Hidden Maxwell's Demon in Cosmological First Order Phase Transition

Juntaro Wada
@PPP2025

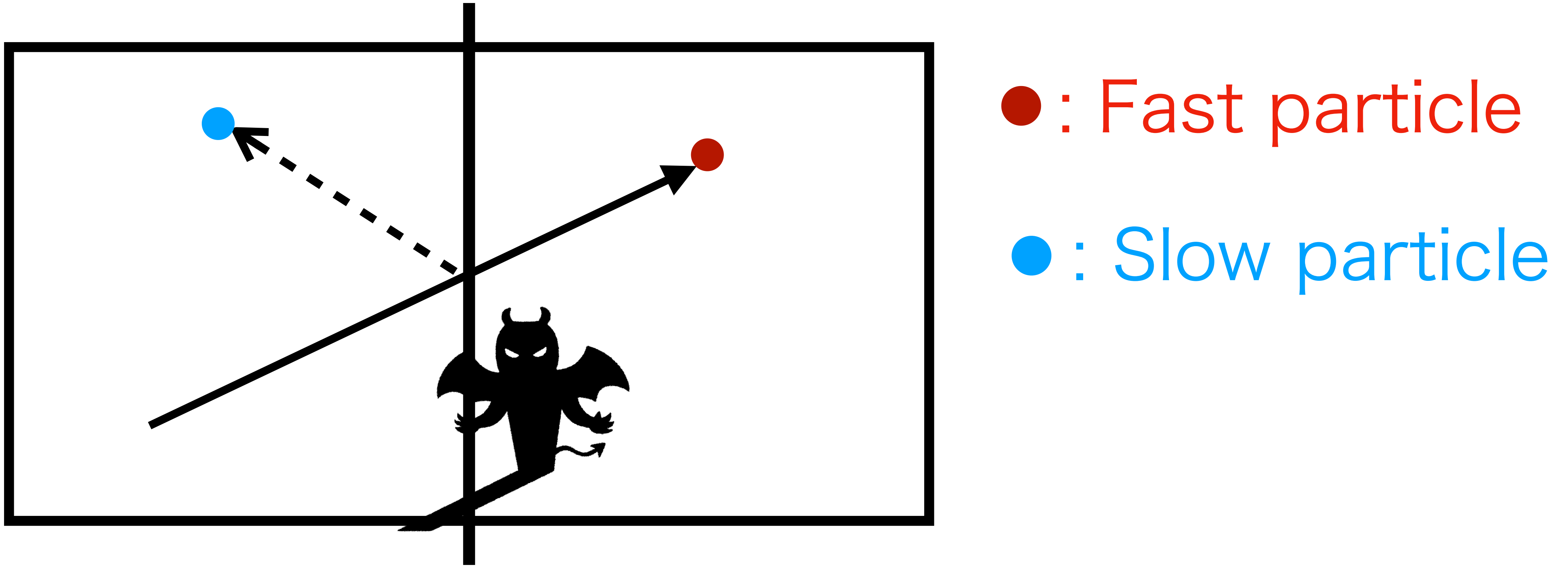
Faculty of Science
Graduate School of Science
TOHOKU UNIVERSITY

(On going work)

Special thanks to Kazuki Enomoto

Introduction & Motivation

- Maxwell's demon J. C. Maxwell, Theory of Heat. Appleton, London, 1871.



Demon performs following operations w/o work:

- Measurement
- Feedback based on the measurements

- These operations can leads Entropy reduction

Pictures are taken from https://www.irasutoya.com/

Introduction & Motivation

ARTICLE

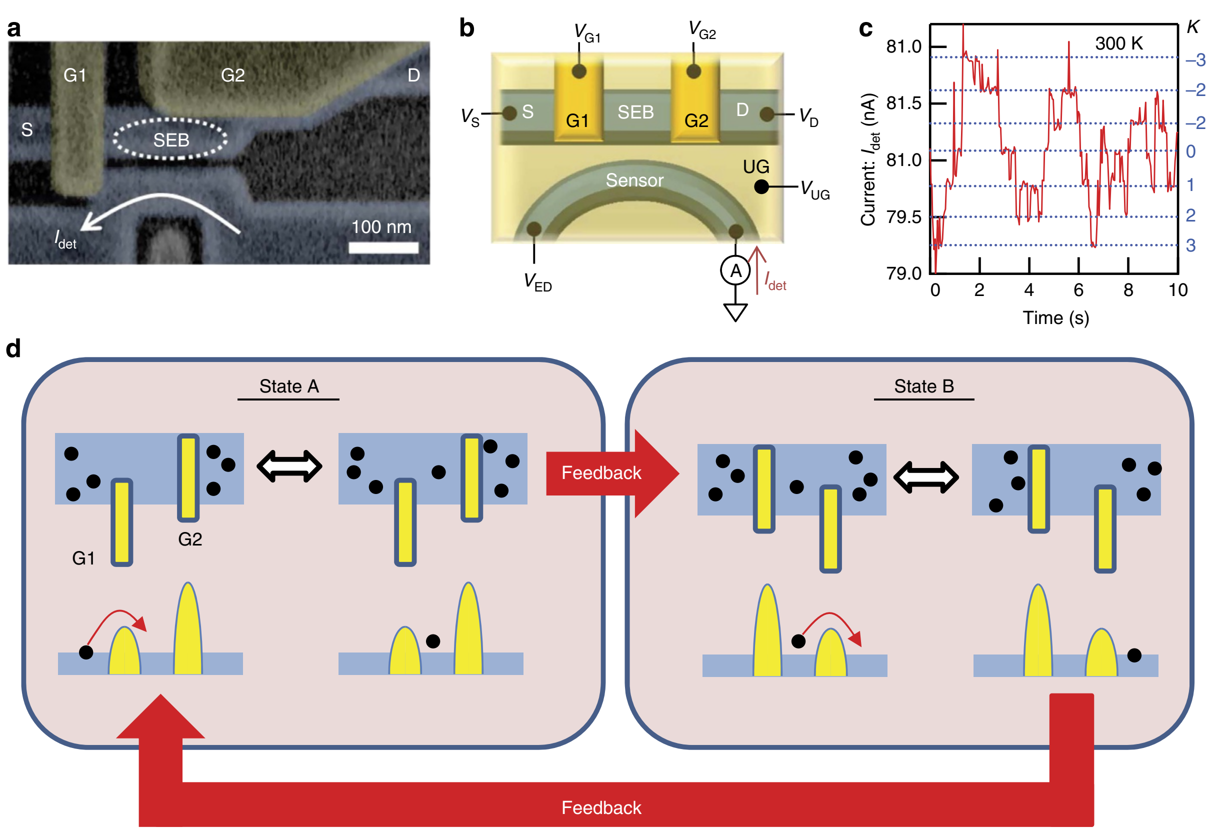
Received 15 Jun 2016 | Accepted 20 Mar 2017 | Published 16 May 2017

DOI: 10.1038/ncomms15301

OPEN

Power generator driven by Maxwell's demon

Kensaku Chida[1], Samarth Desai[1], Katsuhiko Nishiguchi[1] & Akira Fujiwara[1]



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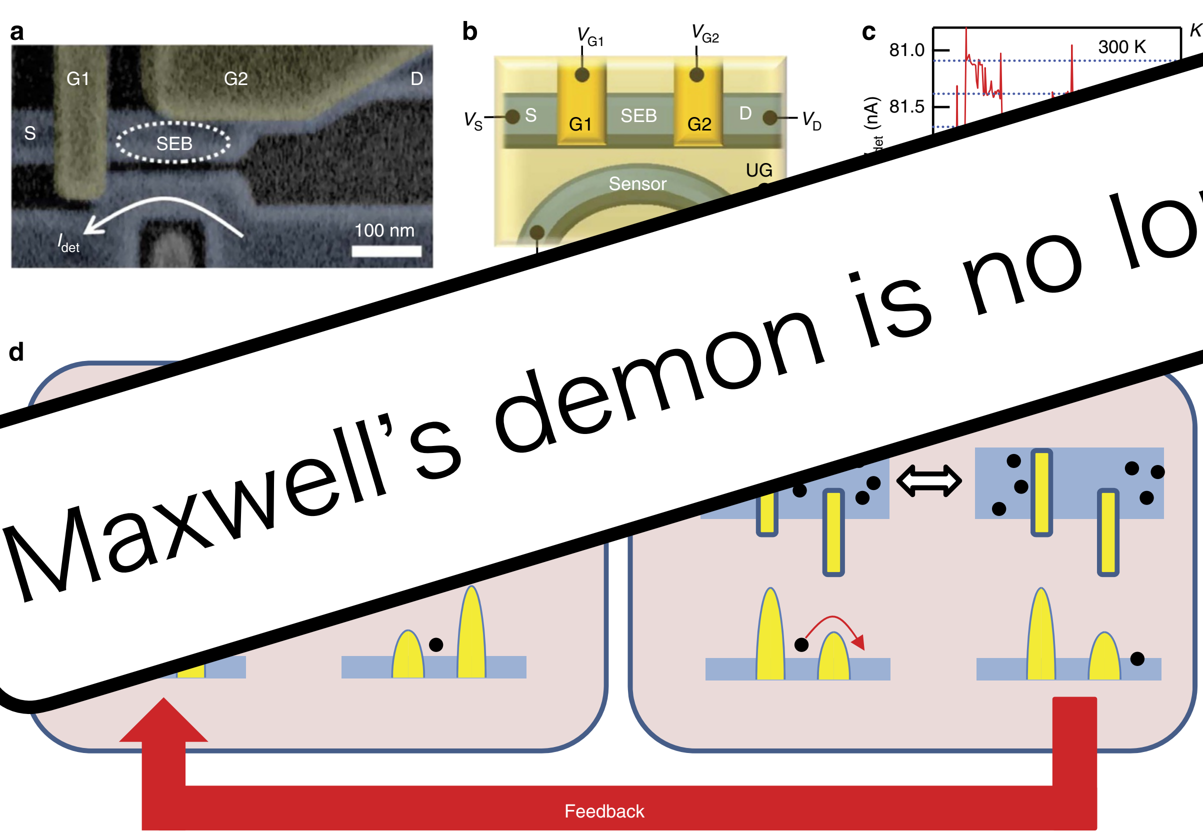


Maxwell's demon is no longer demon !!

Introduction & Motivation

- Maxwell's demon J. C. Maxwell, Theory of Heat. Appleton, London, 1871.

Is there any similar object in HEP literature?

- Some example of FOPT behaves similar.

Demon performs following operations w/o work:

- Measurement
- Feedback based on the measurements

- These operations can leads Entropy reduction

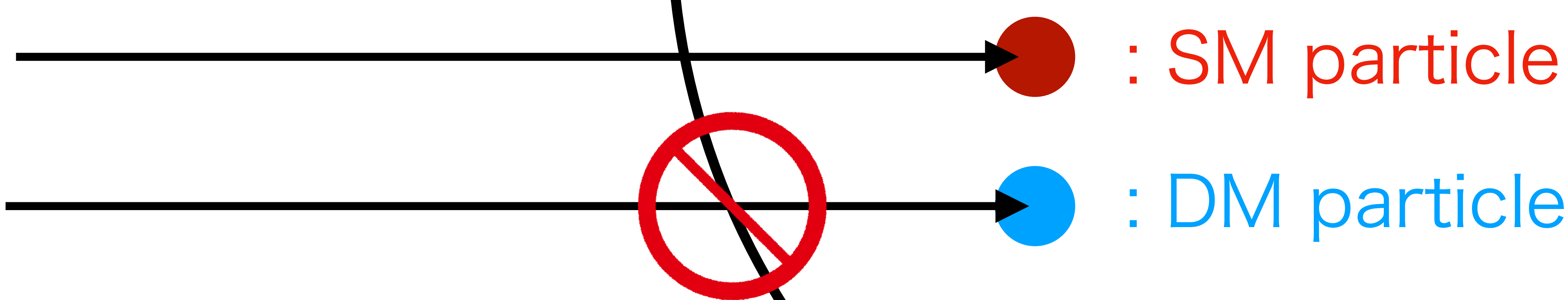
Introduction & Motivation

- Filtered DM scenario

M. J. Baker, J. Kopp, and A. J. Long, Phys. Rev. Lett. 125, 151102 (2020)
D. Chway, T. H. Jung, and C. S. Shin, Phys. Rev. D 101, 095019 (2020)

$$\langle \phi \rangle = 0$$
$$m_{\mathrm{DM}}^{\mathrm{out}} \sim T_n$$

$$\langle \phi \rangle \neq 0$$
$$m_{\mathrm{DM}}^{\mathrm{in}} \simeq M \gg T_n$$



$$\mathcal{L}_{\mathrm{int}} \ni -y_\chi \phi \chi \bar{\chi}$$

Introduction & Motivation

- Filtered DM scenario

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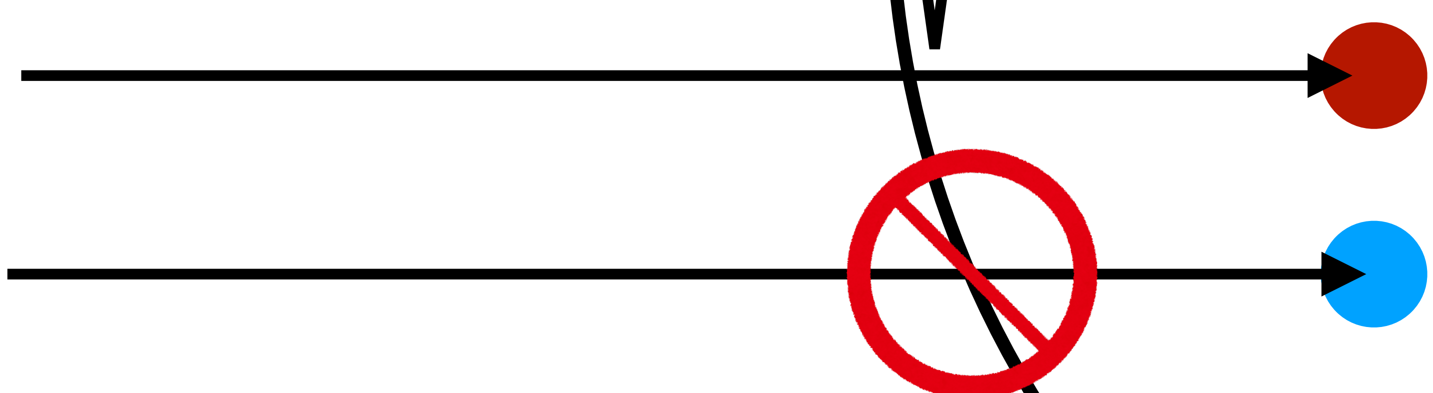
- Measurement
- Feedback



Motivation: Any impact for Hydrodynamics?

Pictures are taken from https://www.irasutoya.com/

Introduction & Motivation

Questions

- What kind of similarities exist?
- What is ``Entropy'' in FOPT? & How it would be modified?
- How is hydrodynamics affected?

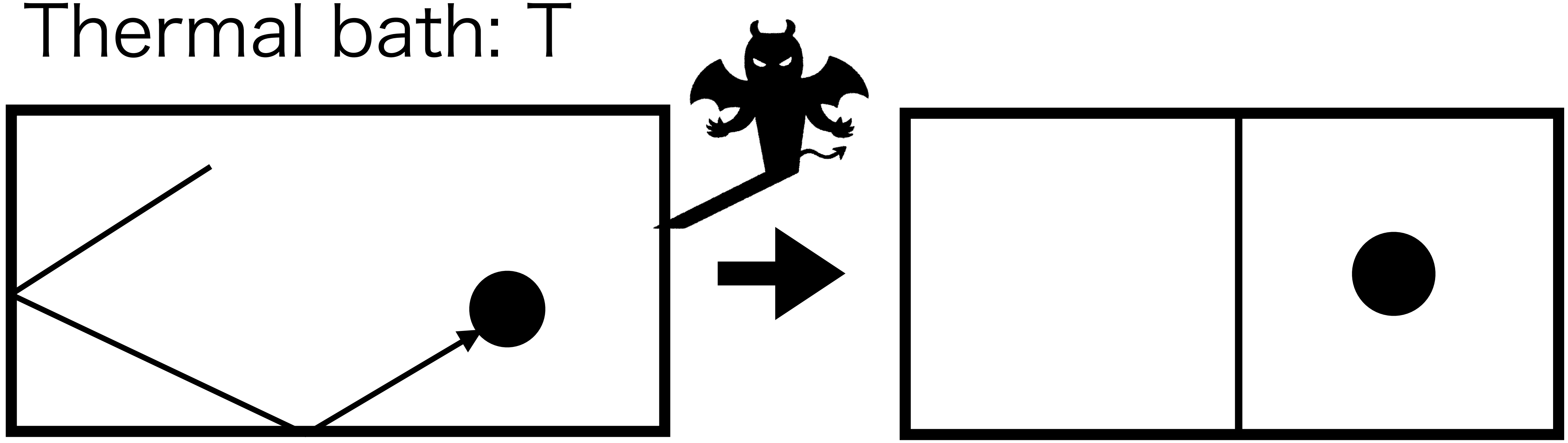
Outline

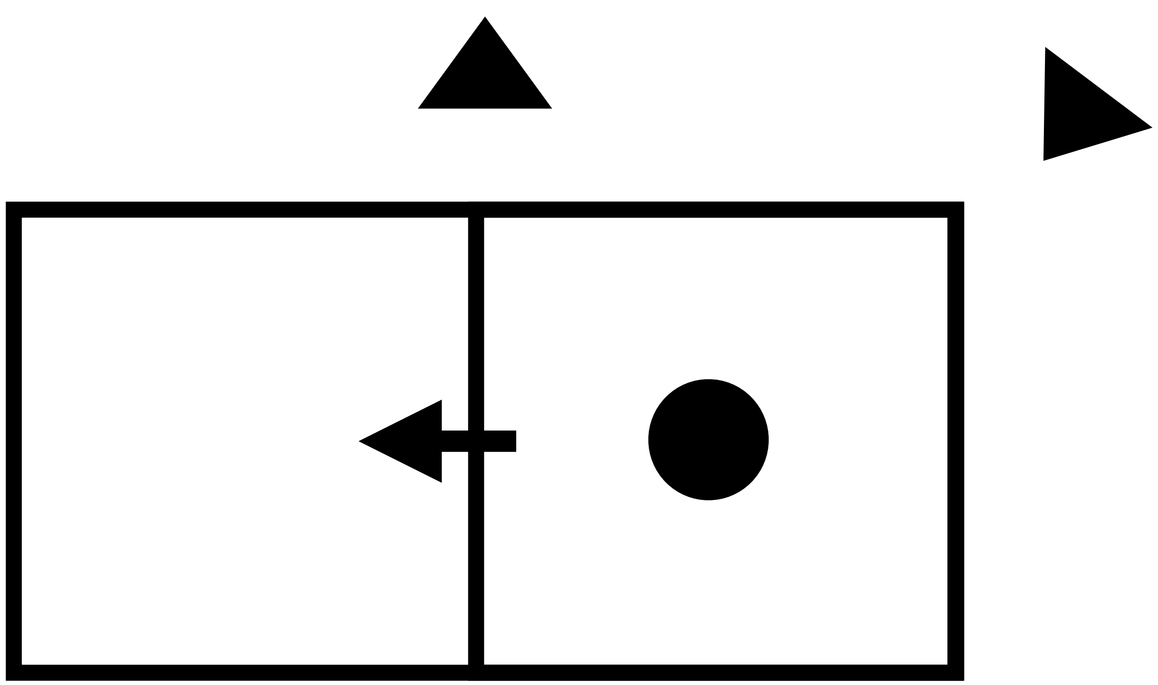
- ✓ Introduction & Motivation
- Correspondence with Szilard engine
- Hydrodynamics & Entropy current for FOPT
- Analysis of hydrodynamics
- Summary

Szilard engine

L. Szilard, Z. Phys. 53, 840 (1929).

Thermal bath: T





For each cycle,

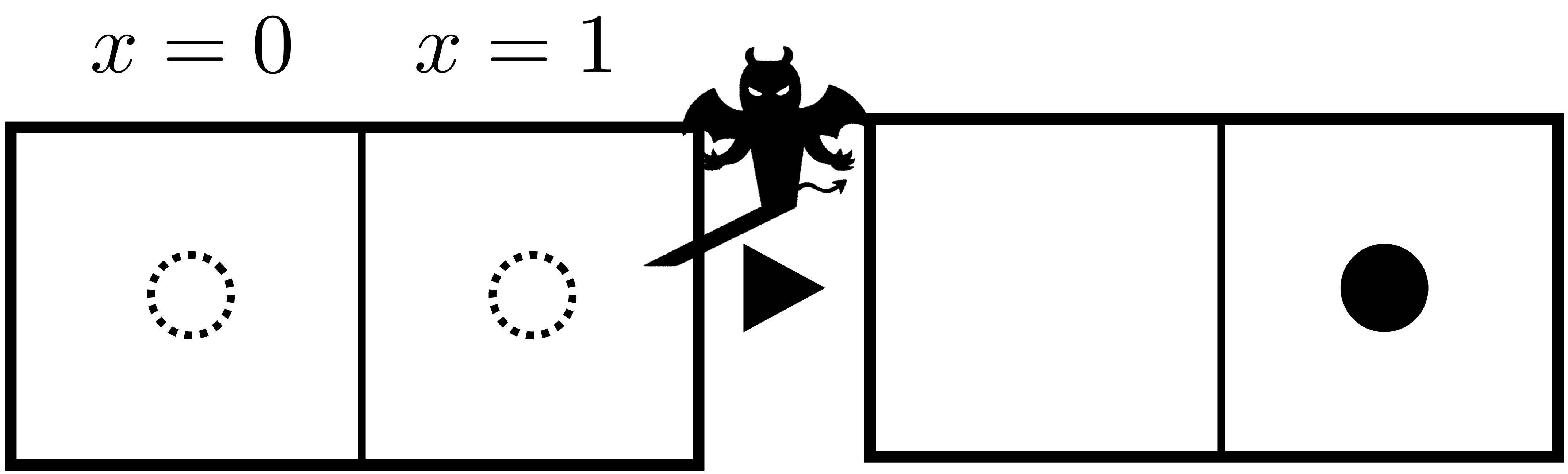
$$W_{ext} = T \ln 2$$

-> 2nd law violation

Szilard engine

L. Szilard, Z. Phys. 53, 840 (1929).

	Initial state (X)	State after feedback (X')
"0" (Left)	$P(X=0)=1/2$	$P(X'=0)=0$
"1" (Right)	$P(X=1)=1/2$	$P(X'=1)=1$



$$I(X:Y)=\ln 2 \qquad\qquad I(X':X)=0$$

Correspondence with Szilard engine

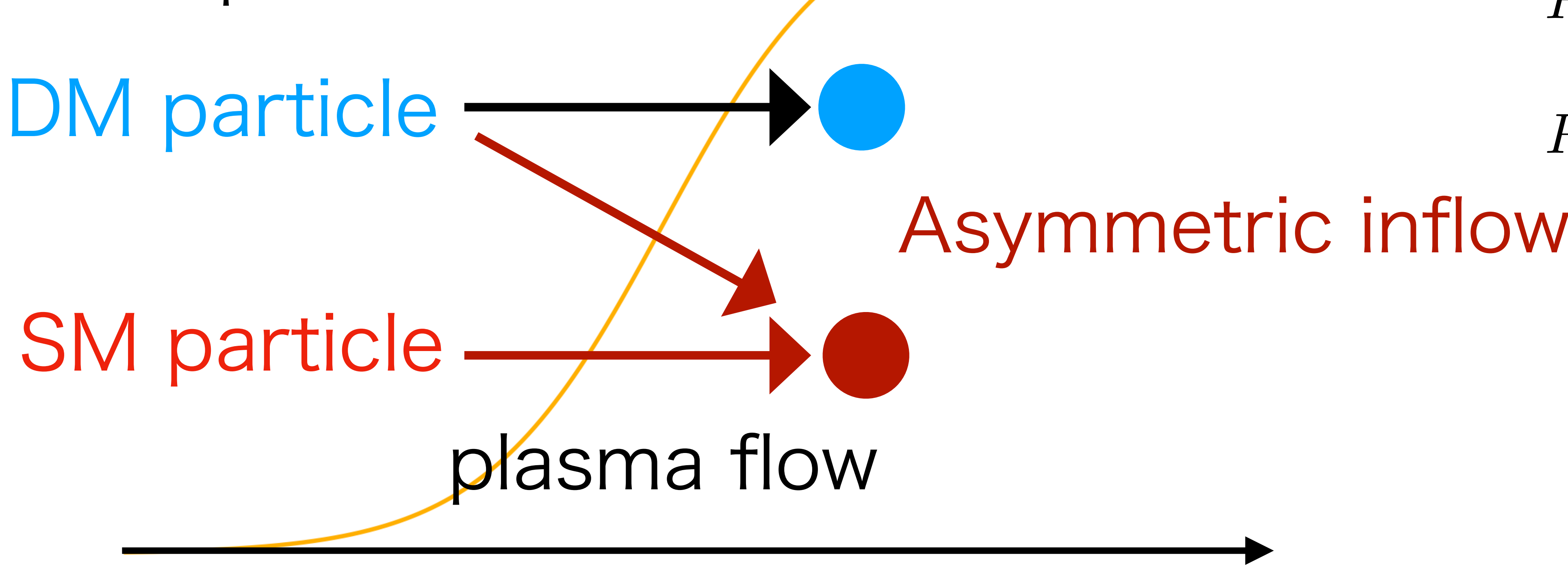
	Outside the wall (X)	Inside the wall (X')
"0" (Dark sector)	$P(X=0)=P_D$	$P(X'=0)=QP_D$
"1" (Standard Model)	$P(X=1)=P_{SM}$	$P(X'=1)=P_{SM}+(1-Q)P_D$

$Q \ll 1$

$$P_S = \frac{g_S}{g_S+g_D}$$

$$P_D = \frac{g_D}{g_S+g_D}$$

Correspondence becomes clear when…



Correspondence with Szilard engine

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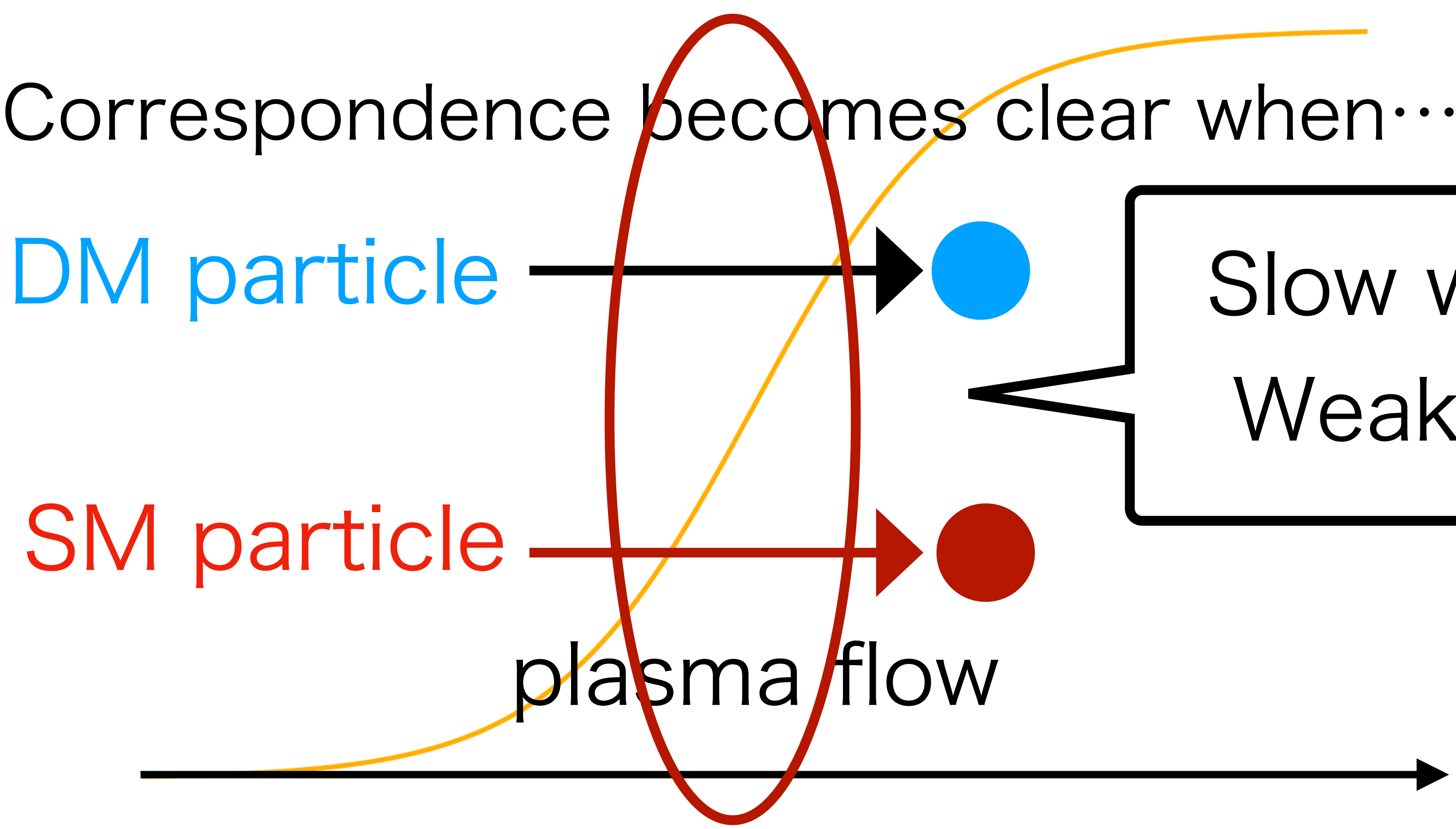

Outline

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Hydrodynamics for FOPT

Conventional approach (Review)

See E.g. M. B. Hindmarsh et.al., SciPost Phys. Lect. Notes 24 (2021) 1 for review

- Energy-momentum conservation (1st law)

$$\nabla_\mu (T_\phi^{\mu\nu} + T_f^{\mu\nu}) \simeq \partial_\mu (T_\phi^{\mu\nu} + T_f^{\mu\nu}) = 0$$

where

$$T_\phi^{\mu\nu} = (\partial^\mu \phi)(\partial^\nu \varphi) - g^{\mu\nu} \left(\frac{1}{2} (\partial\phi)^2 - V_0(\phi) \right)$$

$$T_f^{\mu\nu} = (\rho_f + p_f) u^\mu u^\nu - p_f g^{\mu\nu}$$

Hydrodynamics for FOPT

Conventional approach (Review)

See E.g. M. B. Hindmarsh et.al., SciPost Phys. Lect. Notes 24 (2021) 1 for review

- Entropy current conservation (2nd law)

$$\partial_\mu S^\mu = 0 \quad \text{w/} \quad S^\mu := s u^\mu,\ s = (\rho + p)/T$$

This equation is derived from the equation given below.

$$\partial_\mu T_f^{\mu\nu} = \partial^\nu \phi \frac{dm^2}{d\phi} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{2E} f^{\text{eq}}(x, p, t)$$

$$T_f^{\mu\nu} = (\rho_f + p_f) u^\mu u^\nu - p_f g^{\mu\nu}$$

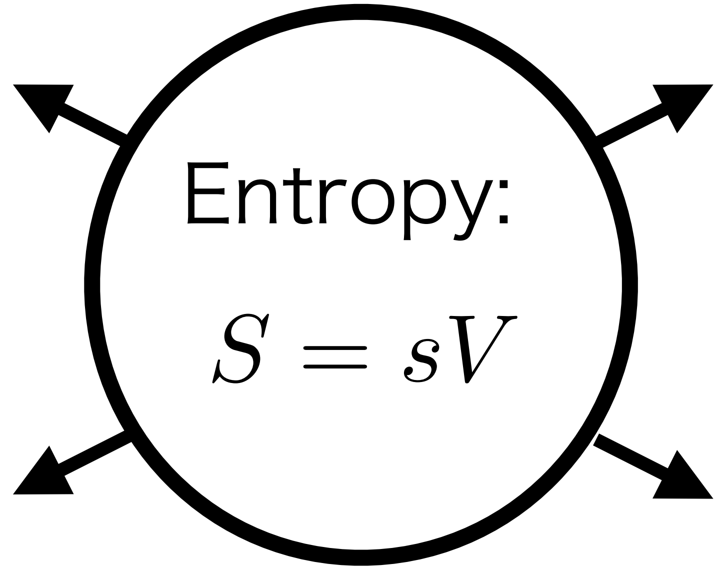
Single perfect fluid

Local Thermal Equilibrium Approximation

PT

Entropy (Heat) current: $\delta Q/T$

$S + \delta Q/T \geq 0$



$$\partial_\mu S^\mu \underline{= 0} \quad \text{w/} \quad S^\mu := s u^\mu, \; s = (\rho + p)/T$$

This equation is derived from the equation given below.

$$\partial_\mu T_f^{\mu\nu} = \partial^\nu \phi \frac{dm^2}{d\phi} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{2E} \underline{f^{\mathrm{eq}}(x, p, t)}$$

$$\underline{T_f^{\mu\nu} = (\rho_f + p_f) u^\mu u^\nu - p_f g^{\mu\nu}}$$

Single perfect fluid

Local Thermal Equilibrium Approximation

Hydrodynamics for FOPT

Novel approach (This work)

- Energy-momentum conservation (1st law)

$$\partial_\mu (T^{\mu\nu}_\phi + T^{\mu\nu}_f) = 0, \quad T^{\mu\nu}_f = T^{\mu\nu}_S + T^{\mu\nu}_D$$

Then, each component can be expressed as

$$\begin{cases} \partial_\mu T^{\mu\nu}_D = \partial^\nu \phi \dfrac{dm_D^2}{d\phi} \displaystyle\int \dfrac{d^3\boldsymbol{p}}{(2\pi)^3} \dfrac{1}{2E} f^{eq}_D + \underline{\displaystyle\int \dfrac{d^3\boldsymbol{p}}{(2\pi)^3} \dfrac{p^\nu}{2E_D} C[f_D]} \\ \partial_\mu T^{\mu\nu}_S = \underline{- \displaystyle\int \dfrac{d^3\boldsymbol{p}}{(2\pi)^3} \dfrac{p^\nu}{2E_D} C[f_D]} \end{cases}$$

Asymmetric inflow

$$\begin{cases} T^{\mu\nu}_D = (\rho_D + p_D) u^\mu u^\nu - p_D g^{\mu\nu} \\ T^{\mu\nu}_S = (\rho_S + p_S) u^\mu u^\nu - p_S g^{\mu\nu} \end{cases}$$

Two perfect fluid

Hydrodynamics for FOPT

Novel approach (This work)

- Entropy current NON-conservation (2nd law)

$$\begin{cases} \partial_\mu S_D^\mu = \dfrac{u_{D\nu}}{T_D} \displaystyle\int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p^\nu}{2E_D} C[f_D] \\ \partial_\mu S_S^\mu = -\dfrac{u_{S\nu}}{T_S} \displaystyle\int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p^\nu}{2E_S} C[f_S] \end{cases}$$

$$\blacktriangleright \quad \partial_\mu S^\mu = \left(\frac{u_{D\nu}}{T_D} - \frac{u_{S\nu}}{T_S} \right) \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p^\nu}{2E_D} C[f_D],$$

$$\simeq \underbrace{\left(\frac{1}{T_D} - \frac{1}{T_S} \right)}_{>0} \underbrace{\int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p^0}{2E_D} C[f_D]}_{<0}$$

Hydrodynamics for FOPT

Novel approach (This work)

- Entropy current NON-conservation (2nd law)

$$
\begin{cases}
\partial_\mu S_D^\mu = \dfrac{u_{D\nu}}{T_D} \displaystyle\int \dfrac{d^3\boldsymbol{p}}{(2\pi)^3} \dfrac{p^\nu}{2E_D} C[f_D] \\
\partial_\mu S_S^\mu = -\dfrac{u_{S\nu}}{T_S} \cdots
\end{cases}
$$

We found novel <u>negative</u> contribution to entropy current, which leading to its non-conservation even under the local thermal equilibrium approximation.

$$
\blacktriangleright\ \partial_\mu S^\mu = \left(\frac{u_{D\cdots}}{T_D} \cdots \frac{\cdots}{T_{\cdots}} \right) \int \cdots \frac{\cdots}{(2\pi)^3\, 2E_D} \cdots
$$

$$
\simeq \underbrace{\left(\frac{1}{T_D} - \frac{1}{T_S} \right)}_{>0} \underbrace{\int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p^0}{2E_D} C[f_D]}_{<0}
$$

Hydrodynamics for FOPT

Novel approach (This work)

- Entropy current NON-conservation (2nd law)

$$\begin{cases} \partial_\mu S_D^\mu = \dfrac{u_{D\nu}}{T_D} \displaystyle\int \dfrac{d^3\boldsymbol{p}}{(2\pi)^3} \dfrac{p^\nu}{2E_D} C[f_D] \\ \partial_\mu S_S^\mu = -\dfrac{u_{S\nu}}{T_S} \displaystyle\int \dfrac{d^3\boldsymbol{p}}{(2\pi)^3} \dfrac{p^\nu}{2E_S} C[f_S] \end{cases}$$

▶ $$\partial_\mu S^\mu \simeq \underline{\left(\frac{1}{T_D} - \frac{1}{T_S} \right) \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p^0}{2E_D} C[f_D]}$$

$\to 0$ If $T_S = T_D$ or $\displaystyle\int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p^0}{2E_D} C[f_D] = 0$

Outline

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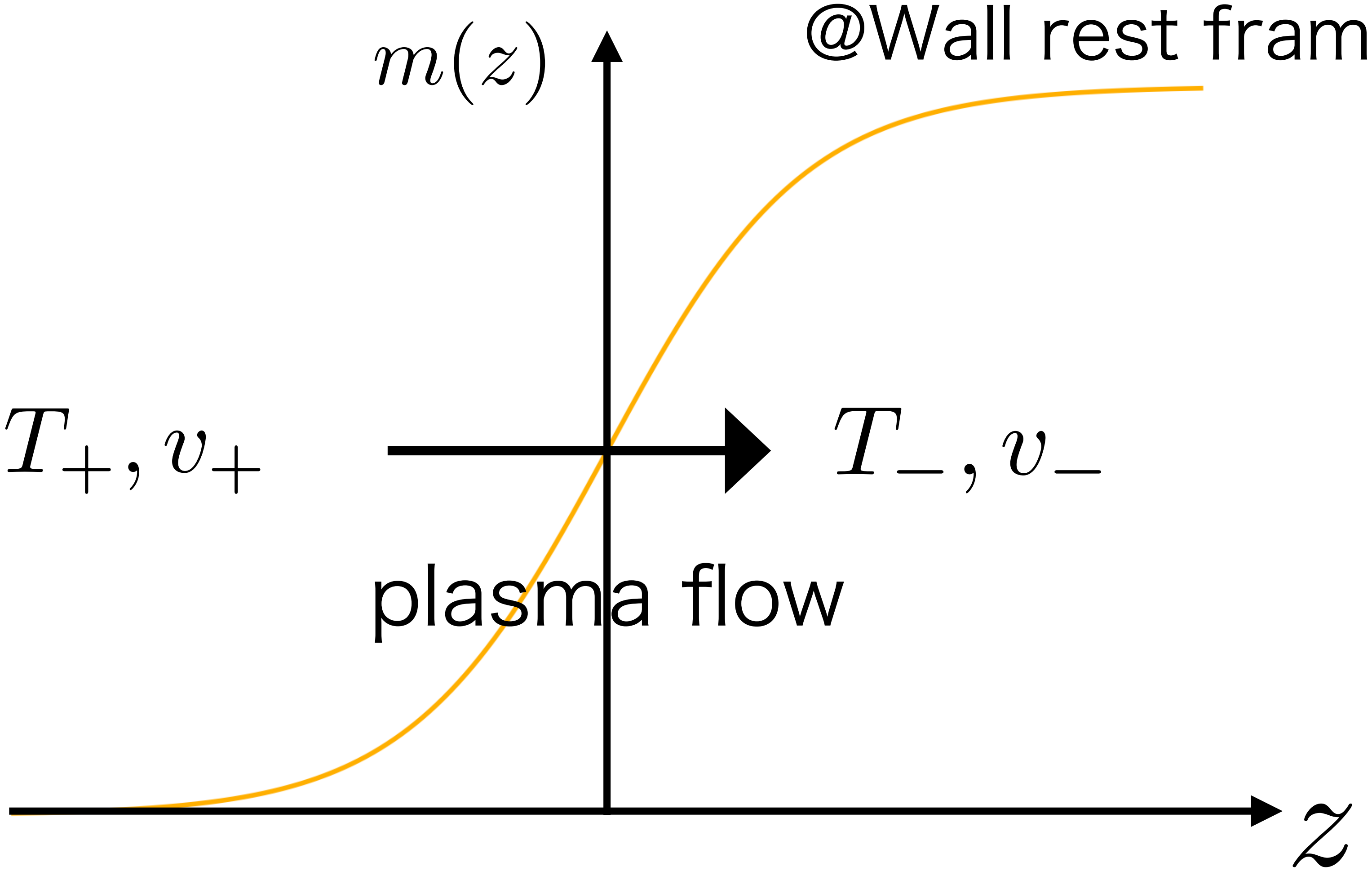
- ▸ Analysis of hydrodynamics
- ▸ Summary

Hydrodynamics

Conventional approach

L. D. Landau and E. M. Lifshitz. Pergamon Press, New York, 1989.
W.-Y. Ai, et.al., JCAP 03 no. 03, 015 (2022)

@Wall rest frame

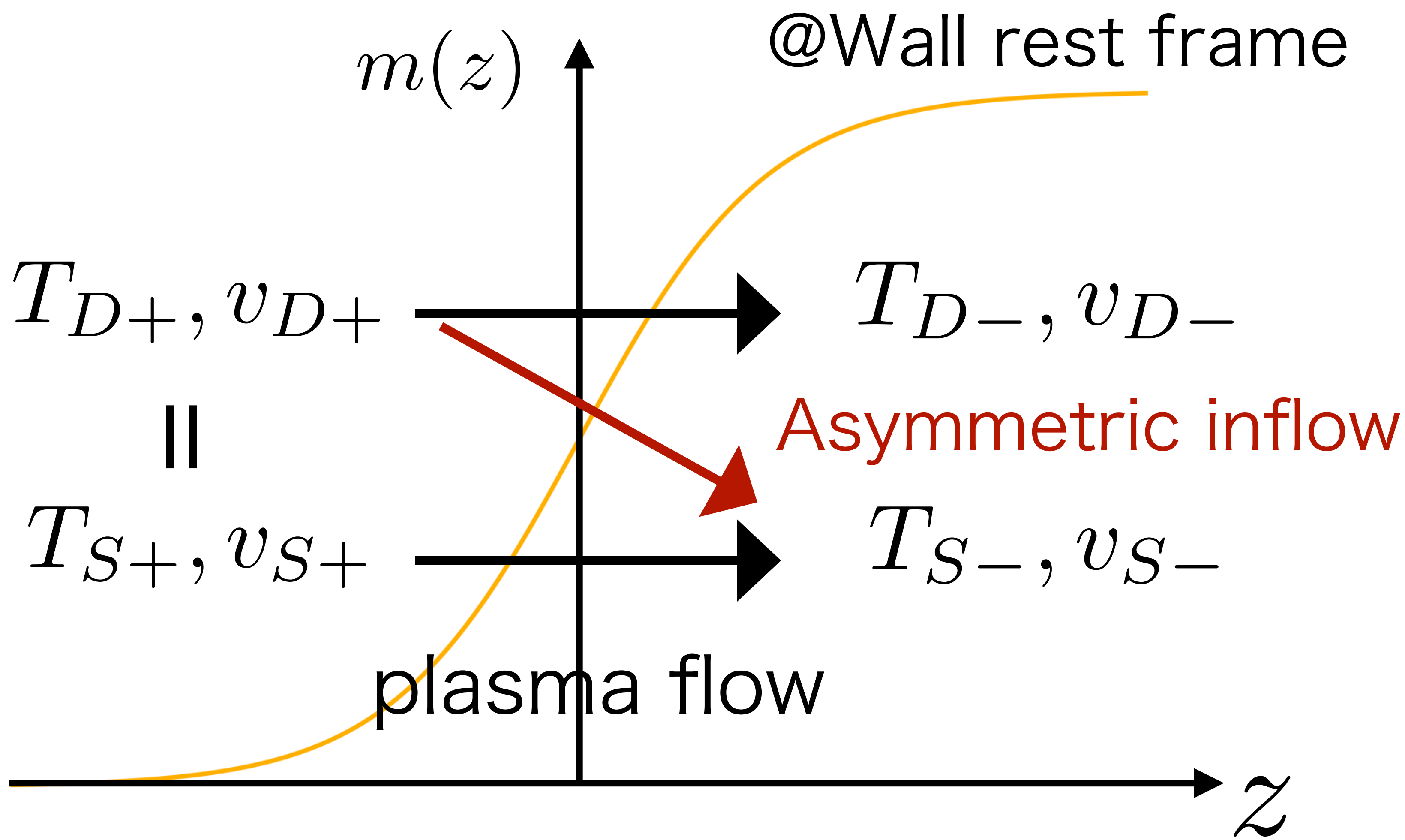


$$\begin{cases} \partial_\mu (T_\phi^{\mu\nu} + T_f^{\mu\nu}) = 0 \\ \partial_\mu S^\mu = 0 \end{cases} \blacktriangleright \begin{cases} \partial_z T^{z0} = 0 \quad (\nu = 0, 3) \\ \partial_z T^{zz} = 0 \\ \partial_z S^z = 0 \end{cases}$$

give matching

Hydrodynamics

Novel approach (This work)

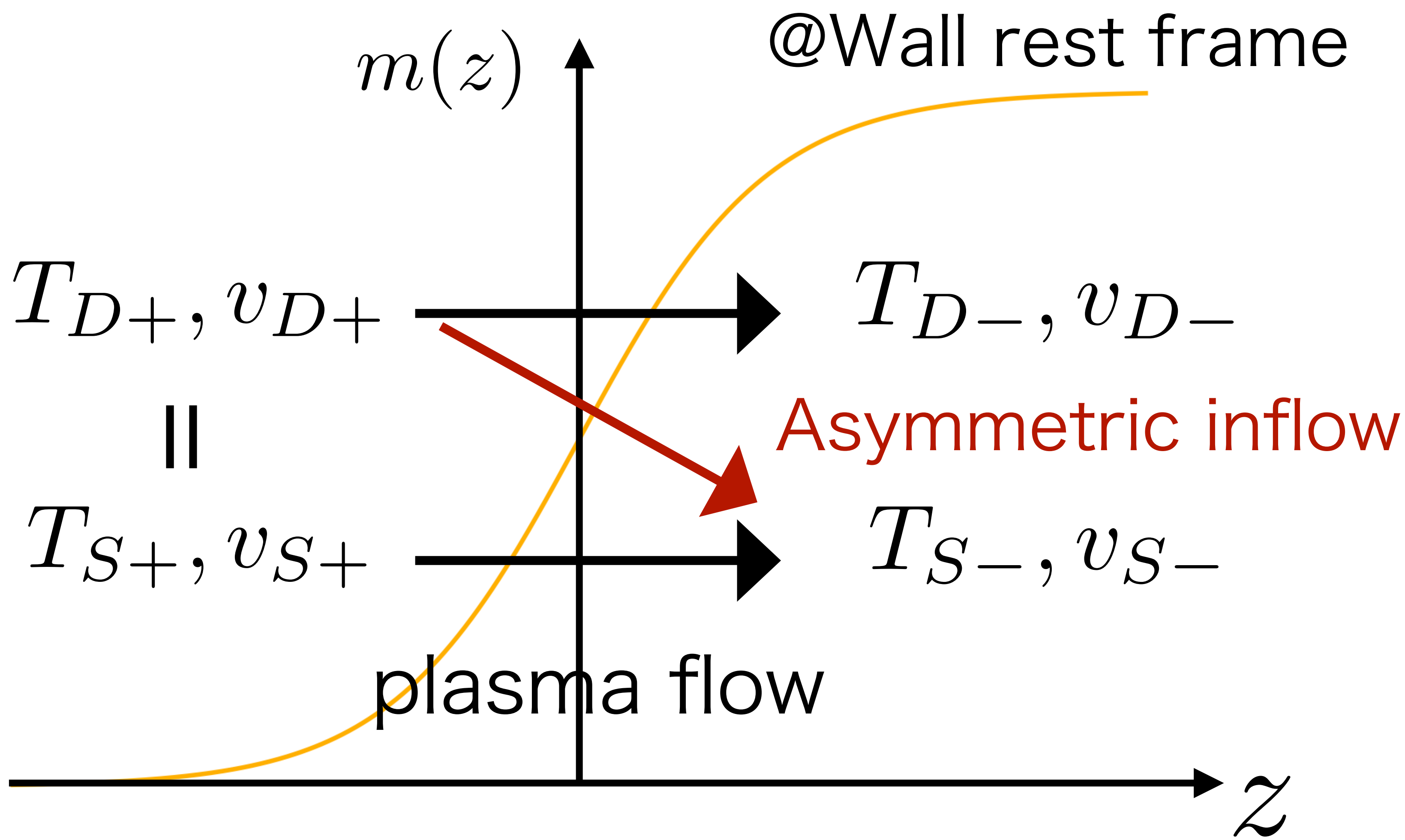


$$
\left\{
\begin{aligned}
&\partial_z (T_\phi^{z\nu} + T_D^{z\nu} + T_S^{z\nu}) = 0 \quad (\nu = 0, 3) \\
&\partial_z S_S^z = -\frac{u_{S\nu}}{T_S} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p^\nu}{2E_D} C[f_D] \\
&\partial_z S_D^z = \frac{u_{D\nu}}{T_D} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p^\nu}{2E_D} C[f_D]
\end{aligned}
\right.
$$

Four independent Eqs

Hydrodynamics

Novel approach (This work)

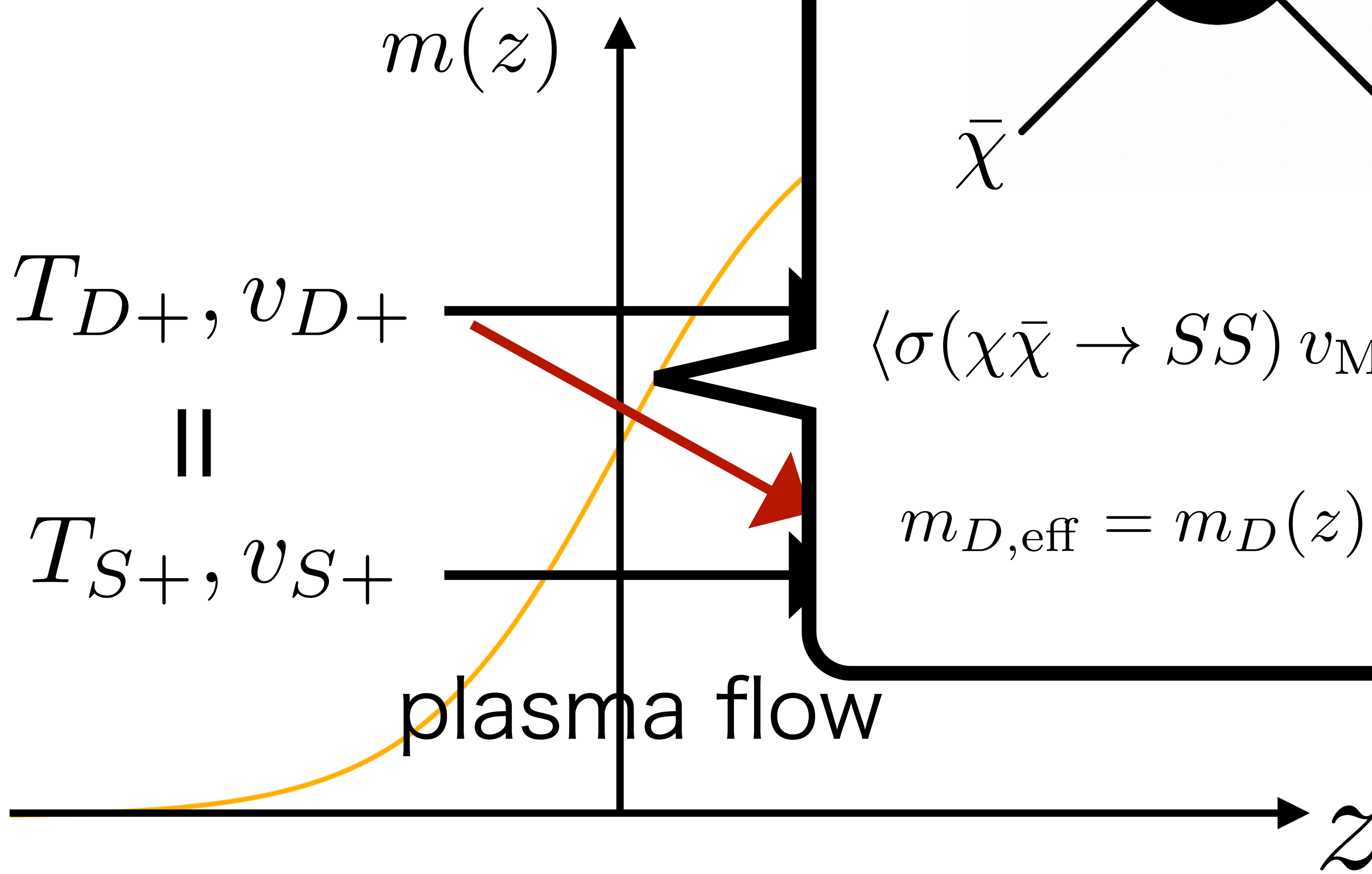


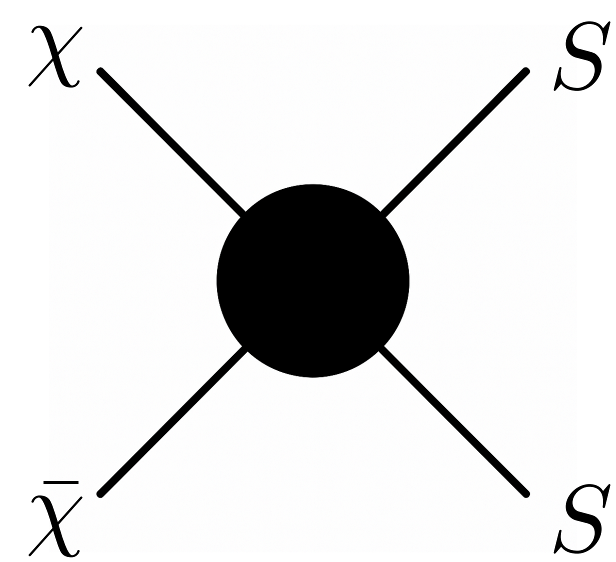
$$\begin{cases} \partial_z T_D^{z\nu} = \partial^\nu \phi \, \dfrac{dm_D^2}{d\phi} \displaystyle\int \dfrac{d^3\boldsymbol{p}}{(2\pi)^3} \dfrac{1}{2E_D} f_D^{eq} + \displaystyle\int \dfrac{d^3\boldsymbol{p}}{(2\pi)^3} \dfrac{p^\nu}{2E_D} C[f_D] \\[2ex] \partial_z T_S^{z\nu} = - \displaystyle\int \dfrac{d^3\boldsymbol{p}}{(2\pi)^3} \dfrac{p^\nu}{2E_D} C[f_D] \end{cases}$$

Four independent Eqs

Hydrodynam

Novel approach (Thi





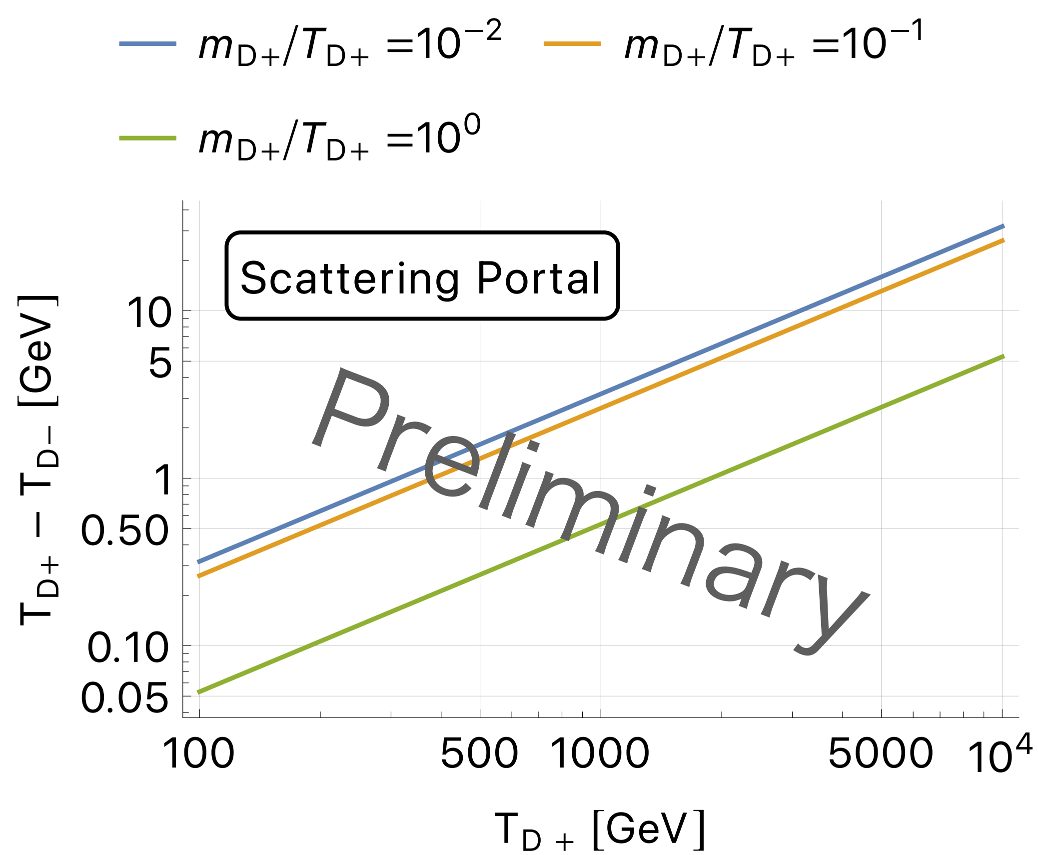
$$\langle \sigma(\chi\bar{\chi} \to SS)\, v_{\text{Møl}} \rangle \simeq \frac{1}{32\pi} \frac{g^4}{m_{D,\text{eff}}^2(z)},$$

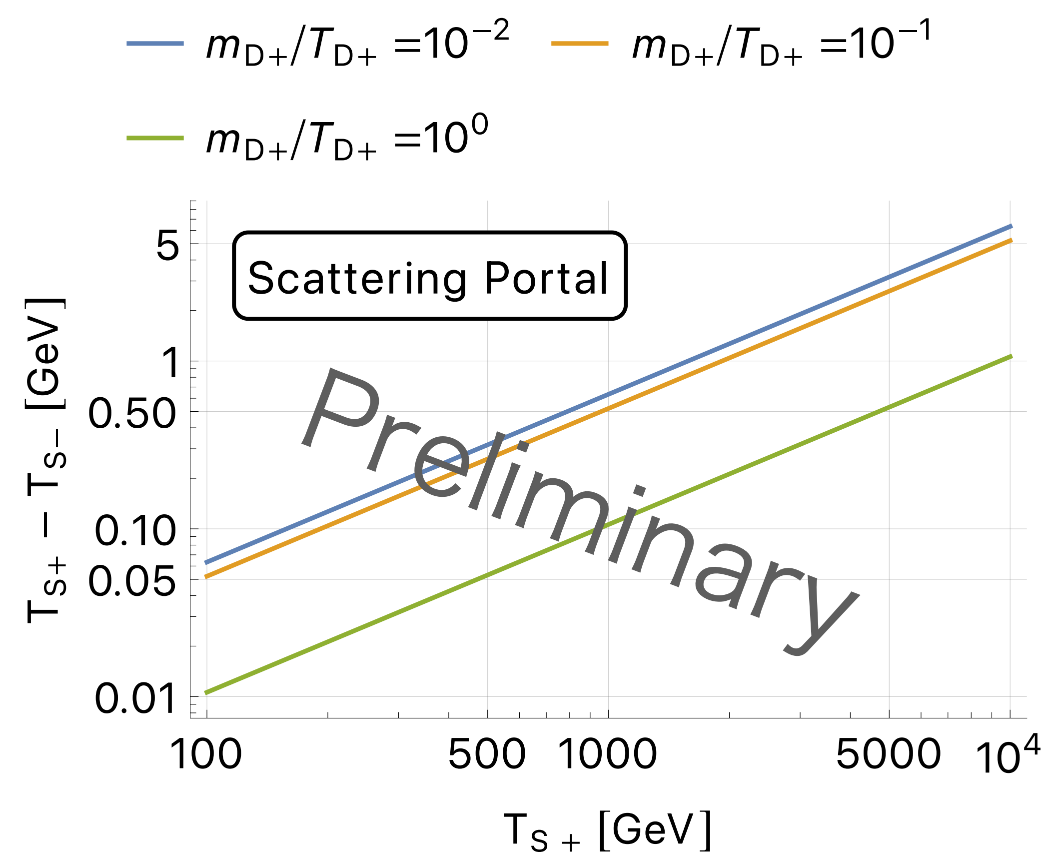
$$m_{D,\text{eff}} = m_D(z) + m_{D,th}(T)$$

$$\begin{cases} \partial_z T_D^{z\nu} = \partial^\nu \phi \, \dfrac{dm_D^2}{d\phi} \displaystyle\int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{2E_D} f_D^{eq} + \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p^\nu}{2E_D} C[f_D] \\ \partial_z T_S^{z\nu} = - \displaystyle\int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p^\nu}{2E_D} C[f_D] \end{cases}$$

Four independent Eqs

Result





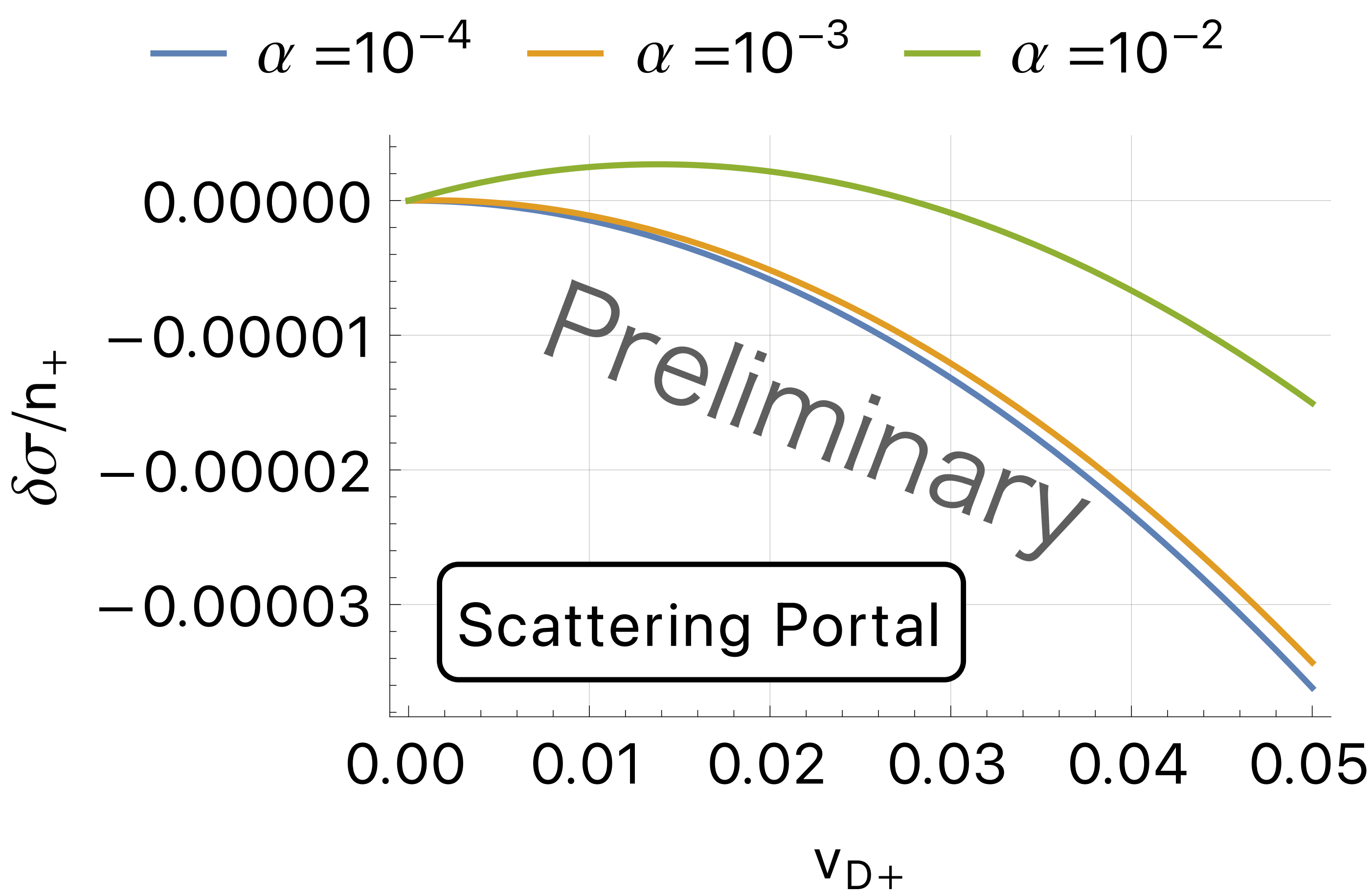
If entropy current & energy flux are conserved,

$T_{D+} = T_{D-}, T_{S+} = T_{S-}$ for non-relativistic fluid

Under asymmetric energy flux inflow

$$T_{D+} > T_{D-}, T_{S+} < T_{S-}$$

Result



$$\delta\sigma := \delta\sigma_D + \delta\sigma_S$$

$$\delta\sigma_D := \int dz \partial_z S^z_D$$

$$\delta\sigma_S := \int dz \partial_z S^z_S$$

$$n_+ := \frac{(g_S + g_D)}{\pi^2} T^3_{D+}$$

Sufficiently small α is required & its lower limit is roughly given by

$$\delta\sigma \gtrsim -10^{-6} \left(\frac{v_{D+}}{10^{-2}}\right)^2 n_+$$

▶ Impact of Hydrodynamics seems to be small at least @ NR limit.

Comments & Implication

In the whole system (scalar + plasma), the second law should be recovered;
otherwise, the FOPT cannot proceed.

$$\Delta(F_f + F_\phi) \leq 0 \qquad F_\phi, F_f : \text{Free energy}$$

This is consistent with fundamental concept of information dynamical system when scalar field has "entropy". (Future prospects)

$$\delta S_\phi \sim I(X:Y) n_+ v_{D+}$$

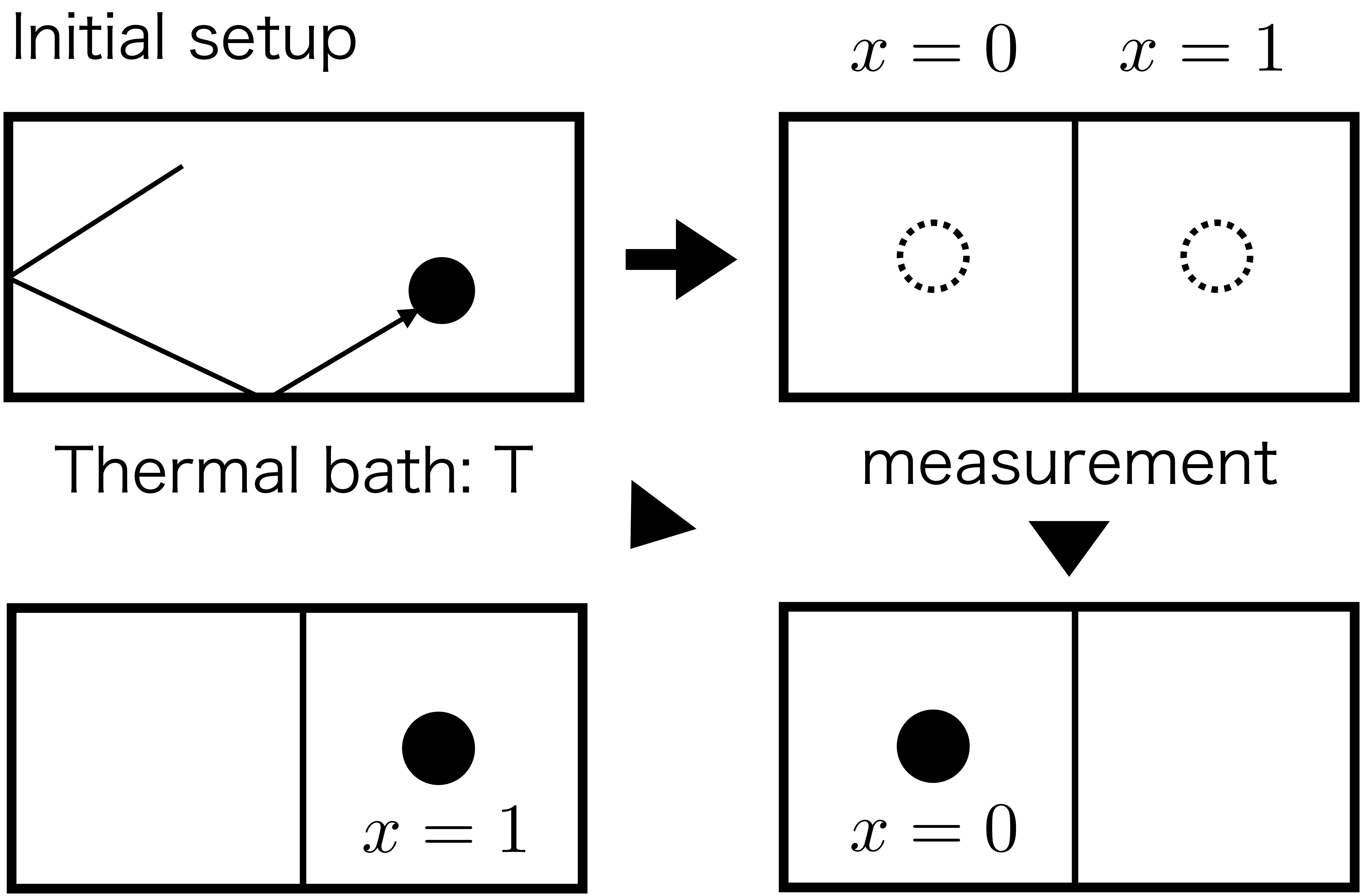
Summary

- We discuss the application of information thermodynamics to FOPT, & found that correspondence with Szilard engine in some specific situation.
- Through asymmetric energy-momentum inflow, novel negative contribution to entropy current appear, and its affect fluid hydrodynamics through matching conditions.
- Unfortunately, impact of hydrodynamics seems to be small at NR limit. Further analysis is future work.

Back up

Szilard engine

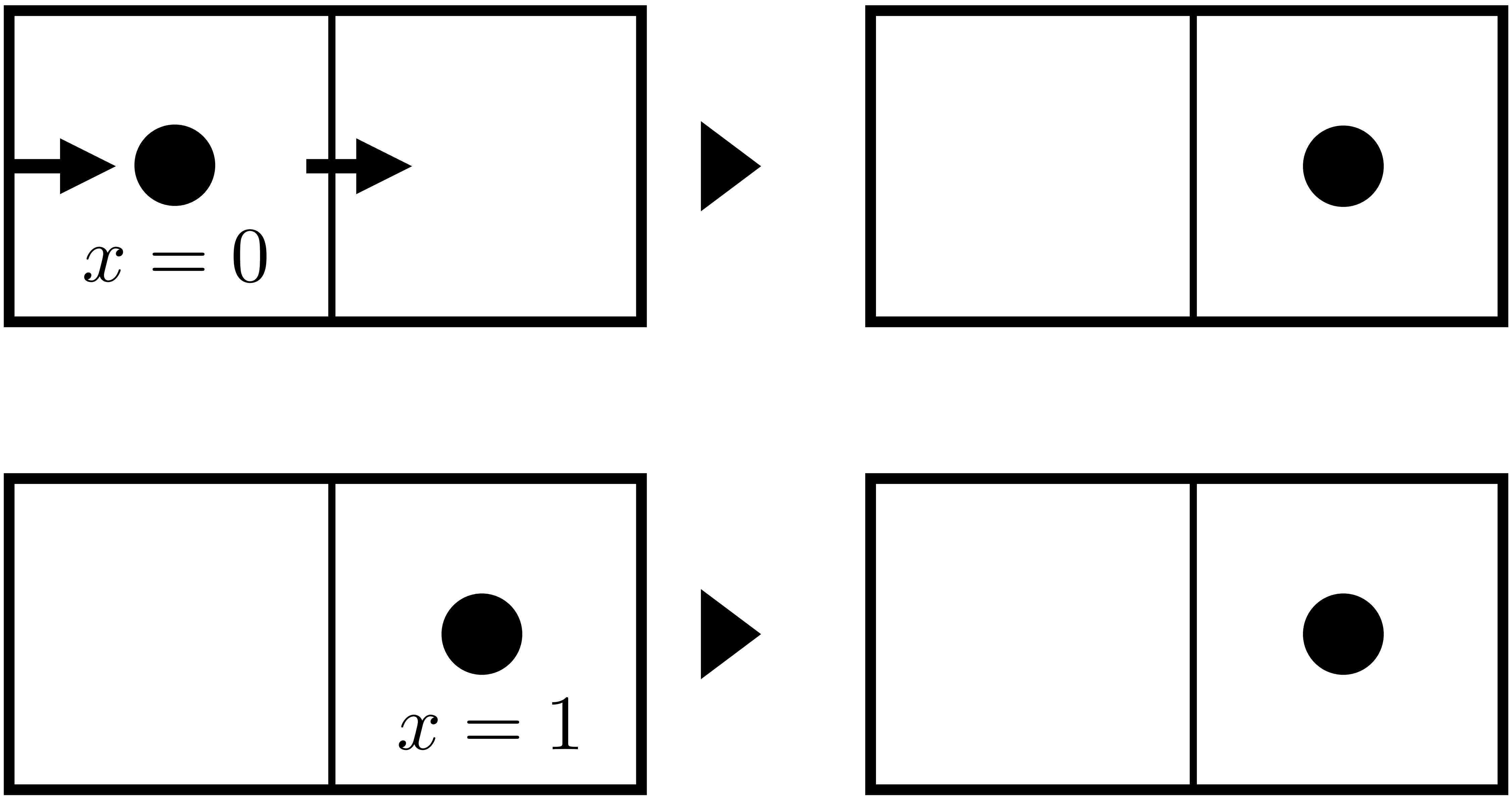
L. Szilard, Z. Phys. 53, 840 (1929).



Szilard engine

L. Szilard, Z. Phys. 53, 840 (1929).

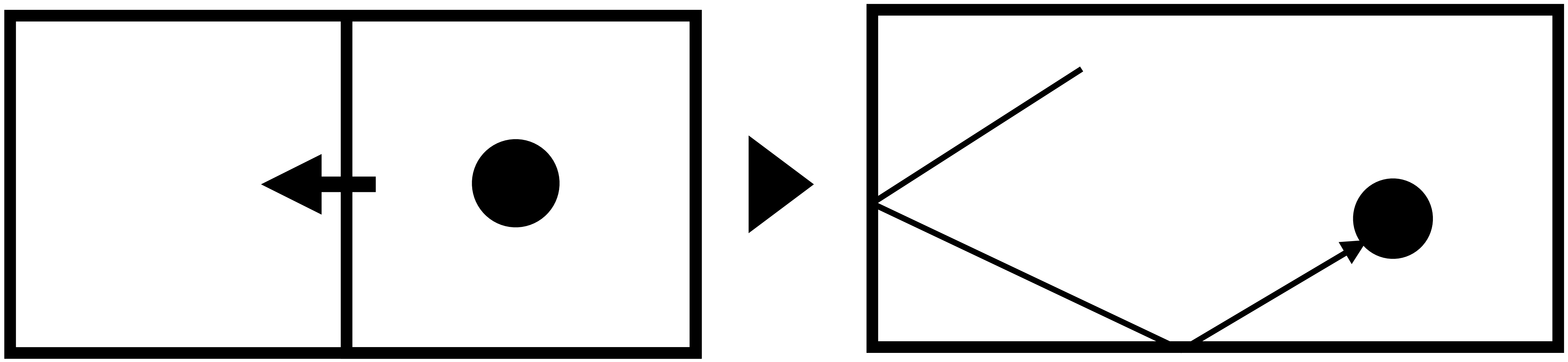
Feedback



Szilard engine

L. Szilard, Z. Phys. 53, 840 (1929).

Quasistatic expansion



Through this process, we have

$$W_{\mathrm{ext}} = \int_{V/2}^{V} p dV' = \int_{V/2}^{V} \frac{k_B T}{V'} dV'$$

$$= k_B T \ln 2$$

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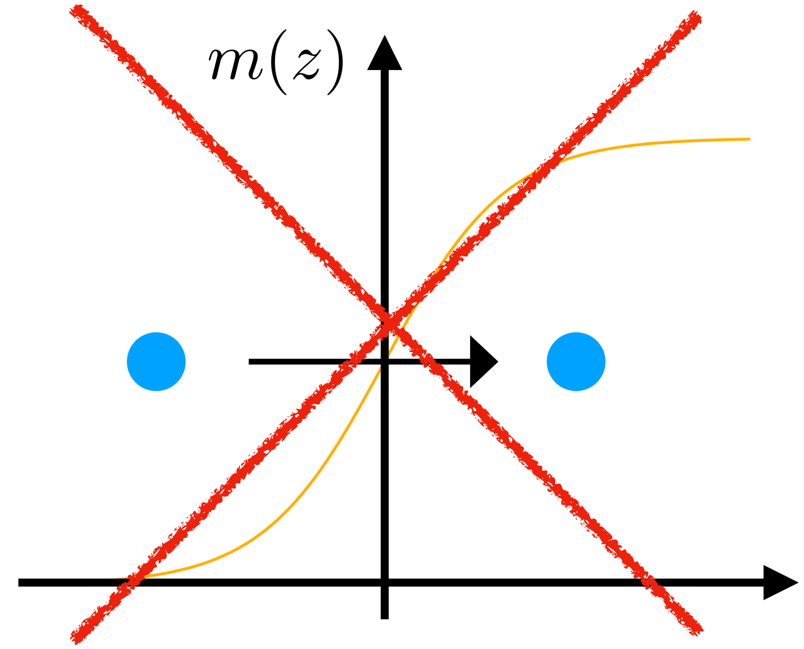
We focus on a slow wall,
which ensures that mean
free path @ wall rest frame
$\lambda \propto \gamma v$
does not become too long.



If wall velocity is higher, energy-momentum exchange between plasma & wall becomes higher.

D. Bodeker and G. D. Moore, JCAP 05 009 (2009), JCAP 05 025 (2017)
S. Höche, J. Kozaczuk, A. J. Long, J. Turner, and Y. Wang, et.al., JCAP 03 009 (2021)
Y. Gouttenoire, R. Jinno, and F. Sala JHEP 05 004 (2022)

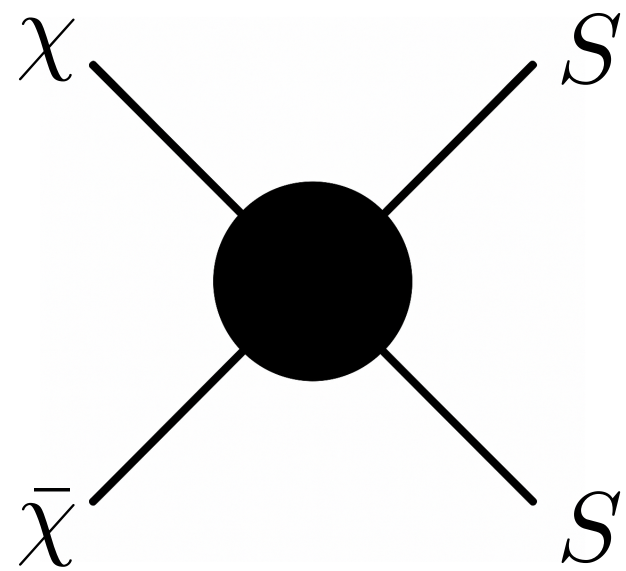
$$\partial_\mu T_f^{\mu\nu} = \partial^\nu \phi \frac{dm^2}{d\phi} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{2E} f + \cancel{\int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p^\nu}{2E_D} C[f]}$$

This can be satisfies when

- Mean free path λ is short
- Gradients of mass profile is small

$$\left.\right\} \quad \lambda \frac{\partial m}{m} \lesssim \mathcal{O}(1)$$

Diagramatic calculation

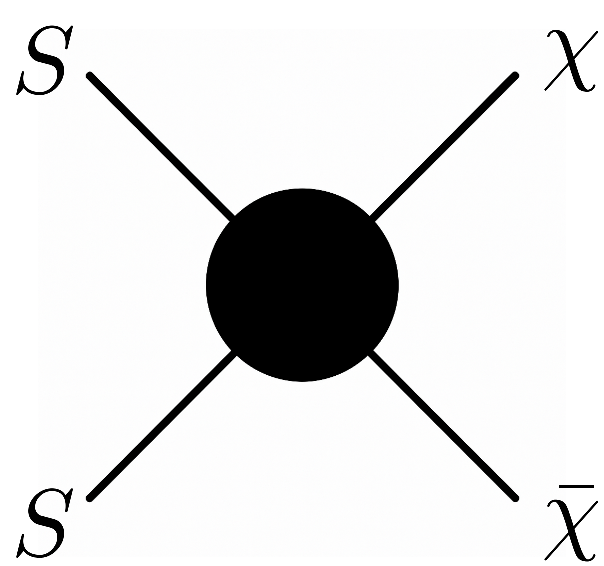


$$\begin{aligned}&\langle p_D^0\, \sigma(\chi\chi \to \varphi\varphi)\, v_{\mathrm{M\o l}}\rangle\ (n_\chi^{\mathrm{eq}})^2 \\ &\simeq -\partial_{T_D^{-1}} \left(\langle \sigma(\chi\chi \to \varphi\varphi)\, v_{\mathrm{M\o l}}\rangle\ (n_\chi^{\mathrm{eq}})^2\right) \\ &=: p^0_{D,\,\mathrm{scattering}}\ \langle \sigma(\chi\chi \to \varphi\varphi)\, v_{\mathrm{M\o l}}\rangle\ (n_\chi^{\mathrm{eq}})^2,\end{aligned}$$

$m_D(z)$

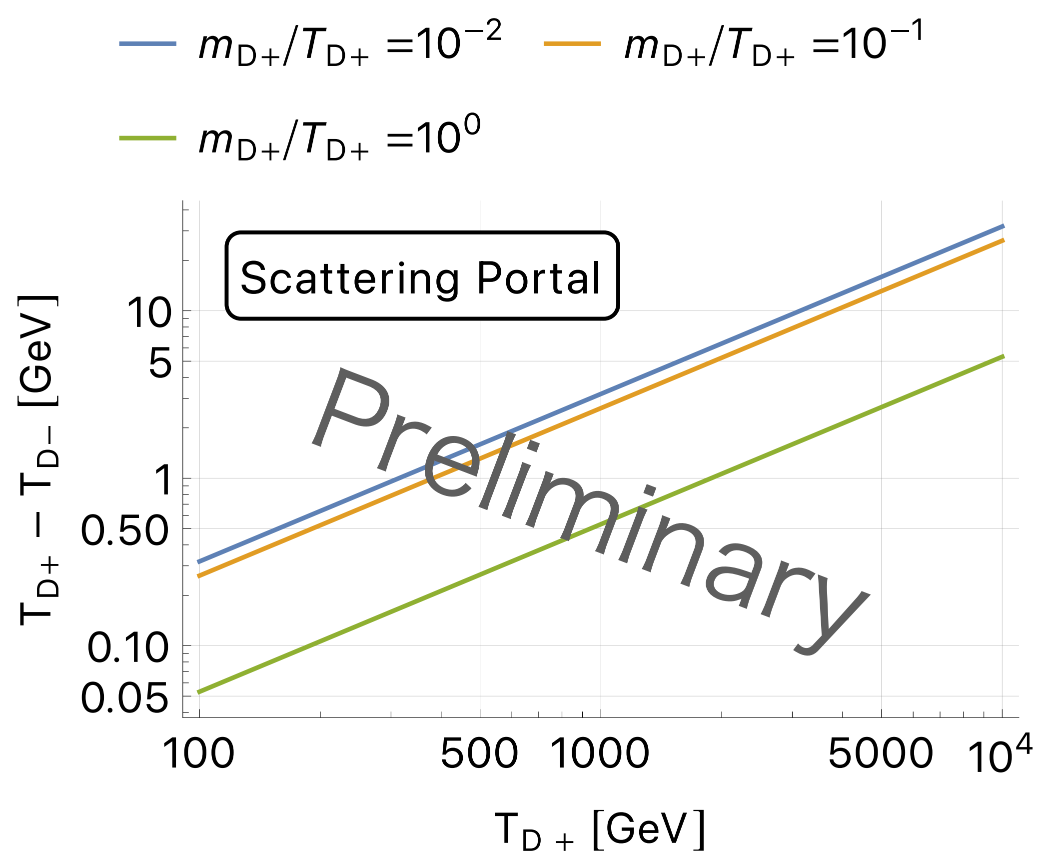
$z \longrightarrow z+\Delta z$ w/ $\Delta z \propto v_{D+}/T_{D+}$

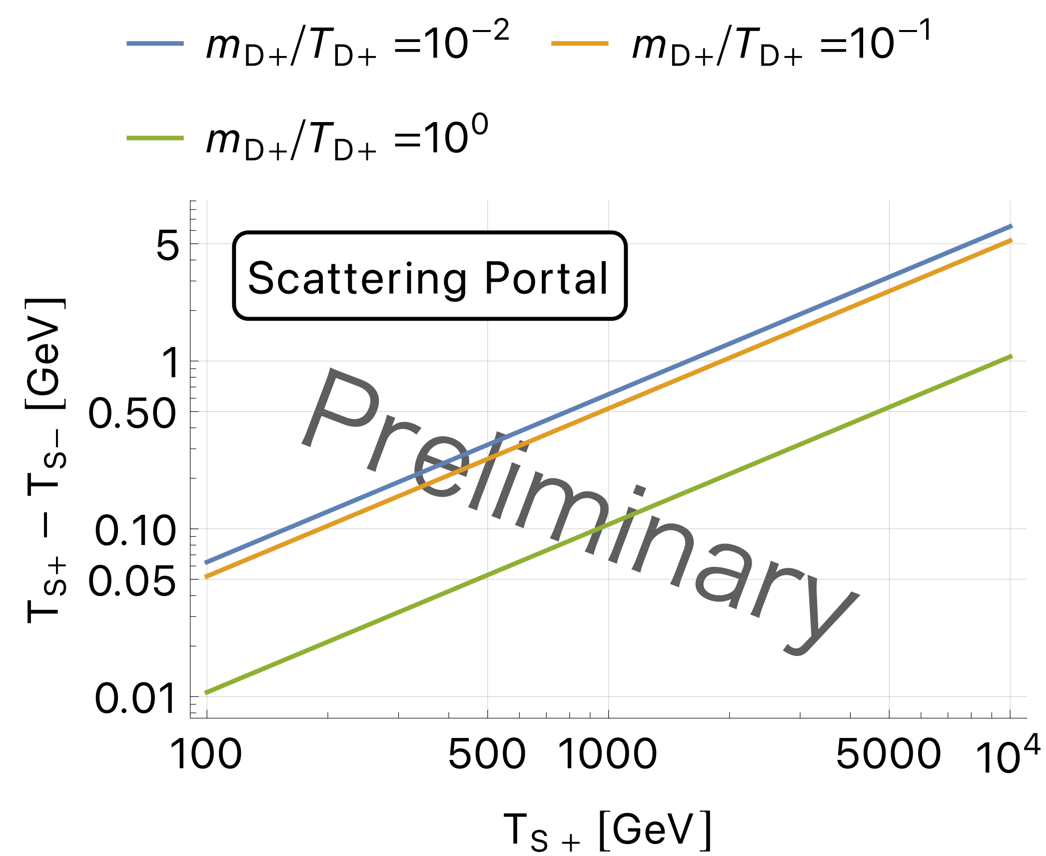
$m_D(z+\Delta z)$



$$\begin{aligned}&\langle p_D^0\, \sigma(\varphi\varphi \to \chi\chi)\, v_{\mathrm{M\o l}}\rangle\ (n_\varphi^{\mathrm{eq}})^2 \\ &\simeq \left(\langle p_D^0\, \sigma(\chi\chi \to \varphi\varphi)\, v_{\mathrm{M\o l}}\rangle\ (n_\chi^{\mathrm{eq}})^2\right)\Big|_{m_D=m_D(z+\Delta z)}\end{aligned}$$

Setup





Fluid system has three independent parameters:

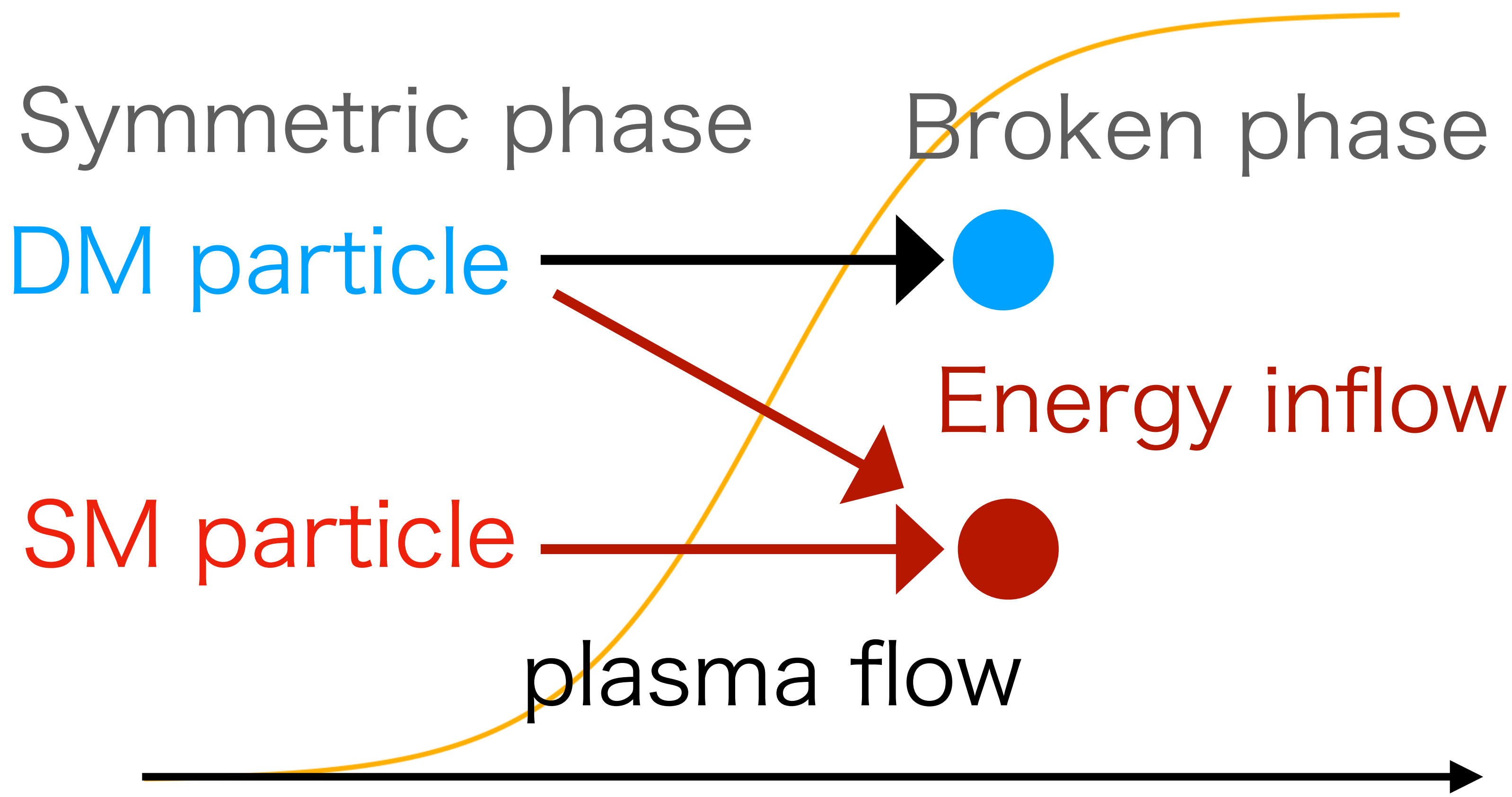
(T_{D+}, v_{D+}, α) α :Dimensionless latent heat

We have taken $g=\sqrt{4\pi}$, $m_{D+} := m_D(z\sim-\infty) \lesssim T_{D+}$

2nd law of information thermodynamics suggests

$$\sigma = (S + \delta Q/T)/V$$

$$\delta\sigma \gtrsim -\underbrace{(I(X:Y) - I(X':Y))}_{\simeq 0.1} \times \underbrace{(\text{Incoming \# flux})}_{\propto v_{wall}}$$



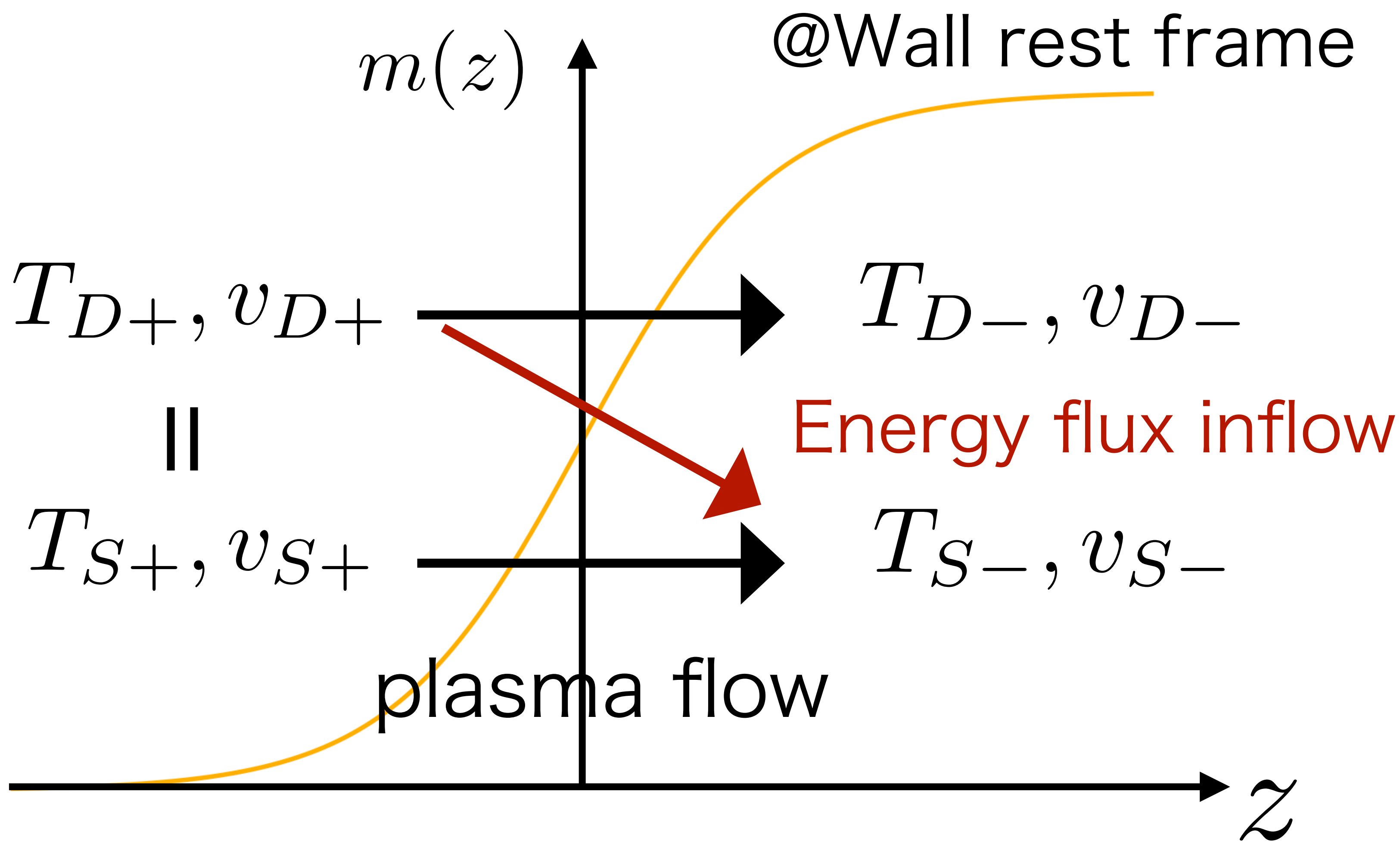
$$Q \ll 1$$

$$P_D = \frac{g_D}{g_S + g_D}$$

$$P_S = \frac{g_S}{g_S + g_D}$$

Hydrodynamics

Novel approach (This work)



$$
\begin{cases}
\partial_z T_D^{z\nu} = \partial^\nu \phi \, \dfrac{dm_D^2}{d\phi} \displaystyle\int \dfrac{d^3\boldsymbol{p}}{(2\pi)^3} \dfrac{1}{2E_D} f_D^{eq} + \displaystyle\int \dfrac{d^3\boldsymbol{p}}{(2\pi)^3} \dfrac{p^\nu}{2E_D} C[f_D] \\
\partial_z T_S^{z\nu} = - \displaystyle\int \dfrac{d^3\boldsymbol{p}}{(2\pi)^3} \dfrac{p^\nu}{2E_D} C[f_D]
\end{cases}
$$

Four independent Eqs

Hydrodynamics

Energy flux & Momentum exchange

$$\begin{cases} \partial_z(\omega_S \gamma_S^2 v_S) = \sum_{a\in S} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p^0}{2E_a} C[f_a], \\ \partial_z\left(\omega_S \gamma_S^2 v_S^2 + p_S\right) = \sum_{a\in S} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p^z}{2E_a} C[f_a], \\ \partial_z(\omega_D \gamma_D^2 v_D) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p^0}{2E_D} C[f_D], \\ \partial_z\left(\omega_D \gamma_D^2 v_D^2 + \frac{1}{2}(\partial_z \phi(z))^2 + p_D^{\mathrm{eff}}\right) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p^z}{2E_D} C[f_D], \end{cases}$$

w/

$$\omega_S := \rho_S + p_S, \quad \omega_D := \rho_D^{\mathrm{eff}} + p_D^{\mathrm{eff}},$$
$$\rho_D^{\mathrm{eff}} := \rho_D + V(\phi), \quad p_D^{\mathrm{eff}} := p_D - V(\phi).$$

Hydrodynamics

Energy flux & Momentum exchange

$$\begin{cases} \omega_{S+}\gamma_{S+}^2 v_{S+} = \omega_{S-}\gamma_{S-}^2 v_{S-} + C_D^0, \\ \omega_{S+}\gamma_{S+}^2 v_{S+}^2 - \omega_{S-}\gamma_{S-}^2 v_{S-}^2 = p_{S-} - p_{S+} + C_D^z, \\ \omega_{D+}\gamma_{D+}^2 v_{D+} = \omega_{D-}\gamma_{D-}^2 v_{D-} - C_D^0, \\ \omega_{D+}\gamma_{D+}^2 v_{D+}^2 - \omega_{D-}\gamma_{D-}^2 v_{D-}^2 = p_{D-}^{\text{eff}} - p_{D+}^{\text{eff}} - C_D^z, \end{cases}$$

w/
$$C_D^0 := \int dz \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p^0}{2E_D} C[f_D],$$
$$C_D^z := \int dz \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p^z}{2E_D} C[f_D],$$

Hydrodynamics

Entropy production

$$\delta\sigma := \delta\sigma_D + \delta\sigma_S$$

$$\delta\sigma \simeq \frac{\omega_{D-}}{T_{D-}} v_{D-} - \frac{\omega_{D+}}{T_{D+}} v_{D+} + \frac{\omega_{S-}}{T_{S-}} v_{S-} - \frac{\omega_{S+}}{T_{S+}} v_{S+}.$$

w/ $$\delta\sigma_D := \int dz \partial_z S^z_D$$

$$\delta\sigma_S := \int dz \partial_z S^z_S$$