On-shell Approach to Black Hole Mergers

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KA, A. Cristofoli (YITP) and Y.-t. Huang (NTU), JHEP 01 (2025) 066, [2410.13632].

Progress of Theoretical Physics, Vol. 46, No. 5, November 1971

Quantum Theory of Gravitation vs. Classical Theory*)

——*Fourth-Order Potential*——

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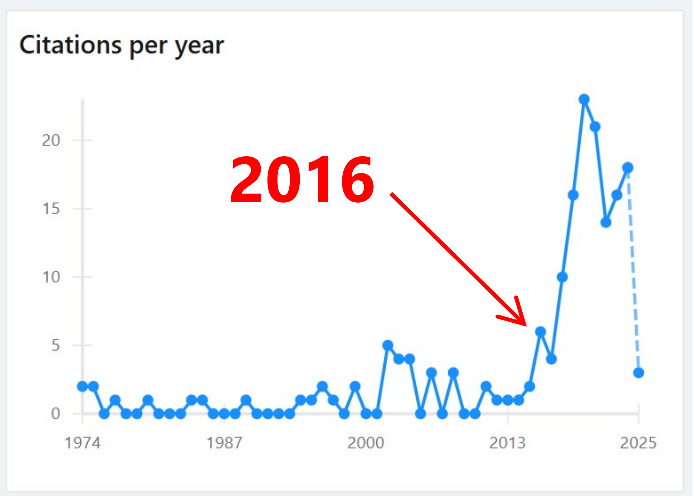
The perihelion-motion of Mercury depends on the fourth-order potential in quantum field theory; it is a "Lamb shift". In spite of the unrenormalizability of the theory, we have extracted a finite and physically meaningful quantity, a fourth-order potential, from fourth-order graphs. We have also discussed briefly renormalization of the Newtonian potential in the fourth-order perturbation.

The Hamiltonian obtained is the same as the classical one and so it cannot explain the Dicke-Goldenberg experiment.

We have calculated fourth-order potential also in Q.E.D.

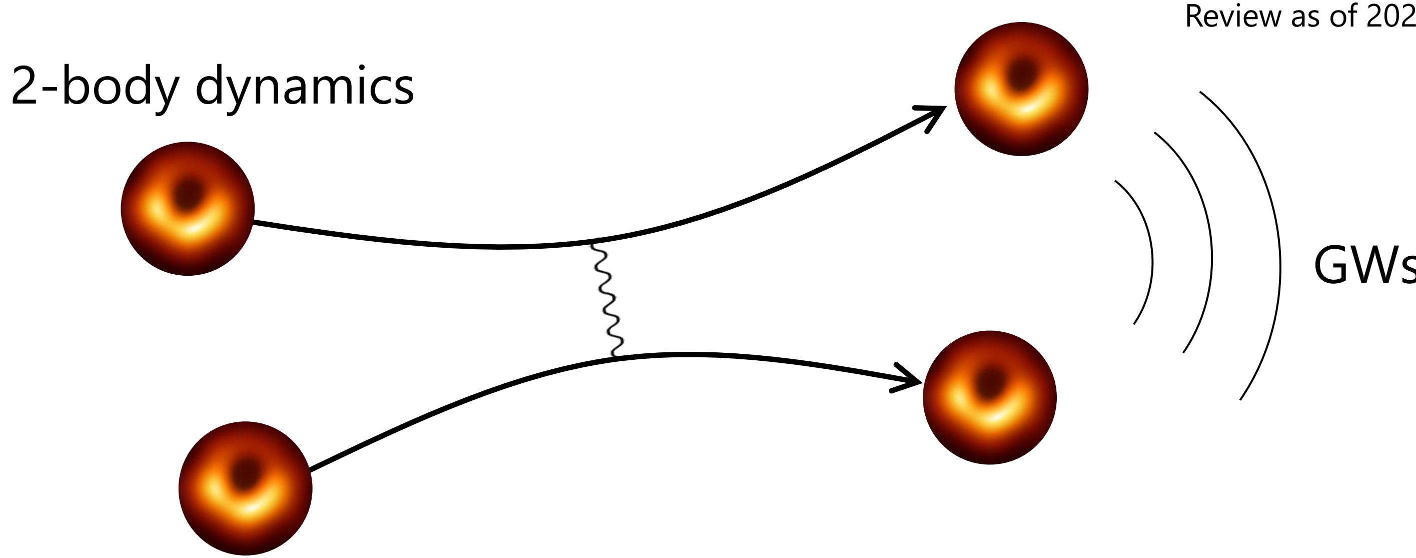


Computing classical physics from scattering amplitudes!

Introduction

❑ **Modern amplitude methods are quite powerful for classical dynamics.**

Review as of 2022: 2204.05194.



Very roughly…

Classical GR (= spacetime)
Solving **10** comp.

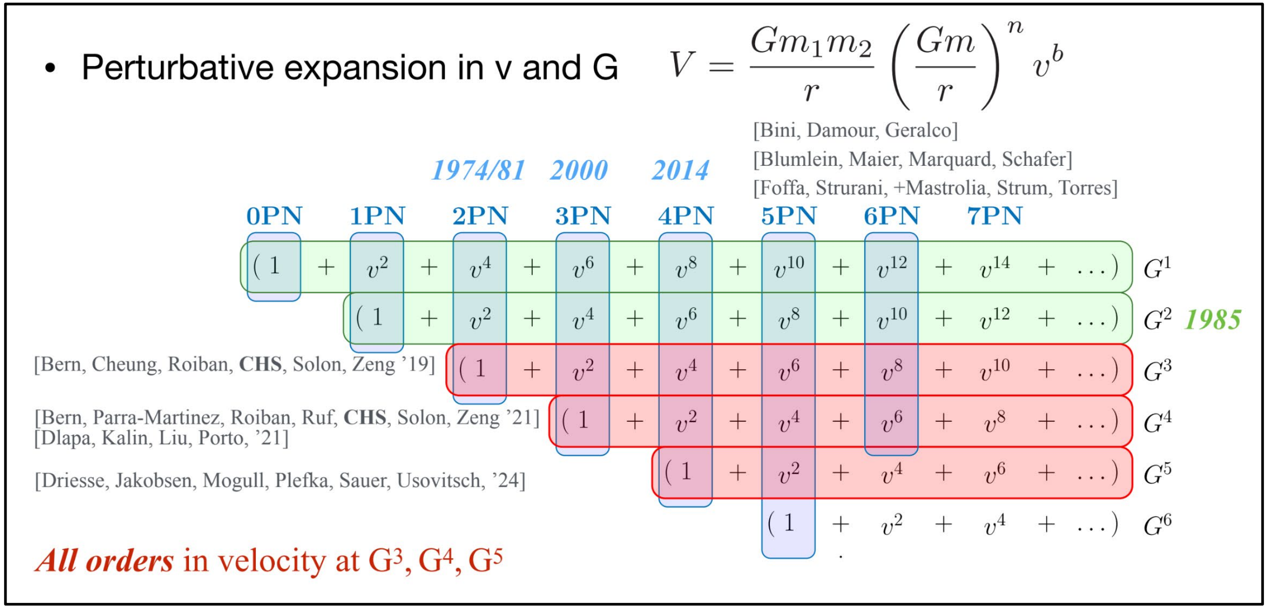
vs.

Quantum GR (= graviton)
Solving **2** dofs.

Which do you think is easier?

Introduction

❑ The amplitude approach has had great success in weak field regime!



Post-Minkowskian (PM) expansion

Taken from Chia-Hsien's talk@Gravity 2025, YITP

✓ Kerr BH dynamics → (massive) higher spin

✓ Radiative observables → higher-point amplitudes

Too many references…

hep-ph × hep-th × gr-qc

- ✓ Unitarity methods,
- ✓ Multiloop techniques,
- ✓ …

- ✓ Doubly copy,
- ✓ Spinor-helicity,
- ✓ …

- ✓ Post-Newton expansion,
- ✓ Black hole perturbations,
- ✓ …

⇓

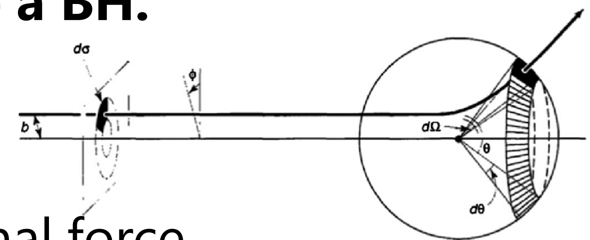
- ❑ **State-of-the-art predictions of gravitational waves,**
- ❑ **Developing more computational techniques,**
- ❑ **More understanding of black holes,**
- ❑ **…**

Point particle vs. particle

Can we go more? The story starts with treating BH as a point particle.

Goldberger and Rothstein, 2006.

- ❑ **Let's consider how a distant observer will see a BH.**



✓ BH is just a localised object with mass & spin.
= one-particle state $|\mathrm{BH}\rangle = |M, J\rangle$

✓ Its structure is seen by how it responds to external force.
= (In)existence of interactions, e.g. BH is minimally coupled to gravity.
→ **BH is just a "particle" with certain interactions.**

A. Guevara+ '18; M.-Z. Chung+ '18; N. Arkani-Hamed+ '19; A. Aoude+ '19

- ❑ **BH mergers are then seen as just a fusion of particles!**

KA, A. Cristofoli, Y.-t. Huang, 2410.13632.

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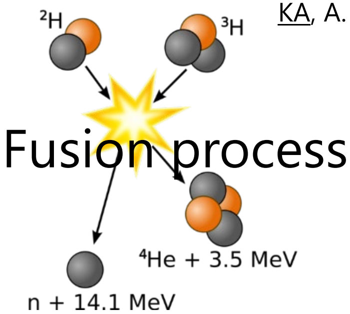
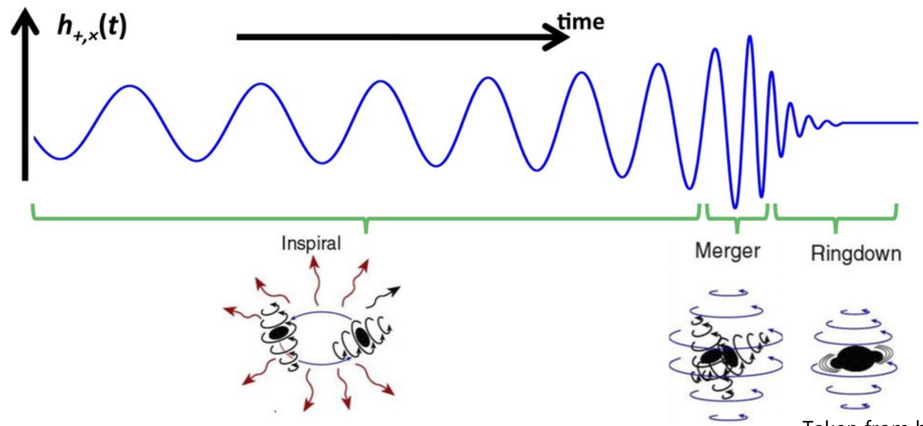
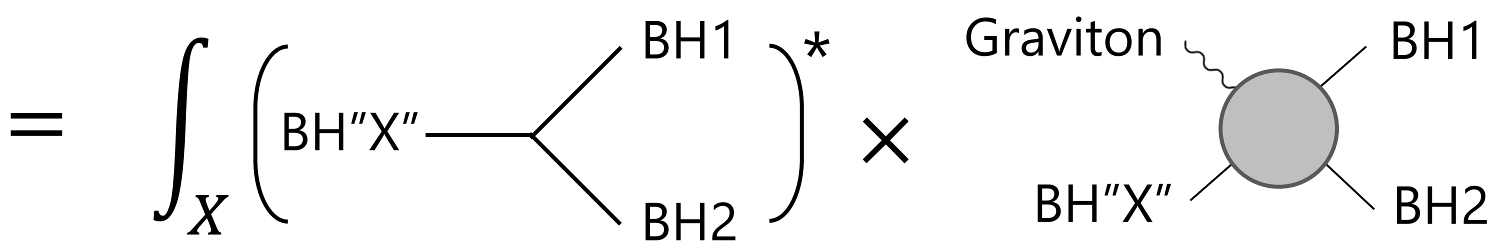


Information of BH merger is in on-shell 3pt & 4pt!

(at least without radiation reaction)



Taken from https://www.soundsofspacetime.org/



Fixed by kinematics

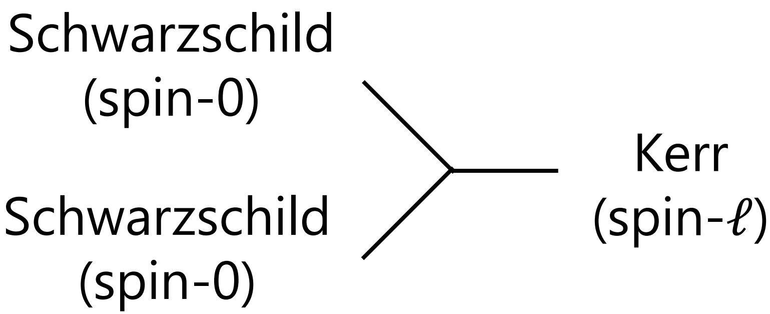
Encodes dynamics

KA, A. Cristofoli, Y.-t. Huang, 2410.13632.

Merger amplitude

- BH merger is non-perturbative. Maybe, no analytic control…
  No, it's just a local reaction and can be described by an effective operator.

c.f. chiral EFT.



$$= g_\ell \langle \boldsymbol{X} | p_1 p_2 | \boldsymbol{X} \rangle^\ell \sim g_\ell Y_{\ell,m}$$

This 3pt is uniquely fixed by kinematics except for overall coupling.

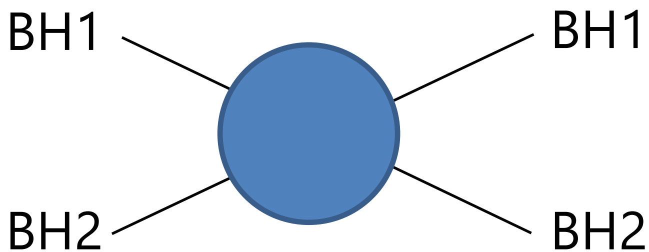
N. Arkani-Hamed, T.-C. Huang, Y.-t. Huang, 2017.

- The coupling is determined to recover the complete absorption of BH.

= black-disk scattering

BH formation = Complete absorption

- Let's consider the elastic scattering of BHs. S.B. Giddings & M. Srednicki 2008; S.B. Giddings & R.A. Porto 2009.



$$\mathcal{A}_{12\to12}(s,\cos\theta) = 16\pi\sum_{\ell=0}^{\infty}(2\ell+1)a_\ell(s)P_\ell(\cos\theta)$$

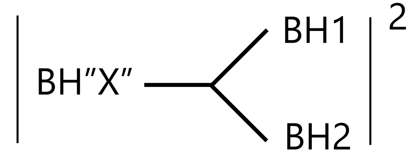
$$a_\ell(s) = \frac{E}{2P}\frac{\eta_\ell e^{2i\delta_\ell}-1}{2i}$$

elasticity $0 \le \eta_\ell \le 1$

- Classically, two BHs collide and form a new BH for $\ell < L_c = O(Gm_1m_2)$.
  The system never comes back to 2-body → $\eta_\ell = 0$ for $\ell < L_c$

- The absorption cross-section for BH formation is computed by the cut.

$$\sigma_\ell^{\rm abs} = \sigma_\ell^{\rm tot} - \sigma_\ell^{\rm el} = \frac{\pi(2\ell+1)}{P^2}, \qquad (\ell \le L_c)$$



$$\sigma_\ell^{\rm abs} = \frac{\pi}{2EP}\sum_{m=-\ell}^{\ell}\int_{(m_1+m_2)^2}^{+\infty} {\rm d}m_X^2\,\delta(s-m_X^2)\rho_\ell(m_X^2)|\mathcal{A}(p_X,\ell,m|p_1;p_2)|^2 \sim |g_\ell|^2\rho_\ell$$

Cf. wave absorption: Aoude+ '23; Jones+ '23; Chen+ '23.

⇒ **The coupling (× spectral density) is completely fixed.**

"Quantum" BH → "Classical" BH

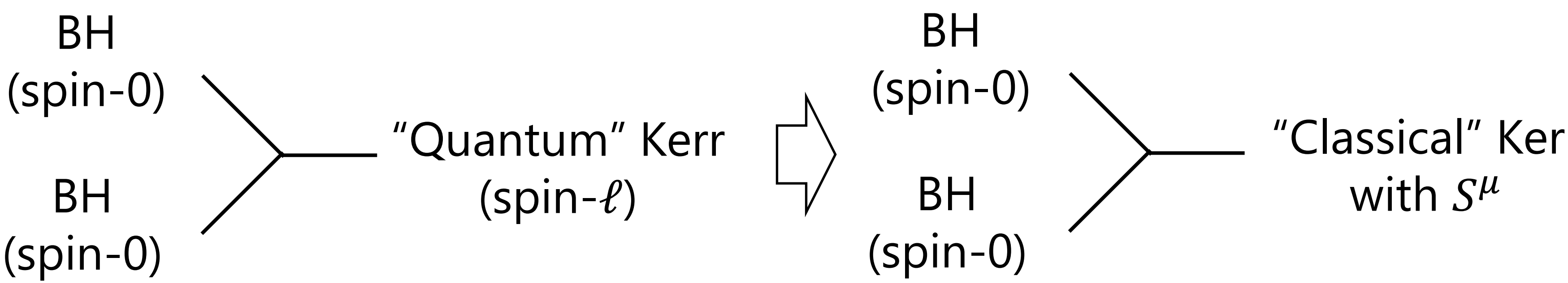
- The final state must be a classical Kerr with classical spin.
  We describe the final state by the coherent spin state R. Aoude & A. Ochirov 2021.

$$|\alpha\rangle := e^{-\frac{1}{2}\tilde{\alpha}_J\alpha^J} e^{\alpha^I \hat{a}_I^\dagger}|0\rangle$$

= a superposition of all spin−ℓ states with a wight α

$$= e^{-\frac{1}{2}\|\alpha\|^2}\sum_{2\ell=0}^{\infty}\frac{1}{\sqrt{(2\ell)!}}\alpha^{I_1}\cdots\alpha^{I_{2\ell}}|\ell,\{I_1\cdots I_{2\ell}\}\rangle$$

$$S_X^\mu = \langle p_X,\tilde{\alpha}|\,\mathbb{S}_X^\mu\,|p_X,\alpha\rangle = \frac{\hbar}{2}\tilde{\alpha}_I[\sigma_X^\mu]^I{}_J\alpha^J$$



$$\mathcal{A}^{I_1\cdots I_{2\ell}}(p_X,\ell|p_1;p_2) = m_X\frac{\sqrt{(2\ell)!}}{\ell!}\left(\frac{\langle\boldsymbol{X}|\,p_1p_2\,|\boldsymbol{X}\rangle}{m_X\lambda^{1/2}(m_1^2,m_2^2,m_X^2)}\right)^\ell$$

$$\mathcal{A}(p_X,\tilde{\alpha}|p_1,p_2) = m_X e^{-\frac{1}{2}\|\alpha\|^2+z}$$

$$z(p_1,p_2) := \frac{\tilde{\alpha}_I\,\langle p_{12}^I|\,p_1p_2\,|p_{12}^J\rangle\,\tilde{\alpha}_J}{m_X\lambda^{1/2}(m_1^2,m_2^2,m_X^2)}$$

We now have the building block to describe BH mergers!

Classical physics from amplitudes

❑ **KMOC formalism** Kosower, Maybee & O'Connel 2018; A. Cristofoli et al, 2021.

✓ Observables in quantum physics are expectation values.

Expectation value at out　　Expectation value at in

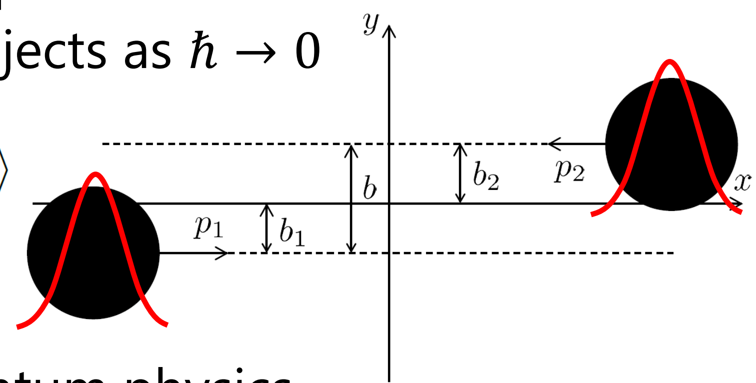
$$\Delta O = \langle\Psi|S^{\dagger}\mathcal{O}S|\Psi\rangle - \langle\Psi|\mathcal{O}|\Psi\rangle \qquad |\mathrm{out}\rangle = S|\Psi\rangle$$

✓ The initial state is given by localised wavepackets.

= classical localised objects as $\hbar \to 0$

$$|\Psi\rangle := \int d\Phi(p_1, p_2)\phi_1(p_1)\phi_2(p_2)e^{i(b_1\cdot p_1 + b_2\cdot p_2)}\,|p_1; p_2\rangle$$

2-body initial states



✓ Classical physics is just $\hbar \to 0$ limit of quantum physics.

Classical physics is recovered by on-shell S-matrix and states only!
No classical equations of motion nor classical fields are needed.

3pt = momentum conservations

- BH + BH → BH amplitude: $\langle p_X, \tilde{\alpha}|S|p_1, p_2\rangle = i\hat{\delta}^{(4)}(p_1 + p_2 - p_X)\mathcal{A}(p_X, \tilde{\alpha}|p_1, p_2)$

$$\mathcal{A}(p_X, \tilde{\alpha}|p_1, p_2) = m_X e^{-\frac{1}{2}\|\alpha\|^2 + z} \qquad z(p_1, p_2) := \frac{\tilde{\alpha}_I \langle p_{12}^I | p_1 p_2 | p_{12}^J \rangle \tilde{\alpha}_J}{m_X \lambda^{1/2}(m_1^2, m_2^2, m_X^2)}$$

with $\alpha \sim \hbar^{-1/2}$

Classical spin

The 3pt vanishes unless $-|\alpha|^2$ is cancelled by the kinematic function z.

→ 3pt is just delta function in classical limit.

$$\mathcal{I}_3(p_1, p_2, p_{12}) = \frac{\hbar^{5/2}}{4}\pi P \delta_\hbar(\mathrm{Im}\,\alpha^1)\delta_\hbar(\mathrm{Re}\,\alpha^2)\delta(S_X^z - bP)\delta(S_X^y)$$

at CoM.

$\sim \delta^{(4)}(S_X^\mu - L_{\rm in}^\mu)$

Fourier transform of 3pt

$$\mathcal{I}_3(p_1, p_2, p_X) := \int \prod_{i=1,2} \hat{\mathrm{d}}^4 q_i \hat{\delta}(2p_i \cdot q_i) e^{-ib_i \cdot q_i} \langle p_1 + q_1; p_2 + q_2| T^\dagger |p_X, \alpha\rangle$$

- Momentum & Spin of final states:

$$p_f^\mu = \langle \Psi | S^\dagger \mathbb{P}_X^\mu S | \Psi \rangle \overset{\hbar \to 0}{=} p_1 + p_2, \;\; S_f^\mu = \langle \Psi | S^\dagger \mathbb{S}_X^\mu S | \Psi \rangle \overset{\hbar \to 0}{=} L_{\rm in}^\mu$$

Classical conservation from microscopic conservation.

GWs from BH merger

$$= \sum_{\sigma} \kappa \int \mathrm{d}\Phi(k) \varepsilon^{-\sigma}_{\mu\nu} e^{-ik\cdot x} iW^{\sigma} + \text{c.c.}$$

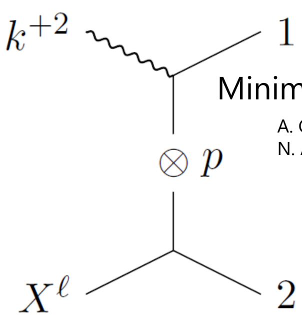
- **Waveform in KMOC:** $h_{\mu\nu} = \langle \text{out} | \mathbb{H}_{\mu\nu} | \text{out} \rangle = \langle \Psi | S^{\dagger} \mathbb{H}_{\mu\nu} S | \Psi \rangle$

  "Leading" order waveform (neglecting radiation reaction) is

$$iW^{\sigma} = \left\langle\!\!\left\langle \int_X \int \prod_{i=1,2} \hat{\mathrm{d}}^4 q_i \hat{\delta}(2p_i \cdot q_i) e^{-ib_i \cdot q_i} \underbrace{\langle p_1 + q_1; p_2 + q_2 | T^{\dagger} | p_X, \alpha \rangle}_{\text{massive 3pt}} \underbrace{\langle p_X, \tilde{\alpha}; k^{\sigma} | T | p_1; p_2 \rangle}_{\text{4pt with one graviton}} \right\rangle\!\!\right\rangle$$

  "Waveform in frequency space = ∫ (Fourier transform of 3pt) × 4pt"

- **The leading (tree-level) waveform**



Minimal coupling

A. Guevara+ '18; M.-Z. Chung+ '18; N. Arkani-Hamed+ '19; A. Aoude+ '19

Merger 3pt

"all-order soft factor"

$$S^{\pm}_{\alpha} = \frac{\kappa}{2} e^{\Delta z^{\pm}} \left[-\frac{(A^{\pm})^2}{(t-m_1^2)(u-m_2^2)(s-m_X^2)} - \frac{A^{\pm} v^{\pm}}{(t-m_1^2)(u-m_2^2)} - \frac{(v^{\pm})^2(1-w_2^{\pm}-e^{-w_2^{\pm}})}{(w_2^{\pm})^2(t-m_1^2)} + \frac{(v^{\pm})^2(1-w_1^{\pm}-e^{-w_1^{\pm}})}{(w_1^{\pm})^2(u-m_2^2)} \right]$$

$$iW^{\pm} = S^{\pm}_{\alpha}|_{S_X = L}$$

$$= \frac{\kappa}{2} \sum_{i=1}^{3} \left[\frac{(\varepsilon^{\pm} \cdot p_i')^2}{k \cdot p_i'} - i \frac{(\varepsilon^{\pm} \cdot p_i')(\varepsilon^{\pm} \cdot J_i \cdot k)}{k \cdot p_i'} - \frac{1}{2} \frac{(\varepsilon^{\pm} \cdot J_i \cdot k)^2}{k \cdot p_i'} \right] + \mathcal{O}(\omega^2 S^3)$$

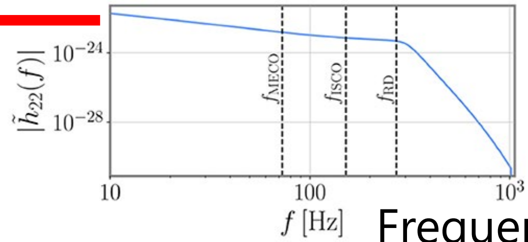
"all-order gravitational spin memory"

LO: Braginsky and Throne, 1987; N(N)LO: Laddha and Sen '19.
Spin memory: Cachazo and Strominger '14; Pasterski and Strominger '14

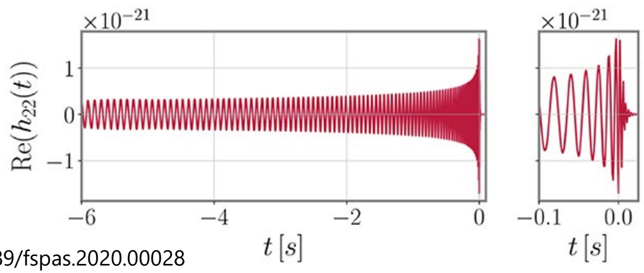
Beyond memory

- **Memory is the waveform in the low-frequency limit**

memory

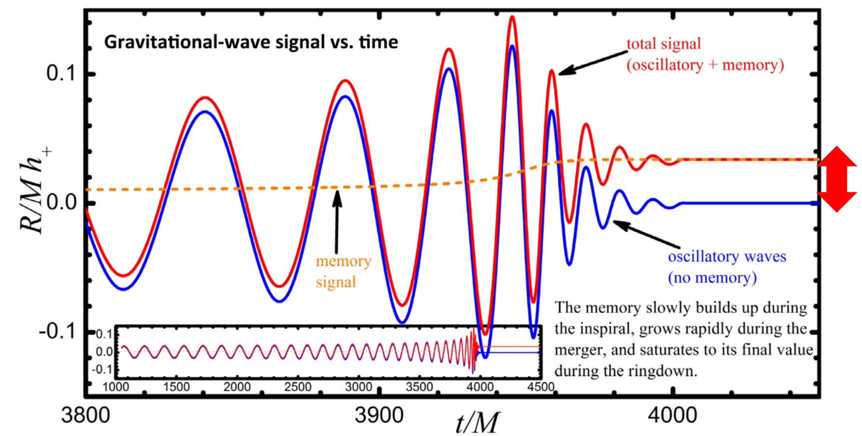


Frequency-domain



https://doi.org/10.3389/fspas.2020.00028

Time-domain



https://www.phy.olemiss.edu/StronGBaD/talks/Favata.pdf

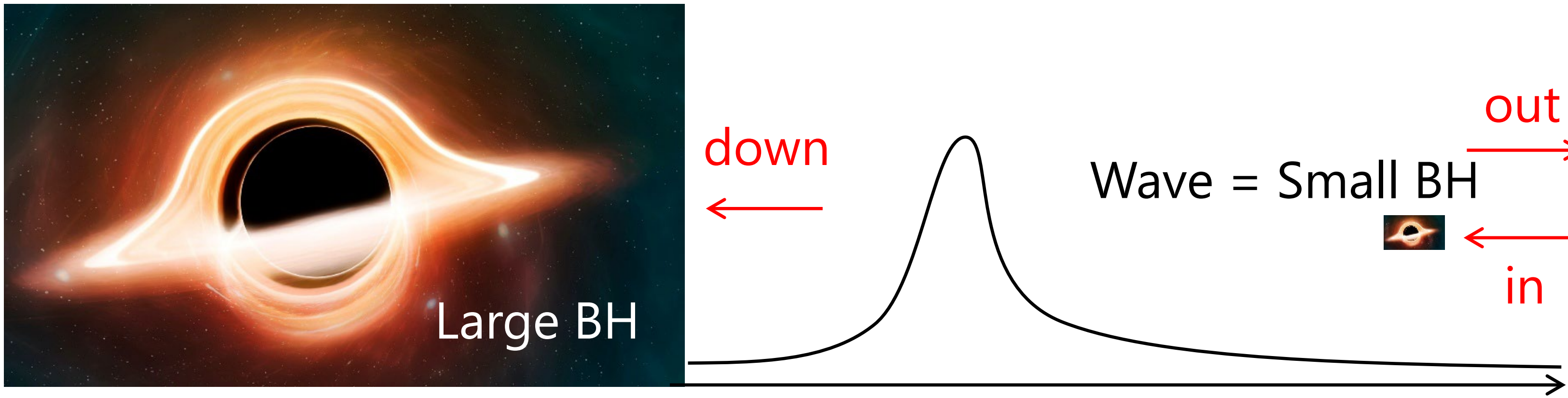
- **Can we actually compute "merger" waveform from amplitudes?**

→ **In principle, yes! Let's look at the connection to BHPT.**

BHPT (Black Hole Perturbation Theory) = quantum field on a fixed BH.

BHPT = Potential Scattering

- ❑ Let's consider a wave scattering on the fixed BH background.



The scattering amplitudes can be computed as in quantum mechanics.

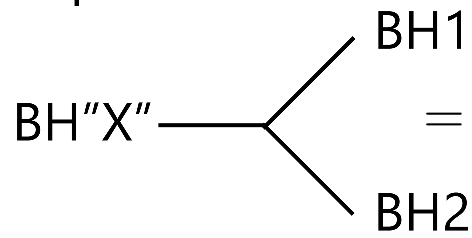
- ❑ How can we understand each process?

in + background → out + background: scattering of two BHs
in + background → down + background = **new BH**: merger of two BHs!

Merger 3pt and Radiation emission 4pt

- We can find the exact agreement of two pictures

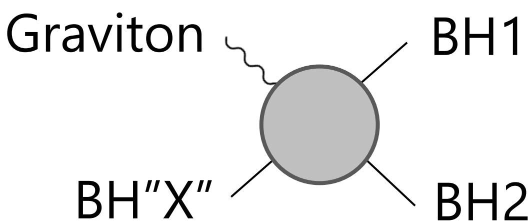
$$\langle\alpha|S|\beta\rangle := \begin{pmatrix} (\varphi^-_{P'}, \varphi^+_P) & (\varphi^-_{P'}, X^+_J) \\ (X^-_{J'}, \varphi^+_P) & (X^-_{J'}, X^+_J) \end{pmatrix} \Leftrightarrow$$



$$= g_\ell \langle \boldsymbol{X} | p_1 p_2 | \boldsymbol{X} \rangle^\ell$$

down ← in transition amplitude

- Waveform computation requires 4pt



$\Leftrightarrow$

down⊗out ← in

$$\langle X^-; h^- | S | \varphi^+ \rangle$$
$$= \lambda \int_{r>r_S} \mathrm{d}^4x \sqrt{-g} (h^-_k)^* (X^-_J)^* (\varphi^+_P)$$

*For simplicity, we consider massless scalar emission.

Flat spacetime perspective

BH spacetime perspective
(or distorted-wave Born approx. in QM)

Classical vs. Quantum calculations

- We can solve the same problem in the classical way (solving EOM):

No radiation reaction

$$(\Box + m_2^2)\varphi \approx 0\,, \qquad \Box h = \frac{\lambda}{2}\varphi^2$$

Motion of lighter BH — Radiation sourced by BH

Point particle ≃ classical wavepacket

- The wave emission is computed by using the retarded Green's function.

- **The classical waveform exactly agrees with the KMOC waveform!**

$$\lambda \int_{r>r_S} \mathrm{d}^4x \sqrt{-g}\,(h_k^-)^* (\varphi_{P'}^+)^* \varphi_P^+ = \int_{P''} \langle P'|\, S^\dagger \,|P''\rangle \langle P''; k|\, S\, |P\rangle + \int_J \langle P'|\, S^\dagger \,|J\rangle \langle J; k|\, S\, |P\rangle$$

EoM-based computation (integral of the source term) — Amplitude-based computation (integral of the on-shell action)

→ **BH mergers can be computed by scattering amplitudes!**

Summary

- ❑ We initiated a program describing BH mergers by on-shell amplitudes.
- ❑ The central idea: **black holes are particles!**

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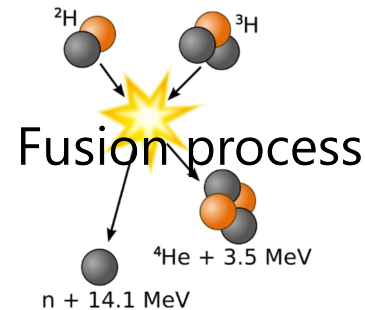

- ❑ Non-perturbative physics of merger can be packaged into massive 3pt.
- ❑ Waveforms are computed in two complementary cases.

1. Linear in G but no assumption about mass ratio (final spin)
   → all-order spin memory waveform (new prediction!).
2. Non-perturbative in G but leading in mass ratio
   → exact agreement with classical physics (proof of concept)

- ❑ **Can we use "new" ideas from phenomenology?**