Emergence of Time

Shinji Mukohyama
(YITP, Kyoto U)

arXiv: 1301.1361 with Jean-Philippe Uzan

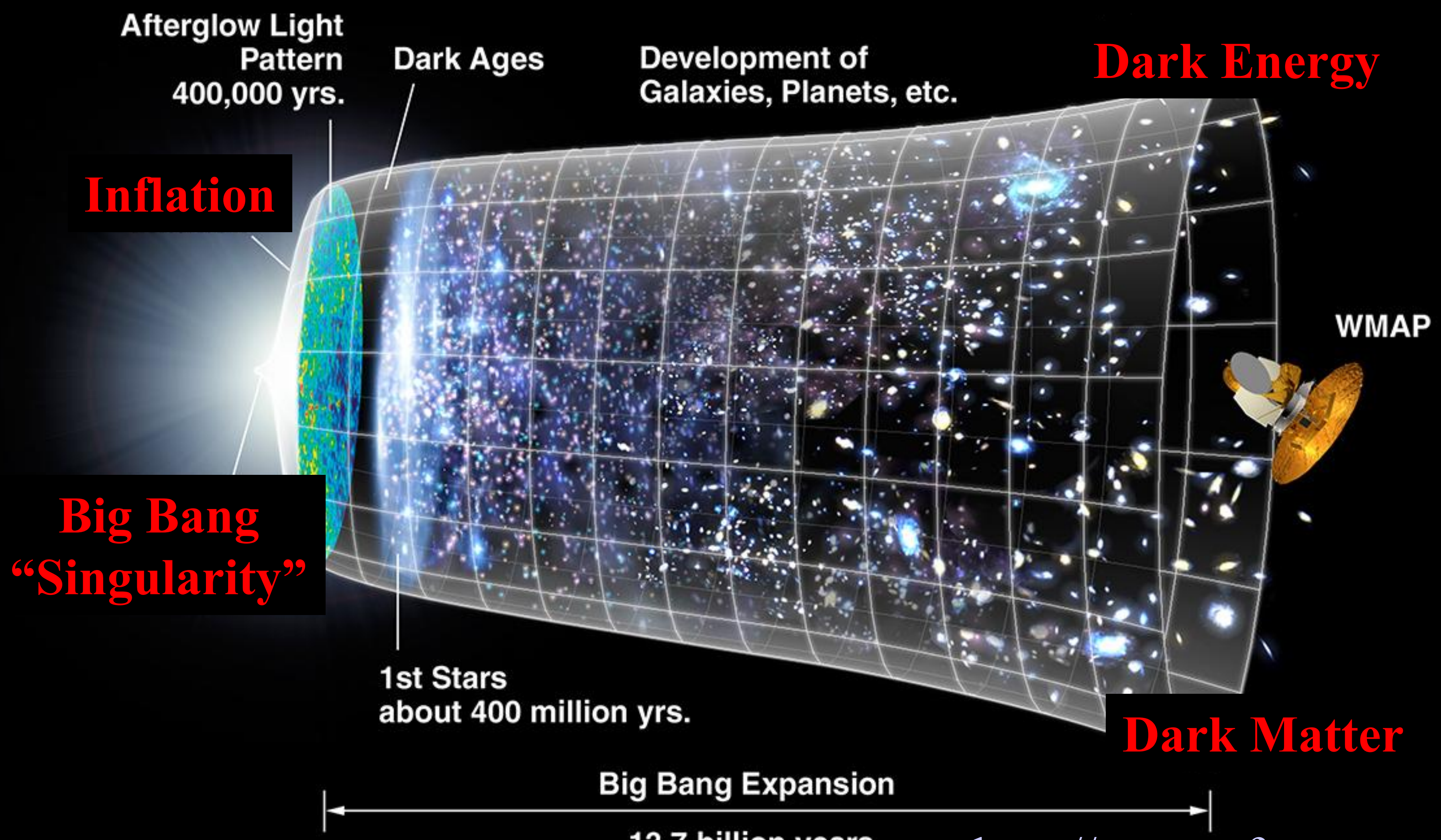
arxiv: 1303.1409

arXiv: 1403.0580 with John Kehayias & Jean-Philippe Uzan

arXiv: 2310.17266 & 2505.00112 with Justin Feng & Sante Carloni

INTRODUCTION

History of the universe



http://map.gsfc.nasa.gov/

History of the universe

- History = dynamics = sequence of configurations parameterized by time
- Beginning of the hot universe @ reheating
- Geometrical description of the universe breaks down @ initial singularity
- Space may be emergent
- How about time? Can time be emergent?

Time and dynamics

- In any diffeo-invariant theories of gravity,
  $H = \Sigma$ constraints = 0 (up to boundary terms)
  → no evolution of quantum state
- Dynamics should be encoded as correlations among various fields
  → one of the fields plays the role of time
  e.g. inflaton during inflation
- In this sense, concept of time and dynamics may be emergent

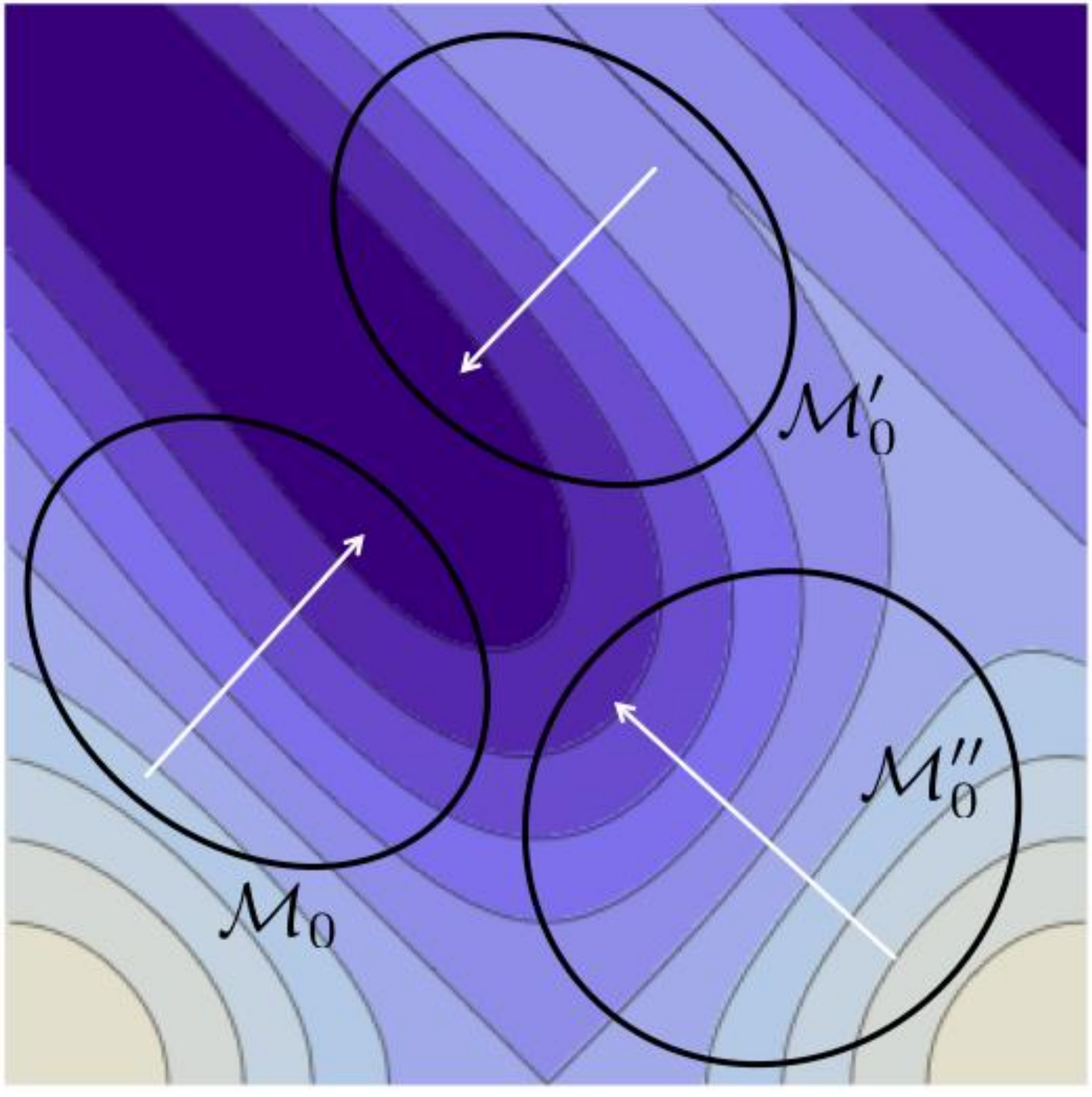
BASIC IDEA

arXiv: 1301.1361 with Jean-Philippe Uzan

Clock field

- Clock field = field playing the role of time
- It must carry at least one number
  ➔ simplest: scalar field ϕ
- Time translational symmetry requires
  shift symmetry: $\phi \rightarrow \phi + c$
- Time reflection symmetry requires
  Z_2 symmetry: $\phi \rightarrow -\phi$

- Clock field does not have to be the same everywhere; multi-clock models also possible


$\mathcal{M}_0'$
$\mathcal{M}_0$
$\mathcal{M}_0''$

Effective metric

- Lorentz symmetry may be emergent (Chadha & Nielsen 1983)
- How about Lorentz signature & time itself?
- Let’s suppose that there is
  no concept of time @ fundamental level
  and start with 4D Riemannian (i.e. locally Euclidean) metric with (++++) signature.
- Physical fields couple to effective metric.
- Can effective metric have signature (-+++) ?

SIMPLE EXAMPLES

arXiv: 1301.1361 with Jean-Philippe Uzan

Scalar field χ in flat space

- Suppose that $\partial_\mu \phi = \text{const.} \neq 0 \text{ in } \mathcal{M}_0$
- Choose one of coordinates t so that $t \equiv \dfrac{\phi}{M^2}$
- Consider the Euclidean action

potential

$$S_\chi = \int d^4x \left[-\frac{1}{2}\delta^{\mu\nu}\partial_\mu \chi \partial_\nu \chi - V(\chi) + \frac{\alpha}{M^4}\left(\delta^{\mu\nu}\partial_\mu \phi \partial_\nu \chi\right)^2 \right]$$

Euclidean kinetic term   coupling to clock field

- This can be rewritten as

$$S_\chi = \int dt d^3x \left[-\frac{1}{2} g_{eff}^{\mu\nu}\partial_\mu \chi \partial_\nu \chi - V(\chi) \right] \qquad g_{eff}^{\mu\nu} = \begin{pmatrix} 1-2\alpha & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

- Lorentz signature emerges if $\alpha > 1/2$!

Vector field A_μ in flat space

- Suppose that $\partial_\mu \phi = \text{const.} \neq 0 \text{ in } \mathcal{M}_0$
- Choose one of coordinates t so that $t \equiv \dfrac{\phi}{M^2}$
- Consider the Euclidean action

$$S_\chi = \int d^4x \left[-\frac{1}{4}\delta^{\mu\rho}\delta^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} + \frac{\alpha}{M^4}\delta^{\mu\rho}\delta^{\nu\sigma}\delta^{\alpha\beta}F_{\rho\alpha}F_{\sigma\beta}\partial_\mu\phi\partial_\nu\phi \right]$$

Euclidean kinetic term — coupling to clock field

- This can be rewritten as

$$S_\chi = \int dt d^3x \left[-\frac{1}{4} g_{eff}^{\mu\rho} g_{eff}^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \right]$$

$$g_{eff}^{\mu\nu} = \begin{pmatrix} 1-2\alpha & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

- Lorentz signature emerges if $\alpha > 1/2$!

Abelian Higgs field ω in flat space

- Suppose that $\partial_\mu \phi = \text{const.} \neq 0 \text{ in } \mathcal{M}_0$
- Choose one of coordinates t so that $t \equiv \dfrac{\phi}{M^2}$
- Consider the Euclidean action

$$D_\mu \equiv \partial_\mu - iqA_\mu$$

potential

$$S_\chi = \int d^4x \left[-\frac{1}{2}\delta^{\mu\nu}\left(D_\mu\omega\right)^*\left(D_\nu\omega\right) - U(|\omega|^2) + \frac{\alpha}{M^4}\left|\delta^{\mu\nu}\partial_\mu\phi D_\nu\omega\right|^2 \right]$$

Euclidean kinetic term

coupling to clock field

- This can be rewritten as

$$S_\chi = \int dt d^3x \left[-\frac{1}{2} g_{eff}^{\mu\nu}\left(D_\mu\omega\right)^*\left(D_\nu\omega\right) - U(|\omega|^2) \right] \qquad g_{eff}^{\mu\nu} = \begin{pmatrix} 1-2\alpha & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

- Lorentz signature emerges if α > 1/2!

GRAVITY @ LOW MOMENTA

arXiv: 1301.1361 with Jean-Philippe Uzan

Model of clock field and gravity

- Shift symmetry: $\phi \rightarrow \phi + c$
- Z_2 symmetry: $\phi \rightarrow - \phi$
- Minimal # of d.o.f. → 2nd-order EOM →
  Riemannian shift- & Z_2-symmetric Horndeski

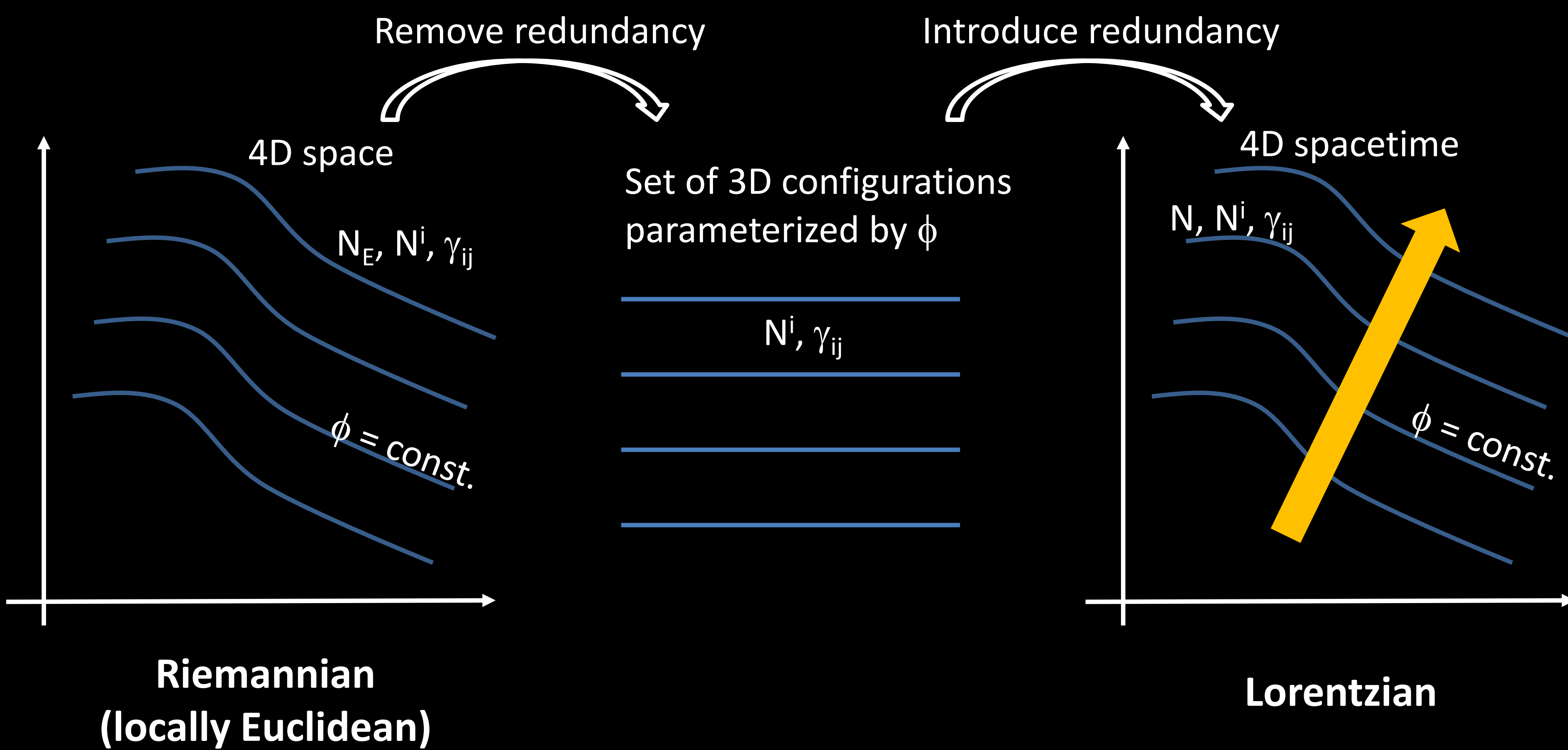
$$S_g = \int \mathrm{d}x^4 \sqrt{g_{\mathrm{E}}} \Big\{ G_4(X_{\mathrm{E}}) R_{\mathrm{E}} - g_5 G_{\mathrm{E}}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \mathcal{K}(X_{\mathrm{E}}) - 2G_4'(X_{\mathrm{E}}) \left[(\nabla_{\mathrm{E}}^2 \phi)^2 - (\nabla_\mu^{\mathrm{E}} \nabla_\nu^{\mathrm{E}} \phi)^2 \right] \Big\}$$

$$X_{\mathrm{E}} \equiv g_{\mathrm{E}}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \qquad (\nabla_\mu^{\mathrm{E}} \nabla_\nu^{\mathrm{E}} \phi)^2 \equiv g_{\mathrm{E}}^{\nu\rho} g_{\mathrm{E}}^{\sigma\mu} (\nabla_\mu^{\mathrm{E}} \nabla_\nu^{\mathrm{E}} \phi)(\nabla_\rho^{\mathrm{E}} \nabla_\sigma^{\mathrm{E}} \phi)$$

- Redefinition of $G_4(X_E)$ → $g_5 = 0$

$$S_g = \int \mathrm{d}x^4 \sqrt{g_{\mathrm{E}}} \Big\{ G_4(X_{\mathrm{E}}) R_{\mathrm{E}} + \mathcal{K}(X_{\mathrm{E}}) - 2G_4'(X_{\mathrm{E}}) \left[(\nabla_{\mathrm{E}}^2 \phi)^2 - (\nabla_\mu^{\mathrm{E}} \nabla_\nu^{\mathrm{E}} \phi)^2 \right] \Big\}$$

Riemannian diffeo. vs Lorentzian diffeo.



Correspondence

$$S_g = \int \mathrm{d}x^4 \sqrt{g_{\mathrm{E}}} \left\{ G_4(X_{\mathrm{E}}) R_{\mathrm{E}} + \mathcal{K}(X_{\mathrm{E}}) - 2G_4'(X_{\mathrm{E}}) \left[(\nabla_{\mathrm{E}}^2 \phi)^2 - (\nabla_\mu^{\mathrm{E}} \nabla_\nu^{\mathrm{E}} \phi)^2 \right] \right\}$$

$$g_{\mu\nu} = g_{\mu\nu}^{\mathrm{E}} - \frac{\partial_\mu \phi \partial_\nu \phi}{X_c}$$

$$g^{\mu\nu} = g_{\mathrm{E}}^{\mu\nu} + \frac{g_{\mathrm{E}}^{\mu\rho} g_{\mathrm{E}}^{\nu\sigma} \partial_\rho \phi \partial_\sigma \phi}{X_c - X_{\mathrm{E}}}$$

$$\frac{1}{X} = \frac{1}{X_c} - \frac{1}{X_{\mathrm{E}}} \qquad X_c = \frac{M^4}{N_c^2}$$

$$\frac{f(X)}{\sqrt{X}} = \frac{G_4(X_{\mathrm{E}})}{\sqrt{X_{\mathrm{E}}}} \qquad \frac{P(X)}{\sqrt{X}} = \frac{\mathcal{K}(X_{\mathrm{E}})}{\sqrt{X_{\mathrm{E}}}}$$

$$S_g = \int \mathrm{d}x^4 \sqrt{-g} \left\{ f(X) R + 2f'(X) \left[(\nabla^2 \phi)^2 - (\nabla^\mu \nabla^\nu \phi)(\nabla_\mu \nabla_\nu \phi) \right] + P(X) \right\}$$

CANDIDATE UV THEORY

arxiv: 1303.1409

Renormalizable theory of clock field & gravity

arxiv: 1303.1409

- Riemannian (i.e. locally Euclidean) theory
- Shift symmetry: $\phi \rightarrow \phi + c$
- Z_2 symmetry: $\phi \rightarrow - \phi$
- 4D parity invariance: $x^\mu \rightarrow -x^\mu$
- Power-counting renormalizable

$$
\begin{aligned}
I_{\rm IR} \;\; &= \;\; \int dx^4 \sqrt{g_{\rm E}}\, [2Z\Lambda_{\rm E} - ZR_{\rm E} \\
&\qquad + X_{\rm E}^2 - 2X_\star X_{\rm E} - 2\gamma(\nabla_{\rm E}^2\phi)^2 + 2\gamma(\nabla_\mu^{\rm E}\nabla_\nu^{\rm E}\phi)^2 + \gamma X_{\rm E} R_{\rm E}]
\end{aligned}
$$

UV action (4^{th} derivatives)

$$I_4 = \int dx^4 \sqrt{g_{\rm E}} \Big[c_1 R_{\rm E}^2 + c_2 R_{\rm E}^{\mu\nu} R^{\rm E}_{\mu\nu} + c_3 R_{\rm E}^{\mu\nu\rho\sigma} R^{\rm E}_{\mu\nu\rho\sigma} + c_4 X_{\rm E} R_{\rm E} + c_5 R_{\rm E}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + c_6 X_{\rm E}^2 + c_7 (\nabla_{\rm E}^2 \phi)^2 + c_8 (\nabla^{\rm E}_\mu \nabla^{\rm E}_\nu \phi)^2 \Big]$$

Relevant deformations (2^{nd} derivatives & 0^{th} derivatives)

$$I_2 = \int dx^4 \sqrt{g_{\rm E}} \left[c_9 R_{\rm E} + c_{10} X_{\rm E} \right] \qquad I_0 = c_{11} \int dx^4 \sqrt{g_{\rm E}}$$

Integration by parts (+ redef of $c_{7,8}$) & rescaling of ϕ

➡ $c_5 = 0 \qquad c_6 = 1$

Total action

$$I = \int dx^4 \sqrt{g_{\rm E}} \Big[2Z\Lambda_{\rm E} - Z R_{\rm E} + \frac{1}{2\lambda} C_{\rm E}^2 - \frac{\omega}{3\lambda} R_{\rm E}^2 + \frac{\theta}{\lambda} E_{\rm E} + X_{\rm E}^2 - 2X_\star X_{\rm E} + \alpha (\nabla_{\rm E}^2 \phi)^2 + \beta (\nabla^{\rm E}_\mu \nabla^{\rm E}_\nu \phi)^2 + \gamma X_{\rm E} R_{\rm E} \Big]$$

$$C_{\rm E}^2 \equiv R_{\rm E}^{\mu\nu\rho\sigma} R^{\rm E}_{\mu\nu\rho\sigma} - 2 R_{\rm E}^{\mu\nu} R^{\rm E}_{\mu\nu} + R_{\rm E}^2/3$$

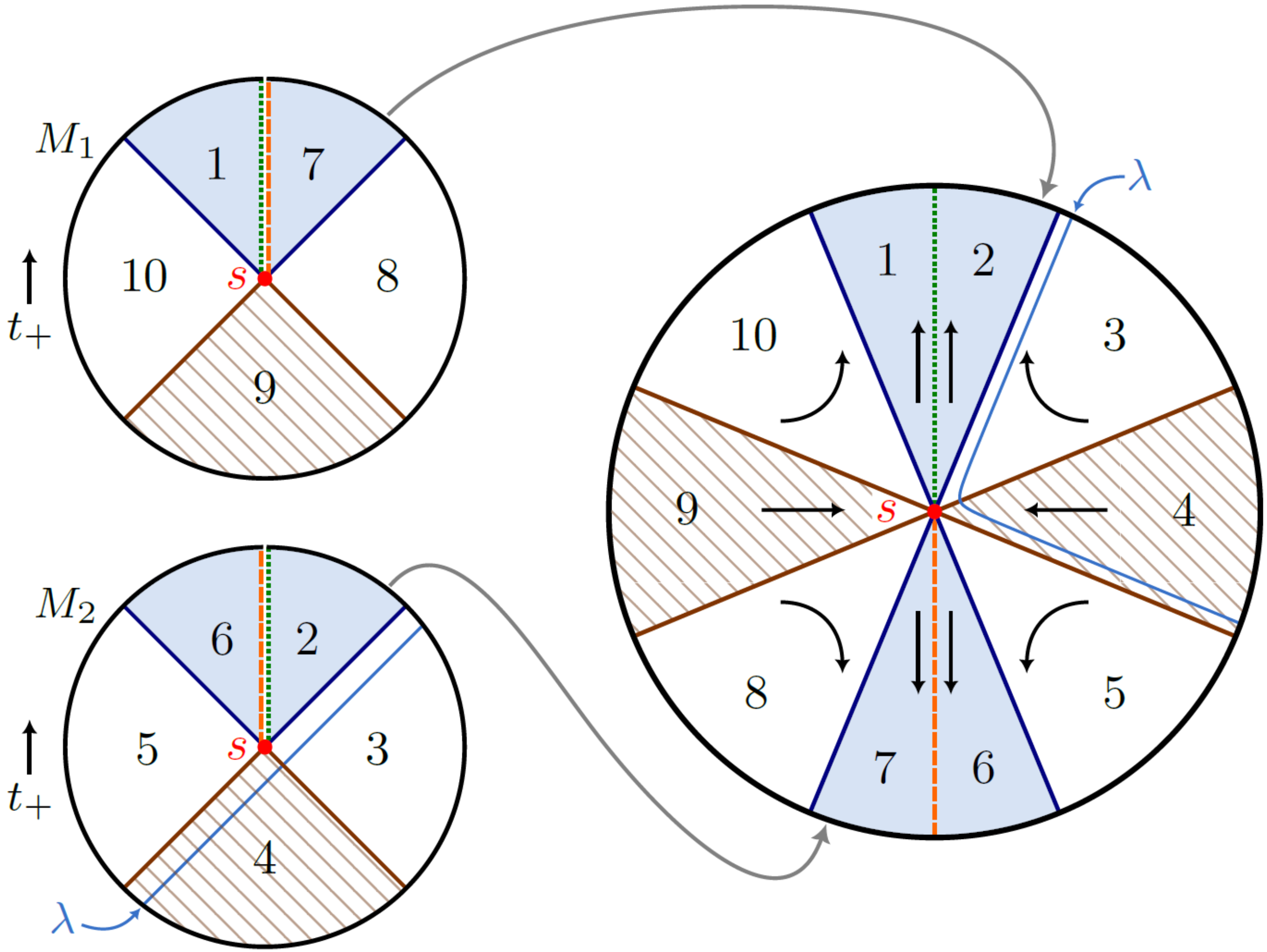
$$E_{\rm E} \equiv R_{\rm E}^{\mu\nu\rho\sigma} R^{\rm E}_{\mu\nu\rho\sigma} - 4 R_{\rm E}^{\mu\nu} R^{\rm E}_{\mu\nu} + R_{\rm E}^2$$

- This theory is indeed renormalizable (Muneyuki & Ohta 2013)
- RG flow has not yet been studied.
- Lorentzian signature is recovered in IR under a certain condition [arXiv: 2505.00112].
- In UV, higher derivative terms become important and physics is purely Euclidean.
- Matter action also needs to be UV completed by Euclidean higher derivative terms.
- Euclidean quantization needs to be developed.
- …

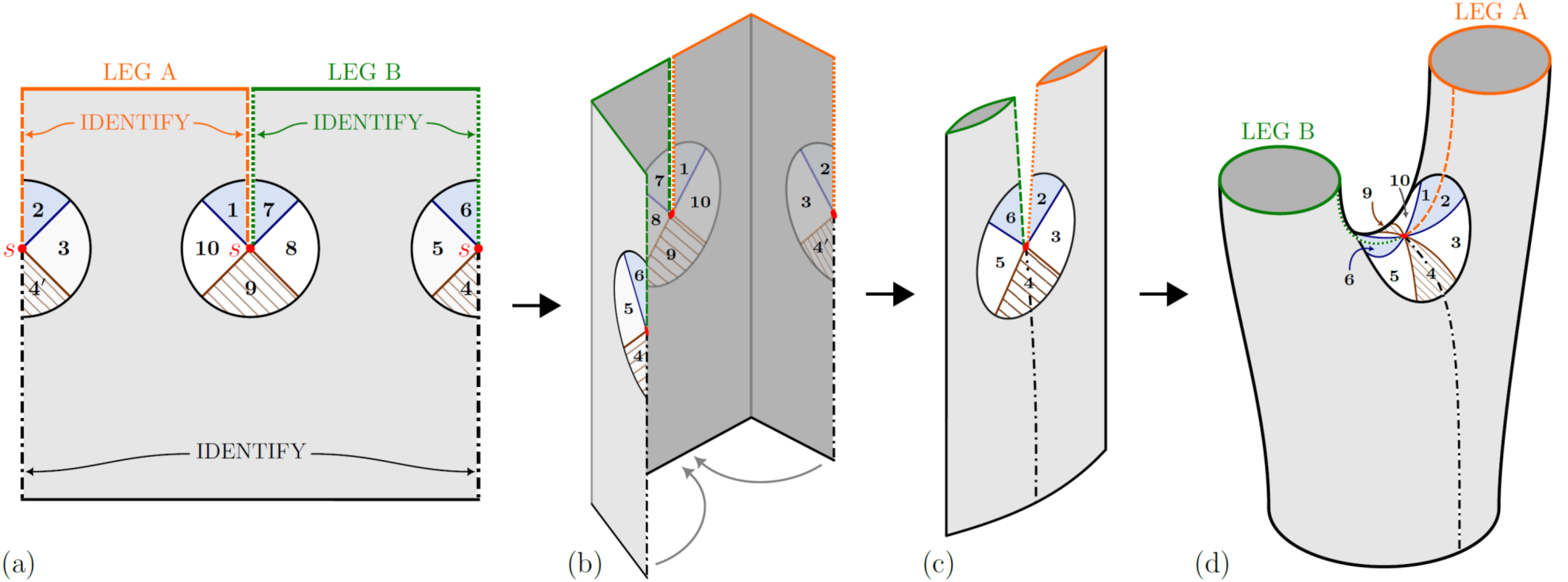
APPLICATION:
END OF BH EVAPORATION

arXiv: 2310.17266 with Justin Feng & Sante Carloni

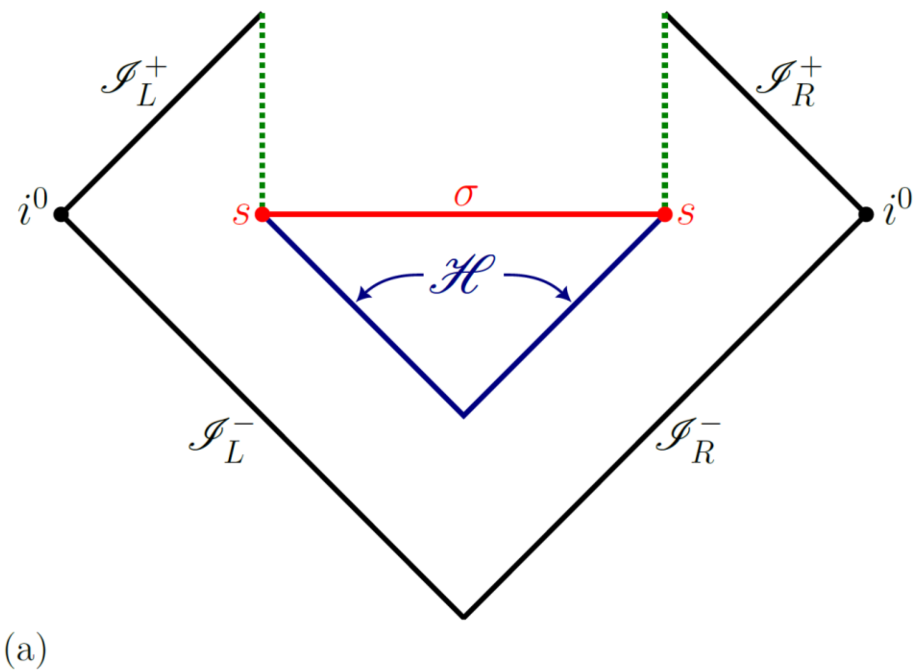
Saddle-like causally discontinuous singularity

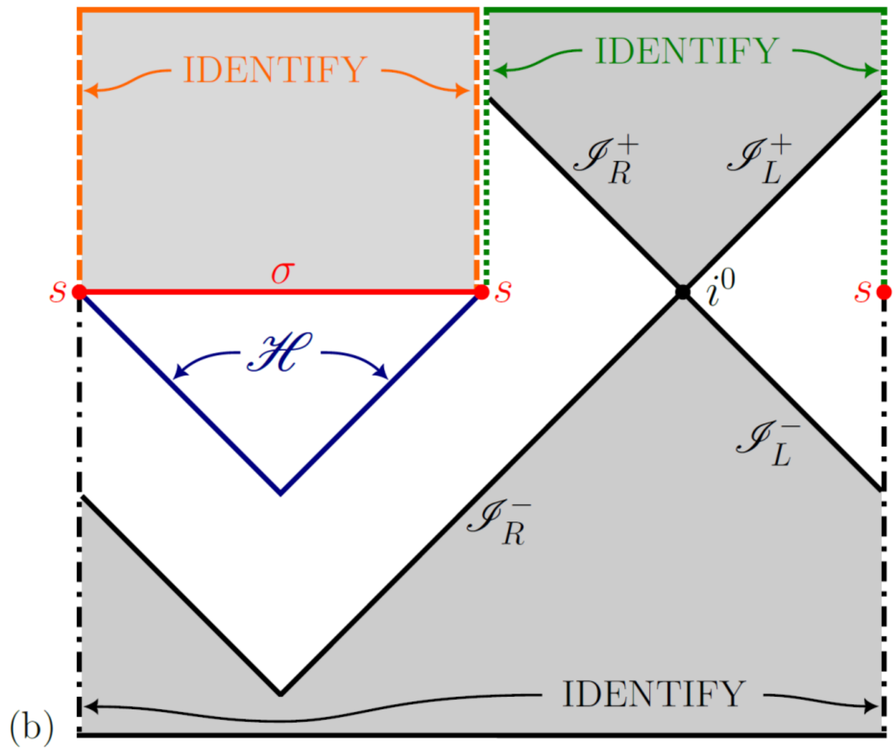


Trousers spacetime

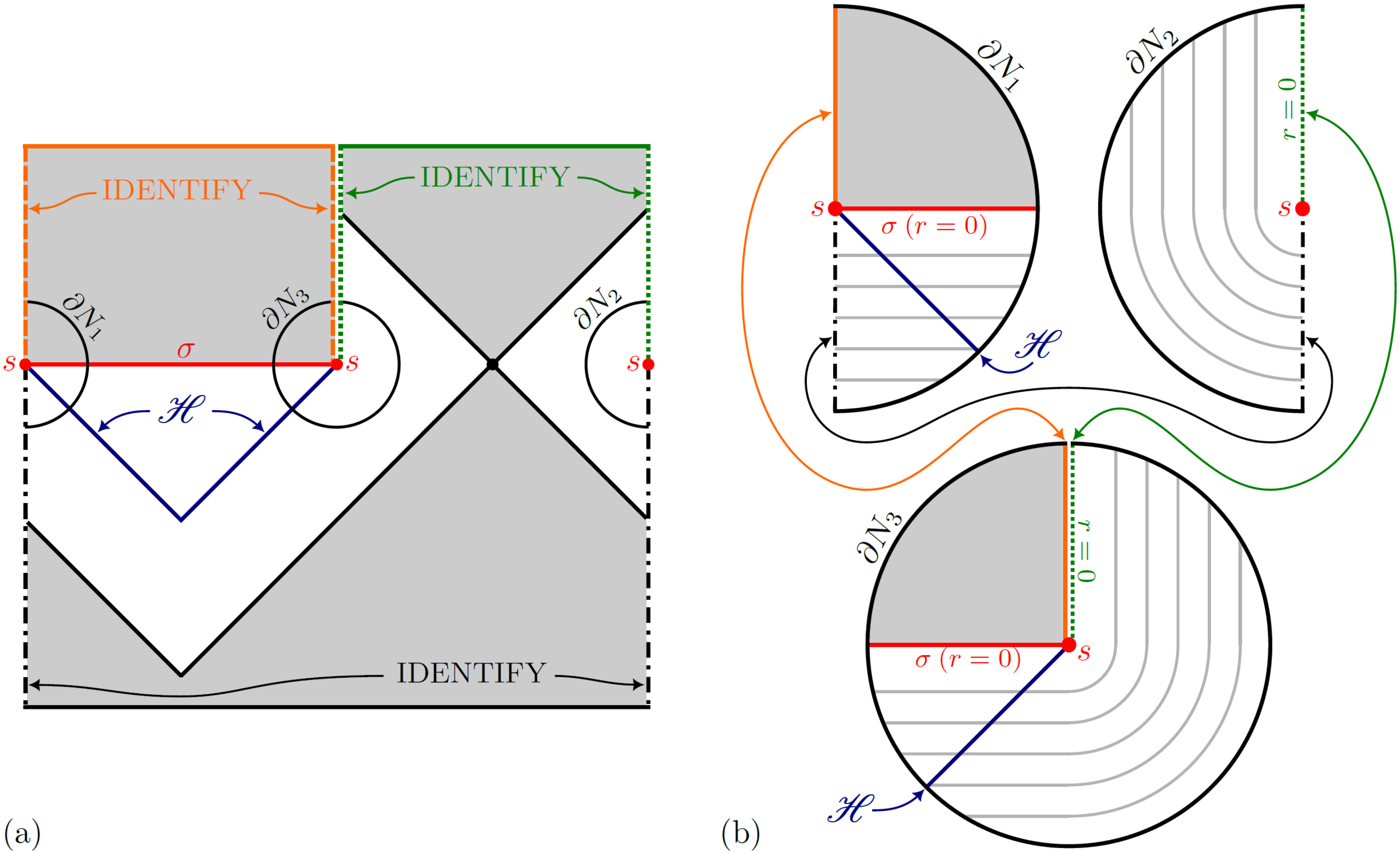


BH evaporation in 2D

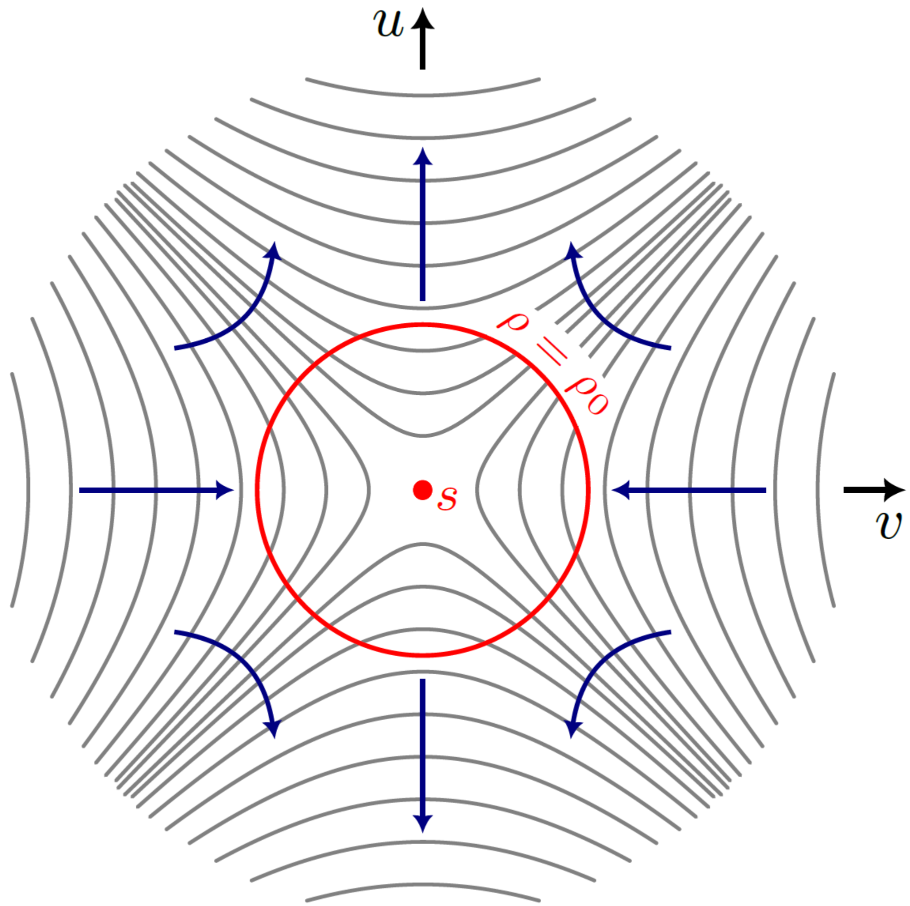




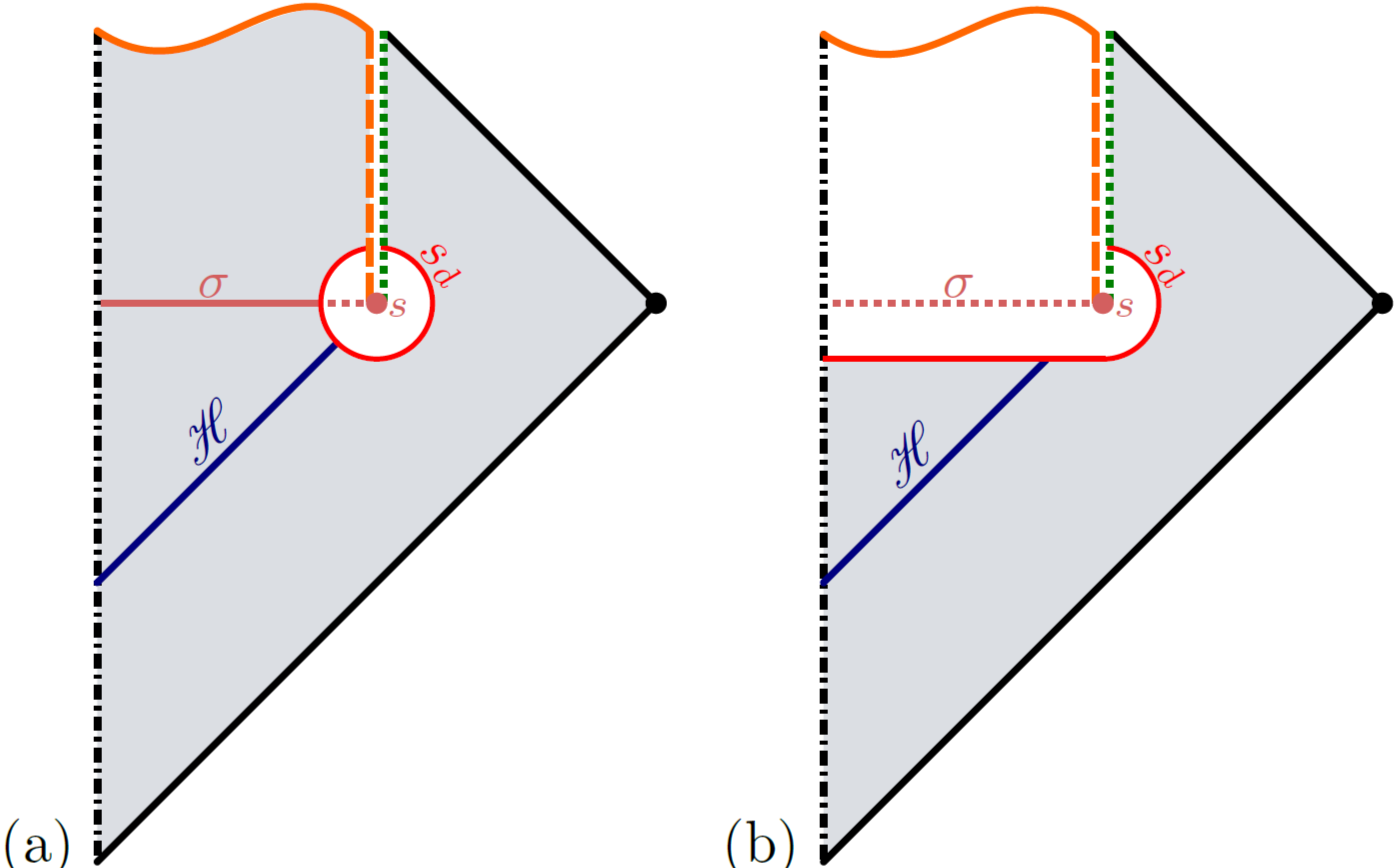
BH evaporation in (d+1)-D



Possible UV completion



Possible global picture



SUMMARY & DISCUSSIONS

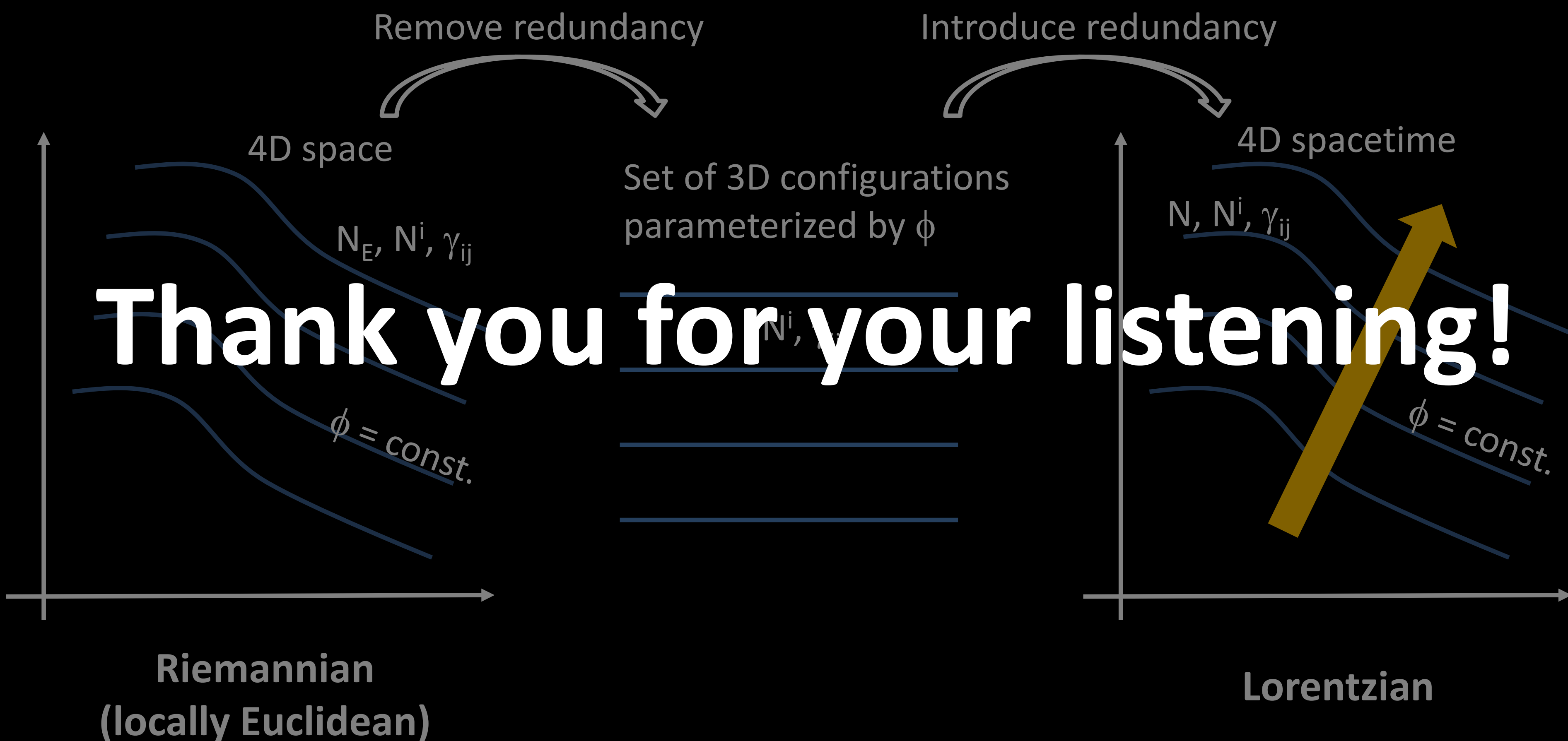
Summary

- Lorentzian dynamics can emerge as an effective property of a fundamentally Riemannian theory.
- This requires introduction of a field playing the role of time, a clock field.
- This idea was applied to scalar, vector, (Dirac, Weyl, Majorana) spinor fields and gravity as explicit examples.
- In a simple realization, the clock field/gravity sector is described by the Riemannian version of a shift- and Z_2-symmetric Horndeski theory.
- We proposed a power-counting renormalizable Riemanian theory as a candidate UV theory.
- We found a FLRW solution and analyzed stability of scalar- and tensor- perturbations.
- Endpoint of BH evaporation may be Euclidean geometry.

Future works

- Big-bang as Euclidean topological defect? [work in progress]
- Cosmological constant problem as boundary value problem?
- Lorentzian dS/CFT from Euclidean AdS/CFT?
- Emergence of Lorentz symmetry at low energy? [Chadha & Nielsen 1983; Groot Nibbelink & Pospelov 2005]
- Development of quantum theory?
- …
- Multi-clock models?
- Time emergence & compactification → landscape with various signatures & dimensions?

Riemannian diffeo. vs Lorentzian diffeo.



Thank you for your listening!

BACKUP SLIDES

COSMOLOGICAL SOLUTION

arXiv: 1301.1361 with Jean-Philippe Uzan

Cosmological solution

- Flat (K=0) FLRW

$$N = 1 \quad N_i = 0 \quad \gamma_{ij} = a(t)^2 \delta_{ij} \qquad \phi = \phi_0(t)$$

- EOM for ϕ = shift charge conservation

$$\dot{J}_\phi + 3HJ_\phi = 0 \quad \Rightarrow \quad \boxed{J_\phi \propto 1/a^3}$$

$$J_\phi \equiv \left[P_0' + 6H^2(2X_0 f_0'' + f_0')\right]\dot{\phi}_0$$

- Metric EOM

$$3M_{\rm eff}^2 H^2 = 2J_\phi \dot{\phi}_0 - P_0 \qquad M_{\rm eff}^2 \equiv 2(f_0 - 2X_0 f_0')$$

- P'(X) and f'(X) near a local minimum of P(X)

$$P'(X) = p_2\delta + \mathcal{O}(\delta^2) \quad f'(X) = \frac{f_1 + f_2\delta}{M^2} + \mathcal{O}(\delta^2) \quad \delta \equiv \frac{X}{M^4} - q$$

$$J_\phi \propto 1/a^3 \Rightarrow \quad \delta + \mathcal{O}(H^2/M^2) \propto 1/a^3 \rightarrow 0$$

$\Rightarrow$ $3M_{\rm eff}^2H^2$ = DM ($\propto 1/a^3$) + DE (∼const)

Stability of tensor perturbation

- Tensor-type perturbation

$$N = 1 \quad N_i = 0 \quad \gamma_{ij} = a(t)^2 \left[\mathrm{e}^{h} \right]_{ij}$$

$$\phi = \phi_0(t) \qquad \boxed{\partial_i h^i_k = 0 = \delta^{ij} h_{ij}}$$

- Quadratic action in Fourier space

$$\delta S^{(2)}_{\mathrm{T},\boldsymbol{k}} = \frac{1}{8} \int \mathrm{d}t a^3 \left[\boxed{M^2_{\mathrm{eff}}} \dot{h}^2_{\boldsymbol{k}} - 2 \boxed{f_0} \frac{\boldsymbol{k}^2}{a^2} h^2_{\boldsymbol{k}} \right]$$

$$M^2_{\mathrm{eff}} \equiv 2(f_0 - 2X_0 f'_0)$$

- Stability condition

$$\boxed{M^2_{\mathrm{eff}} > 0 \qquad f_0 > 0}$$

Stability of scalar perturbation

- Scalar-type perturbation in unitary gauge

$$N = 1 + \alpha \quad N_i = \partial_i \beta \quad \gamma_{ij} = a(t)^2 \mathrm{e}^{2\zeta} \delta_{ij}$$

$$\phi = \phi_0(t)$$

- Quadratic action after eliminating α and β

$$\delta S^{(2)}_{\mathrm{S},\boldsymbol{k}} = \frac{1}{2} \int \mathrm{d}t a^3 \left[\mathcal{A} \dot{\zeta}_{\boldsymbol{k}}^2 - \mathcal{B} \frac{\boldsymbol{k}^2}{a^2} \zeta_{\boldsymbol{k}}^2 \right]$$

$$\mathcal{A} = \frac{M_{\mathrm{eff}}^2}{H^2 \mathcal{G}^2} \left(6 + M_{\mathrm{eff}}^2 \mathcal{F}\right) \qquad \mathcal{B} = \frac{1}{a} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{a M_{\mathrm{eff}}^4}{H \mathcal{G}^2} \right) + 4 f_0$$

$$\mathcal{F} = P_0'' X_0^2 + \frac{1}{2} J_\phi \dot{\phi}_0 + 3H^2 \left[4 f_0''' X_0^3 + 14 f_0'' X_0^2 + 6 f_0' X_0 - f_0 \right]$$

$$\mathcal{G} = 4 f_0'' X_0^2 + 4 f_0' X_0 - f_0 \qquad M_{\mathrm{eff}}^2 \equiv 2(f_0 - 2 X_0 f_0')$$

- Stability condition

$$\mathcal{A} > 0 \qquad \mathcal{B} > 0$$

PHENOMENOLOGY

arXiv: 1301.1361 with Jean-Philippe Uzan

Free functions/parameters

- Gravity sector: $G_4(X_E)$, $K(X_E)$ [or $f(X)$, $P(X)$]

$$S_g = \int dx^4 \sqrt{g_E} \Big\{ G_4(X_E) R_E - g_5 G_E^{\mu\nu} \partial_\mu\phi \partial_\nu\phi + \mathcal{K}(X_E) - 2 G_4'(X_E) \left[(\nabla_E^2 \phi)^2 - (\nabla_\mu^E \nabla_\nu^E \phi)^2 \right] \Big\}$$

- Matter sector: $(\kappa_\chi, \alpha_\chi)$, (κ_A, α_A), $(\kappa_\omega, \alpha_\omega)$

$$S_\chi = \int dx^4 \sqrt{g_E} \left[-\frac{\kappa_\chi}{2} g_E^{\mu\nu} \partial_\mu\chi \partial_\nu\chi - \tilde{V}(\chi) + \frac{\alpha_\chi}{2M^4} (g_E^{\mu\nu} \partial_\mu\phi \partial_\nu\chi)^2 \right]$$

$$S_A = \frac{1}{4} \int dx^4 \sqrt{g_E} \left[-\kappa_A F_E^{\mu\nu} F_{\mu\nu} + 2 \frac{\alpha_A}{M^4} F_E^{\mu\rho} F_{E\rho}^{\nu} \partial_\mu\phi \partial_\nu\phi \right]$$

$$S_\psi = \int dx^4 \sqrt{g_E} \Big\{ -\frac{\kappa_\psi}{2} g_E^{\mu\nu} (\partial_\mu + iqA_\mu)\psi^* (\partial_\nu - iqA_\nu)\psi + \frac{\alpha_\psi}{2M^4} |g_E^{\mu\nu} \partial_\mu\phi (\partial_\nu - iqA_\nu)\psi|^2 - \tilde{U}(|\psi|^2) \Big\}$$

- Clock field configuration: $\phi(x)$

$$X_E \equiv g_E^{\mu\nu} \partial_\mu\phi \partial_\nu\phi$$

Constraints

- Stability of clock field/gravity sector

$$M_{\mathrm{eff}}^2 > 0 \qquad f_0 > 0 \qquad \mathcal{A} > 0 \qquad \mathcal{B} > 0$$

- Amount of DE/DM

$$P_0 \sim -3\Omega_{\Lambda 0} M_{\mathrm{eff}}^2 H_0^2 \sim -2.1 M_{\mathrm{eff}}^2 H_0^2$$

$$\frac{2}{3}\frac{J_{\phi_0}}{M_{\mathrm{eff}}^2}\sqrt{q}\frac{M^2}{H_0^2} \leq \Omega_{\mathrm{m}0} \sim 0.3$$

- Stability of matter sector

$$\frac{\alpha_\chi}{N_{\mathrm{E}}^2} > \kappa_\chi > 0 \qquad \frac{\alpha_A}{N_{\mathrm{E}}^2} > \kappa_A > 0$$

- Coincidence of speed limits in matter sector

$$\frac{\kappa_A}{\alpha_A} = \frac{\kappa_\chi}{\alpha_\chi}$$ independently from clock field configuration

- Speed of GW [LIGO/Virgo 2017]

$$-3\times 10^{-16} < \frac{c_{GW}}{c_\gamma} - 1 < 7\times 10^{-16} \qquad c_\gamma^2 = \left[\frac{\alpha_A X_{\mathrm{E}}}{\kappa_A M^4} - 1\right]^{-1} \quad c_{\mathrm{GW}}^2 = \left[\frac{2G_4' X_{\mathrm{E}}}{G_4} - 1\right]^{-1}$$

FERMIONS IN FLAT SPACE

arXiv: 1403.0580 with John Kehayias & Jean-Philippe Uzan

Dirac spinor ψ in flat space

- Consider the Euclidean action

$$S_\psi = \int dx^4 \left\{ \bar{\psi} \left(\frac{i}{2} \gamma_{\mathrm{E}}^{\mu} \overleftrightarrow{\partial_\mu} - m \right) \psi \right.$$

Euclidean kinetic and mass terms

$$\left. + \frac{1}{2M^2} \delta^{\mu\nu} \left[(i\bar{\psi}\gamma_{\mathrm{E}}^{5} \overleftrightarrow{\partial_\mu} \psi) - (i\bar{\psi}\gamma_{\mathrm{E}}^{\rho} \overleftrightarrow{\partial_\mu} \psi) \partial_\rho \phi \right] \partial_\nu \phi \right\}$$

couplings to clock field

- This can be rewritten as

$$S_\psi = \int dx^4 \bar{\psi} \left[\frac{i}{2} \gamma^0 \overleftrightarrow{\partial_0} + \frac{i}{2} \gamma^i \overleftrightarrow{\partial_i} - m \right] \psi$$

- Lorentzian & Euclidean gamma matrices

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} \qquad \gamma^5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\gamma_{\mathrm{E}}^{0} \equiv i\gamma^5 \qquad \gamma_{\mathrm{E}}^{i} \equiv \gamma^i \qquad \gamma_{\mathrm{E}}^{5} \equiv \gamma_{\mathrm{E}}^{0}\gamma_{\mathrm{E}}^{1}\gamma_{\mathrm{E}}^{2}\gamma_{\mathrm{E}}^{3} = \gamma^0$$

$$\{\gamma_{\mathrm{E}}^{\mu}, \gamma_{\mathrm{E}}^{\nu}\} = -2\delta^{\mu\nu} \qquad (\gamma_{\mathrm{E}}^{5})^2 = 1 \qquad \{\gamma_{\mathrm{E}}^{5}, \gamma_{\mathrm{E}}^{\mu}\} = 0$$

MATTER FIELDS IN CURVED SPACE

arXiv: 1301.1361 with Jean-Philippe Uzan

Scalar field χ in curved space

- ADM decomposition in unitary gauge $t \equiv \dfrac{\phi}{M^2}$

$$g^{\mathrm{E}}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = N_{\mathrm{E}}^2\mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}x^i + N^i\mathrm{d}t)(\mathrm{d}x^j + N^j\mathrm{d}t)$$

- Riemannian action

$$\begin{aligned} S_\chi &= \int \mathrm{d}x^4\sqrt{g_{\mathrm{E}}}\left[-\frac{\kappa_\chi}{2}g_{\mathrm{E}}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - \tilde{V}(\chi) + \frac{\alpha_\chi}{2M^4}(g_{\mathrm{E}}^{\mu\nu}\partial_\mu\phi\partial_\nu\chi)^2\right] \\ &= \int \mathrm{d}t\mathrm{d}x^3 N_{\mathrm{E}}\sqrt{\gamma}\left[\frac{1}{2}\left(\frac{\alpha_\chi}{N_{\mathrm{E}}^2} - \kappa_\chi\right)(\partial_\perp^{\mathrm{E}}\chi)^2 - \tilde{V}(\chi) - \frac{\kappa_\chi}{2}\gamma^{ij}\partial_i\chi\partial_j\chi\right] \end{aligned}$$

- If $\dfrac{\alpha_\chi}{N_{\mathrm{E}}^2} > \kappa_\chi > 0$, then the action is rewritten as

$$S_\chi = -\int \mathrm{d}x^4\sqrt{-g^\chi}\left[\frac{1}{2}g_\chi^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + V(\chi, X)\right]$$

$$V(\chi, X) = \tilde{V}(\chi)\left[\kappa_\chi^3\left(\frac{\alpha_\chi X_{\mathrm{E}}}{M^4} - \kappa_\chi\right)\right]^{-1/2}$$

with Lorentzian metric

$$g^\chi_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -N_\chi^2\mathrm{d}t^2 + \Omega_\chi^2\gamma_{ij}(\mathrm{d}x^i + N^i\mathrm{d}t)(\mathrm{d}x^j + N^j\mathrm{d}t)$$

$$N_\chi = N_{\mathrm{E}}\left[\frac{\kappa_\chi^3}{\frac{\alpha_\chi}{N_{\mathrm{E}}^2} - \kappa_\chi}\right]^{1/4} \qquad \Omega_\chi = \left[\kappa_\chi\left(\frac{\alpha_\chi}{N_{\mathrm{E}}^2} - \kappa_\chi\right)\right]^{1/4}$$

Vector field A_μ in curved space

- ADM decomposition in unitary gauge $\quad t \equiv \frac{\phi}{M^2}$

$$g^{\mathrm{E}}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = N_{\mathrm{E}}^2 \mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}x^i + N^i \mathrm{d}t)(\mathrm{d}x^j + N^j \mathrm{d}t)$$

- Riemannian action $\quad \tilde{F}_{\perp i} \equiv \frac{1}{N_{\mathrm{E}}}(F_{ti} - N^j F_{ji})$

$$S_A = \frac{1}{4}\int dx^4 \sqrt{g_{\mathrm{E}}}\left[-\kappa_A F^{\mu\nu}_{\mathrm{E}} F_{\mu\nu} + 2\frac{\alpha_A}{M^4} F^{\mu\rho}_{\mathrm{E}} F^{\nu}_{\mathrm{E}\rho}\partial_\mu\phi\partial_\nu\phi\right]$$

$$= \frac{1}{4}\int \mathrm{d}t\mathrm{d}x^3 N_{\mathrm{E}}\sqrt{\gamma}\left[2\left(\frac{\alpha_A}{N_{\mathrm{E}}^2} - \kappa_A\right)\gamma^{ij}\tilde{F}_{\perp i}\tilde{F}_{\perp j} - \kappa_A \gamma^{ik}\gamma^{jl}F_{ij}F_{kl}\right]$$

- If $\frac{\alpha_A}{N_{\mathrm{E}}^2} > \kappa_A > 0$, then the action is rewritten as

$$S_A = -\int \mathrm{d}x^4 \sqrt{-g^A}\frac{1}{4e^2} g^{\mu\rho}_A g^{\nu\sigma}_A F_{\mu\nu}F_{\rho\sigma} \qquad e^2 = \left[\kappa_A\left(\frac{\alpha_A}{N_{\mathrm{E}}^2} - \kappa_A\right)\right]^{-1/2}$$

with Lorentzian metric

$$g^{A}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -N_A^2 dt^2 + \Omega_A^2 \gamma_{ij}(\mathrm{d}x^i + N^i \mathrm{d}t)(\mathrm{d}x^j + N^j \mathrm{d}t)$$

$$N_A = N_{\mathrm{E}}\Omega_A\left[\frac{\kappa_A}{\frac{\alpha_A}{N_{\mathrm{E}}^2} - \kappa_A}\right]^{1/2} \qquad \Omega_A > 0$$

Abelian Higgs field ψ in curved space

- ADM decomposition in unitary gauge $t \equiv \frac{\phi}{M^2}$

$$g^{\mathrm{E}}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = N_{\mathrm{E}}^2 \mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}x^i + N^i \mathrm{d}t)(\mathrm{d}x^j + N^j \mathrm{d}t)$$

- Riemannian action

$$S_\psi = \int \mathrm{d}x^4 \sqrt{g_{\mathrm{E}}} \left\{ -\frac{\kappa_\psi}{2} g_{\mathrm{E}}^{\mu\nu}(\partial_\mu + iqA_\mu)\psi^*(\partial_\nu - iqA_\nu)\psi + \frac{\alpha_\psi}{2M^4} |g_{\mathrm{E}}^{\mu\nu}\partial_\mu\phi(\partial_\nu - iqA_\nu)\psi|^2 - \tilde{U}(|\psi|^2) \right\}$$

- If $\frac{\alpha_\psi}{N_{\mathrm{E}}^2} > \kappa_\psi > 0$, then the action is rewritten as

$$S_\psi = -\int \mathrm{d}x^4 \sqrt{-g^\psi} \left[\frac{1}{2} g_\psi^{\mu\nu}(\partial_\mu + iqA_\mu)\psi^*(\partial_\nu - iqA_\nu)\psi + U(|\psi|^2, X) \right]$$

$$U(|\psi|^2, X) = \tilde{U}(|\psi|^2) \left[\kappa_\psi^3 \left(\frac{\alpha_\psi X_{\mathrm{E}}}{M^4} - \kappa_\psi \right) \right]^{-1/2}$$

with Lorentzian metric

$$g^\psi_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -N_\psi^2 \mathrm{d}t^2 + \Omega_\psi^2 \gamma_{ij}(\mathrm{d}x^i + N^i \mathrm{d}t)(\mathrm{d}x^j + N^j \mathrm{d}t)$$

$$N_\psi = N_{\mathrm{E}} \left[\frac{\kappa_\psi^3}{\frac{\alpha_\psi}{N_{\mathrm{E}}^2} - \kappa_\psi} \right]^{1/4} \qquad \Omega_\psi = \left[\kappa_\psi \left(\frac{\alpha_\psi}{N_{\mathrm{E}}^2} - \kappa_\psi \right) \right]^{1/4}$$