t-channel 特異点に対する解析的処方

横浜国立大学 博士課程１年、理研仁科センター

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Plan

- Introduction
- Analytic prescription
- Application : $L_\mu - L_\tau$ model
  majoron production in the early universe
- Summary

Introduction

Interaction in QFT

model building (construct Lagrangian)

↓

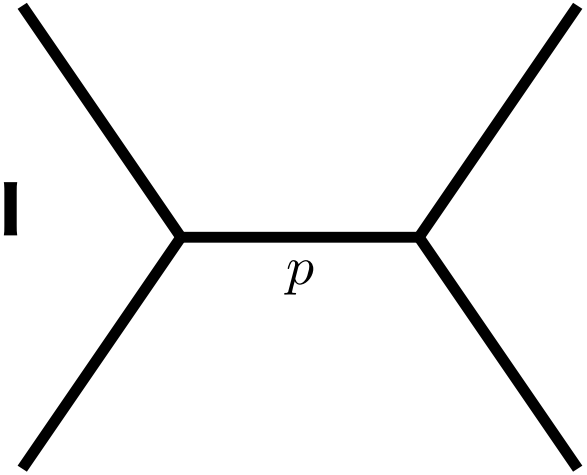
Check the Lagrangian by scattering

↓

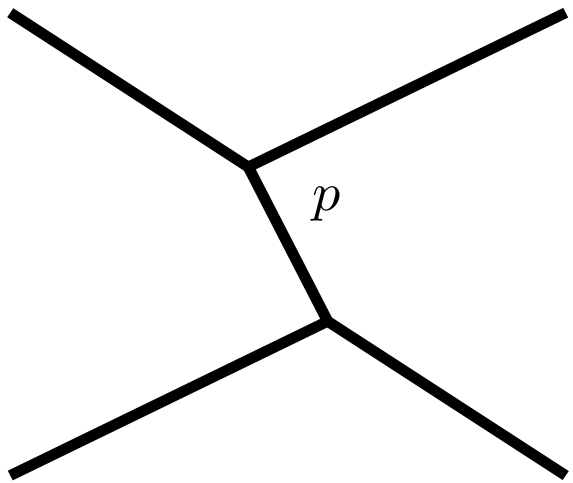
scattering amplitude $|\mathcal{M}|^2$ is important

Divergence in scattering amplitude

s-channel



t-channel



propagator $\dfrac{1}{p^2 - m^2 + i\varepsilon}$

- Off-shell (Virtual)

$$p^2 - m^2 \neq 0 \rightarrow \text{finite}$$

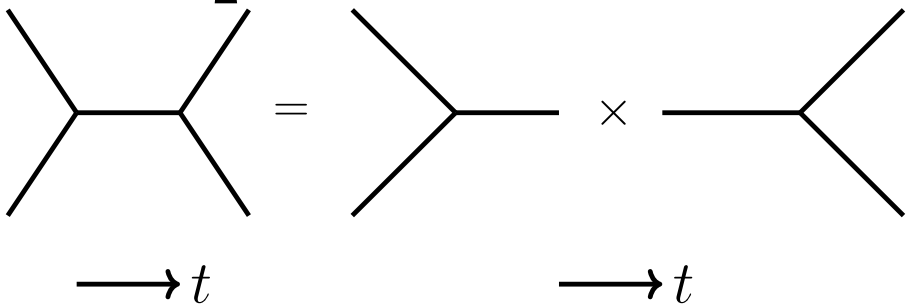
- On-shell

$$p^2 - m^2 = 0 \rightarrow \textbf{divergence}$$

$$\sim \frac{1}{i\varepsilon}$$

Physical interpretation

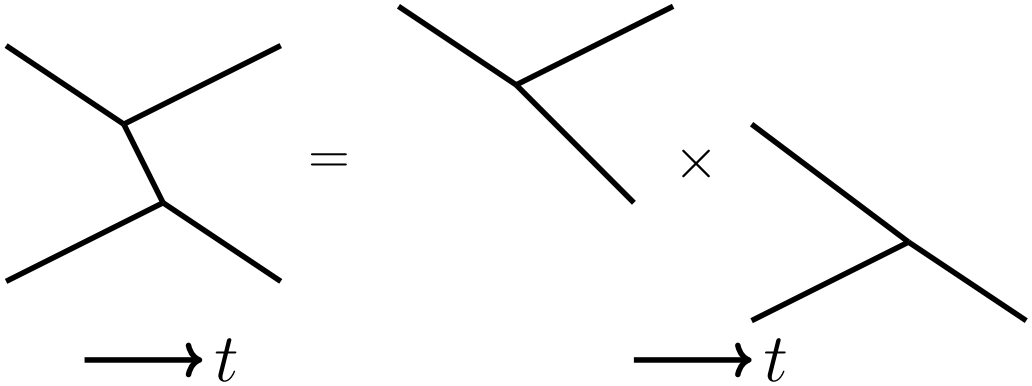
s-channel



Pole of s-channel = inverse decay × decay

➡ Mediator is unstable

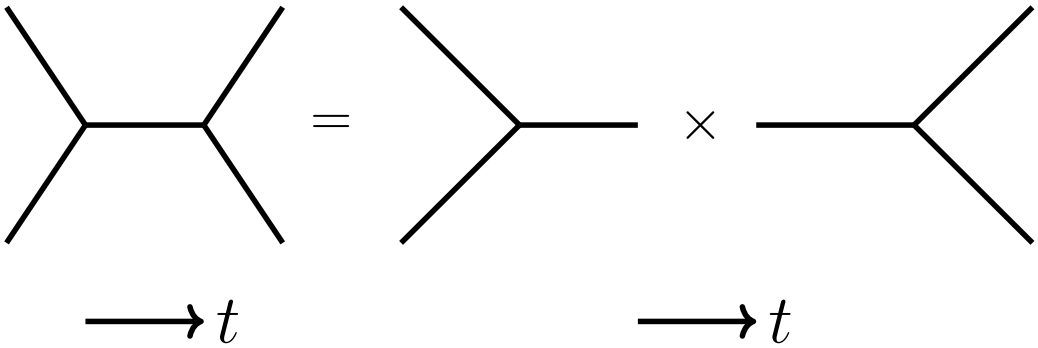
t-channel



Pole of t-channel = decay × inverse decay

➡ Mediator **can** be **stable**

s-channel amplitude

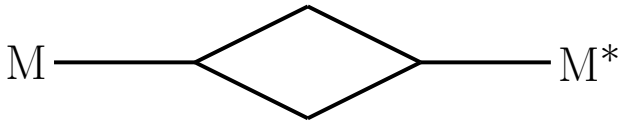


Inverse decay and decay
→Intermediate particles are unstable

Tree propagator

$$\frac{1}{p^2 - m^2 + i\varepsilon}$$

One-loop propagator



$$\frac{1}{p^2 - m^2 + im\Gamma}$$

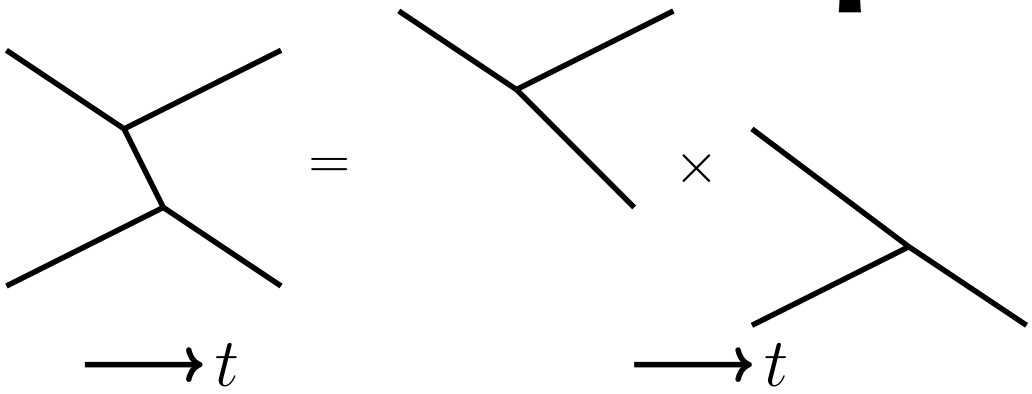
Imaginary part is decay width Γ **Breit–Wigner distribution**

propagator is modified by finite decay width!

Does not diverge at the pole.

Resonance rather than singularity

t-channel amplitude



Decay and Inverse decay
→ **Intermediate particles can be stable.**

Tree propagator

$$\frac{1}{p^2 - m^2 + i\varepsilon}$$

Stable mediator → no decay width!
true singularity ?

How can we regularize the amplitude ?

Previous works & our focus

Previous work

- introducing effective scattering width (collider)

  K. Melnikov, G. L. Kotkin, and V. G. Serbo. PRD, 54:3289–3295, 1996.

- Just to ignore as it(divergent part) is included in inverse decay(cosmology)

  M. Escudero, Dan Hooper, G. Krnjaic, M. Pierre. JHEP, 03:071, 2019

- numerical subtraction of divergent part

  Kento Asai, Tomoya Asano, Joe Sato, and Masaki J. S. Yang. PTEP, 2024(7):073E01, 2024.

Previous works & our focus

Previous work

- Not applicable to cosmology

  K. Melnikov, G. L. Kotkin, and V. G. Serbo. PRD, 54:3289–3295, 1996.

- Is it realistic?

  M. Escudero, Dan Hooper, G. Krnjaic, M. Pierre. JHEP, 03:071, 2019

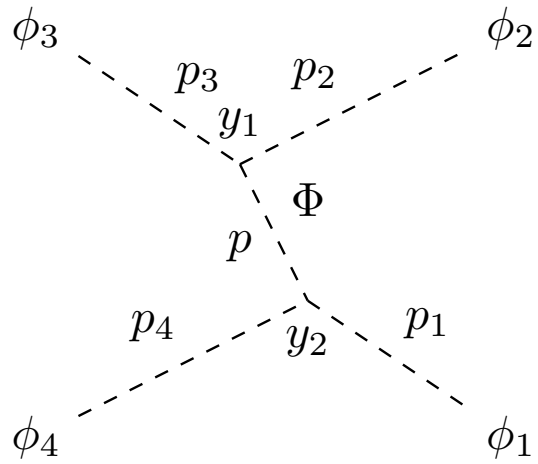
- High computational cost

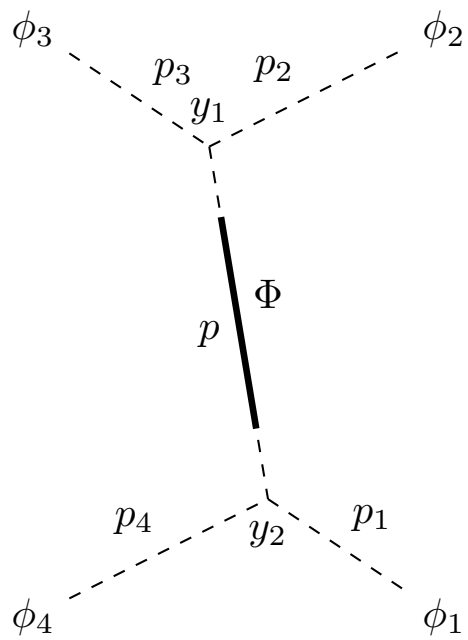
  Kento Asai, Tomoya Asano, Joe Sato, and Masaki J. S. Yang. PTEP, 2024(7):073E01, 2024.

Analytic prescription

Structure of divergence

$$\sigma = \sigma_{\text{off-shell}} + \sigma_{\text{on-shell}}$$





$$\sigma_{\text{on-shell}} = \frac{1}{\varepsilon} E \cdot \Gamma(\phi_3 \to \Phi\phi_1) \cdot \sigma(\phi_4\Phi \to \phi_2)$$

The on-shell contribution is a divergence of $1/\varepsilon$.

Structure of divergence

Calculation of t-channel singularity part

$$\int_a^b \mathrm{d}X |\mathcal{M}^t|^2 \sim \int_a^b \mathrm{d}X \frac{1}{X^2+\epsilon^2} \qquad X = (p_3 - p_2)^2 - m^2$$

$$\int_{a/\epsilon}^{b/\epsilon} \mathrm{d}(\epsilon \bar{X}) \frac{1}{\epsilon^2} \frac{1}{\bar{X}^2+1} = \frac{1}{\epsilon}\left(\arctan\left(\frac{b}{\epsilon}\right) - \arctan\left(\frac{a}{\epsilon}\right)\right)$$

$$= \frac{1}{a} - \frac{1}{b} + \mathcal{O}(\epsilon) \qquad \text{If} \quad ab > 0 \quad \textbf{No on-shell terms}$$

$$= -\boxed{\frac{\pi}{\epsilon}} + \frac{1}{a} - \frac{1}{b} + \mathcal{O}(\epsilon) \qquad \text{If} \quad a < 0 < b \quad \textbf{on-shell terms}$$

on-shell term that should be subtracted

To avoid double counting : already included by **Decay & Inverse Decay**

Application

Majoron production ($L_\mu - L_\tau$ model)

- Interaction of Z'

e.g. Phys.Rev.D 100 (2019) 9, 095012

$$\mathcal{L}_{Z'} = -\frac{1}{4} Z'^{\rho\sigma} Z'_{\rho\sigma} + \frac{1}{2} m_Z^2 Z'^{\rho} Z'_{\rho} + g_{\mu-\tau} Z'_{\rho} J^{\rho}_{\mu-\tau}$$

$$Z'_{\rho\sigma} \equiv \partial_\rho Z'_\sigma - \partial_\sigma Z'_\rho$$

$$J^{\rho}_{\mu-\tau} \equiv \bar{\mu}\gamma^\rho \mu + \bar{\upsilon}_\mu \gamma^\rho P_L \upsilon_\mu - \bar{\tau}\gamma^\rho \tau - \bar{\upsilon}_\tau \gamma^\rho P_L \upsilon_\tau$$

- Interaction of majoron ϕ (pNGB)

$$\mathcal{L}_\phi = h_{\alpha\beta} \bar{\nu}_{L,\alpha} \nu^c_{L,\beta} \phi + \mathrm{H.c.}$$

decay & inverse decay

$$Z' \leftrightarrow \nu_{\alpha'} \bar{\nu}_{\alpha'} \quad (\alpha' = \mu, \tau)$$

$$\phi \leftrightarrow \nu_\alpha \nu_\beta$$

scattering

$$Z' \nu_\alpha \leftrightarrow \phi \bar{\nu}_\beta (\nu \leftrightarrow \bar{\nu})$$

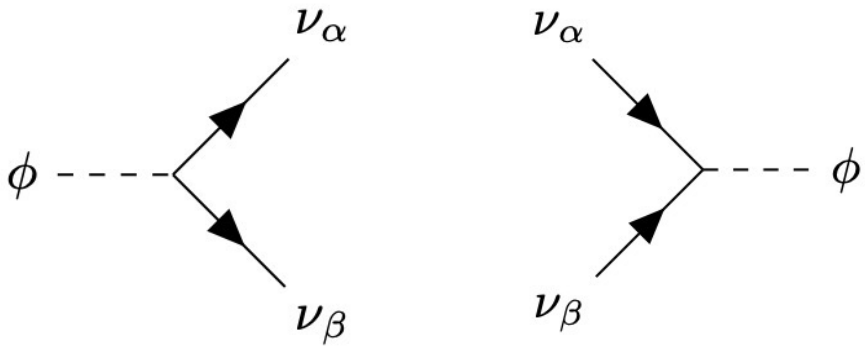
$$Z' \phi \leftrightarrow \nu_\alpha \nu_\beta (\nu \leftrightarrow \bar{\nu})$$

Majoron production in the early universe

Majoron (pNGB of lepton number) may have been created in the early universe.
→ The effective d.o.f. of neutrinos and cosmic expansion rate may change.

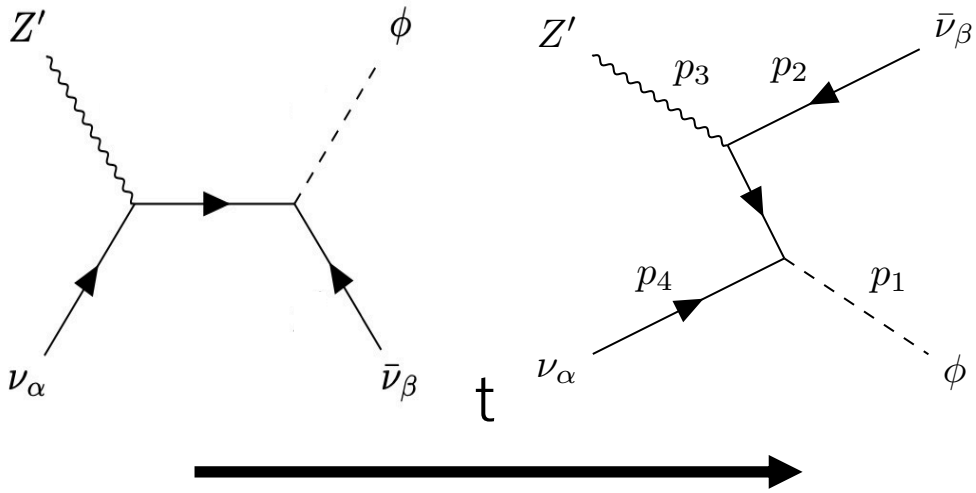
Let's discuss $U(1)_{L_\mu - L_\tau}$ model

(inverse) decay



M. Escudero, Dan Hooper, G. Krnjaic, M. Pierre.
JHEP, 03:071, 2019

$Z' - \phi$ scattering (New!)



t-channel singularity in Majoron production

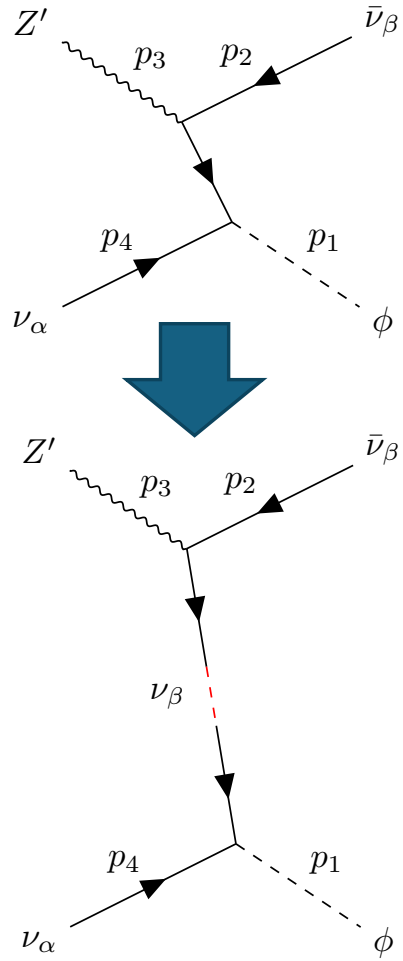
$Z'\nu_\alpha \leftrightarrow \phi\bar{\nu}_\beta$ in t-channel diagram

Intermediate particle can be on-shell

→ Amplitude. $|\mathcal{M}|^2$ **is divergent**

$$\left|\mathcal{M}^t_{Z'v\leftrightarrow\phi\bar{v}}\right|^2 \propto \frac{1}{(p_3-p_2)^4+\varepsilon^2}$$

Regularized by our new method



Net production rate

Generation of Majoron ϕ in the early universe
is evaluated by Boltzmann eq.

$$\frac{\mathrm{d}n_a}{\mathrm{d}t} = -3Hn_a + \frac{\delta n_a}{\delta t}$$

thermal averaged event rates of scattering and inverse decay

$$\langle \Gamma_{\mathrm{S}} \rangle = \frac{1}{n_{\phi,\mathrm{eq}}} \left.\frac{\delta n_\phi}{\delta t}\right|_{\mathrm{S}}, \quad \langle \Gamma_{\mathrm{ID}} \rangle = \frac{1}{n_{\phi,\mathrm{eq}}} \left.\frac{\delta n_\phi}{\delta t}\right|_{\mathrm{ID}}, \quad \mathrm{d}\Pi_a = \frac{\mathrm{d}^3\boldsymbol{p}_a}{(2\pi)^3 2E_a}$$

$$\left.\frac{\delta n_a}{\delta t}\right|_{\mathrm{ID}} = -\frac{1}{S_{ij}} \int \mathrm{d}\Pi_a \int \mathrm{d}\Pi_i\, \mathrm{d}\Pi_j (2\pi)^4 \delta^{(4)}\left(p_a - p_i - p_j\right) \sum_{\mathrm{spins}} \left|\mathcal{M}_{a\leftrightarrow ij}\right|^2 \Lambda_{\mathrm{ID}} \left\{f_i^{\mathrm{MB}}\right\}$$

$$\left.\frac{\delta n_a}{\delta t}\right|_{\mathrm{S}} = -\int \mathrm{d}\Pi_a\, \mathrm{d}\Pi_b\, \mathrm{d}\Pi_c\, \mathrm{d}\Pi_d \Lambda_S \left\{f_i^{\mathrm{MB}}\right\} (2\pi)^4 \delta^{(4)}\left(\boldsymbol{p}_a + \boldsymbol{p}_b - \boldsymbol{p}_c + \boldsymbol{p}_d\right) \sum_{\mathrm{spins}} \left|\mathcal{M}_{ab\leftrightarrow cd}\right|^2$$

Calculation procedure

$$\left.\frac{\delta n_a}{\delta t}\right|_{\mathrm{S}} = -\int \mathrm{d}\Pi_a\ \mathrm{d}\Pi_b\ \mathrm{d}\Pi_c\ \mathrm{d}\Pi_d \Lambda_S \left\{f_i^{\mathrm{MB}}\right\} (2\pi)^4 \delta^{(4)}\left(\boldsymbol{p}_a + \boldsymbol{p}_b - \boldsymbol{p}_c + \boldsymbol{p}_d\right) \sum_{\text{spins}} \left|\mathcal{M}_{ab\leftrightarrow cd}\right|^2$$

$$I = \int d\Pi_3 d\Pi_4 (2\pi)^4 \delta^{(4)}\left(p_1 + p_2 - p_3 - p_4\right) \sum \left|\mathcal{M}_{\nu_\alpha \phi \leftrightarrow Z' \nu_\beta}\right|^2 \qquad \mathrm{d}\Pi_a = \frac{\mathrm{d}^3 \boldsymbol{p}_a}{(2\pi)^3 2E_a}$$

$$I^t = \frac{16 g_\beta^2 \left|h_{\alpha\beta}\right|^2}{2\pi\sqrt{s}\mathrm{p}_2} \theta(\sqrt{s} - m_{Z'}) \left[\underline{\frac{1}{\varepsilon}\tan^{-1}\frac{x}{\varepsilon} m_{Z'}^2 (s - 2m_{Z'}^2 + 2\mathrm{p}_2\sqrt{s})}\right.$$

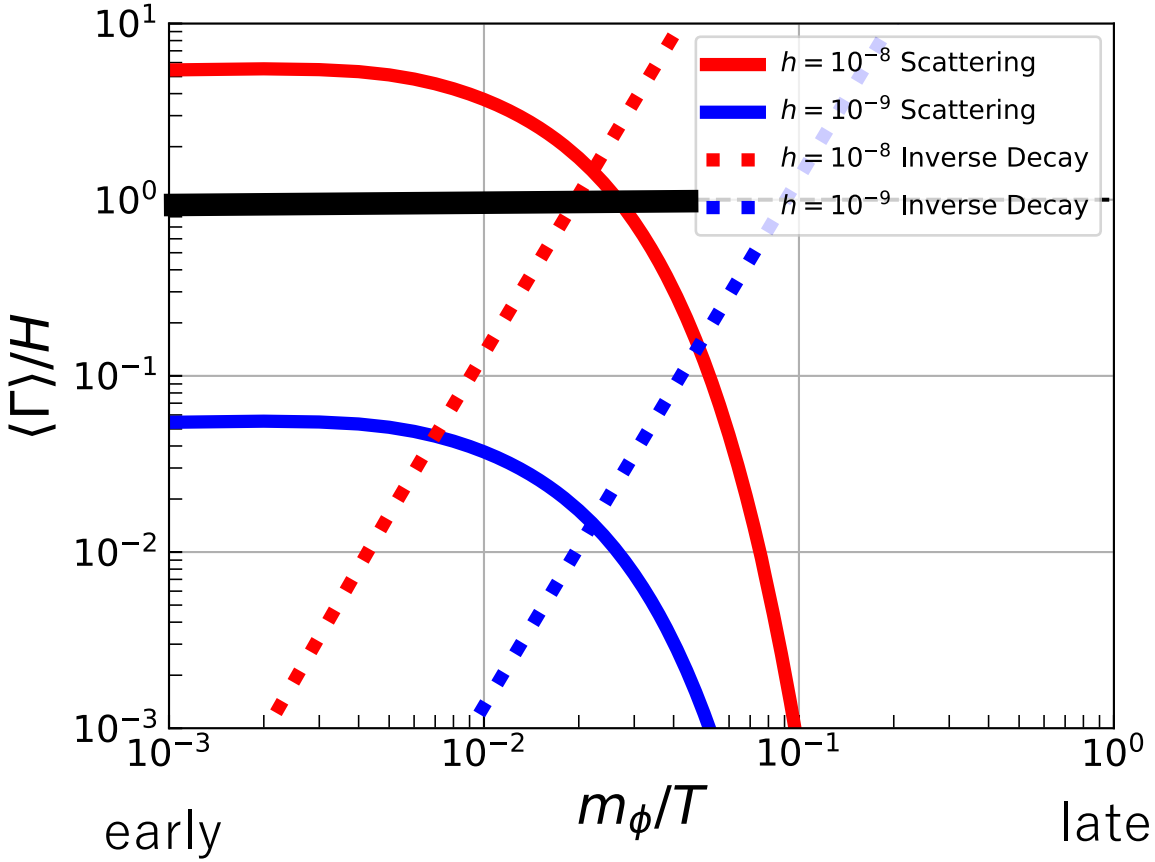
By on-shell subtraction

$$\left. -\frac{1}{2}\log(x^2 + \varepsilon^2)(s - 2m_{Z'}^2) + \left(x - \varepsilon\tan^{-1}\frac{x}{\varepsilon}\right)\frac{1}{2m_{Z'}^2}(m_{Z'}^2 - 2\mathrm{p}_2\sqrt{s} - x)\right]$$

$$I_0 = \frac{16 g_\beta^2 \left|h_{\alpha\beta}\right|^2}{2\pi\sqrt{s}\mathrm{p}_2} \theta(\sqrt{s} - m_{Z'}) \left[-\frac{1}{x} m_{Z'}^2 (s - 2m_{Z'}^2 + 2\mathrm{p}_2\sqrt{s})\right.$$

$$\left. -\frac{1}{2}\log(x^2 + \varepsilon^2)(s - 2m_{Z'}^2) + \frac{x}{2m_{Z'}^2}(m_{Z'}^2 - 2\mathrm{p}_2\sqrt{s} - x)\right]$$

Production efficiency



$$\langle\Gamma_{\rm S}\rangle = \frac{1}{n_{\phi,\rm eq}}\frac{\delta n_\phi}{\delta t}\bigg|_{\rm S}, \quad \langle\Gamma_{\rm ID}\rangle = \frac{1}{n_{\phi,\rm eq}}\frac{\delta n_\phi}{\delta t}\bigg|_{\rm ID},$$

Gamov Criterion

$$\langle\Gamma_{\rm S,ID}\rangle/H > 1$$

For Majoron
to be thermalized !

Majoron ϕ can have been made a lot in the early universe?
→previous works could underestimate Majoron effects

Summary

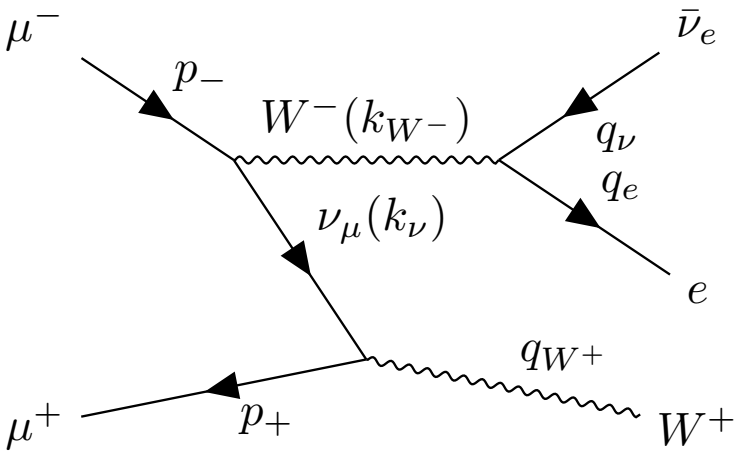
- Generally, divergence occurs when intermediate particles become on-shell.
  - s-channel: regulated by the decay width.
  - t-channel: divergence can appear.

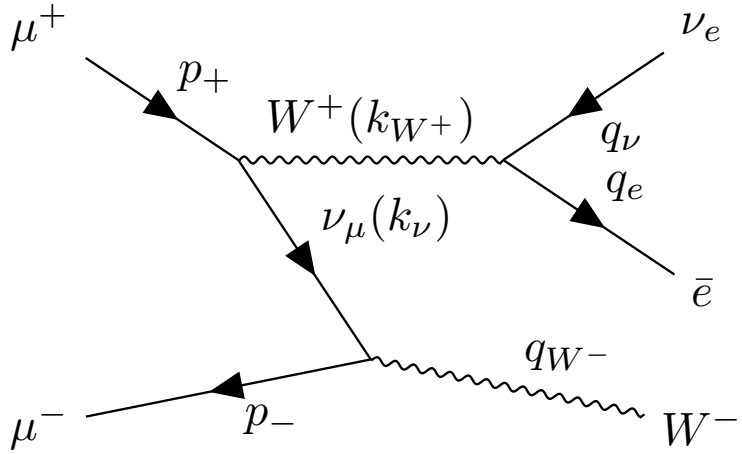
- In this study, we proposed an analytical method to eliminate the t-channel singularity.
  - divergent part $\sim 1/\varepsilon$ ： subtracted
    real particle contribution = decay & Inverse decay
  - scattering part ：remains

Future Work

- We plan to perform parameter scan for cosmological systems.
  - Constraints on new particles such as Majoron can be more stringent.

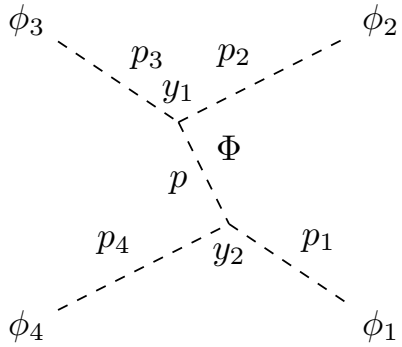
- We apply our method to $\mu^+\mu^- \to \bar{\nu} e^- W^+$



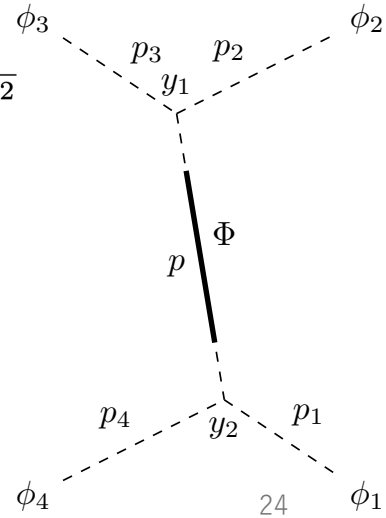


Back up

Structure of divergence



$$
\begin{aligned}
\sigma &= \frac{y_1^2 y_2^2}{2E_3 2E_4} \int \mathrm{d}\Pi_1 \mathrm{d}\Pi_2 (2\pi)^4 \mathrm{d}^4 p\, \delta^{(4)}\left(p_3 - p - p_2\right) \delta^{(4)}\left(p + p_4 - p_1\right) \left| \frac{i}{p^2 - m^2 + i\varepsilon} \right|^2 \\
&= \frac{y_1^2 y_2^2}{2E_3 2E_4} \int \mathrm{d}\Pi_1 \mathrm{d}\Pi_2 (2\pi)^4 \,\mathrm{p.v.} \Big|_{p^2 \neq m^2} \left(\int \mathrm{d}^4 p\, \delta^{(4)}\left(p_3 - p - p_2\right) \delta^{(4)}\left(p + p_4 - p_1\right) \frac{1}{(p^2 - m^2)^2} \right) \\
&\quad + \frac{y_1^2 y_2^2}{2E_3 2E_4} \lim_{\eta \to 0^+} \int \mathrm{d}\Pi_1 \mathrm{d}\Pi_2 (2\pi)^4 \int_{m^2 - \eta < p^2 < m^2 + \eta} \mathrm{d}^4 p\, \delta^{(4)}\left(p_3 - p - p_2\right) \delta^{(4)}\left(p + p_4 - p_1\right) \frac{1}{(p^2 - m^2)^2 + \varepsilon^2} \\
&\equiv \sigma_{\text{off-shell}} + \sigma_{\text{on-shell}} \,,
\end{aligned}
$$



$$
\begin{aligned}
\sigma_{\text{on-shell}} &= \frac{y_1^2 y_2^2}{2E_3 2E_4} \int \mathrm{d}\Pi_1 \mathrm{d}\Pi_2 (2\pi)^4 \int_{m^2 - \eta < p^2 < m^2 + \eta} \mathrm{d}^4 p\, \delta^{(4)}\left(p_3 - p - p_2\right) \delta^{(4)}\left(p + p_4 - p_1\right) \frac{\pi}{\varepsilon} \frac{1}{\pi} \frac{\varepsilon}{(p^2 - m^2)^2 + \varepsilon^2} \\
&= \frac{y_1^2 y_2^2}{2E_3 2E_4} \int \mathrm{d}\Pi_1 \mathrm{d}\Pi_2 (2\pi)^4 \int_{m^2 - \eta < p^2 < m^2 + \eta} \mathrm{d}^4 p\, \delta^{(4)}\left(p_3 - p - p_2\right) \delta^{(4)}\left(p + p_4 - p_1\right) \frac{\pi}{\varepsilon} \delta(p^2 - m^2) \\
&= \frac{\pi}{\varepsilon} \frac{y_1^2 y_2^2}{2E_3 2E_4} \int \mathrm{d}\Pi_1 \mathrm{d}\Pi_2 \frac{\mathrm{d}^3 \boldsymbol{p}}{2E} (2\pi)^4 \delta^{(4)}\left(p_3 - p - p_2\right) \delta^{(4)}\left(p + p_4 - p_1\right) \\
&= \frac{1}{\varepsilon} E \cdot \frac{y_1^2}{2E_3} \int \mathrm{d}\Pi_2 \frac{\mathrm{d}^3 \boldsymbol{p}}{2E (2\pi)^3} (2\pi)^4 \delta^{(4)}\left(p_3 - p - p_2\right) \cdot \frac{y_2^2}{2E_4 2E} \int \mathrm{d}\Pi_1 (2\pi)^4 \delta^{(4)}\left(p + p_4 - p_1\right) \\
&= \frac{1}{\varepsilon} E \cdot \Gamma(\phi_3 \to \Phi \phi_1) \cdot \sigma(\phi_4 \Phi \to \phi_2) \,.
\end{aligned}
$$