrix.

2.7. Scattering from the boundary condition

We are interested in the annihilation cross section when the language here defines the wavefunction from a finite size. We assume that the long-range force does not provide any annihilation effect directly, i.e. the corresponding potential is zero. This is useful to separate the short-range interactions which can include inelastic, absorptive channels and the long-range interactions which are well-described by a real potential at a short distance $r = a$.

$$V(r) = \begin{cases} V_0(r) & (r < a) \\ 0 & (r \geq a) \end{cases} \quad (2.10)$$

It is impossible to get the relation between the coefficient of r^2 and $r^{2\ell}$ only from the asymptotic behavior near the origin. This is because the sum of an infinite series of r^2 and $r^{2\ell}$ is another irregular solution, violating the same Schrödinger equation. The coefficient of $r^{2\ell}$ is determined by imposing the boundary condition at infinity (Eq. 2.10). For us, we just keep both the r^2 term and $r^{2\ell}$ term and write $C_{\ell} = a$ as

$$C_{\ell} = \frac{2\pi}{3} (2\ell + 1) \left[\frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} + \frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} \exp(2\nu^2) \right]^{-1} \quad (2.20)$$

Note that we also keep track of the term scaling as $|p|^{2\ell} \log(r)/r$, because it comes a regular solution which we will discuss later. Again, C_{ℓ} cannot be determined by the asymptotic behavior at the origin. On the other hand, σ_{el} can be determined by Eq. 2.10. There is a parameter which has been introduced to make the argument of the short-range $V_0(r)$ can be taken to be any value because the difference of r_0 can be absorbed by redefining C_{ℓ} . However, we will find it useful to take $r_0 = a$. We will discuss the behavior of C_{ℓ} for some examples of $V_0(r)$ in Sec. 2.6.

2.8. Momentum scaling of terms in $h(p)$

Now let us discuss the momentum dependence of the coefficient $h(p)$. We only keep the leading term and drop terms of $O(p^2/a^2)$ in Eq. 2.10. Thus, the boundary condition from the short-range physics can be expressed by two parameters: κ_0 and ζ_0 . By using Eq. 2.10-2.15, we obtain

$$h(p) = \kappa_0 + \zeta_0 p + i\sigma(p) \quad (2.31)$$

where κ_0 and ζ_0 are momentum-independent constants which are defined as

$$\kappa_0 = \frac{2\pi}{3} (2\ell + 1) \left[\frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} + \frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} \exp(2\nu^2) \right]^{-1} \quad (2.32)$$

$$\zeta_0 = \frac{2\pi}{3} (2\ell + 1) \left[\frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} + \frac{h(\nu)}{h(\nu) + \Delta\nu(\nu, p, \nu)} \exp(2\nu^2) \right]^{-1} \quad (2.33)$$

Instead of κ_0 and ζ_0 , we can use the two parameters k_0 and l_0 to parameterize the effects of the short-range physics. k_0 is $O(p^{2\ell})$ and the leading term in $h(p)$ in most of the cases. As we will see in explicit examples, $\sigma(p)$ can be large at small p if there exists a resonance or a bound state. l_0 can be determined by a reference solution given in Eq. A.1, and it is a polynomial of momentum p in general.

We can extract the momentum-dependent

S-matrix

がんばって計算すると、各部分波の S-matrix が得られる。

$$S_\ell \simeq \underbrace{\exp\left(2i\delta_\ell^{(L)}(p)\right)}_{\substack{\text{Phase-shift} \\ \text{by long-range force}}} \times \underbrace{\frac{k_{\ell,0} + z_\ell(p) - ip^{2\ell+1}C_\ell^2(p)}{k_{\ell,0} + z_\ell(p) + ip^{2\ell+1}C_\ell^2(p)}}_{\text{Relevant part for annihilation}}$$

一つの複素パラメーター

$k_{\ell,0}$

三つの実の関数

$C_\ell^2(p)$, $z_\ell(p)$, $\delta_\ell^{(L)}(p)$

S-matrix

がんばって計算すると、各部分波の S-matrix が得られる。

$$S_\ell \simeq \underbrace{\exp\left(2i\delta_\ell^{(L)}(p)\right)}_{\text{Phase-shift by long-range force}} \times \underbrace{\frac{k_{\ell,0} + z_\ell(p) - ip^{2\ell+1}C_\ell^2(p)}{k_{\ell,0} + z_\ell(p) + ip^{2\ell+1}C_\ell^2(p)}}_{\text{Relevant part for annihilation}}$$

対消滅断面積：

$$\sigma_{ann,\ell} = \frac{\pi}{p^2} (2\ell + 1)(1 - |S_\ell|^2) < \frac{(2\ell + 1)\pi}{p^2} \quad \text{Unitarity bound } \checkmark$$

S-matrix

がんばって計算すると、各部分波の S-matrix が得られる。

$$S_\ell \simeq \underbrace{\exp\left(2i\delta_\ell^{(L)}(p)\right)}_{\substack{\text{Phase-shift} \\ \text{by long-range force}}} \times \underbrace{\frac{k_{\ell,0} + z_\ell(p) - ip^{2\ell+1}C_\ell^2(p)}{k_{\ell,0} + z_\ell(p) + ip^{2\ell+1}C_\ell^2(p)}}_{\text{Relevant part for annihilation}}$$

対消滅断面積：

Unitarity bound ✓

$$\begin{aligned}\sigma_{ann,\ell} &= \frac{\pi}{p^2} (2\ell + 1)(1 - |S_\ell|^2) < \frac{(2\ell + 1)\pi}{p^2} \\ &= \frac{\pi}{p^2} (2\ell + 1) \times 4\text{Re}\left[\frac{ip^{2\ell+1}C_\ell^2}{k_{\ell,0}}\right] \times \left|1 + \frac{z_\ell + ip^{2\ell+1}C_\ell^2}{k_{\ell,0}}\right|^{-2}\end{aligned}$$

S-matrix

がんばって計算すると、各部分波の S-matrix が得られる。

$$S_\ell \simeq \underbrace{\exp\left(2i\delta_\ell^{(L)}(p)\right)}_{\substack{\text{Phase-shift} \\ \text{by long-range force}}} \times \underbrace{\frac{k_{\ell,0} + z_\ell(p) - ip^{2\ell+1}C_\ell^2(p)}{k_{\ell,0} + z_\ell(p) + ip^{2\ell+1}C_\ell^2(p)}}_{\text{Relevant part for annihilation}}$$

対消滅断面積：

Unitarity bound ✓

$$\begin{aligned} \sigma_{ann,\ell} &= \frac{\pi}{p^2} (2\ell + 1)(1 - |S_\ell|^2) < \frac{(2\ell + 1)\pi}{p^2} \\ &= 4\pi(2\ell + 1)p^{2\ell-1} \operatorname{Im}\left[-\frac{1}{k_{\ell,0}}\right] \times C_\ell^2 \times \left|1 + \frac{z_\ell + ip^{2\ell+1}C_\ell^2}{k_{\ell,0}}\right|^{-2} \end{aligned}$$

短距離の対消滅の効果
通常の Sommerfeld factor
補正項

計算例

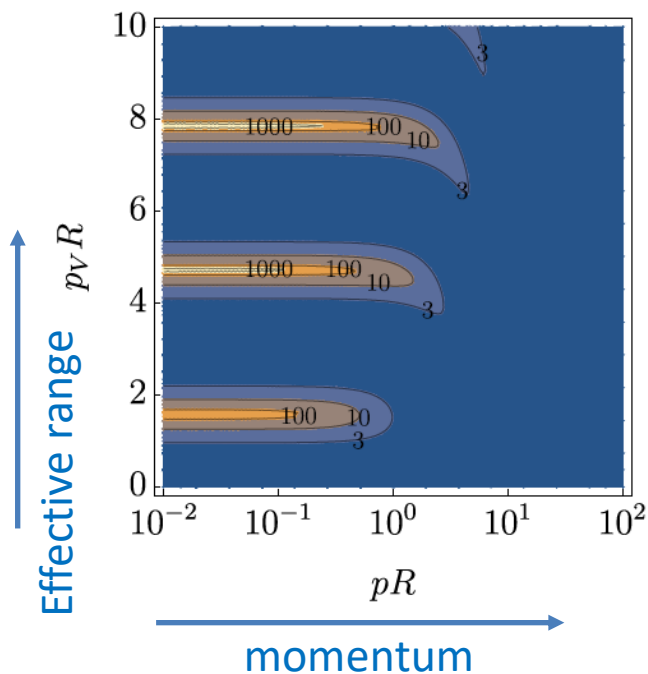
[Parikh, Sato, Slatyer (2024)]

球形井戸型ポテンシャル

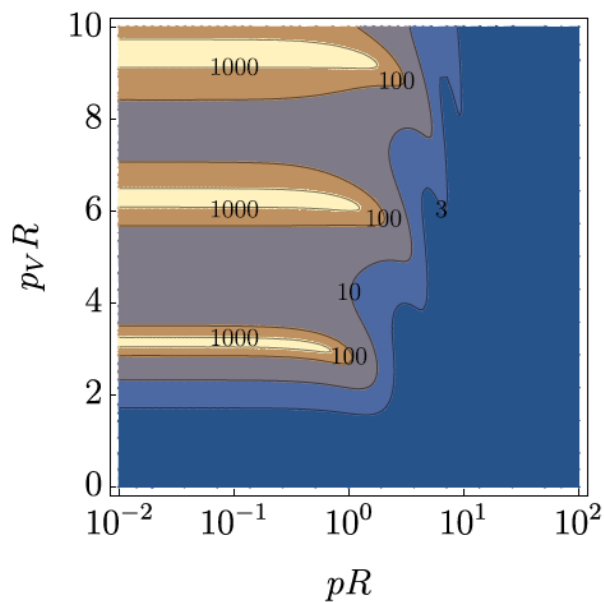
$$V(r) = -\frac{p_V^2}{2\mu} \theta(R - r)$$

(通常の) Sommerfeld factor : C_ℓ^2

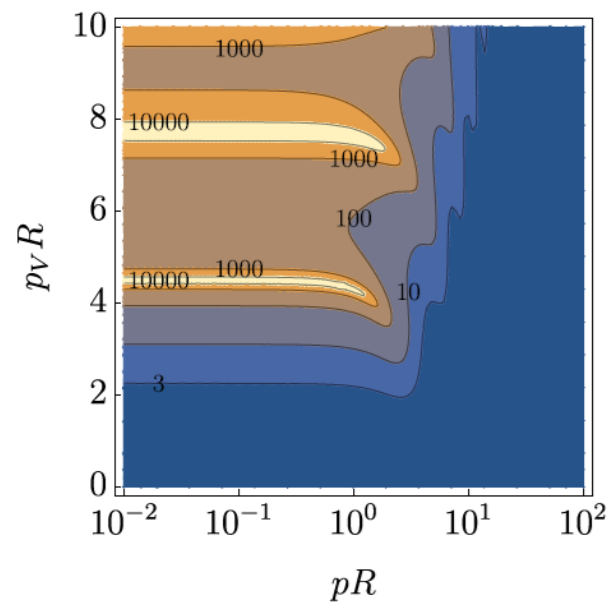
S-wave



P-wave



D-wave



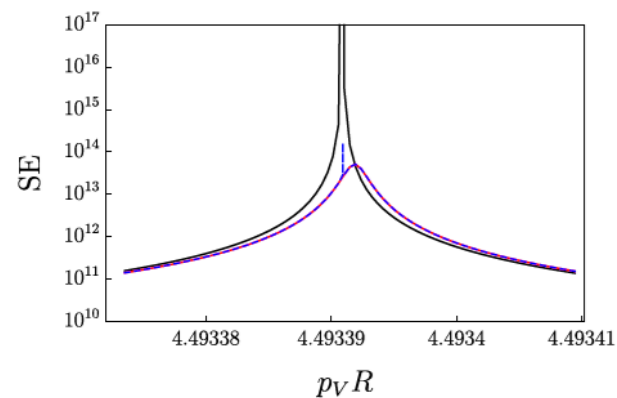
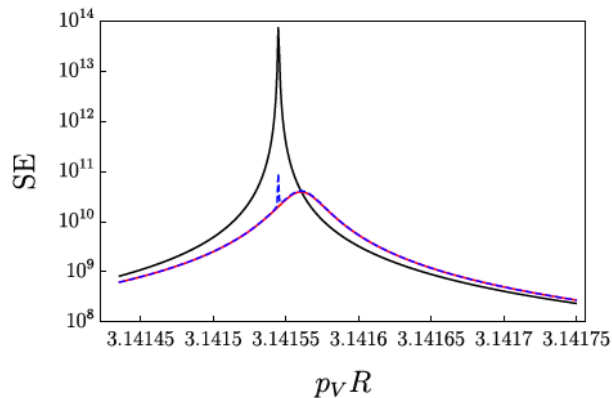
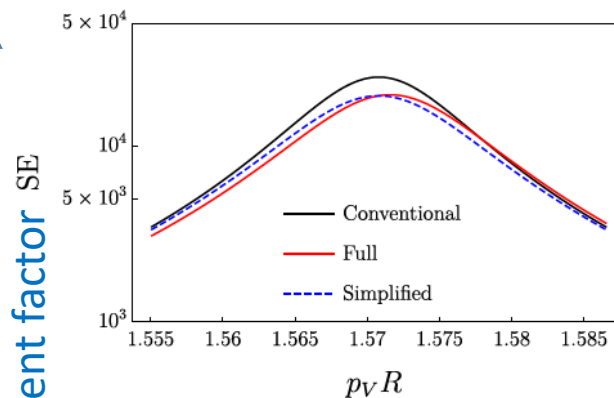
計算例

[Parikh, Sato, Slatyer (2024)]

球形井戸型ポテンシャル

$$V(r) = -\frac{p_V^2}{2\mu} \theta(R - r)$$

$$\frac{\sigma_{ann}}{\sigma_{ann,LO}}$$



Effective range

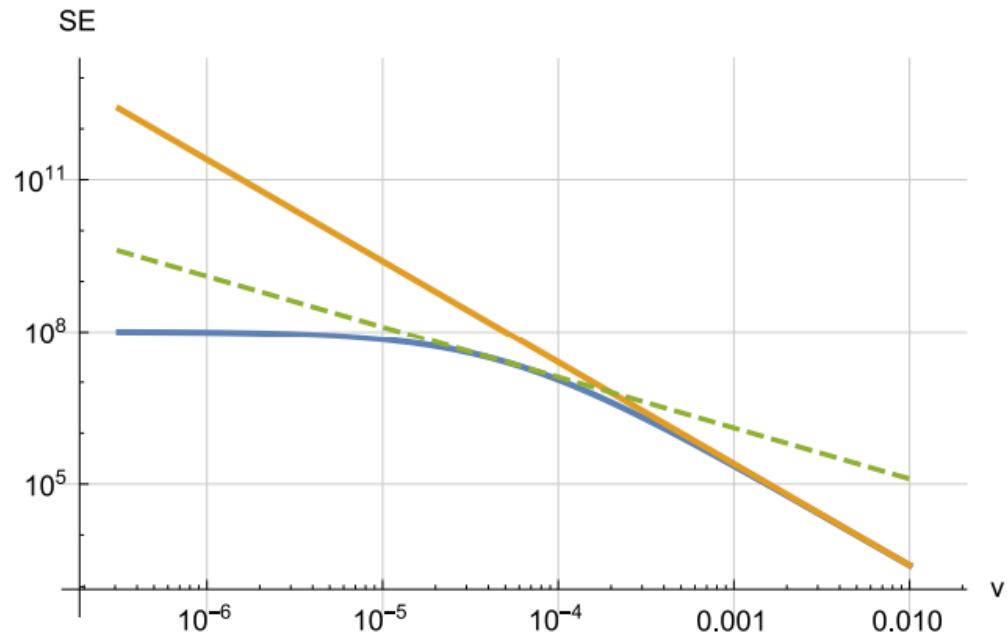
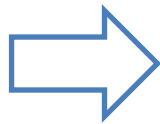
計算例

[Blum, Sato, Slatyer (2016)]

Hulthen potential : $V(r) = -\frac{\alpha m_* e^{-m_* r}}{1 - e^{-m_* r}}$ ($V(r) = -\frac{\alpha e^{-mr}}{r}$, $m_* = \frac{\pi^2}{6} m$ の良い近似)

$$\alpha = 1, \quad \sigma v = \frac{1}{32\pi M^2}, \quad \sigma_{sc} = \frac{\mu^2}{4\pi} (\sigma v)^2$$

$$m_* = 0.0625M$$



黄色実線 : 通常の公式

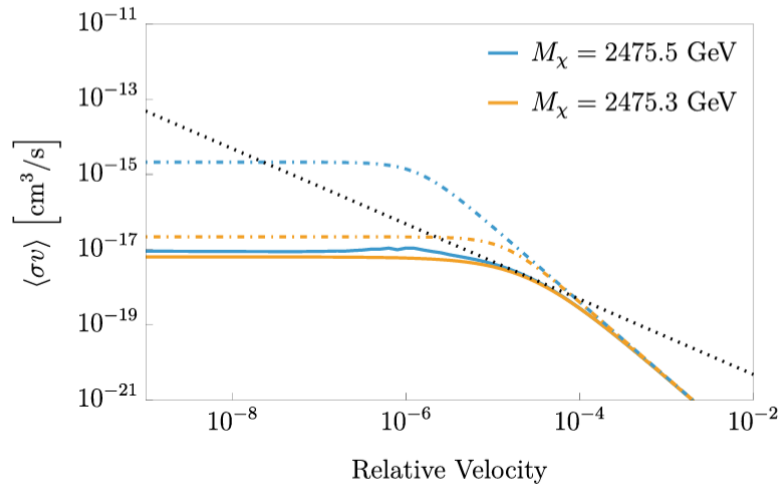
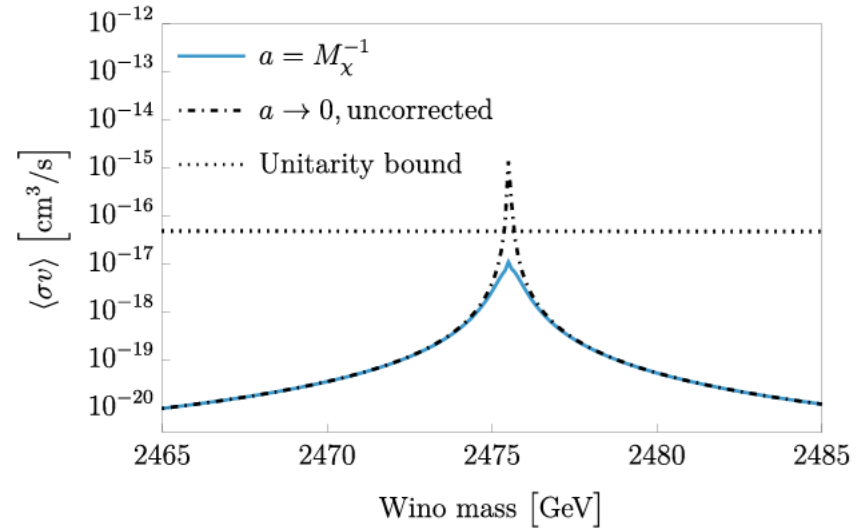
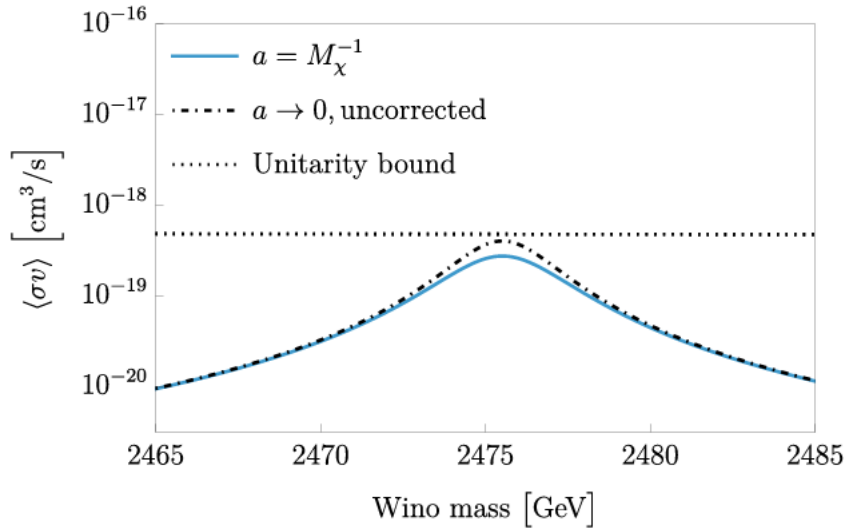
青の実線 : 我々の公式

緑の破線 : ユニタリティによる上限

計算例

[Blum, Sato, Slatyer (2016)]

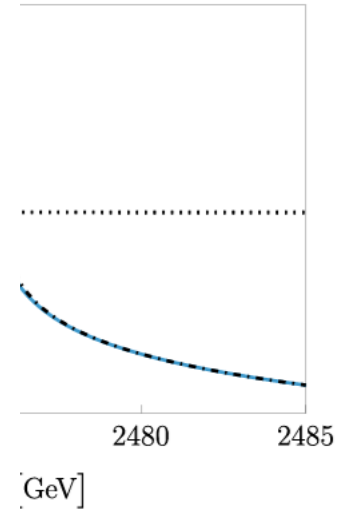
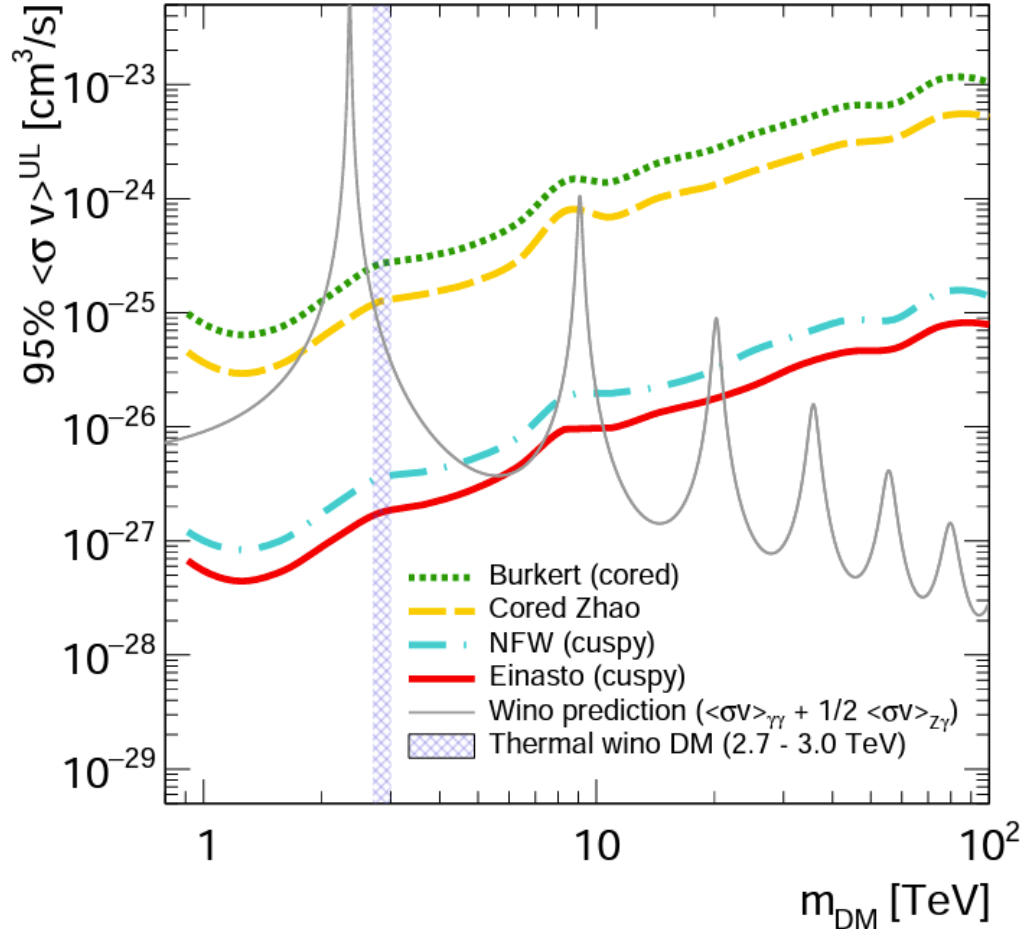
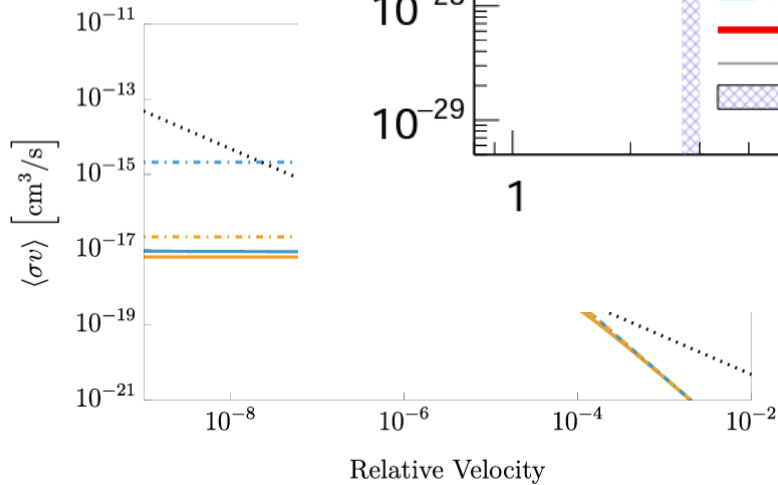
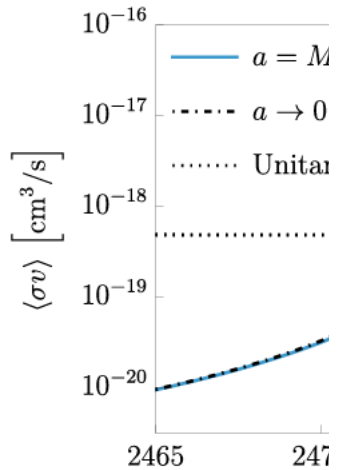
Wino



計算例

[Blum, Sato, Slatyer (2016)]

Wino



[MAGIC, 2212.10527]

まとめ

- Schroedinger方程式により、軽いボゾンの効果を取り込める
- 対消滅の効果は、ポテンシャルの複素の項であらわせる
- 量子力学のユニタリティの上限とも矛盾しない公式ができた
- 対消滅断面積が大きい場合には、通常の公式からの補正が重要

Backup

Unitarity bound と zero-energy resonance

zero-energy resonance があるときの対消滅断面積

$$|\psi_{\text{long}}|^2 \propto p^{-2} \quad (\text{for s-wave}) \quad \Rightarrow \quad \sigma_{\text{ann},s} = \frac{1}{v} \times |\psi_{\text{long}}|^2 \times (\sigma v)_0 \propto \frac{1}{p^3}$$

部分波展開

$$E\psi = -\frac{1}{2\mu}\nabla^2\psi + V(x)\psi$$

$$\text{with } \psi \rightarrow e^{ipz} + f(\theta)\frac{e^{ikr}}{r} = \sum_{\ell} P_{\ell}(\cos\theta)\frac{S_{\ell}e^{ipr} - (-1)^{\ell}e^{-ipr}}{2ipr}$$

pが小さいと
やばい

対消滅断面積

$$\sigma_{\text{ann}} = \frac{\pi}{p^2} \sum_{\ell} (2\ell + 1)(1 - |S_{\ell}|^2)$$



$$\sigma_{\text{ann},s} \leq \frac{\pi}{p^2}$$

Wave function w/ long-range force

$$\left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + 2\mu V(r) - p^2 \right] H_\ell^+(r) = 0, \quad H_\ell^+(r) \rightarrow (-i)^\ell \exp(ipr + \delta_\ell^{(L)})$$

$$F_\ell(r) \equiv \text{Im}H_\ell^+ \simeq C_\ell p^{\ell+1} \times \left[\frac{r^{\ell+1}}{(2\ell+1)!!} + \dots \right]$$

Leading term

$$G_\ell(r) \equiv \text{Re}H_\ell^+ \simeq \frac{1}{C_\ell p^\ell} \times \left[\frac{(2\ell-1)!!}{r^\ell} + \dots + z_\ell(p) \frac{r^{\ell+1}}{(2\ell+1)!!} + \dots \right]$$

(basically) leading term

Sizable in some cases

On zero energy resonance,

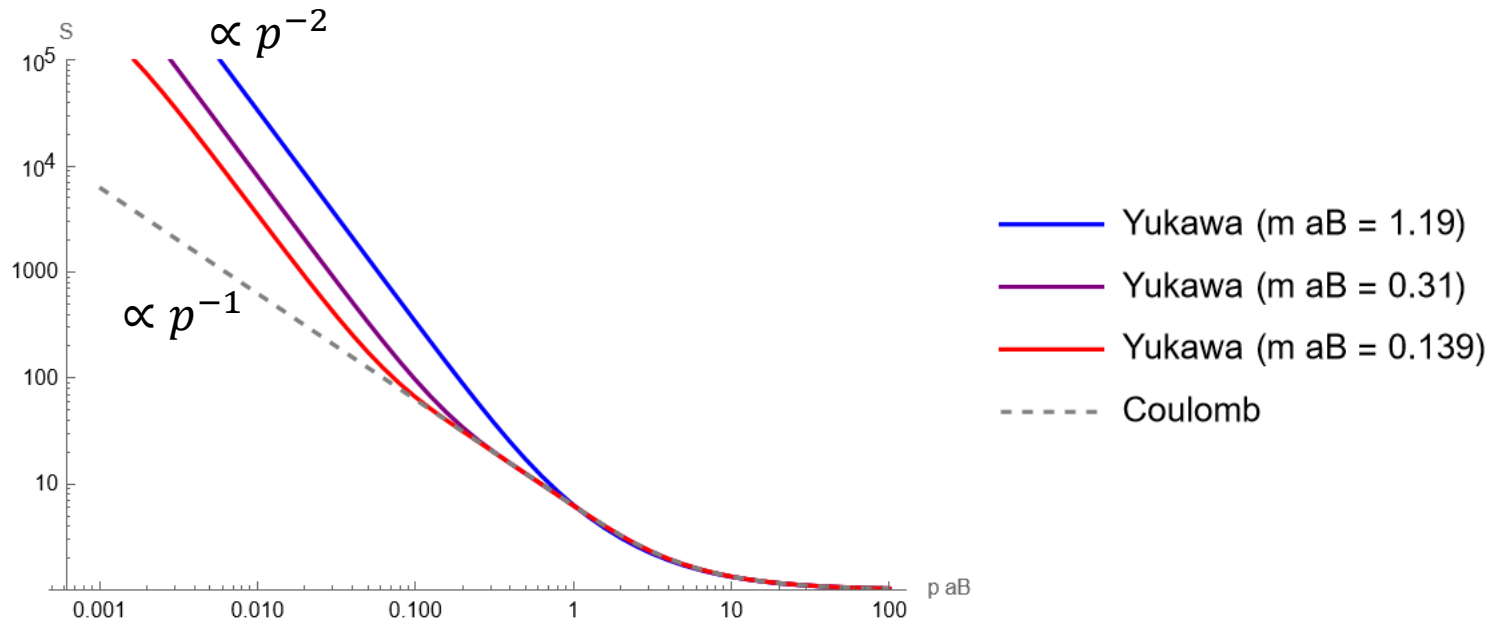
$$C_\ell^2(p) \propto \begin{cases} p^{-2} & (\ell = 0) \\ p^{-4} & (\ell \geq 1) \end{cases} \quad z_\ell(p) \propto \begin{cases} p^0 & (\ell = 0) \\ p^{-2} & (\ell \geq 1) \end{cases}$$

See also [Kamada, Kuwahara, Patel (2023)]

Resonant points

At some specific points ($ma_B = 1.19, 0.31, 0.139, \dots$), $|\psi_{\text{long}}|^2 \propto \frac{1}{p^2}$

(zero energy resonance : bound state with zero binding energy)



$$\sigma = \frac{1}{v} \times |\psi_{\text{long}}|^2 \times (\sigma v)_0 \propto \frac{1}{p^3}$$

Zero energy resonance

Schroedinger eq

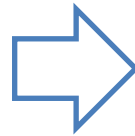
$$\left[-\frac{1}{2\mu} \frac{d^2}{dr^2} + V(r) - \frac{p^2}{2\mu} \right] \chi(r) = 0,$$

Boundary cond.

$$\chi(r) \rightarrow \frac{S \exp(ipr) - \exp(-ipr)}{2i}$$

Bound state w/ $E = -\frac{\kappa^2}{2m}$

$$\chi(r) \rightarrow \exp(-\kappa r)$$



pole in $S(k)$ at $p = i\kappa$

$$\rightarrow S = e^{2i\delta} = -\frac{p + i\kappa}{p - i\kappa}$$

$$\rightarrow \sin \delta = -\frac{\kappa}{p}$$

$$\psi(\vec{x}) = \frac{\chi(r)}{pr}$$

$$|\psi_{\text{long}}(0)|^2 \propto \frac{\chi^2(0)}{p^2} = \frac{\sin^2 \delta}{p^2} = \frac{1}{p^2 + \kappa^2}$$

Solution of Schroedinger eq.

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

Schroedinger equation:

$$\left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + 2\mu V(r) - p^2 \right] u_\ell(r) = 0,$$

$$V(r) = \begin{cases} V_{\text{short}}(r) & (r < a) & \text{complex} \\ V_{\text{long}}(r) & (r \geq a) & \text{real} \end{cases}$$

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u_ℓ should be a linear combination of $\begin{cases} \bullet F_\ell(r) & \text{(regular solution)} \\ \bullet G_\ell(r) & \text{(irregular solution)} \end{cases}$ at $r > a$

$$\left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + 2\mu V_{\text{long}}(r) - p^2 \right] F_\ell(r) = 0 \quad \& \quad \left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + 2\mu V_{\text{long}}(r) - p^2 \right] G_\ell(r) = 0$$

	$r \simeq 0$	$r \rightarrow \infty$
$F_\ell(r)$	$\frac{C_\ell}{(2\ell+1)!!} (pr)^{\ell+1}$	$\sin\left(pr + \delta_\ell^{(L)} - \frac{\pi\ell}{2}\right)$
$G_\ell(r)$	$\frac{(2\ell-1)!!}{C_\ell} (pr)^{-\ell}$	$\cos\left(pr + \delta_\ell^{(L)} - \frac{\pi\ell}{2}\right)$

$C_\ell \simeq 1, \delta_\ell^{(L)} \simeq 0$ for $V_{\text{long}}(r) \simeq 0$

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$$\rightarrow \frac{1}{2i} \left((-i)^\ell \exp(2i\delta_\ell^{(L)} + 2i\delta_\ell^{(S)}) e^{ipr} - i^\ell e^{-ipr} \right)$$

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S_ℓ

S-matrix

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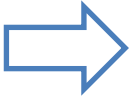
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S_ℓ

$\frac{u_{\ell'}'}{u_\ell}$ is continuous at $r = a$ 

$$\frac{u_{\ell,<}'}{u_{\ell,<}} = \frac{\cos \delta_\ell^{(S)} F_\ell' + \sin \delta_\ell^{(S)} G_\ell'}{\cos \delta_\ell^{(S)} F_\ell + \sin \delta_\ell^{(S)} G_\ell}$$

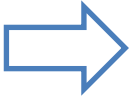
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S_ℓ

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$k_\ell(p)$ is *almost* independent on p
(will be discussed later)

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- $u_{\ell,<}(r)$: p independent
- $f_\ell(r)$: p independent
- $g_\ell(r)$: *almost* p independent



$k_\ell(p)$ is *almost* independent on p
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(basically) leading term

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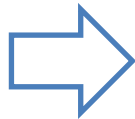
(basically) leading term

Sizable in some cases

$$k_\ell(p) \simeq k_{\ell,0} + z_\ell(p)$$

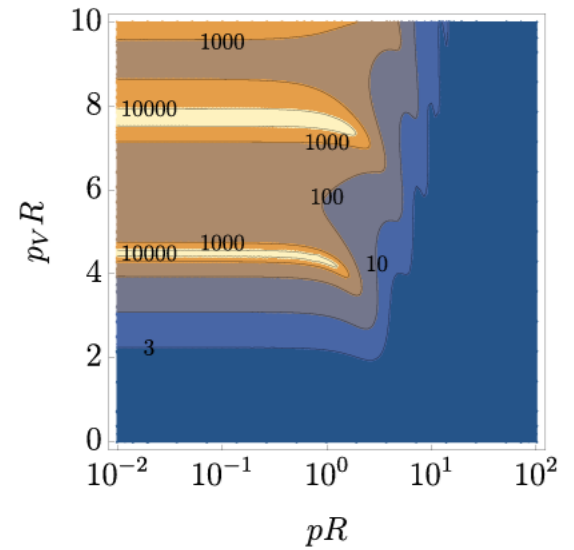
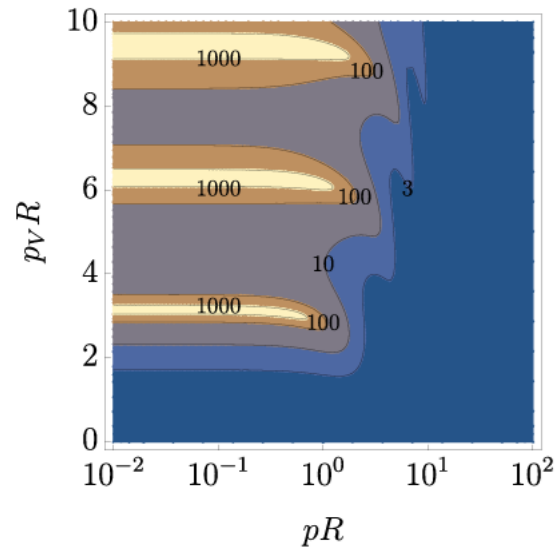
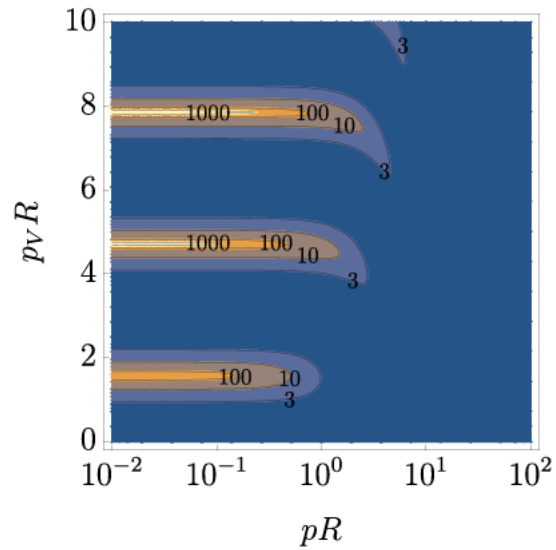
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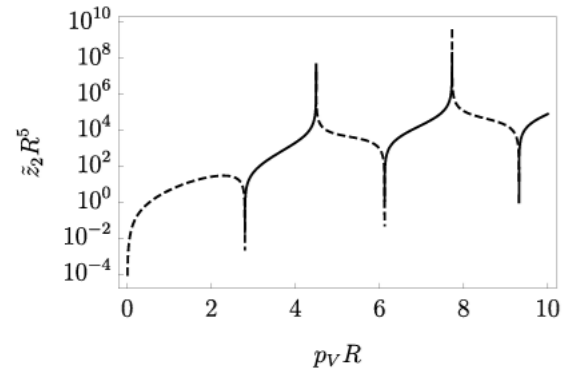
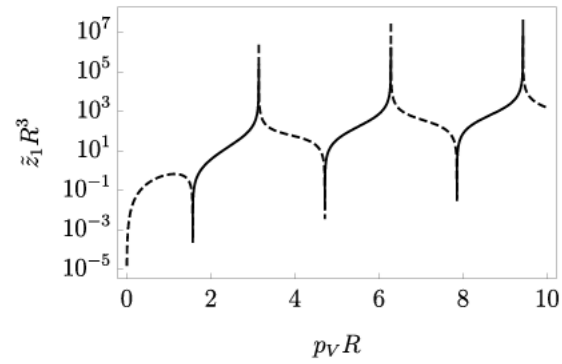
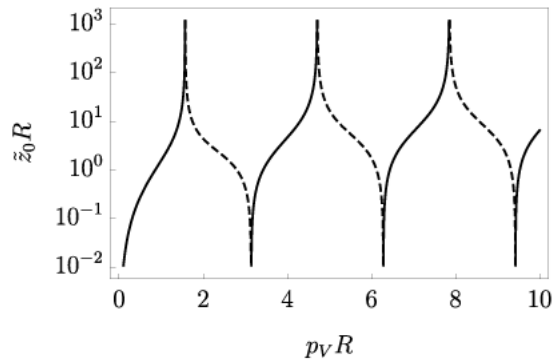


$$S_\ell \simeq \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{k_{\ell,0} + z_\ell(p) - ip^{2\ell+1}C_\ell^2}{k_{\ell,0} + z_\ell(p) + ip^{2\ell+1}C_\ell^2}$$

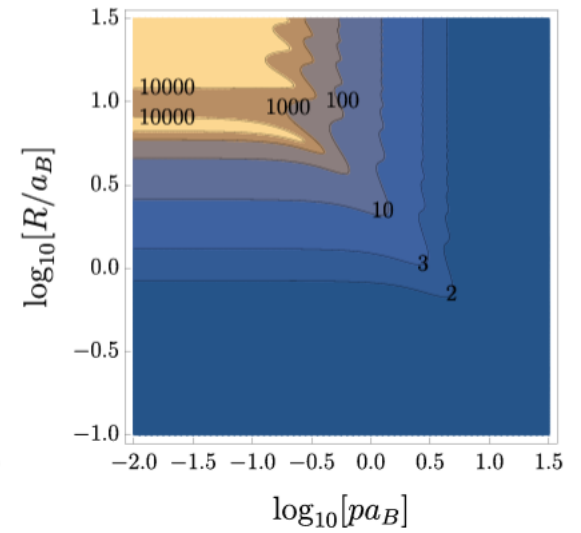
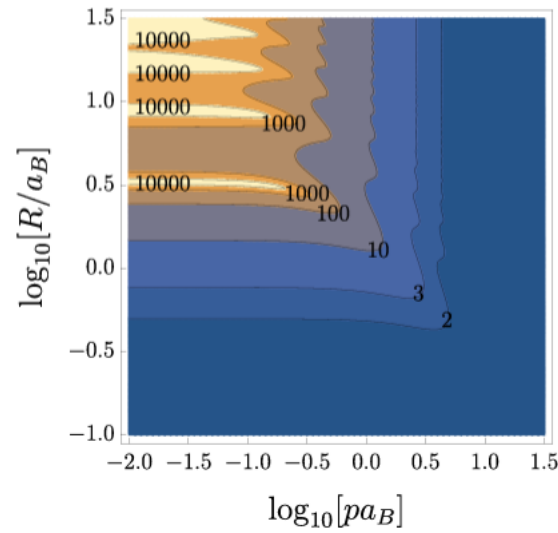
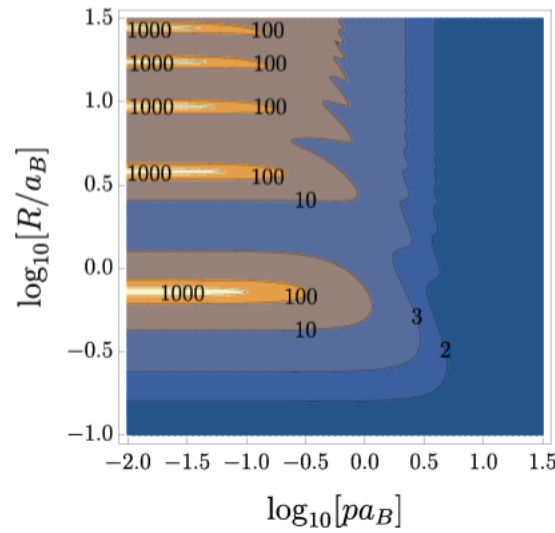
SE for Spherical-well potential



Z function for Spherical-well potential



SE for finite range Coulomb potential



Z function for finite range Coulomb potential

