ユニタリティと矛盾しない Sommerfeld 効果の計算

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K. Blum, R. Sato, T. R. Slatyer, 1603.01383, JCAP 06 (2016) 021
A. Parikh, R. Sato, T. R. Slatyer, 2410.18168

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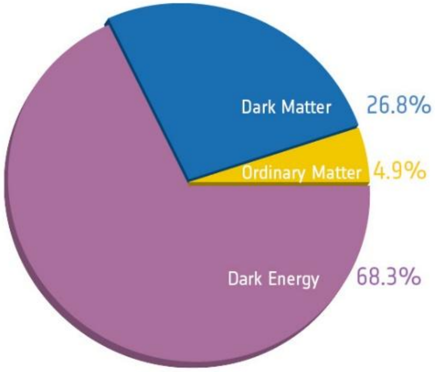
Plan

1. Dark matter and Sommerfeld effect
2. Sommerfeld effect and unitarity

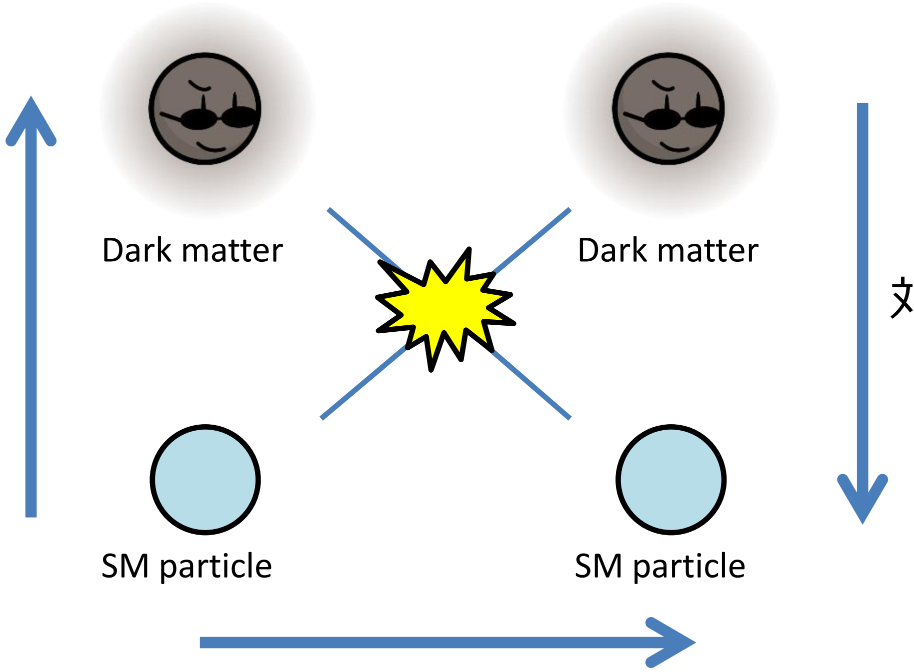
暗黒物質の対消滅

宇宙のエネルギーの27%は暗黒物質
その正体は未だ不明



[Planck collaboration]

暗黒物質が標準模型粒子と相互作用を持つと…



対生成

対消滅

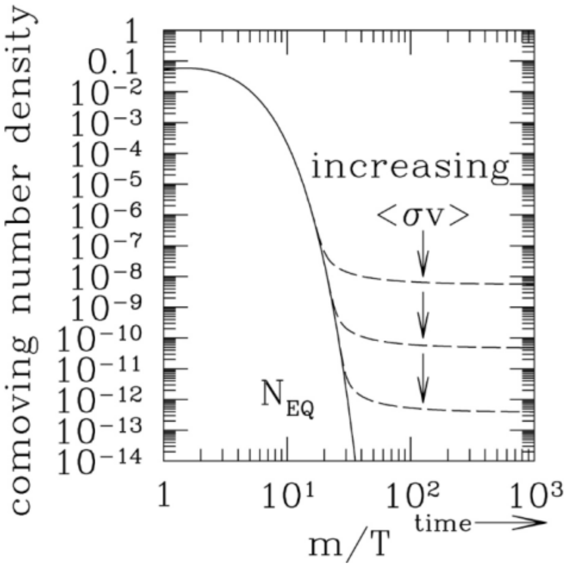
弾性散乱

暗黒物質の対消滅

- **Freeze-out 機構**

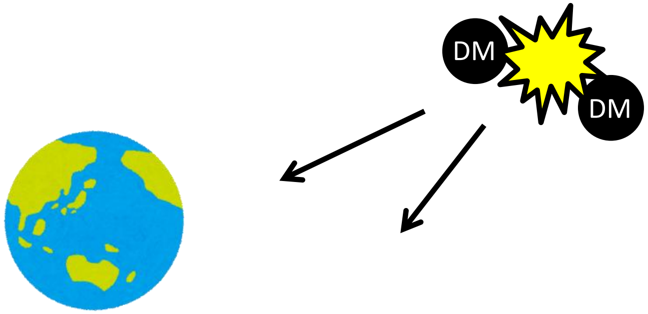
現在の宇宙の残存量

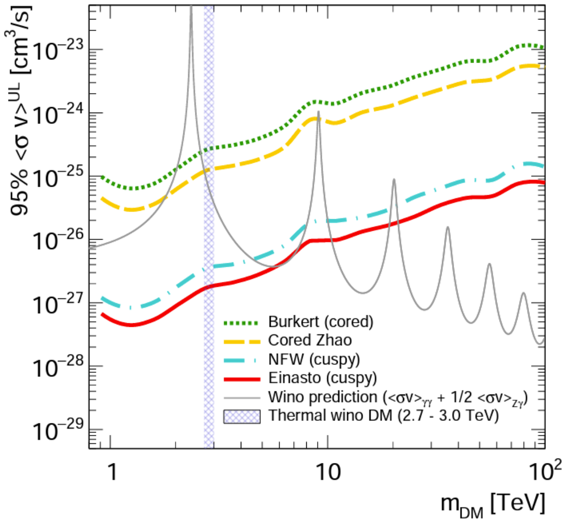
$$\rightarrow \quad \Omega_{\mathrm{DM}} h^2 \sim 0.1 \times \frac{3 \times 10^{-26} cm^3/s}{\sigma v}$$



- **暗黒物質の間接探索**

暗黒物質由来の高エネルギー宇宙線



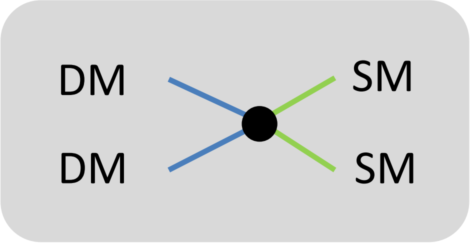


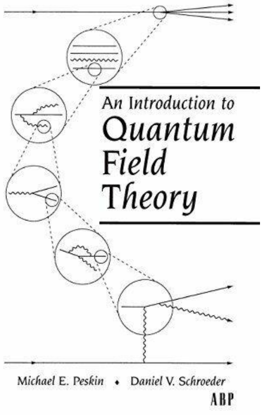
[MAGIC, 2212.10527]

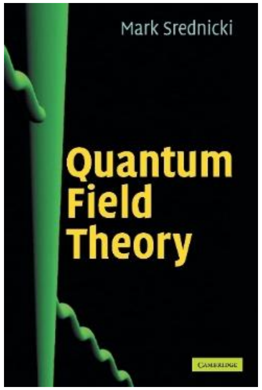
- …

σv の計算方法

結合定数がO(1)未満なら、基本的には摂動論で計算ができる







などなど

ヒッグスポータル暗黒物質の例

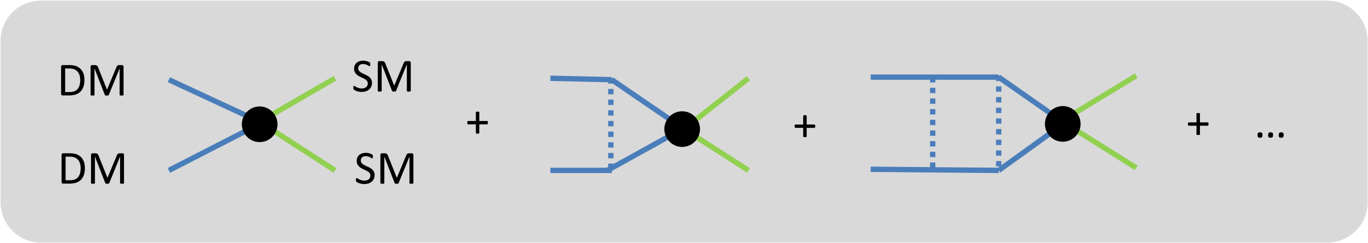
$$L \ni -\frac{\lambda}{4}\phi^2 h^2 \quad \Rightarrow \quad iM(\phi\phi \to hh) = -i\lambda \quad \Rightarrow \quad \sigma v_{rel} = \frac{\lambda^2}{64\pi^2 m_\phi^2}\sqrt{1-\frac{m_h^2}{m_\phi^2}}$$

σv の計算方法

結合定数がO(1)未満でも、非摂動的な効果が効く場合がある！

暗黒物質が軽いボゾンと相互作用していると…

$(m_{\mathrm{boson}} \ll m_{\mathrm{DM}})$



M $\quad\sim M\dfrac{\alpha}{v}\quad$ $\sim M\left(\dfrac{\alpha}{v}\right)^2$

ex)
- Wino / Higgsino dark matter
- SU(2) 5-plet dark matter
- …

Sommerfeld効果

（しきい値共鳴効果）

[Sommerfeld (1931)]
[Hisano, Matsumoto, Nojiri (2003)]

計算の方針

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

See also [Agrawal, Parikh, Reece (2020)]

非相対論的な状況の対消滅に興味がある！

- Freeze-out ($T \simeq m/20$)
- $v \simeq 10^{-3} c$ in galaxy

Schroedinger方程式が有効なはず。

$$E\psi = -\frac{1}{2\mu}\nabla^2\psi + V(x)\psi$$

ポテンシャル項が有効理論的な記述を担う

$$V(r) \;=\; \left\{ \begin{array}{lll} V_{\text{short}}(r) & (r < a) & \text{complex} \\ \quad \text{短距離の効果} & \text{：} & \text{対消滅（など）} \\ V_{\text{long}}(r) & (r \geq a) & \text{real} \\ \quad \text{長距離の効果} & \text{：} & \text{軽いボゾンの交換} \end{array} \right.$$

ex) $V(r) \;\sim\; u\,\delta^3(x) \,+\, \frac{\alpha}{r}\, e^{-mr}$

aは（とりあえず）手で決めた境界

Schroedinger方程式と対消滅

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

学部で習うような二体散乱問題

$$E\psi = -\frac{1}{2\mu}\nabla^2\psi + V(x)\psi \qquad \text{with} \qquad \psi \to\ e^{ipz}\ +\ f(\theta)\frac{e^{ikr}}{r}$$

確率の流速

$$\vec{j}(x) = \frac{1}{\mu}\mathrm{Im}\left[\psi^*(x)\vec{\nabla}\psi(x)\right]$$

⇒

確率の”非保存”

$$\vec{\nabla}\cdot\vec{j}(x) = 2\mathrm{Im}V(x)|\psi(x)|^2$$

$$= 2\mathrm{Im}V_{\mathrm{short}}(x)|\psi(x)|^2$$

Schroedinger方程式と対消滅

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

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断面積の定義

$$\sigma \times j_{in} = -\frac{dP}{dt}$$

入射波の流速

$$j_{in} = \frac{1}{\mu}\mathrm{Im}\left[\psi_{\mathrm{in}}^*(x)\vec{\nabla}\psi_{\mathrm{in}}(x)\right] = \frac{p}{\mu} = v$$

対消滅のレート

$$-\frac{dP}{dt} = \int_{r<a} d^3x\ 2\mathrm{Im}V_{\mathrm{short}}(x)|\psi(x)|^2$$

Schroedinger方程式と対消滅

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

学部で習うような二体散乱問題

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対消滅断面積

$$\sigma v = \int_{r<a} d^3x\, 2\mathrm{Im}V_{\mathrm{short}}(x)|\psi(x)|^2$$

Schroedinger方程式と対消滅

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

$$\sigma v \ = \int_{r<a} d^3x \ 2\mathrm{Im}V_{\mathrm{short}}(x)|\psi(x)|^2 \quad \Rightarrow \quad \simeq \int_{r<a} d^3x \ 2\mathrm{Im}V_{\mathrm{short}}(x)\left|\psi_{\mathrm{long}}(x)\right|^2$$

$$\psi \simeq \psi_{\mathrm{long}}$$

（歪曲波ボルン近似）
（Distorted Wave Born Approximation）

σvが大きくなければ悪くないはず…
（あとで議論します）

s. t. $\left[-\frac{1}{2\mu}\nabla^2 + V_{\mathrm{long}}(x) - E\right]\psi_{\mathrm{long}} = 0$

s-wave の場合

$$\sigma v \ \simeq \left|\psi_{\mathrm{long}}(0)\right|^2 \ \times \int_{r<a} d^3x \ 2\mathrm{Im}V_{\mathrm{short}}(x)$$

Enhancement factor

Schroedinger方程式と対消滅

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

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s-wave の場合

[Hisano, Matsumoto, Nojiri (2002)]
[Arkani-hamed, Finkbeiner, Slatyer, Weiner (2008)]
etc

$$\sigma v \;\simeq\; \left|\psi_{\mathrm{long}}(0)\right|^2 \;\times\; (\sigma v)_0$$

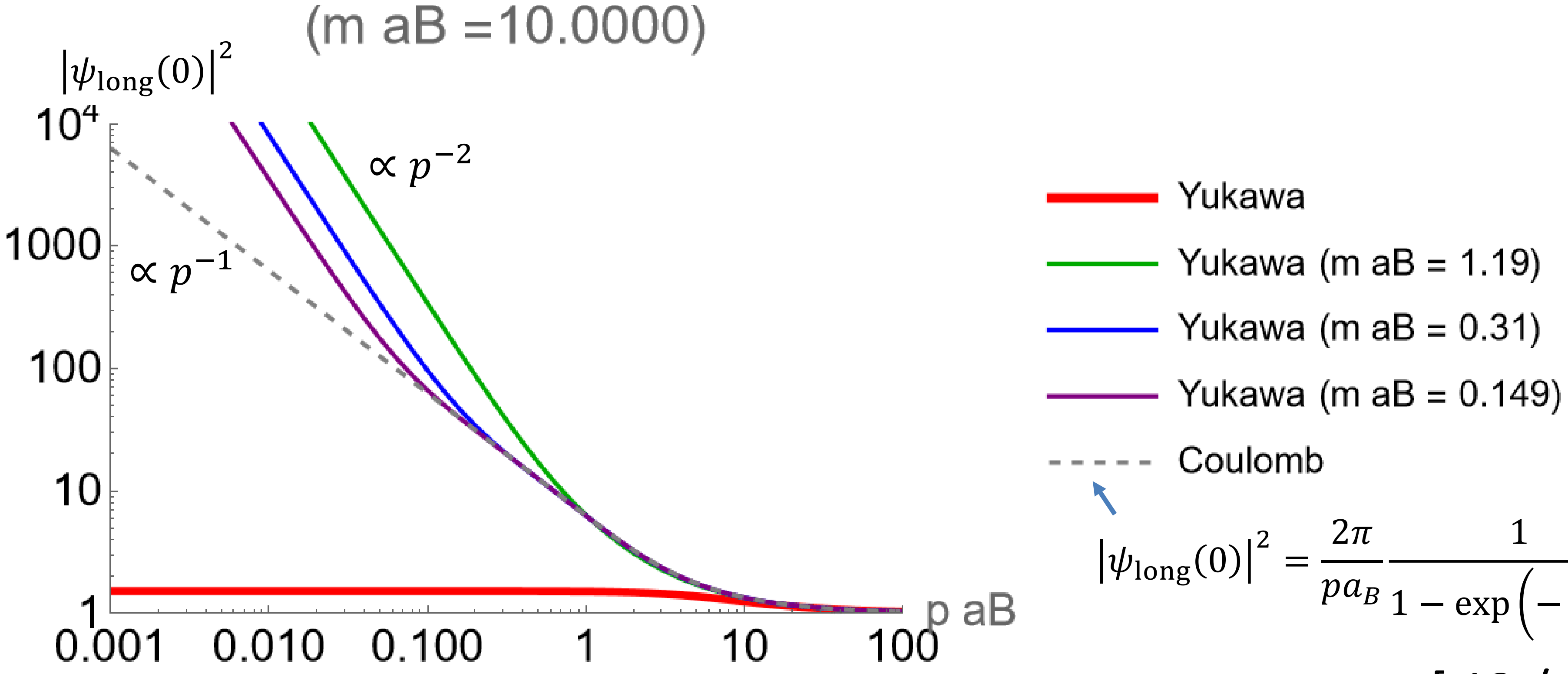
Enhancement factor

長距離力を考慮しない
対消滅断面積

Sommerfeld factor

$V(r) = -\frac{\alpha}{r}\exp(-mr)$　　ボーア半径：$a_B \equiv \frac{1}{\alpha\mu}$

- $m < \frac{1}{a_B}$ & $p < \frac{1}{a_B}$ のとき σv は大きい
- 特定の ma_B のとき特に σv は大きい



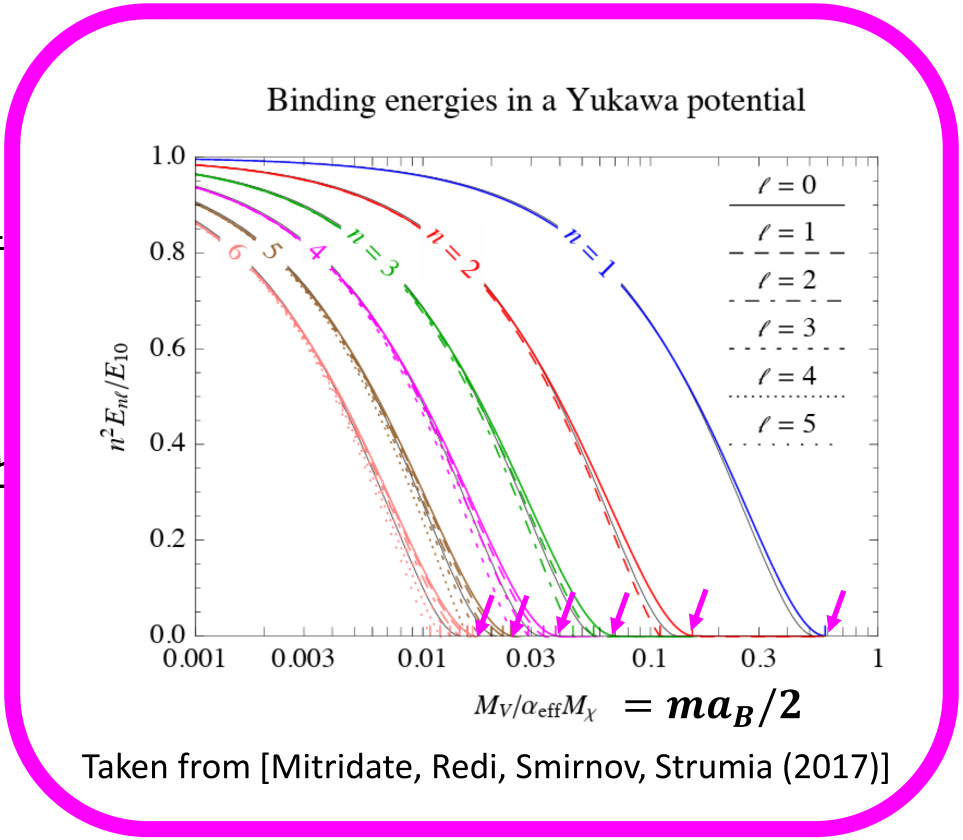
$$|\psi_{\rm long}(0)|^2 = \frac{2\pi}{pa_B}\frac{1}{1-\exp\left(-\frac{2\pi}{pa_B}\right)}$$

Sommerfeld factor

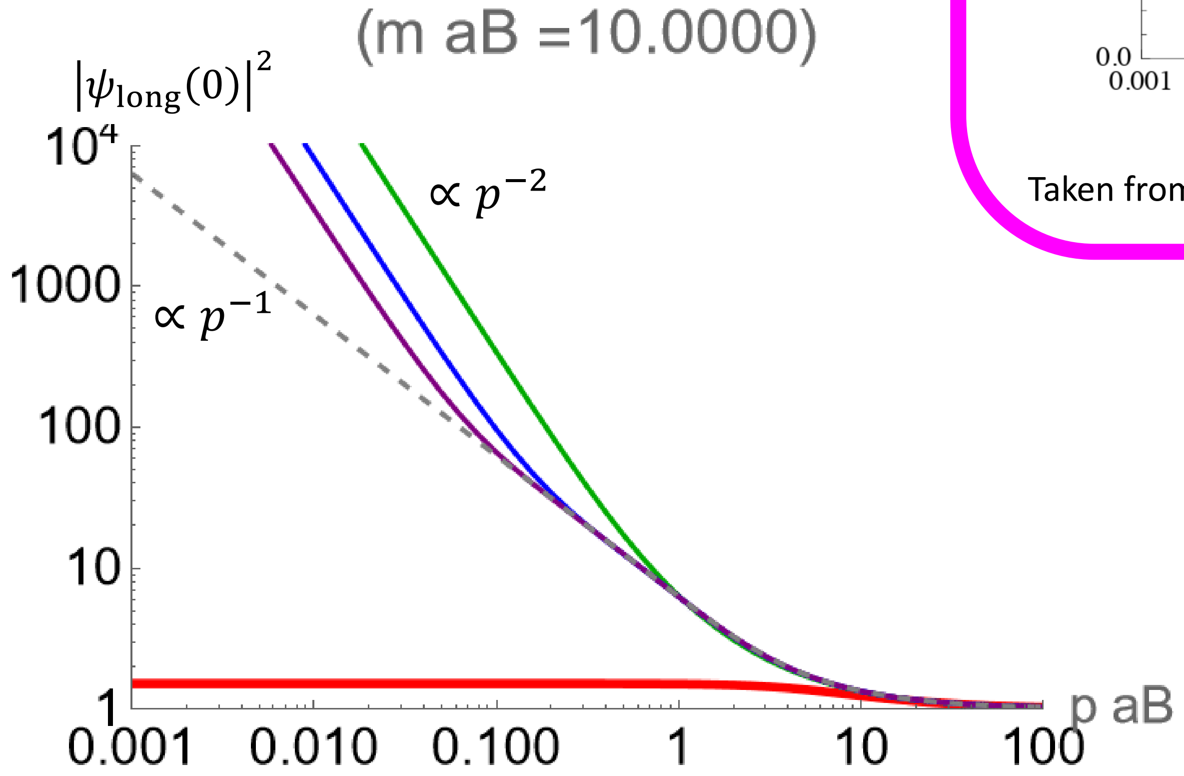
$$V(r) = -\frac{\alpha}{r}\exp(-mr)$$

ボーア半

- $m < \frac{1}{a_B}$ & $p <$
- 特定の ma_B のと



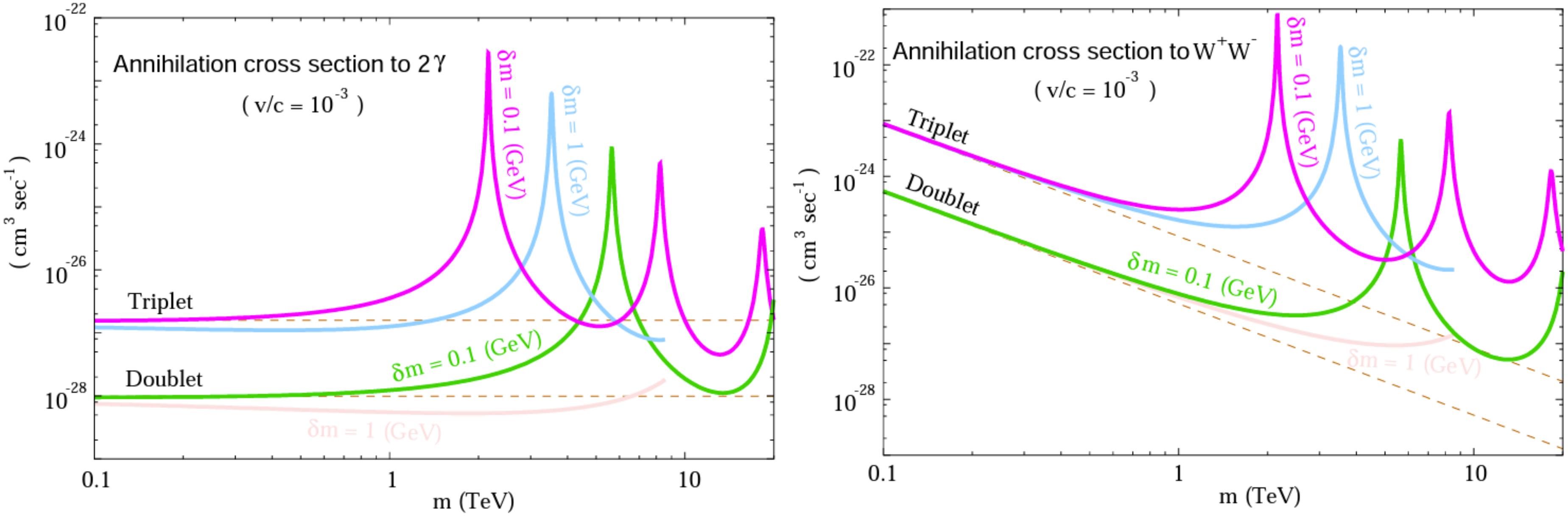
Taken from [Mitridate, Redi, Smirnov, Strumia (2017)]



$$|\psi_{\rm long}(0)|^2 = \frac{2\pi}{pa_B}\frac{1}{1-\exp\left(-\frac{2\pi}{pa_B}\right)}$$

Wino / Higgsino 暗黒物質

破線： 普通の摂動計算 (w/o Sommerfeld effect)
実線： 非摂動的な計算 (w/ Sommerfeld effect)



[Hisano, Matsumoto, Nojiri (2003)]

大きくちがう！

Plan

1. Dark matter and Sommerfeld effect

2. Sommerfeld effect and unitarity

Unitarity bound

$$\boldsymbol{\sigma} = \boldsymbol{\sigma_0} \times \boldsymbol{S(v)}$$

LO cross section : σ_0

Enhancement factor : $S(v) = |\psi(\mathbf{0})|^2$

σ_0 と $S(v)$ は無関係

$\Updownarrow$

s-wave 断面積の上限 : $\sigma \leq \frac{\pi}{p^2}$

[Griest, Kamionkowski (1992)]
[Landau-Lifshits’s textbook]

こんなときに上限が気になる

1. $\sigma_0 v$ がそもそも大きい
2. zero energy resonance がある場合 ($S(v) \propto v^{-2} \rightarrow \sigma \propto v^{-3}$)
(for s-wave)

Unitarity boun

$\boldsymbol{\sigma} = \boldsymbol{\sigma_0} \times \boldsymbol{S(v)}$

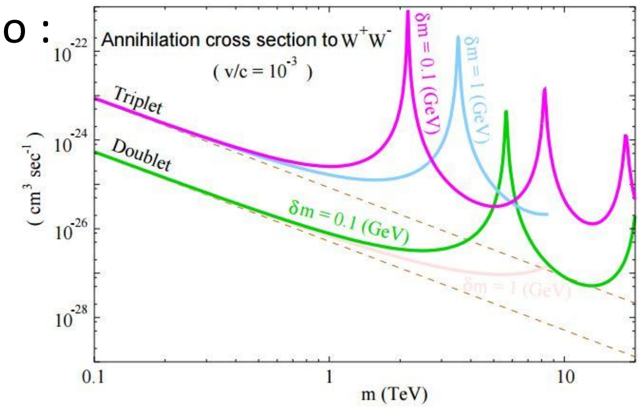
LO cros

Enhan

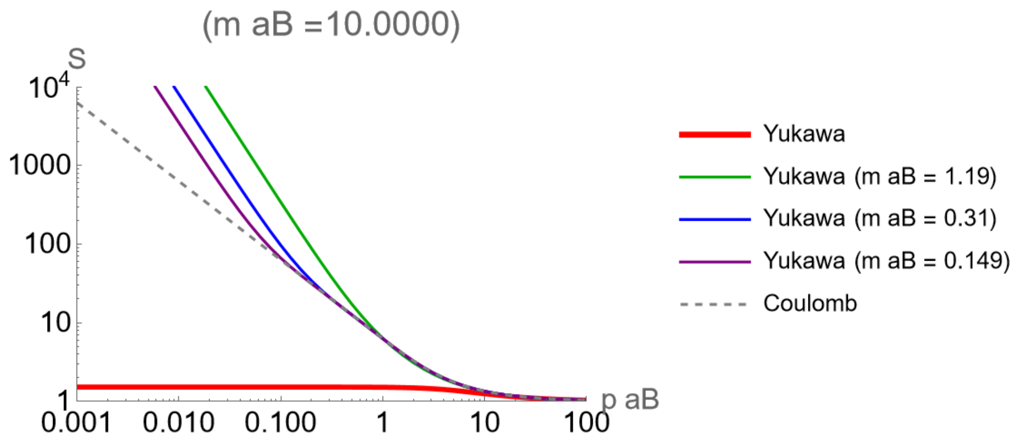
σ_0 と $S(v)$ (

s-wave 断面

Wino / Higgisno :



Yukawa potential :



こんなときに上限が気になる

1. $\sigma_0 v$ がそもそも大きい
2. zero energy resonance がある場合 $(S(v) \propto v^{-2} \rightarrow \sigma \propto v^{-3})$

(for s-wave)

Schroedinger方程式をちゃんと解かなくちゃ

何かいけないことをしただろうか…　公式の導出まで戻ってみる。

$$\sigma v = \int_{r<a} d^3x \; 2\mathrm{Im}V_{\mathrm{short}}(x)|\psi(x)|^2 \Rightarrow \simeq \int_{r<a} d^3x \; 2\mathrm{Im}V_{\mathrm{short}}(x)\left|\psi_{\mathrm{long}}(x)\right|^2$$

$$\psi \simeq \psi_{\mathrm{long}}$$

（歪曲波ボルン近似）
（Distorted Wave Born Approximation）

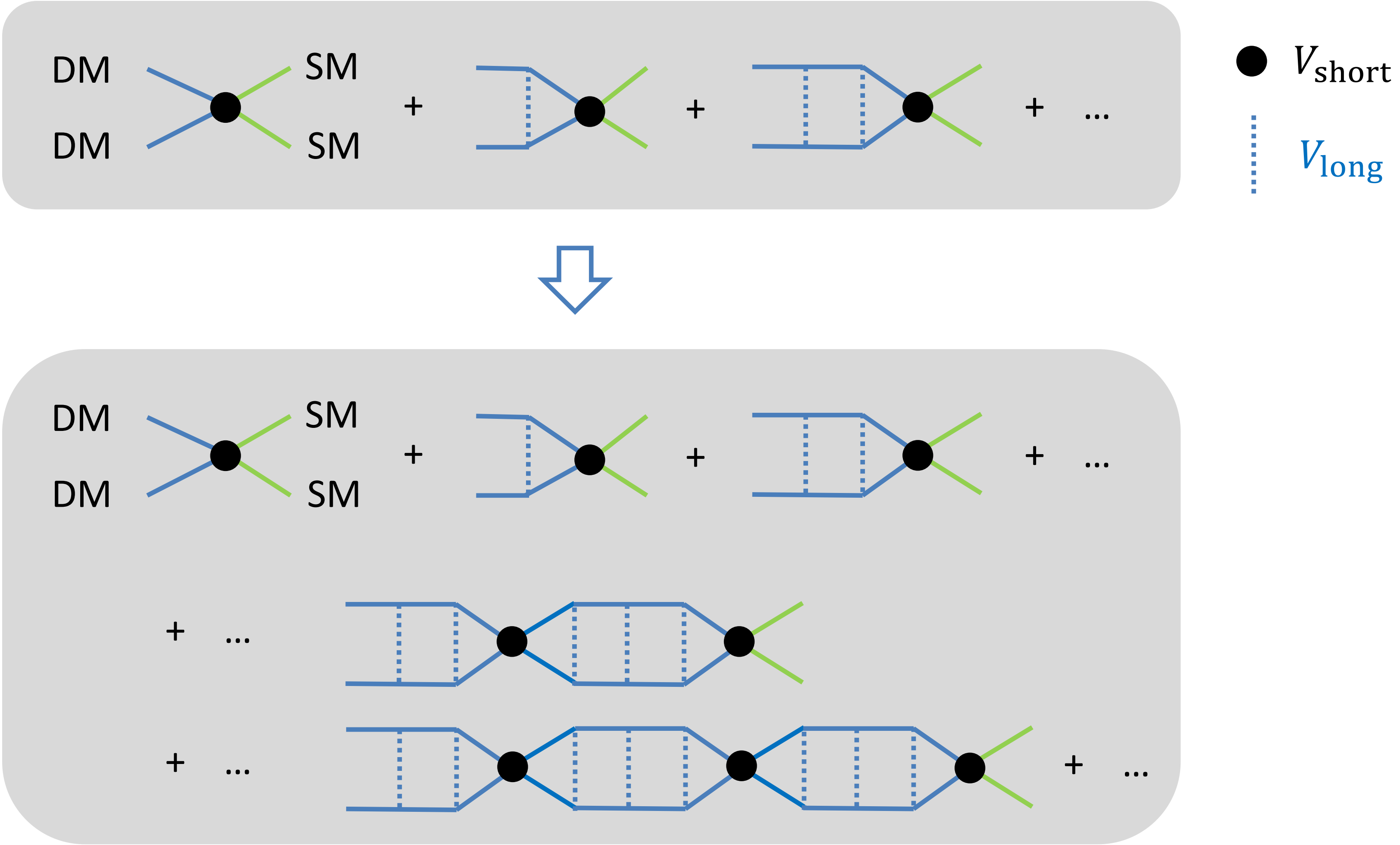
σvが大きくなければ悪くないはず…
（あとで議論します）

s. t. $\left[-\frac{1}{2\mu}\nabla^2 + V_{\mathrm{long}}(x) - E\right]\psi_{\mathrm{long}} = 0$

σvが大きいとき、 ψ と ψ_{long} はだいぶ違うはず

Schroedinger方程式をちゃんと解かなくちゃ

ファインマンダイアグラム的な解釈：



S-matrixの計算

必要なのは、ちゃんと計算した波動関数：　$\psi = \sum_{\ell} P_{\ell}(\cos\theta)\,\frac{(-1)^{\ell}\chi_{\ell}(r)}{pr}$

Schroedinger eq.

$$\left[-\frac{1}{2\mu}\nabla^2 + V(r) - \frac{p^2}{2\mu}\right]\psi(r) = 0 \quad \Rightarrow \quad \left[-\frac{1}{2\mu}\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{2\mu r^2} + V(r) - \frac{p^2}{2\mu}\right]\chi_{\ell}(r) = 0$$

$$V(r) \;=\; \begin{cases} V_{\text{short}}(r) & (r < a) \quad \text{complex} \\ V_{\text{long}}(r) & (r \geq a) \quad \text{real} \end{cases}$$

$r \to \infty$ における境界条件

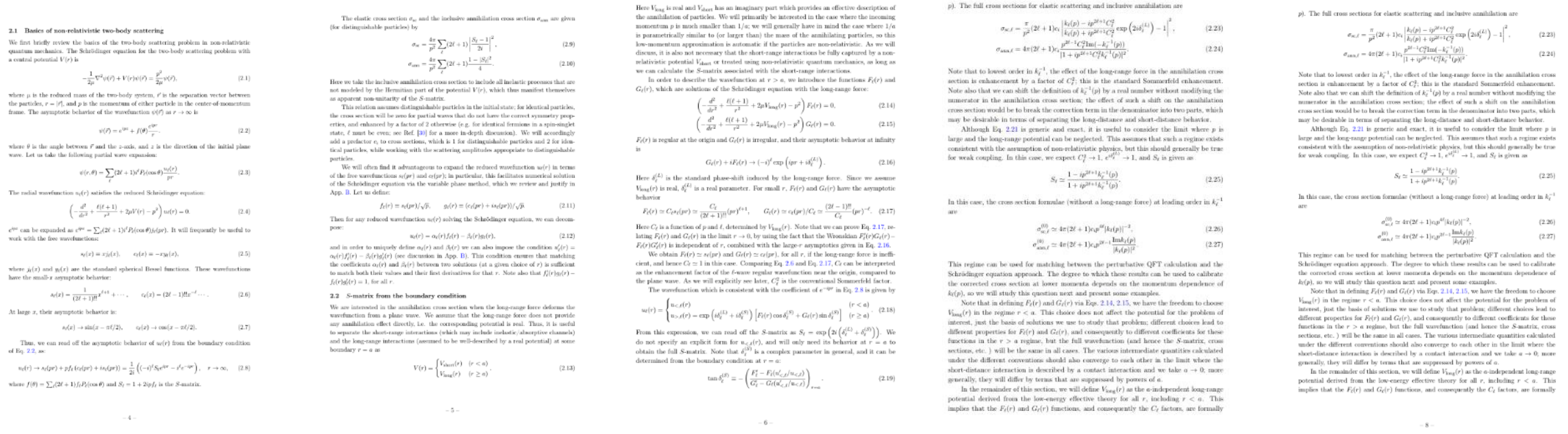
$$\chi_{\ell}(r) \quad \to \quad \frac{S_{\ell}\exp(ipr) \;-\; \exp(-ipr)}{2i}$$

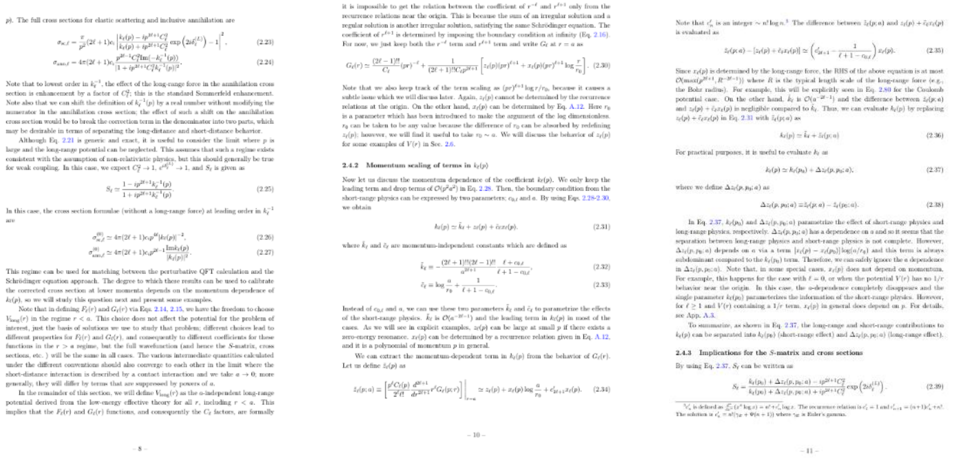
$r = a$ における境界条件（V_{short} により決まる短距離の効果）

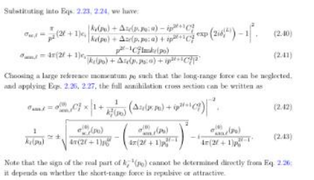
$\chi_{\ell}{}'(r)/\chi_{\ell}(r)$ at $r = a$ は p に依らないとする

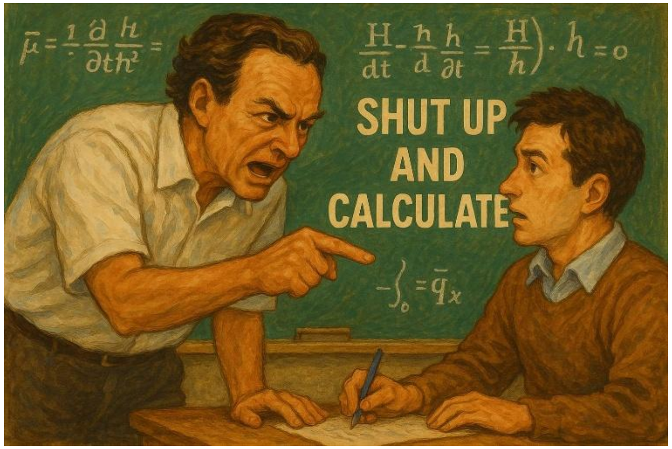
Straightforward な計算

あとは、腕力でがんばる。









[Parikh, Sato, Slatyer (2024)]

S-matrix

がんばって計算すると、各部分波の S-matrix が得られる。

$$S_\ell \simeq \underbrace{\exp\left(2i\delta_\ell^{(L)}(p)\right)}_{\text{Phase-shift by long-range force}} \times \underbrace{\frac{k_{\ell,0}+z_\ell(p)-ip^{2\ell+1}C_\ell^2(p)}{k_{\ell,0}+z_\ell(p)+ip^{2\ell+1}C_\ell^2(p)}}_{\text{Relevant part for annihilation}}$$

一つの複素パラメーター	$k_{\ell,0}$
三つの実の関数	$C_\ell^2(p)$, $z_\ell(p)$, $\delta_\ell^{(L)}(p)$

S-matrix

がんばって計算すると、各部分波の S-matrix が得られる。

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対消滅断面積：

$$\sigma_{ann,\ell} = \frac{\pi}{p^2}(2\ell+1)(1-|S_\ell|^2) \overset{\text{Unitarity bound}\checkmark}{<} \frac{(2\ell+1)\pi}{p^2}$$

S-matrix

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対消滅断面積：

Unitarity bound ✓

$$\sigma_{ann,\ell} = \frac{\pi}{p^2}(2\ell+1)(1-|S_\ell|^2) < \frac{(2\ell+1)\pi}{p^2}$$

$$= \frac{\pi}{p^2}(2\ell+1) \times 4\mathrm{Re}\left[\frac{ip^{2\ell+1}C_\ell^2}{k_{\ell,0}}\right] \times \left|1+\frac{z_\ell + ip^{2\ell+1}C_\ell^2}{k_{\ell,0}}\right|^{-2}$$

S-matrix

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対消滅断面積：

Unitarity bound ✓

$$\sigma_{ann,\ell} \quad = \quad \frac{\pi}{p^2}(2\ell+1)(1-|S_\ell|^2) \qquad < \quad \frac{(2\ell+1)\pi}{p^2}$$

$$= \quad 4\pi(2\ell+1)p^{2\ell-1}\,\mathrm{Im}\left[-\frac{1}{k_{\ell,0}}\right] \quad \times \quad C_\ell^2 \quad \times \quad \left|1 + \frac{z_\ell + ip^{2\ell+1}C_\ell^2}{k_{\ell,0}}\right|^{-2}$$

短距離の対消滅の効果

通常の
Sommerfeld factor

補正項

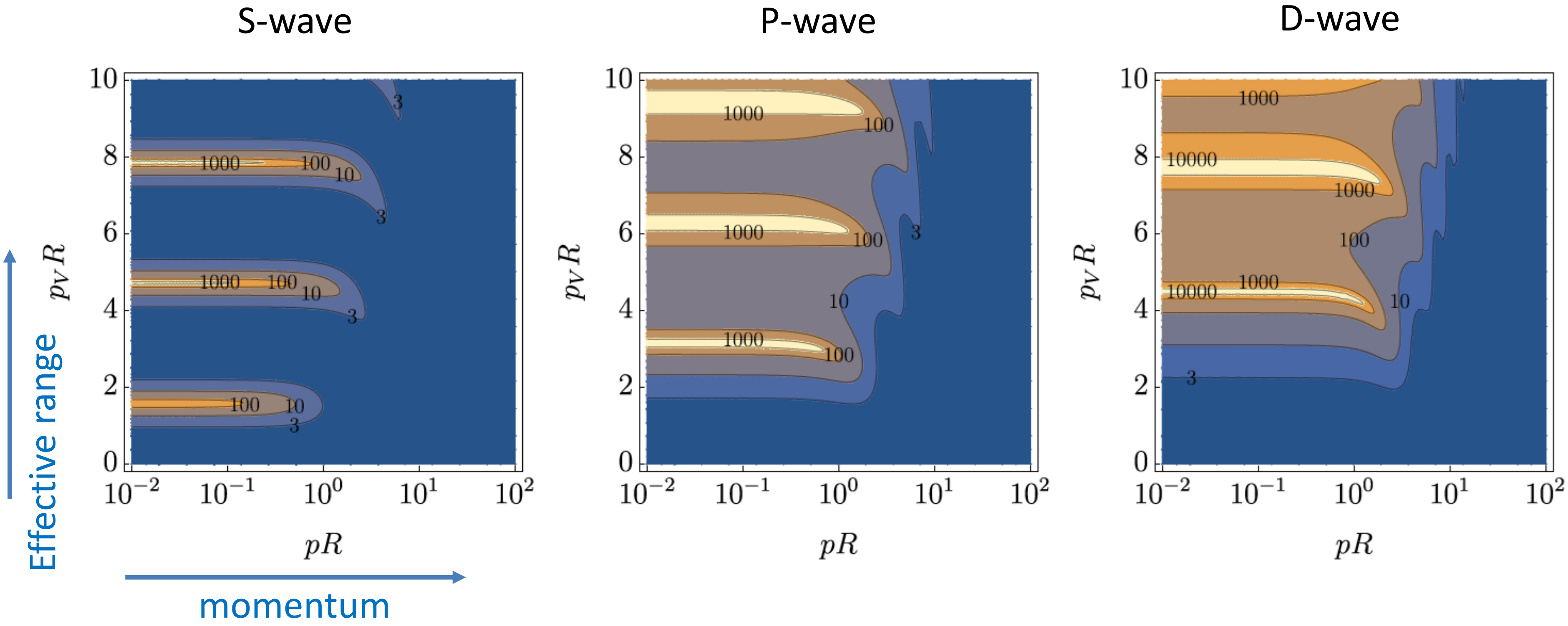
計算例

[Parikh, Sato, Slatyer (2024)]

球形井戸型ポテンシャル

$$V(r) = -\frac{p_V^2}{2\mu}\theta(R-r)$$

（通常の）Sommerfeld factor : C_ℓ^2



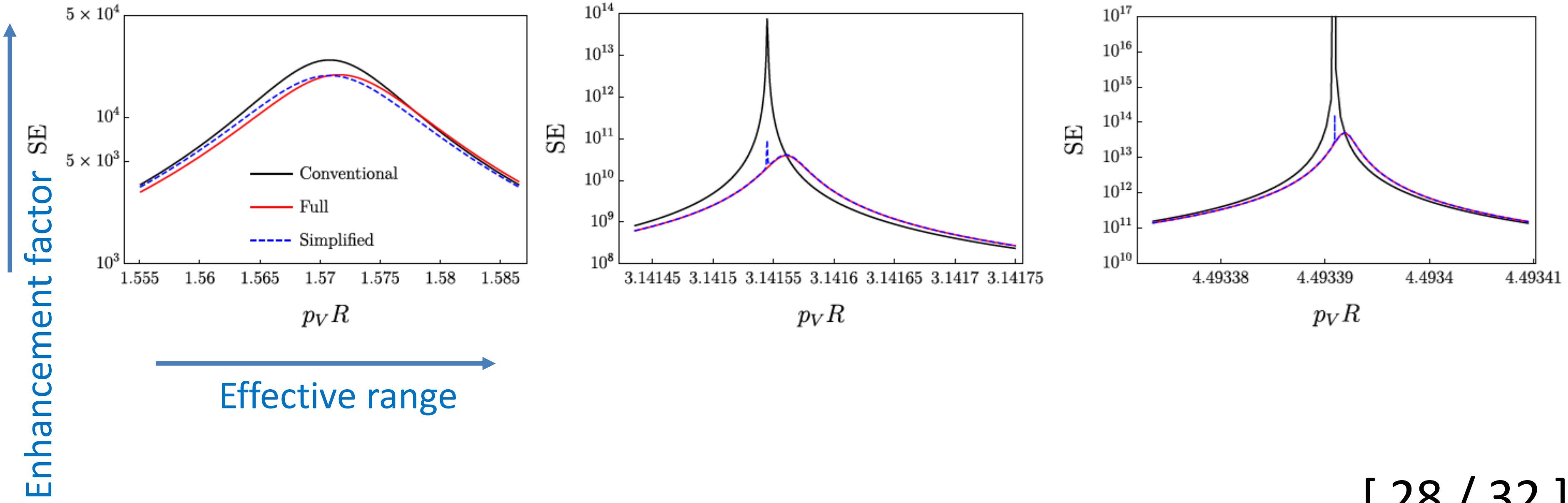
計算例

[Parikh, Sato, Slatyer (2024)]

球形井戸型ポテンシャル

$$V(r) = -\frac{p_V^2}{2\mu}\theta(R-r)$$

$$\frac{\sigma_{ann}}{\sigma_{ann,\mathrm{LO}}}$$



Enhancement factor

Effective range

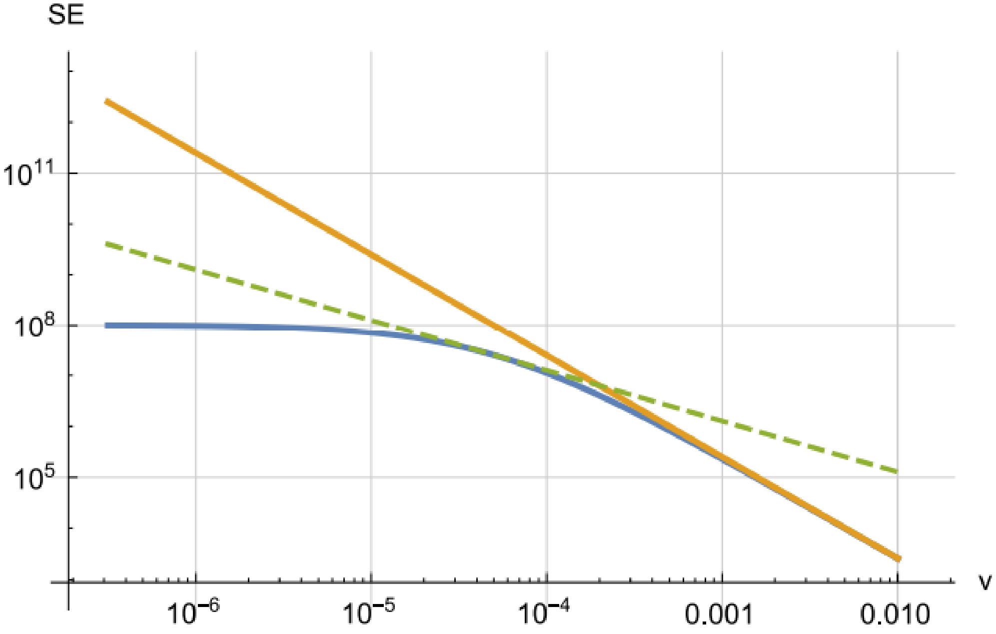
計算例

[Blum, Sato, Slatyer (2016)]

Hulthen potential : $V(r) = -\frac{\alpha m_* e^{-m_* r}}{1 - e^{-m_* r}}$ $\left(V(r) = -\frac{\alpha e^{-mr}}{r}, m_* = \frac{\pi^2}{6} m\right.$ の良い近似)

$$\alpha = 1, \qquad \sigma v = \frac{1}{32\pi M^2}, \qquad \sigma_{sc} = \frac{\mu^2}{4\pi}(\sigma v)^2$$

$$m_* = 0.0625M$$

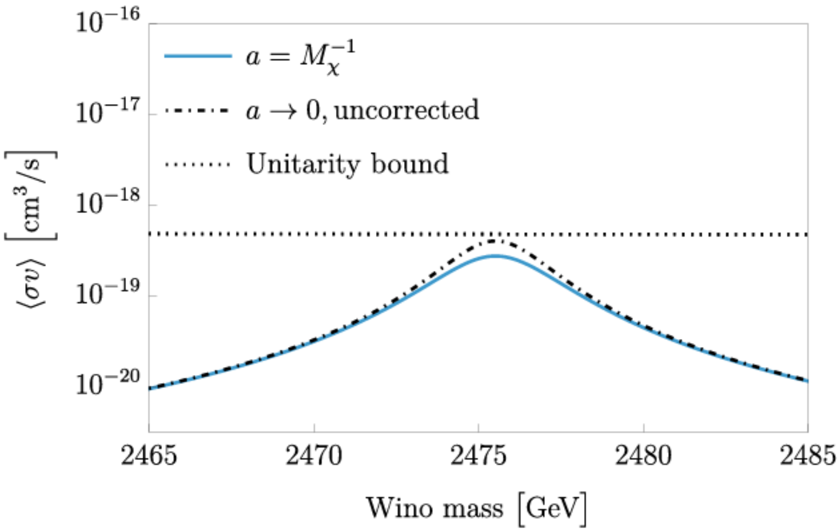


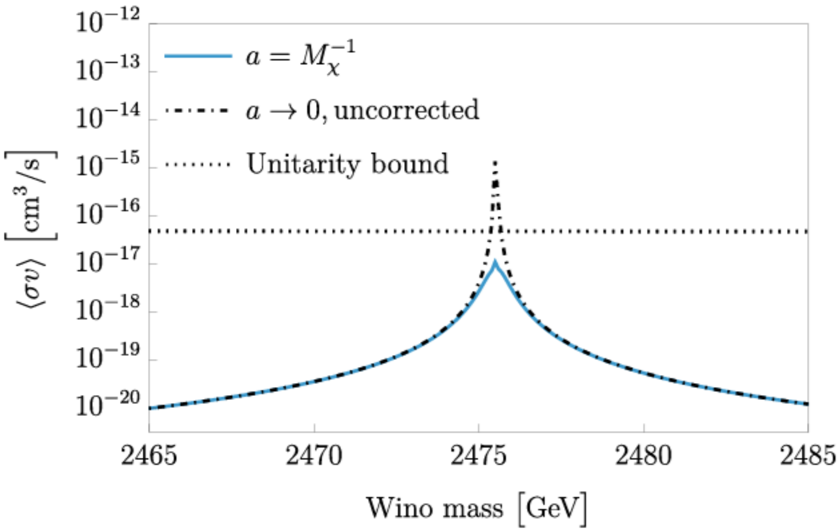
黄色実線 : 通常の公式
青の実線 : 我々の公式
緑の破線 : ユニタリティによる上限

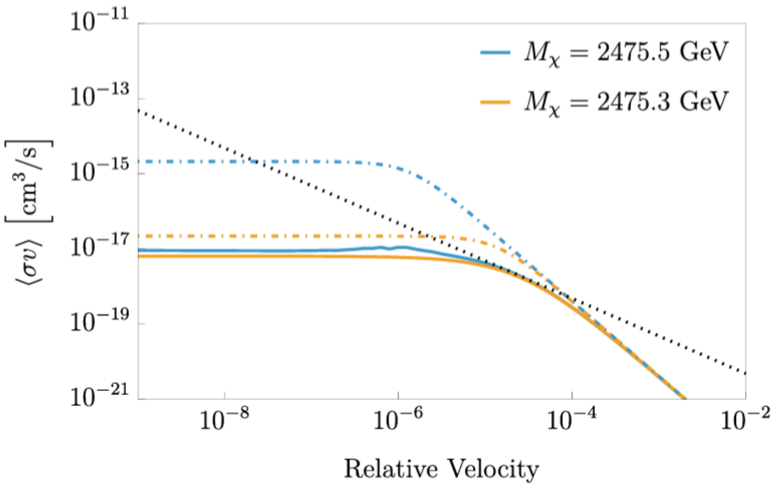
計算例

[Blum, Sato, Slatyer (2016)]

Wino



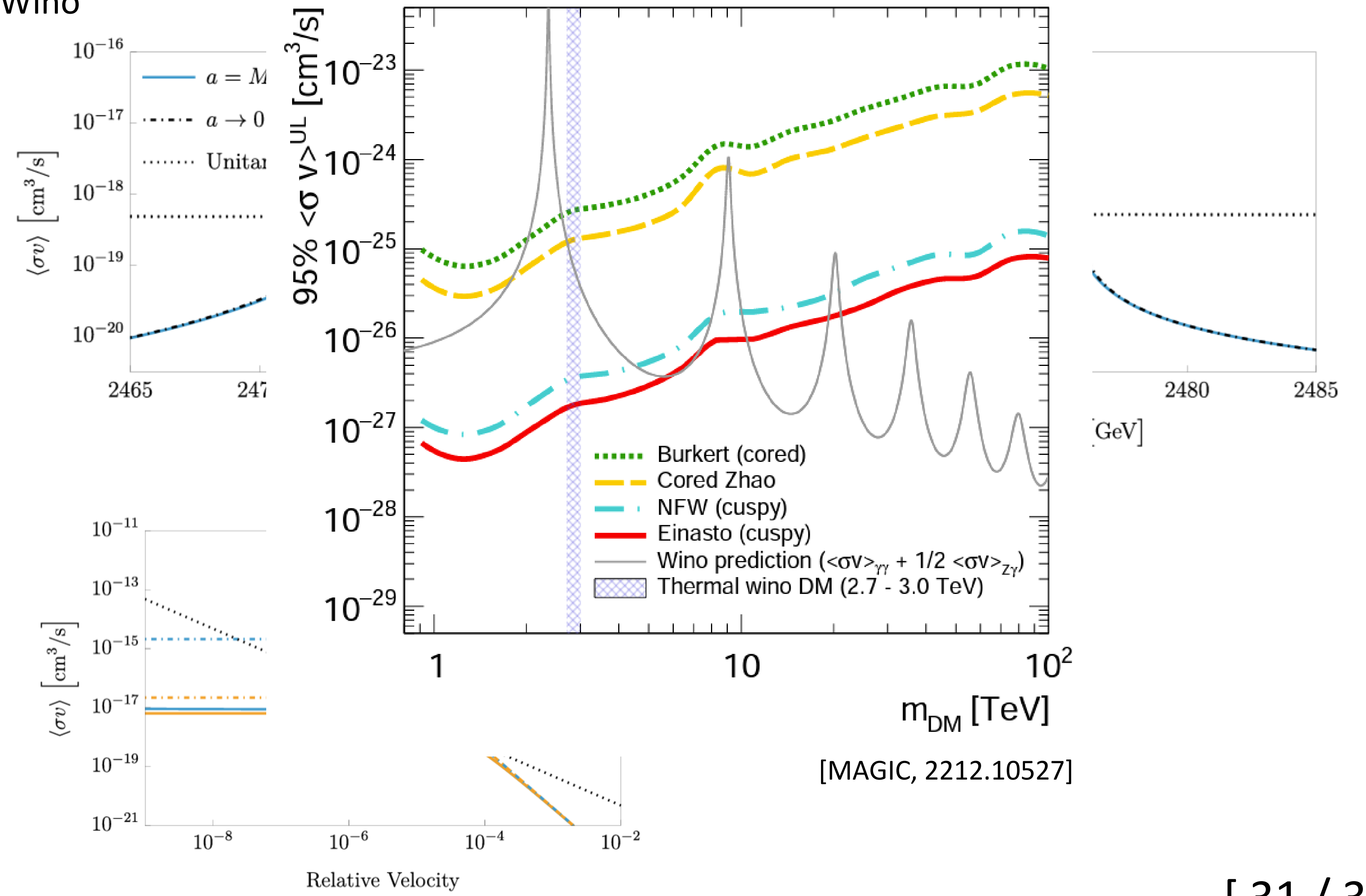




計算例

[Blum, Sato, Slatyer (2016)]

Wino



[MAGIC, 2212.10527]

まとめ

- Schroedinger方程式により、軽いボゾンの効果を取り込める
- 対消滅の効果は、ポテンシャルの複素の項であらわせる
- 量子力学のユニタリティの上限とも矛盾しない公式ができた
- 対消滅断面積が大きい場合には、通常の公式からの補正が重要

Backup

Unitarity bound と zero-energy resonance

zero-energy resonance があるときの対消滅断面積

$$|\psi_{\mathrm{long}}|^2 \propto p^{-2} \quad \text{(for s-wave)} \quad \Rightarrow \quad \sigma_{ann,s} = \frac{1}{v} \times |\psi_{long}|^2 \times (\sigma v)_0 \propto \frac{1}{p^3}$$

pが小さいと
やばい

部分波展開

$$E\psi = -\frac{1}{2\mu}\nabla^2\psi + V(x)\psi$$

with $\psi \rightarrow e^{ipz} + f(\theta)\frac{e^{ikr}}{r} = \sum_\ell P_\ell(\cos\theta)\frac{S_\ell e^{ipr} - (-1)^\ell e^{-ipr}}{2ipr}$

対消滅断面積

$$\sigma_{ann} = \frac{\pi}{p^2}\sum_\ell (2\ell+1)(1-|S_\ell|^2) \quad \Rightarrow \quad \sigma_{ann,s} \le \frac{\pi}{p^2}$$

Wave function w/ long-range force

$$\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+2\mu V(r)-p^2\right]H_\ell^+(r)=0, \qquad H_\ell^+(r)\to(-i)^\ell\exp\left(ipr+\delta_\ell^{(L)}\right)$$

$$F_\ell(r)\ \equiv\ \mathrm{Im}H_\ell^+\ \simeq\ C_\ell p^{\ell+1}\times\left[\underbrace{\frac{r^{\ell+1}}{(2\ell+1)\,!!}}_{\text{Leading term}}+\ \dots\right]$$

$$G_\ell(r)\ \equiv\ \mathrm{Re}H_\ell^+\ \simeq\ \frac{1}{C_\ell p^\ell}\times\left[\underbrace{\frac{(2\ell-1)\,!!}{r^\ell}}_{\text{(basically) leading term}}+\ \dots\ +\underbrace{z_\ell(p)\frac{r^{\ell+1}}{(2\ell+1)\,!!}}_{\text{Sizable in some cases}}+\ \dots\right]$$

On zero energy resonance,

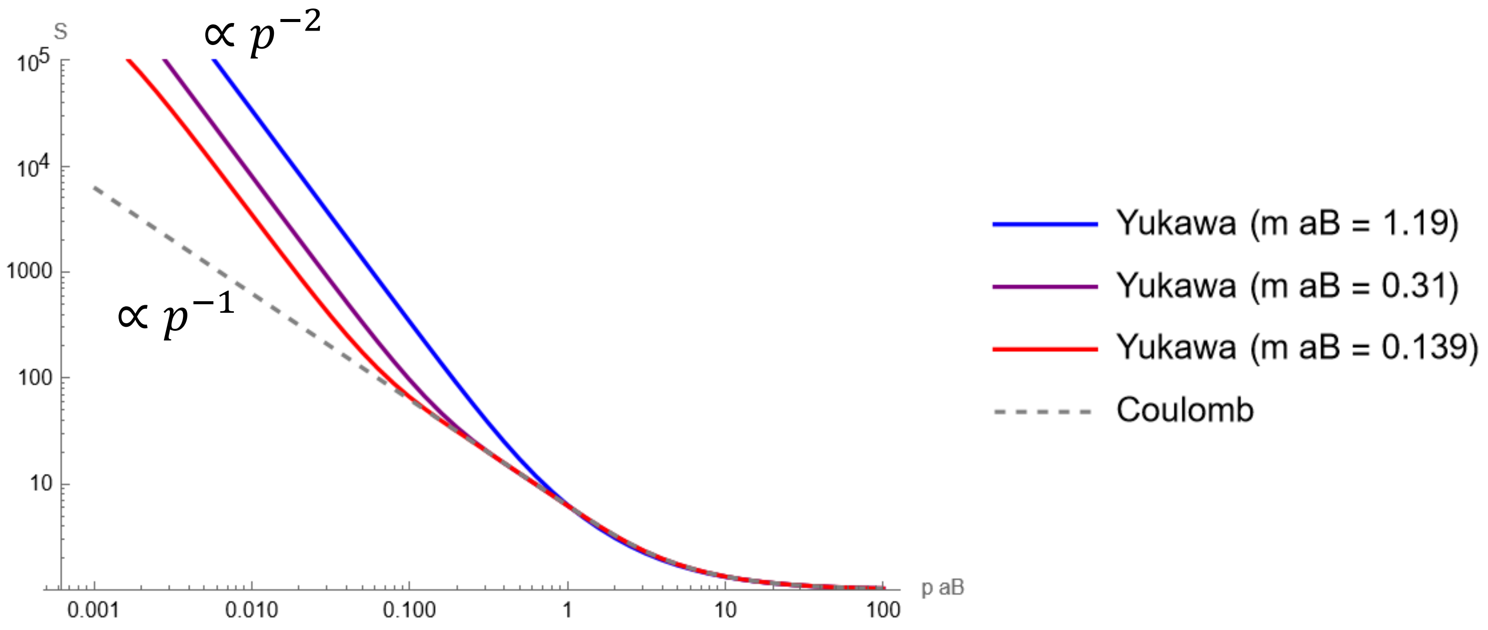
$$C_\ell^2(p)\ \propto\ \begin{cases} p^{-2} & (\ell=0)\\ p^{-4} & (\ell\geq 1)\end{cases} \qquad z_\ell(p)\ \propto\ \begin{cases} p^{0} & (\ell=0)\\ p^{-2} & (\ell\geq 1)\end{cases}$$

See also [Kamada, Kuwahara, Patel (2023)]

Resonant points

At some specific points ($ma_B = 1.19, \;\; 0.31, \;\; 0.139, \ldots$), $\quad |\psi_{\mathrm{long}}|^2 \;\; \propto \;\; \dfrac{1}{p^2}$

(zero energy resonance : bound state with zero binding energy)



$$\sigma \;\; = \;\; \frac{1}{v} \times |\psi_{long}|^2 \times (\sigma v)_0 \quad \propto \quad \frac{1}{p^3}$$

Zero energy resonance

Schroedinger eq

$$\left[-\frac{1}{2\mu}\frac{d^2}{dr^2}+V(r)-\frac{p^2}{2\mu}\right]\chi(r)=0,$$

Boundary cond.

$$\chi(r) \to \frac{S\exp(ipr) \quad - \quad \exp(-ipr)}{2i}$$

Bound state w/ $E=-\frac{\kappa^2}{2m}$

$$\chi(r) \to \exp(-\kappa r)$$

$\Rightarrow$

pole in $S(k)$ at $p=i\kappa$

$$\to \qquad S=e^{2i\delta}=-\frac{p+i\kappa}{p-i\kappa}$$

$$\to \qquad \sin\delta=-\frac{\kappa}{p}$$

$$\psi(\vec{x})=\frac{\chi(r)}{pr}$$

$$\left|\psi_{\text{long}}(0)\right|^2 \propto \frac{\chi^2(0)}{p^2}=\frac{\sin^2\delta}{p^2}=\frac{1}{p^2+\kappa^2}$$

Solution of Schroedinger eq.

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

Schroedinger equation:

$$\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+2\mu V(r)-p^2\right]u_\ell(r)=0, \qquad V(r)\ =\begin{cases} V_{\mathrm{short}}(r) \quad (r<a) & \text{complex} \\ V_{\mathrm{long}}(r) \quad (r\geq a) & \text{real} \end{cases}$$

Solution of Schroedinger eq.

[Blum, Sato, Slatyer (2016)]
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$$\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+2\mu V(r)-p^2\right]u_\ell(r)=0, \qquad V(r) = \begin{cases} V_{\text{short}}(r) & (r<a) \quad \text{complex} \\ V_{\text{long}}(r) & (r\geq a) \quad \text{real} \end{cases}$$

u_ℓ should be a linear combination of

- $F_\ell(r)$ (regular solution)
- $G_\ell(r)$ (irregular solution)

at $r > a$

$$\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+2\mu V_{long}(r)-p^2\right]F_\ell(r)=0 \quad \& \quad \left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+2\mu V_{long}(r)-p^2\right]G_\ell(r)=0$$

	$r \simeq 0$	$r \to \infty$
$F_\ell(r)$	$\dfrac{C_\ell}{(2\ell+1)\,!!}(pr)^{\ell+1}$	$\sin\left(pr+\delta_\ell^{(L)}-\dfrac{\pi\ell}{2}\right)$
$G_\ell(r)$	$\dfrac{(2\ell-1)\,!!}{C_\ell}(pr)^{-\ell}$	$\cos\left(pr+\delta_\ell^{(L)}-\dfrac{\pi\ell}{2}\right)$

$C_\ell \simeq 1,\ \ \delta_\ell^{(L)} \simeq 0$ for $V_{long}(r) \simeq 0$

Solution of Schroedinger eq.

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

Schroedinger equation:

$$\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+2\mu V(r)-p^2\right]u_\ell(r)=0, \qquad V(r) = \begin{cases} V_{\text{short}}(r) & (r<a) \quad \text{complex} \\ V_{\text{long}}(r) & (r \geq a) \quad \text{real} \end{cases}$$

u_ℓ should be a linear combination of $\left\{ \begin{array}{l} \bullet \ \ F_\ell(r) \ \ \text{(regular solution)} \\ \bullet \ \ G_\ell(r) \ \ \text{(irregular solution)} \end{array} \right.$ at $r > a$

$$u_\ell(r) = \begin{cases} u_{\ell,<}(r) & (r<a) \\ \exp\left(i\delta_\ell^{(L)}+i\delta_\ell^{(S)}\right)\left[\cos\delta_\ell^{(S)}F_\ell(r)+\sin\delta_\ell^{(S)}G_\ell(r)\right] & (r \geq a) \end{cases}$$

$$\to \quad \frac{1}{2i}\left((-i)^\ell \exp\left(2i\delta_\ell^{(L)}+2i\delta_\ell^{(S)}\right)e^{ipr}-i^\ell e^{-ipr}\right)$$

Solution of Schroedinger eq.

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

Schroedinger equation:

$$\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+2\mu V(r)-p^2\right]u_\ell(r)=0, \qquad V(r) = \begin{cases} V_{\text{short}}(r) & (r<a) \quad \text{complex} \\ V_{\text{long}}(r) & (r\geq a) \quad \text{real} \end{cases}$$

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$$\to \quad \frac{1}{2i}\left(\underbrace{(-i)^\ell \exp\left(2i\delta_\ell^{(L)}+2i\delta_\ell^{(S)}\right)e^{ipr}}_{\text{Outgoing wave}}-\underbrace{i^\ell e^{-ipr}}_{\text{Incoming wave}}\right)$$

Solution of Schroedinger eq.

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

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$$\rightarrow \quad \frac{1}{2i}\left((-i)^\ell \underbrace{\exp\left(2i\delta_\ell^{(L)}+2i\delta_\ell^{(S)}\right)}_{S_\ell} e^{ipr} - i^\ell e^{-ipr}\right)$$

Outgoing wave

Incoming wave

S-matrix

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

$$u_\ell(r) = \begin{cases} u_{\ell,<}(r) & (r < a) \\ \exp\left(i\delta_\ell^{(L)} + i\delta_\ell^{(S)}\right)\left[\cos\delta_\ell^{(S)} F_\ell(r) + \sin\delta_\ell^{(S)} G_\ell(r)\right] & (r \geq a) \end{cases}$$

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Outgoing wave

Incoming wave

$\frac{u_\ell{}'}{u_\ell}$ is continuous at $r = a$ ⇨ $$\frac{u_{\ell,<}{}'}{u_{\ell,<}} = \frac{\cos\delta_\ell^{(S)} F_\ell' + \sin\delta_\ell^{(S)} G_\ell'}{\cos\delta_\ell^{(S)} F_\ell + \sin\delta_\ell^{(S)} G_\ell}$$

S-matrix

[Blum, Sato, Slatyer (2016)]
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Outgoing wave

Incoming wave

$\dfrac{u_\ell{}'}{u_\ell}$ is continuous at $r = a$ ⇒

$$\frac{u_{\ell,<}{}'}{u_{\ell,<}} = \frac{F_\ell' + \tan\delta_\ell^{(S)} G_\ell'}{F_\ell + \tan\delta_\ell^{(S)} G_\ell}$$

S-matrix

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

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Outgoing wave

Incoming wave

$$\frac{u_\ell{}'}{u_\ell} \text{ is continuous at } r = a \quad \Rightarrow \quad \tan\delta_\ell^{(S)} = -\frac{F_\ell' - F_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}{G_\ell' - G_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}$$

$$S_\ell \;=\; \exp\left(2i\delta_\ell^{(L)}\right) \times \exp\left(2i\delta_\ell^{(S)}\right)$$

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Incoming wave

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$$S_\ell = \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{1 + i\tan\delta_\ell^{(S)}}{1 - i\tan\delta_\ell^{(S)}}$$

S-matrix

[Blum, Sato, Slatyer (2016)]
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Solutions for $V(r) = V_{long}(r)$

$$F_\ell(r) \;\simeq\; \frac{C_\ell}{(2\ell+1)\,!!}(pr)^{\ell+1}$$

$$G_\ell(r) \simeq \frac{(2\ell-1)\,!!}{C_\ell}(pr)^{\ell+1}$$

S-matrix

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$$S_\ell = \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{1 + i\tan\delta_\ell^{(S)}}{1 - i\tan\delta_\ell^{(S)}}$$

$$\tan\delta_\ell^{(S)} = -p^{2\ell+1} C_\ell^2 \frac{f_\ell' - f_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}{g_\ell' - g_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}$$

Solutions for $V(r) = V_{long}(r)$

$$f_\ell(r) \equiv \frac{F_\ell}{C_\ell p^{\ell+1}} \simeq \frac{r^{\ell+1}}{(2\ell+1)!!}$$

$$g_\ell(r) \equiv \frac{C_\ell}{p^\ell} G_\ell \simeq \frac{(2\ell-1)!!}{r^\ell}$$

S-matrix

[Blum, Sato, Slatyer (2016)]
[Parikh, Sato, Slatyer (2024)]

$$S_\ell \;=\; \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{1 + i\tan\delta_\ell^{(S)}}{1 - i\tan\delta_\ell^{(S)}}$$

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$$k_\ell(p) \;\equiv\; -\frac{p^{2\ell+1} C_\ell^2}{\tan\delta_\ell^{(S)}} \;=\; \frac{g_\ell' - g_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}{f_\ell' - f_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}$$

$k_\ell(p)$ is *almost* independent on p
(will be discussed later)

S-matrix

[Blum, Sato, Slatyer (2016)]
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$$S_\ell \;=\; \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{1 + i\tan\delta_\ell^{(S)}}{1 - i\tan\delta_\ell^{(S)}}$$

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$u_{\ell,<}(r)$: p independent
$f_\ell(r)$: p independent
$g_\ell(r)$: *almost* p independent

⇨ $k_\ell(p)$ is *almost* independent on p
(will be discussed later)

$$S_\ell \;=\; \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{k_\ell(p) - ip^{2\ell+1}C_\ell^2}{k_\ell(p) + ip^{2\ell+1}C_\ell^2}$$

S-matrix

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

$$S_\ell \quad = \quad \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{k_\ell(p) - ip^{2\ell+1}C_\ell^2}{k_\ell(p) + ip^{2\ell+1}C_\ell^2}$$

$$k_\ell(p) \quad = \quad \frac{g'_\ell - g_\ell\left(u'_{\ell,<}/u_{\ell,<}\right)}{f'_\ell - f_\ell\left(u'_{\ell,<}/u_{\ell,<}\right)}$$

S-matrix

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$$k_\ell(p) = \frac{g'_\ell - g_\ell\left(u'_{\ell,<}/u_{\ell,<}\right)}{f'_\ell - f_\ell\left(u'_{\ell,<}/u_{\ell,<}\right)}$$

Solutions for $V(r) = V_{long}(r)$

$$f_\ell(r) \simeq \frac{r^{\ell+1}}{(2\ell+1)!!} + \cdots$$

$$g_\ell(r) \simeq \frac{(2\ell-1)!!}{r^\ell} + \cdots + z_\ell(p)\frac{r^{\ell+1}}{(2\ell+1)!!} + \cdots$$

S-matrix

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

$$S_\ell = \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{k_\ell(p) - ip^{2\ell+1}C_\ell^2}{k_\ell(p) + ip^{2\ell+1}C_\ell^2}$$

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Solutions for $V(r) = V_{long}(r)$

$$f_\ell(r) \simeq \frac{r^{\ell+1}}{(2\ell+1)\,!!} + \ldots$$

Leading term

$$g_\ell(r) \simeq \frac{(2\ell-1)\,!!}{r^\ell} + \ldots + z_\ell(p)\frac{r^{\ell+1}}{(2\ell+1)\,!!} + \ldots$$

(basically) leading term

Sizable in some cases

S-matrix

[Blum, Sato, Slatyer (2016)] [Parikh, Sato, Slatyer (2024)]

$$S_\ell \;=\; \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{k_\ell(p) - ip^{2\ell+1}C_\ell^2}{k_\ell(p) + ip^{2\ell+1}C_\ell^2}$$

$$k_\ell(p) \;=\; \frac{g_\ell' - g_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}{f_\ell' - f_\ell\left(u_{\ell,<}'/u_{\ell,<}\right)}$$

Solutions for $V(r) = V_{long}(r)$

$$f_\ell(r) \;\simeq\; \underbrace{\frac{r^{\ell+1}}{(2\ell+1)\,!!}}_{\text{Leading term}} + \;\dots$$

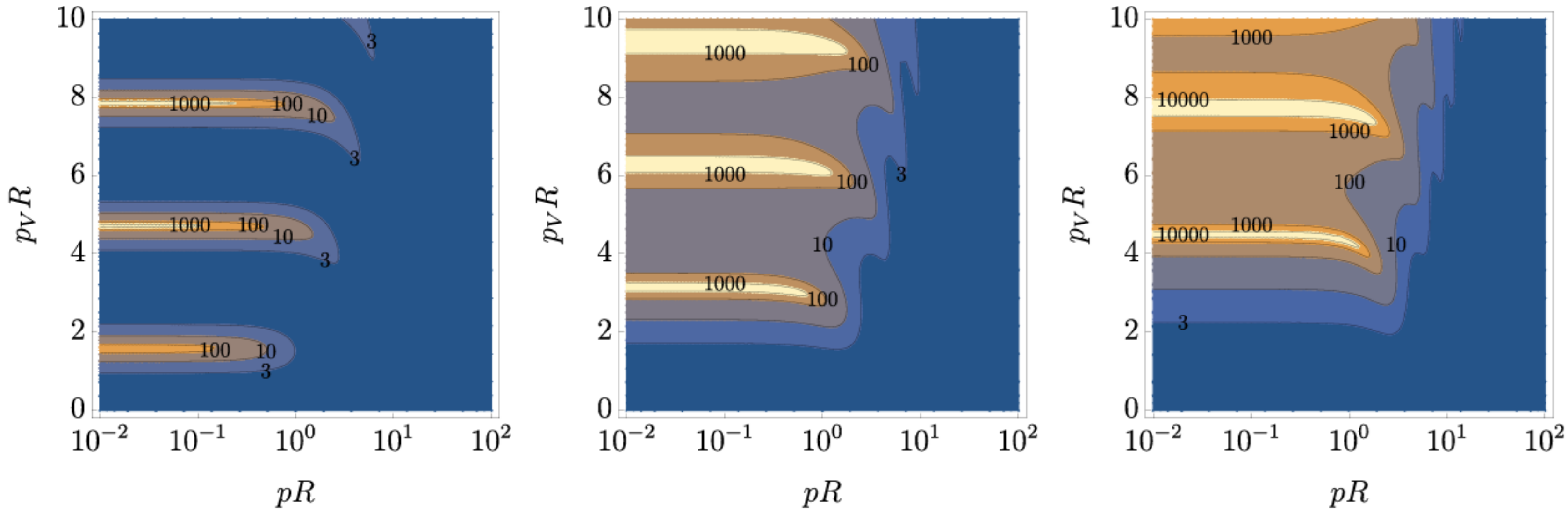
$$g_\ell(r) \;\simeq\; \underbrace{\frac{(2\ell-1)\,!!}{r^\ell}}_{\text{(basically) leading term}} + \;\dots\; + \underbrace{z_\ell(p)\frac{r^{\ell+1}}{(2\ell+1)\,!!}}_{\text{Sizable in some cases}} + \;\dots$$

$$k_\ell(p) \;\simeq\; \underbrace{k_{\ell,0}}_{\substack{\text{(basically)}\\ \text{Leading term}}} + \underbrace{z_\ell(p)}_{\text{Sizable in some cases}}$$

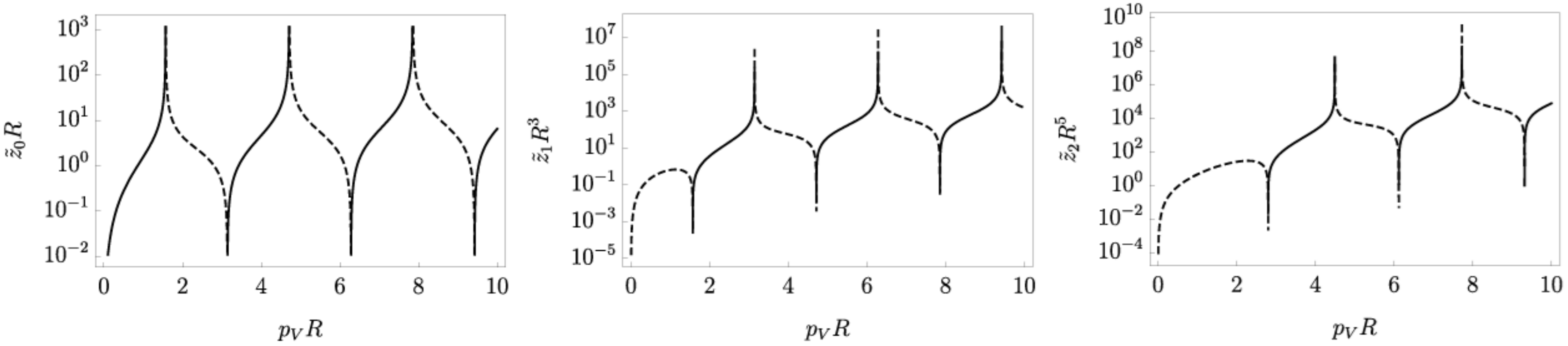
⇒

$$\boxed{S_\ell \;\simeq\; \exp\left(2i\delta_\ell^{(L)}\right) \times \frac{k_{\ell,0} + z_\ell(p) - ip^{2\ell+1}C_\ell^2}{k_{\ell,0} + z_\ell(p) + ip^{2\ell+1}C_\ell^2}}$$

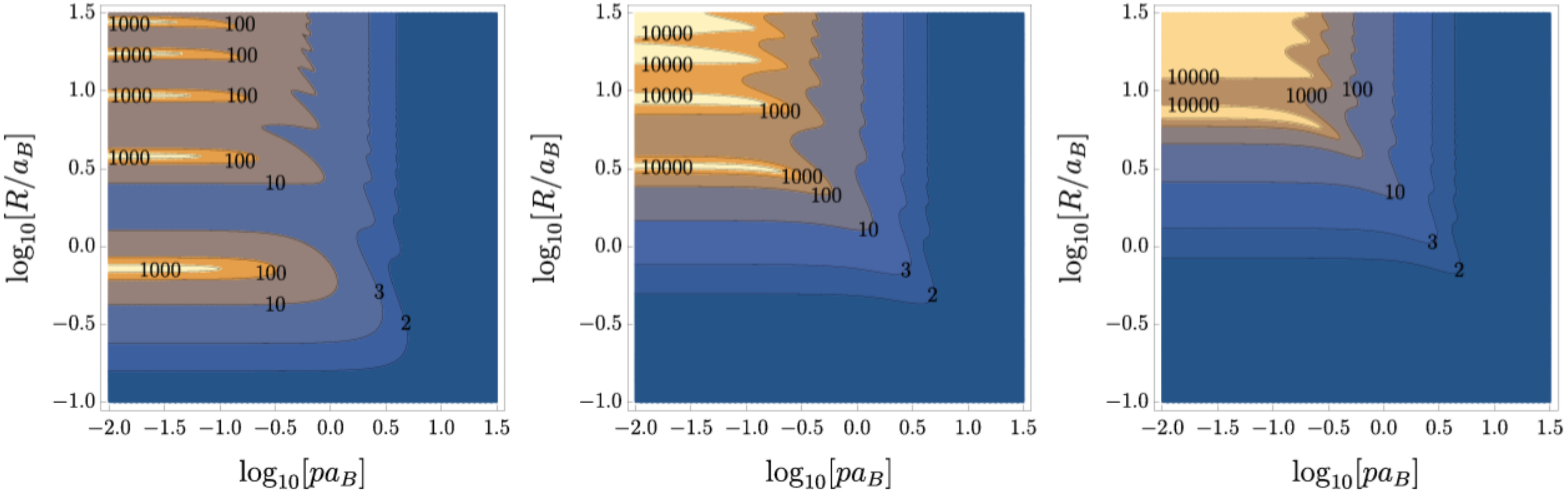
SE for Spherical-well potential



Z function for Spherical-well potential



SE for finite range Coulomb potential



Z function for finite range Coulomb potential