Phase transitions in monitored oscillator chains and Josephson junction arrays

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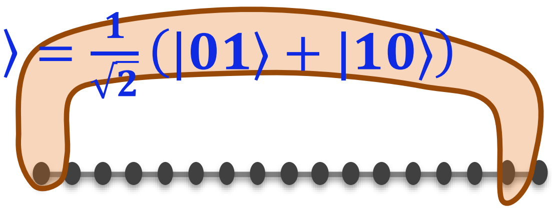
Frontiers in Nonequilibrium Physics
Yukawa Institute for Theoretical Physics, Kyoto, May 11, 2026

Superposition

$|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ **Measurement** → Collapses to $|0\rangle$ or $|1\rangle$ with probability 1/2

Entanglement

$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$



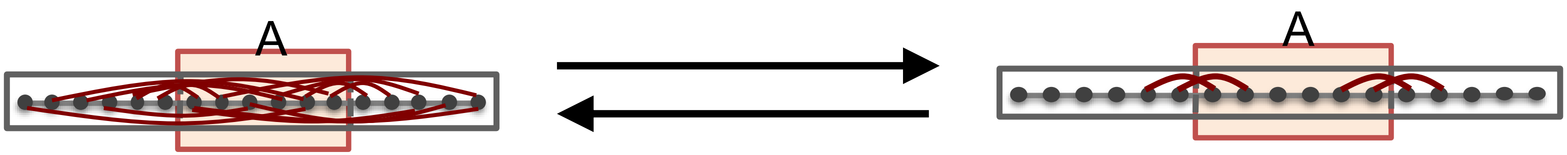
Entanglement entropy $S_1 = \ln 2$

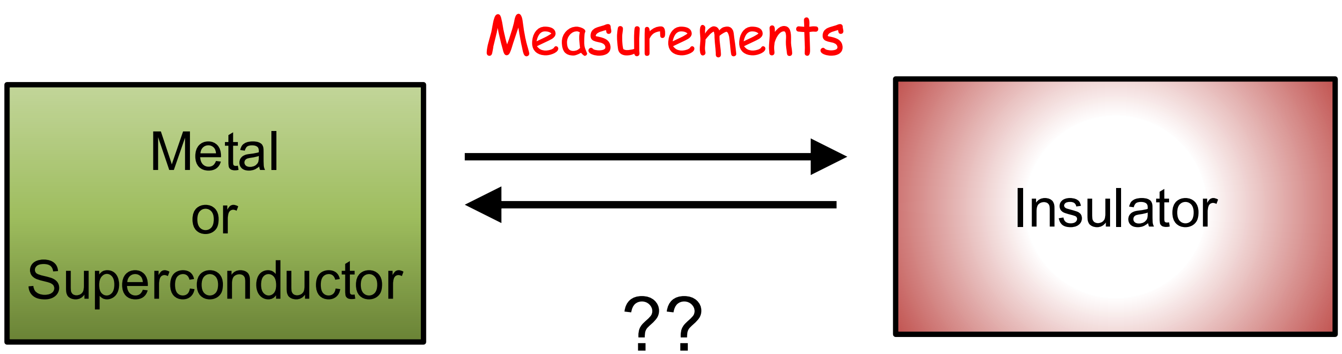
→ Measure state of one qubit

$|01\rangle$ or $|10\rangle$ with probability 1/2

Unentangled product state

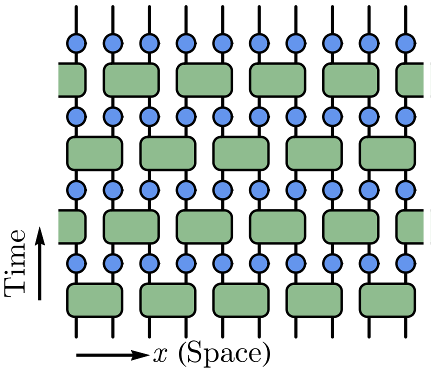
Can measurements change the nature of many-body systems and induce phase transitions?





Measurement-induced phase transition (MIPT)

Random quantum circuits with non-unitary evolution



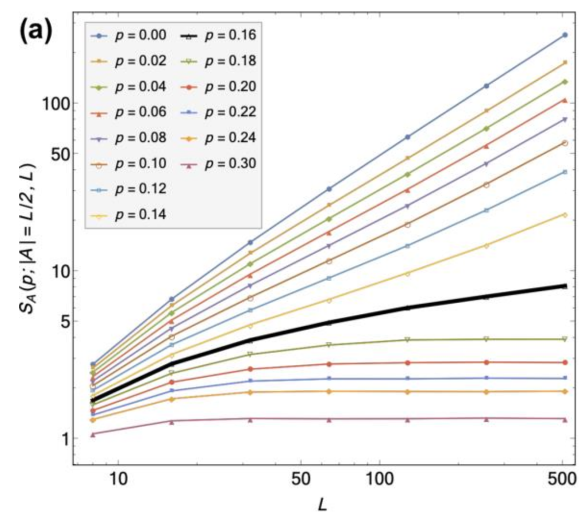
Unitary evolution + measurements

local projective or weak measurements

Tune measurement rate, measurement strength, etc., ..

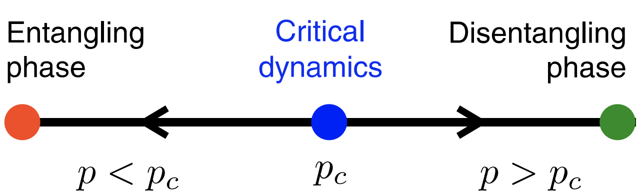
Other systems – non-interacting fermions, interacting bosons, Luttinger liquids, ..

Skinner et al. (2019), Bao et al. (2019);
Li et al. (2019), Jian et al. (2019),
Gullans et al. (2020), Nahum et al. (2021)
Sang et al. (2021), ...



Transition in steady state from volume-law to area-law entanglement

“Chaotic”	“Non chaotic”
Volume law	Area law
$S_A \sim L^d$	$S_A \sim L^{d-1}$

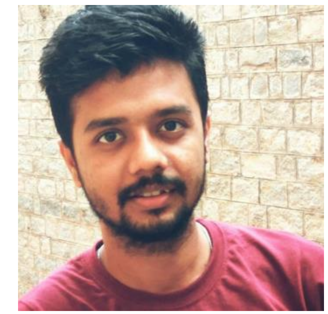


Chaotic to non-chaotic phase transitions

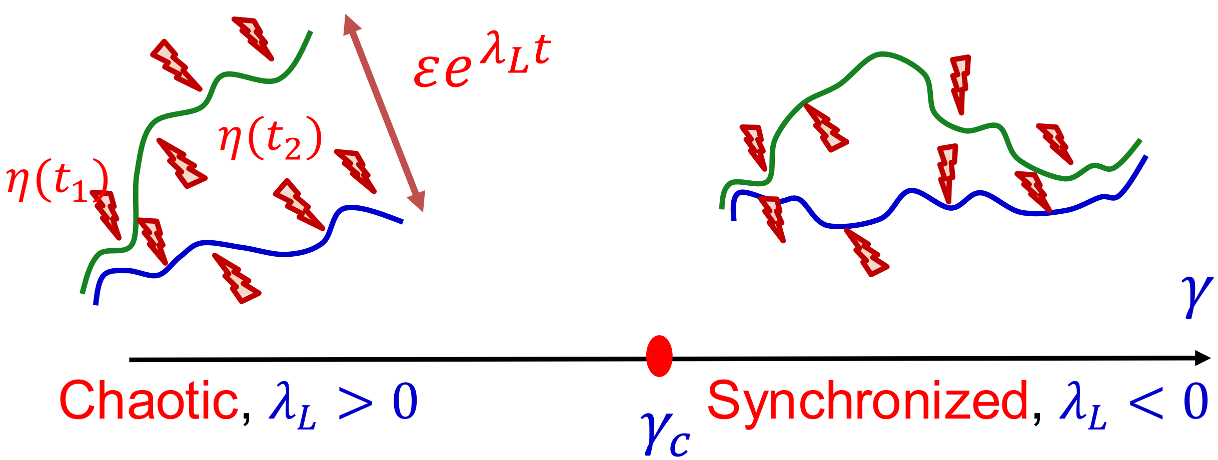
Can there be any “measurement-induced phase transition” in the “semi-classical limit”?

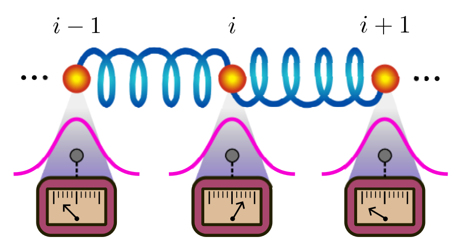
Semiclassical Limit of a Measurement-Induced Transition in Many-Body Chaos in Oscillator Chains

S. Ruidas & SB,
Phys. Rev. Lett. 132, 030402 (2024)



Sibaram Ruidas
(ICTS)





Noise/dissipation (measurement) strength

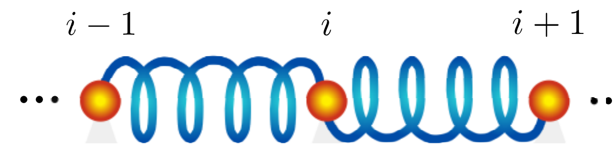
Well-known stochastic synchronization transition in non-linear dynamics

Quantum model of weak position measurements + feedback

Caves and Milburn, Phys. Rev. A 36 (1987)

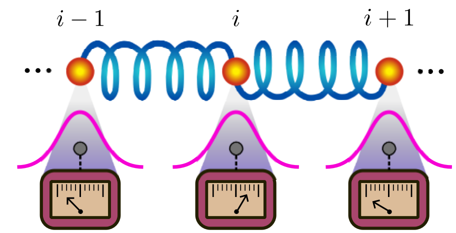
System under repeated weak measurements in intervals of τ

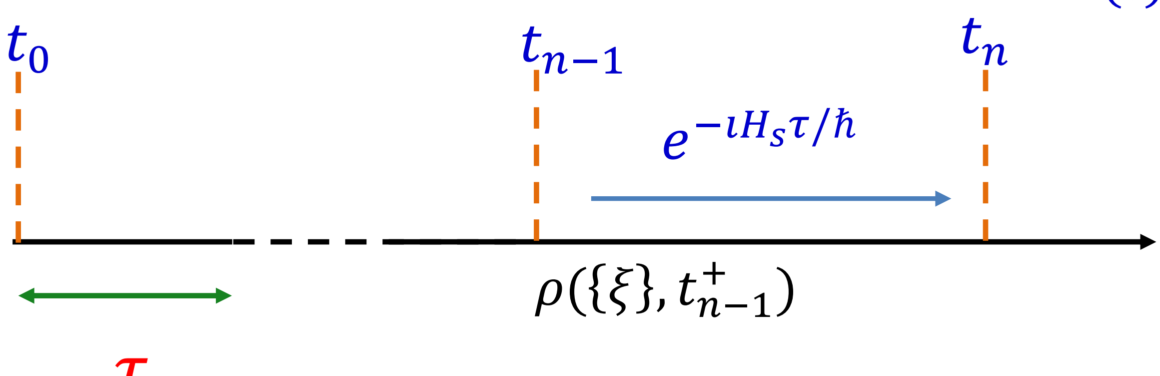
$t_n = n\tau$



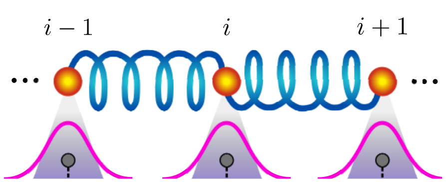
$$H_s = \sum_{i=1}^{N} \frac{\hat{p}_i^2}{2m} + V(\{\hat{x}_i\})$$

$$H(t) = H_s + \sum_{i,n} \delta(t - t_n)\, \hat{x}_i\, \hat{p}_{in}$$





t_n^-

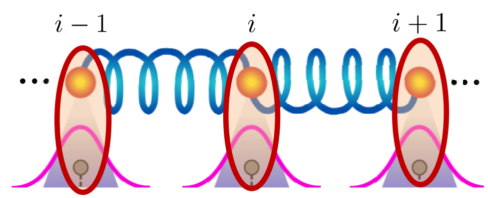


Detector

$\hat{\xi}_{in}, \hat{p}_{in}$

$$\psi(\xi_{in}) \sim \exp\left(-\frac{\xi_{in}^2}{2\sigma}\right)$$

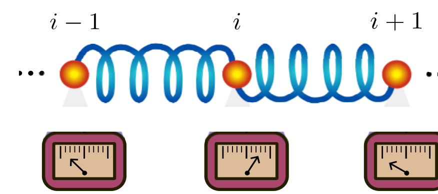
t_n



Apply

$$\sum_i \delta(t - t_n)\, \hat{x}_i\, \hat{p}_{in}$$

t_n^+

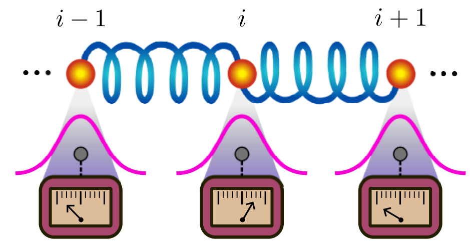


Readings

$\{\xi_{in}\}$

Projective measurements

Non-unitary time evolution of system density matrix



$$\rho(\{\xi\}_n, t_n^+) = M(\xi_n) e^{-\frac{\iota H_S \tau}{\hbar}} \rho(\{\xi\}_{n-1}, t_{n-1}^+) e^{\frac{\iota H_S \tau}{\hbar}} M^\dagger(\xi_n)$$

Measurement + feedback

$$M(\xi_n) \sim \prod_i \exp\left(\frac{\iota \gamma \tau \xi_{in} \hat{p}_i}{\hbar}\right) \quad \exp\left(-\frac{(\xi_{in} - \hat{x}_i)^2}{2\Delta}\tau\right)$$

Caves and Milburn, Phys. Rev. A (1987)

Momentum “feedback” $\gamma \sim \sqrt{\hbar/\Delta}$, acts like dissipation

Limit of continuous weak measurement

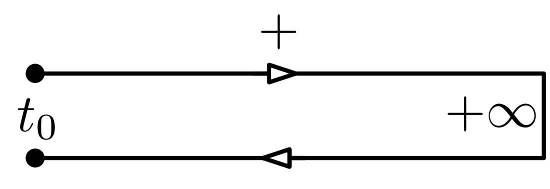
$\sigma \to \infty, \tau \to 0$ with Δ finite

$$\Delta = \sigma\tau$$

Measurement strength Δ^{-1}

Schwinger-Keldysh path integral

$$Tr[\rho(\{\xi(t)\})] = \int \mathcal{D}x \, e^{\frac{\iota S[\{\xi(t)\}, x(t)]}{\hbar}}$$



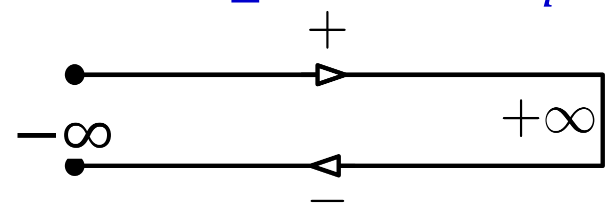
Action

$$S[\{\xi\}, x] = \int_{-\infty}^{\infty} dt \sum_{s=\pm} s \left[\sum_i \left\{ \frac{m}{2} \dot{x}_{is}^2 + m\gamma \, \dot{x}_{is} \xi_i + \frac{\iota s \hbar}{2\Delta} (x_{is} - \xi_i)^2 \right\} - V(\{x_{is}\}) \right]$$

Action

$$S[\{\xi\},x] = \int_{-\infty}^{\infty} dt \sum_{s=\pm} s \left[\sum_i \left\{ \frac{m}{2} \dot{x}_{is}^2 + m\gamma\, \dot{x}_{is}\xi_i + \frac{\imath s\hbar}{2\Delta}(x_{is}-\xi_i)^2 \right\} - V(\{x_{is}\}) \right]$$

Classical $(x_{ic} \equiv x_i)$ and quantum (x_{iq}) components $x_{i\pm} = x_i \pm x_{iq}$



Semiclassical limit, small $\hbar$

Expand in x_{iq} or $\hbar$ keeping $\mathcal{O}\left(\frac{1}{\sqrt{\hbar}}\right), \mathcal{O}(1)$ while scaling $\Delta \sim \hbar^2$

$\Rightarrow$ Stochastic Langevin equation

$$\frac{d^2 x_i}{dt^2} + \gamma \frac{dx_i}{dt} = \frac{1}{m}\left[-\frac{\partial V(\{x_i\})}{\partial x_i} + \eta_i(t) \right]$$

$\langle \eta_i(t)\eta_j(t') \rangle = 2m\gamma T_{eff}\delta_{ij}\delta(t-t')$

Noise strength $\sim \gamma T_{eff} \sim \frac{\hbar^2}{\Delta}$

$\propto$ measurement strength

Effective temperature $T_{eff} \sim \frac{\hbar^{\frac{3}{2}}}{\sqrt{\Delta}} \sim \sqrt{\hbar}$

Long-time steady state (non-equilibrium pure steady state) $\Rightarrow$

Effective classical Boltzmann distribution $\sim \exp\left[-\frac{H_S(\{x_i,p_i\})}{T_{eff}} \right]$ for x and p

"Chaotic-to-non-chaotic" transition in Langevin dynamics $\Rightarrow$

Classical many-body chaos

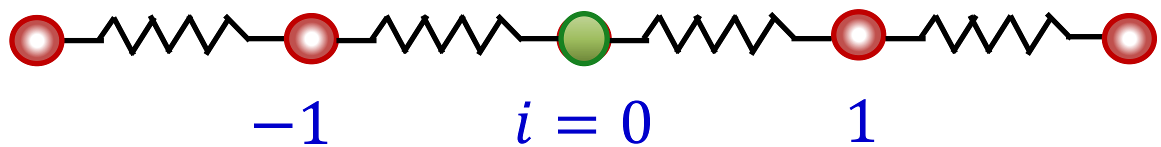
Example: Anharmonic coupled oscillator chain

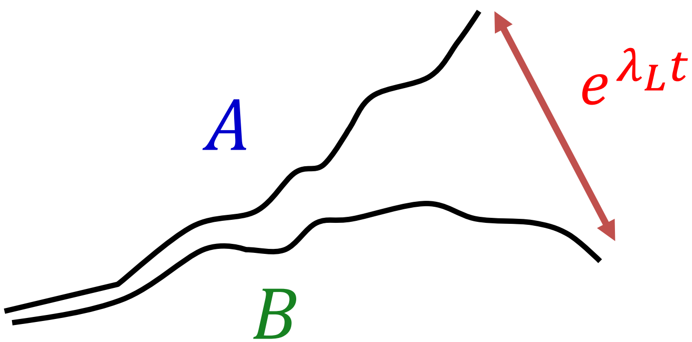
Newtonian dynamics

$$\ddot{x}_i = -\frac{\partial V(\{x_i\})}{\partial x_i}$$

$$i = 1, \dots, N$$

$$V(\{x_i\}) = \sum_i \left[\frac{k}{2}(x_{i+1} - x_i)^2 + \frac{u}{4}(x_{i+1} - x_i)^4\right]$$





$$x_i^A(0) - x_i^B(0) = \varepsilon \delta_{i,0}$$

Two trajectories with slightly different initial conditions at $i = 0$ at time $t = 0$

Classical OTOC or decorrelation function

$$D(i,t) = \left\langle \left(x_i^A(t) - x_i^B(t) \right)^2 \right\rangle_T$$

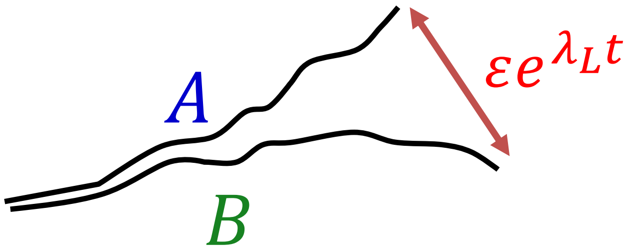
$\langle \cdots \rangle$
Thermal initial condition at temperature T

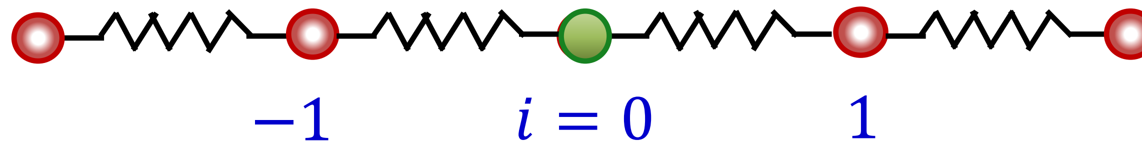
$$\sim \varepsilon^2 e^{\lambda_L t} \quad \Rightarrow \text{Chaotic}$$

Newtonian dynamics

$$V(\{x_i\}) = \sum_i \left[\frac{k}{2}(x_{i+1}-x_i)^2 + \frac{u}{4}(x_{i+1}-x_i)^4\right]$$

$$\ddot{x}_i = -\frac{\partial V(\{x_i\})}{\partial x_i}$$





$$x_i^A(0) - x_i^B(0) = \varepsilon\delta_{i,0}$$

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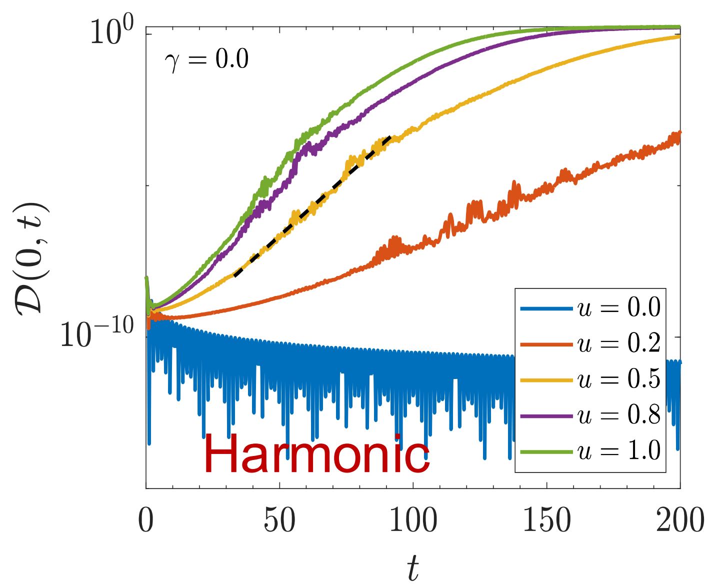
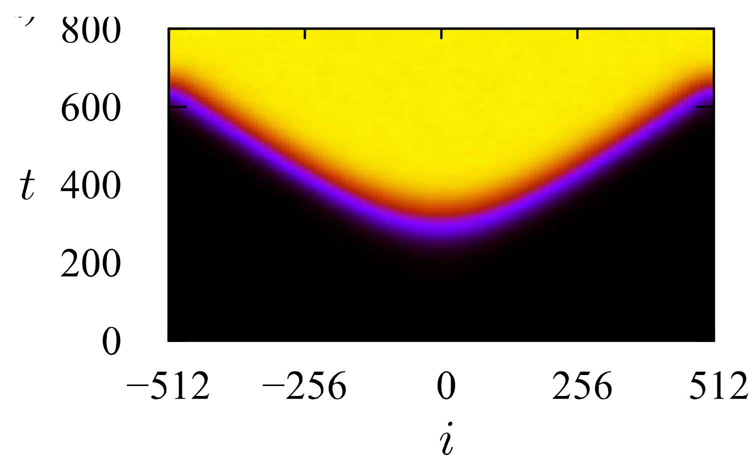
$$D(i,t) = \left\langle \left(x_i^A(t) - x_i^B(t)\right)^2 \right\rangle_T$$





Light cone spread with butterfly velocity $v_B \neq 0$

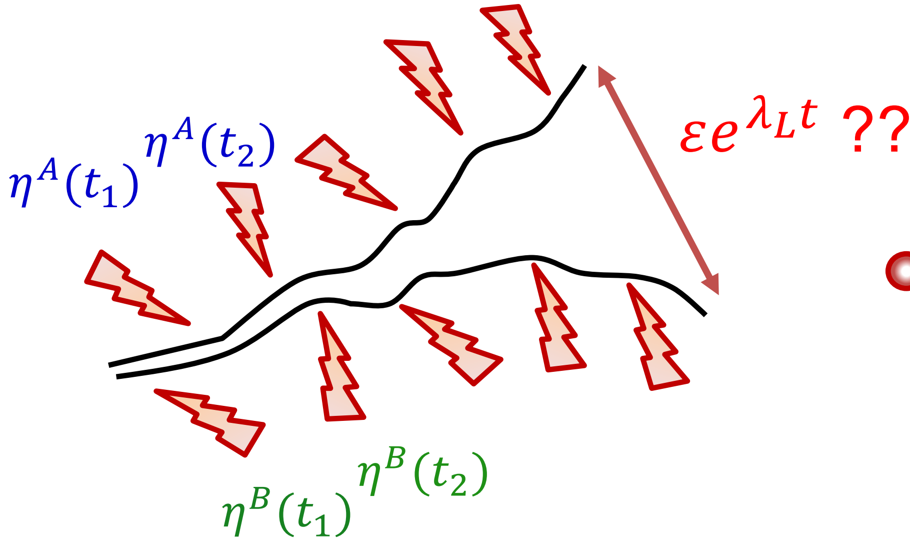
- $\lambda_L > 0$ in the interacting case $u \neq 0$.

Two chaos parameters

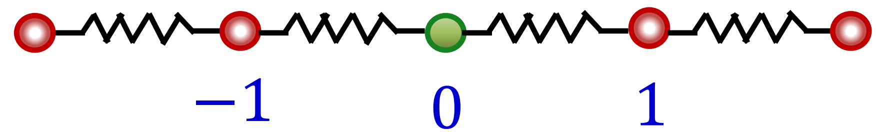
λ_L, v_B

Can one meaningfully define chaos in the presence of noise?

System is randomly kicked at each instant of time.



Noise strength, $\gamma \neq 0$



$$x_i^A(0) - x_i^B(0) = \varepsilon \delta_{i,0}$$

Take exactly the same noise realizations for the two copies

$$\{\eta_i^A(t)\} = \{\eta_i^B(t)\} \quad \forall t$$

Momentum OTOC

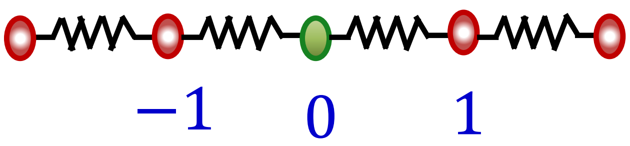
$$D(i,t) = \left\langle \left(p_i^A(t) - p_i^B(t) \right)^2 \right\rangle_{T,\{\eta\}}$$

with perturbation at $i = 0, t = 0$

- Thermal initial condition at temperature T is generated using Langevin dynamics

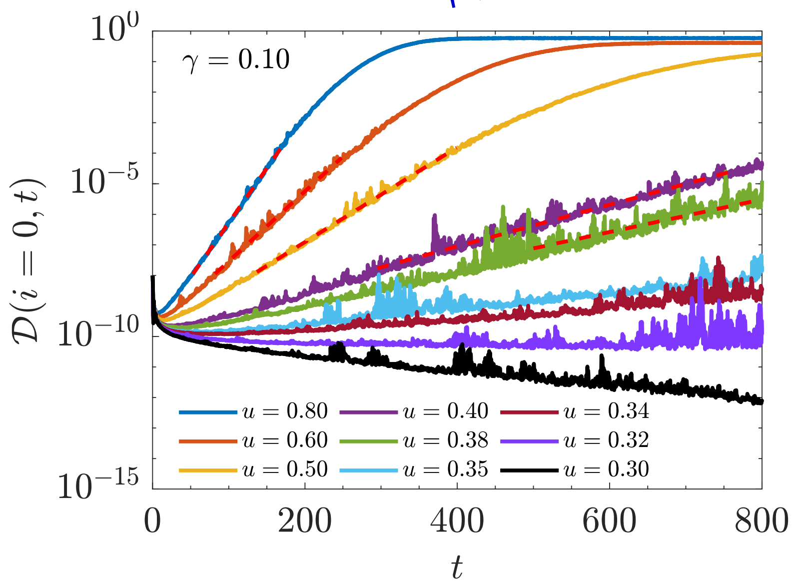
Noise-induced chaotic to non-chaotic transition

Anharmonic oscillators



$$V(\{x_i\}) = \sum_i \left[\frac{k}{2}(x_{i+1}-x_i)^2 + \frac{u}{4}(x_{i+1}-x_i)^4\right]$$

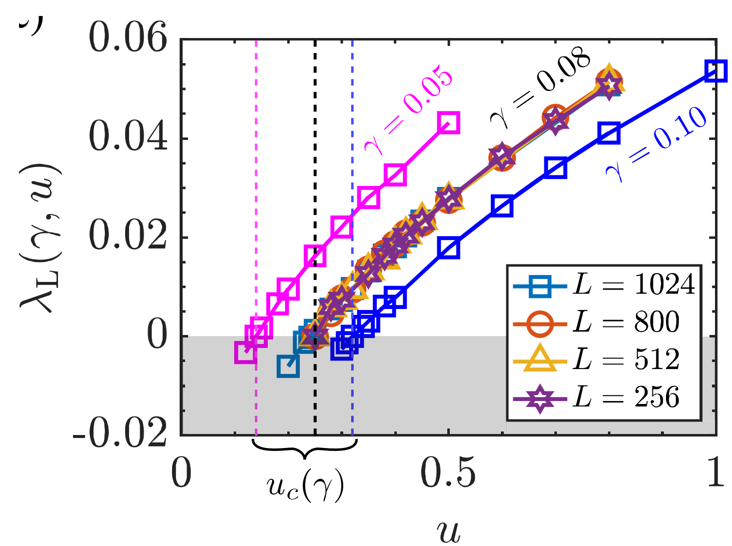
OTOC $D(i,t) = \left\langle \left(p_i^A(t) - p_i^B(t)\right)^2 \right\rangle_{T,\{\eta\}}$



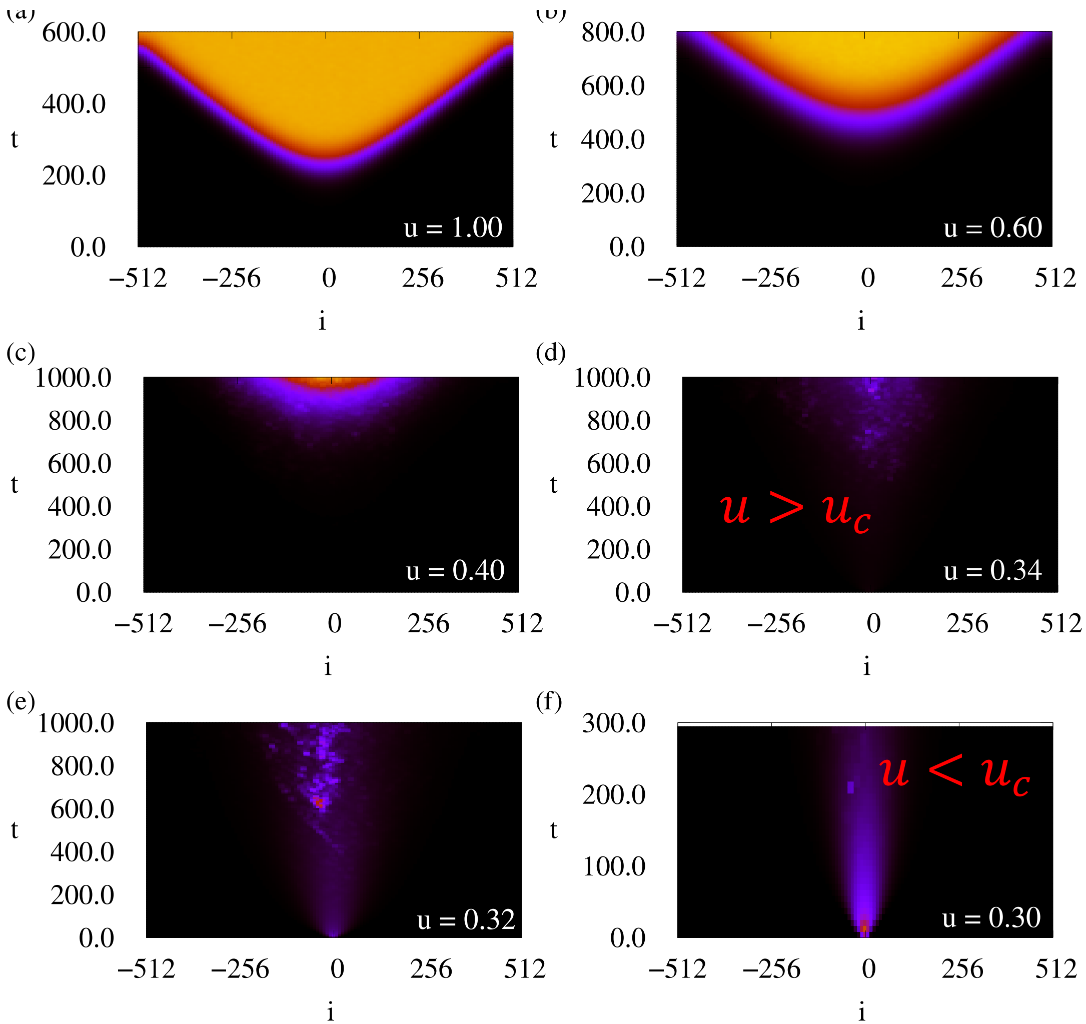
- Transition from exponential growth to exponential decay as a function of decreasing u or u/γ

Lyapunov exponent

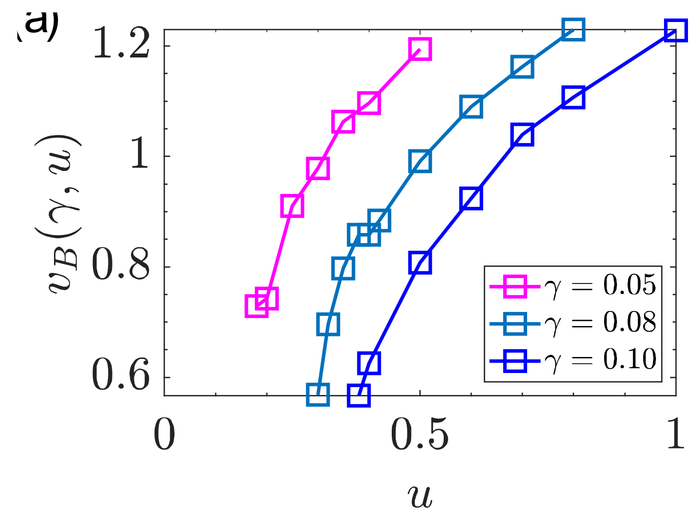
- $\lambda_L > 0 \to \lambda_L < 0$ for $u < u_c(\gamma)$ or $\gamma < \gamma_c(u)$



Light cone and butterfly velocity



Butterfly velocity



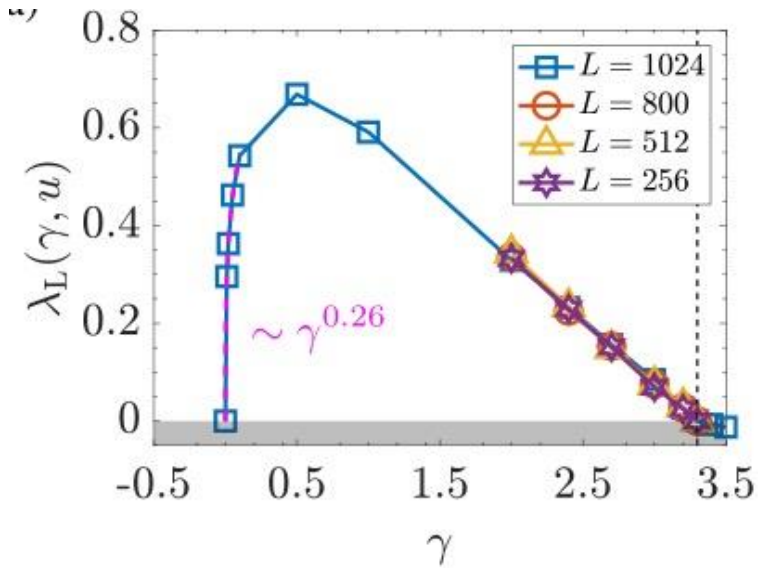
- $v_B \to 0$ for $u \lesssim u_c(\gamma)$.

- Light cone is destroyed for $u < u_c(\gamma)$.

Noise-induced chaotic to non-chaotic transitions in Toda chain

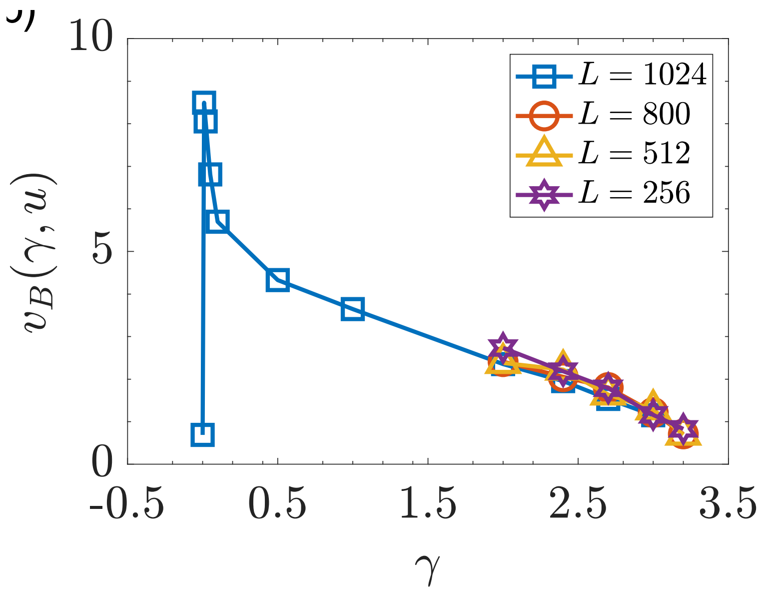
Integrable model $V(\{x_i\}) = \sum_i \left[\frac{a}{b} e^{-b(x_{j+1}-x_j)} + a(x_{j+1}-x_j) - \frac{a}{b}\right]$

Lyapunov exponent



- $\lambda_L \to 0$, $v_B \to$ large in the integrable limit $\gamma \to 0$.
- $\lambda_L, v_B \to 0$ for $\gamma > \gamma_c$.

Butterfly velocity



- Weak noise induces weak chaos in integrable model

Lam and Kurchan, J. Stat. Phys. 156 (2014)

What is this chaotic-non chaotic transition?

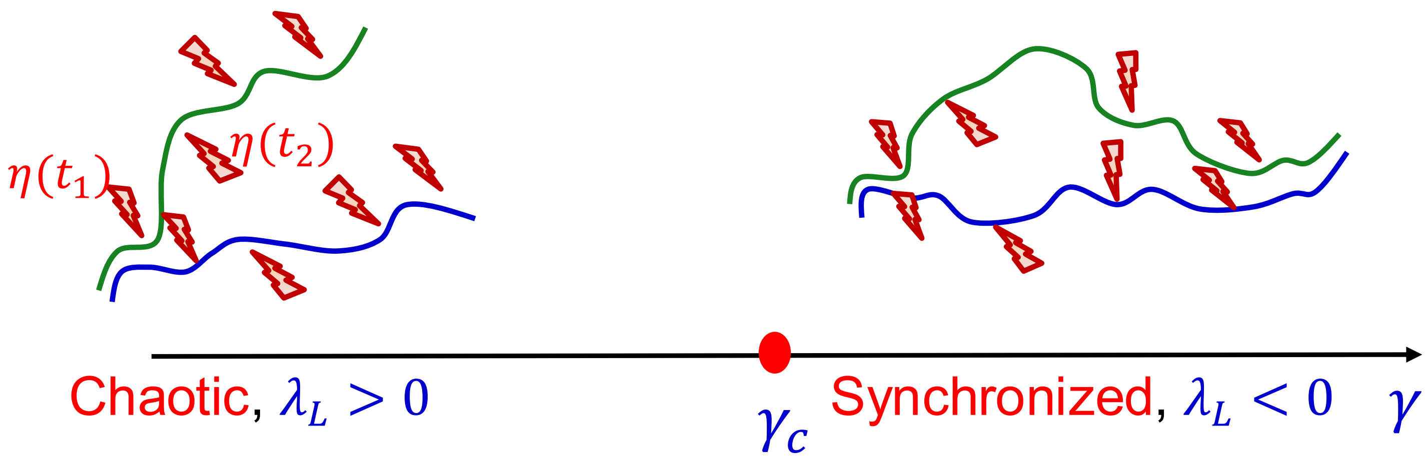
Stochastic synchronization transition (ST) in extended systems

Coupled map lattices (CML)

Bagnoli et al. PRE (1999); Baroni et al. PRE (2001); Cencini et al. PRE (2001); Ginelli et al. PRE (2003), ...

Multiplicative noise/KPZ and Directed percolation universality classes

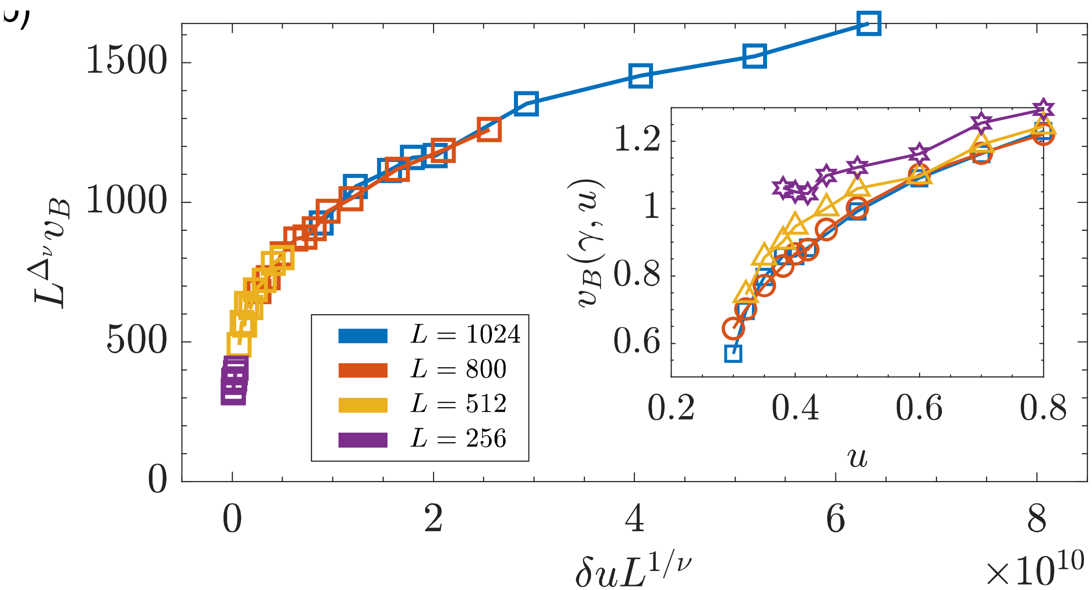
Ahlers and Pikovsky, PRL (2002); Munoz et al. PRL (2003); ...



Noise/dissipation (measurement) strength

Stochastic ST as an MIPT in the semiclassical limit

Dynamical transition and finite-size scaling



$\delta u = u - u_c > 0$

$v_B(u, L) = L^{-\frac{\beta}{\nu}} \mathcal{F}(\delta u \, L^{1/\nu})$

$v_B \sim (\delta u)^{\beta}$
$\xi \sim (\delta u)^{-\nu}$

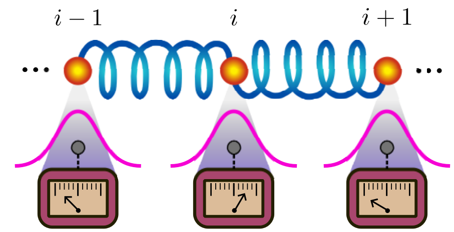
$\beta \simeq 0.28, \qquad \nu \simeq 0.3$

- The transition shows critical scaling.

- The critical exponents do not match with known universality classes like directed percolation (DP) or multiplicative noise (MN)

Recent works on chaotic transition in classical systems
Willsher et al. PRB (2022); Deger et al. PRLs (2022)
- DP universality class

Summary (Part 1)



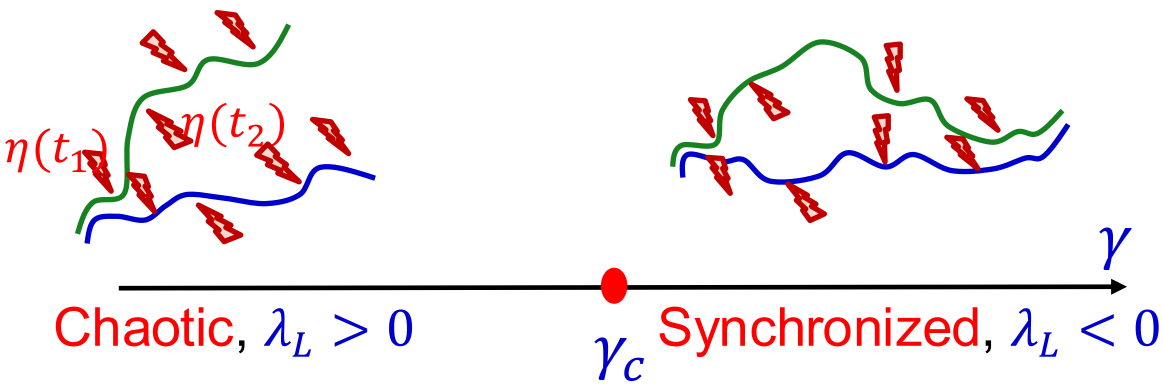
- Semiclassical limit of a model of continuous weak measurements

  ⇒ Stochastic Langevin equation

  noise/dissipation ∝ "measurement strength"

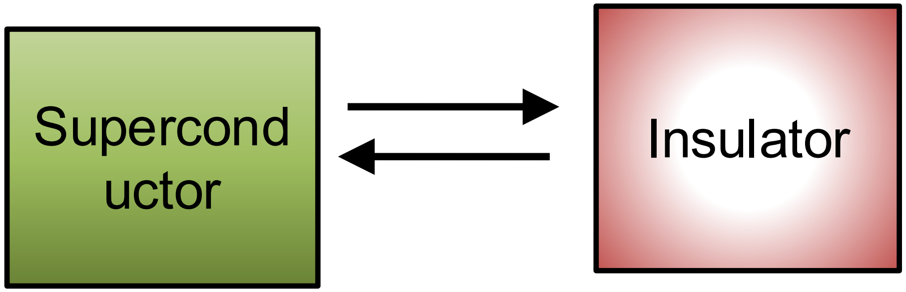
- Noise/measurement induced chaotic to non-chaotic transition

  Stochastic synchronization transition

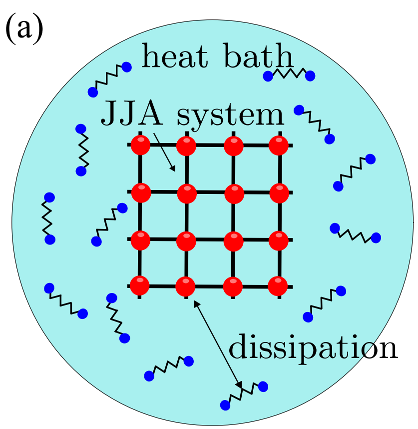


Noise/dissipation (measurement) strength

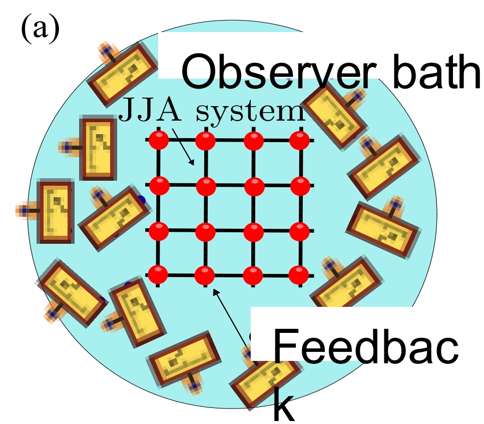
Inverse Superconductor-Insulator Transition in Weakly Monitored Josephson Junction Arrays (JJAs)



- Do repeated measurements and feedback by *observer* behave like a *bath*? How similar or dissimilar are the *observer baths* to usual equilibrium thermal/quantum baths?



Temperature T



Effective temperature T_{eff}?

Compare fluctuation-dissipation relation (FDR)

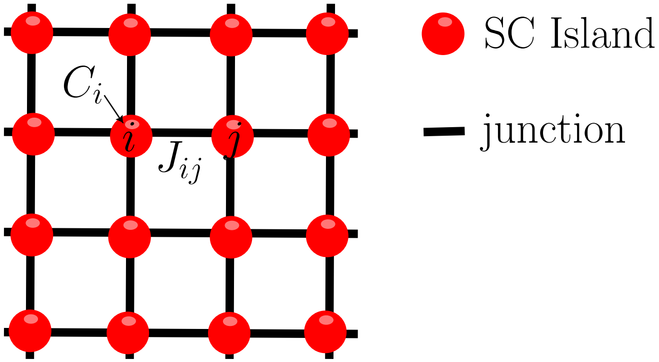
Thermal equilibrium at temperature T

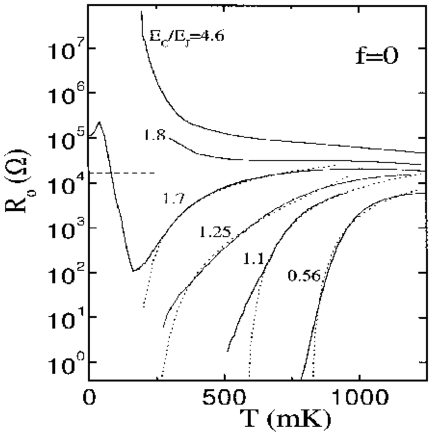
$$\frac{\text{Correlation}}{\text{Response}} = \coth\left(\frac{\hbar\omega}{2T}\right)$$

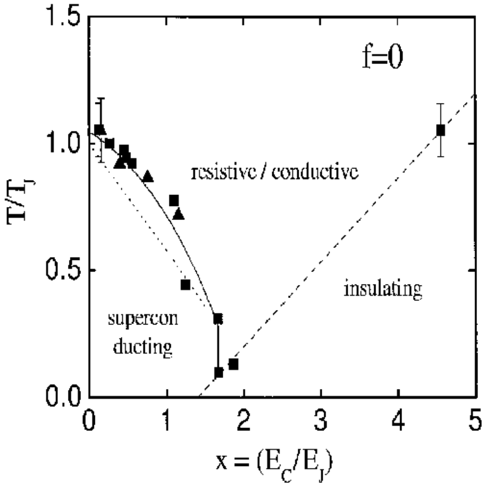
Inverse Superconductor-Insulator Transition in Weakly Monitored Josephson Junction Arrays (JJAs)

Purnendu Das
(IISc UG→ TU Munich)

P. Das & SB
Phys. Rev. B (Letter)
112, L180503 (2025)







$$H_{JJA} = \sum_i \frac{1}{2} E_c n_i^2 + J \sum_{<ij>} (1 - \cos(\theta_i - \theta_j))$$

$$[\theta_i, n_j] = i\delta_{ij} \qquad E_c = \frac{4e^2}{C}$$

H. van der Zant, et al.
Phys. Rev. B (1996)

R. Fazio & H. van der Zant,
Phys. Rep. (2001)

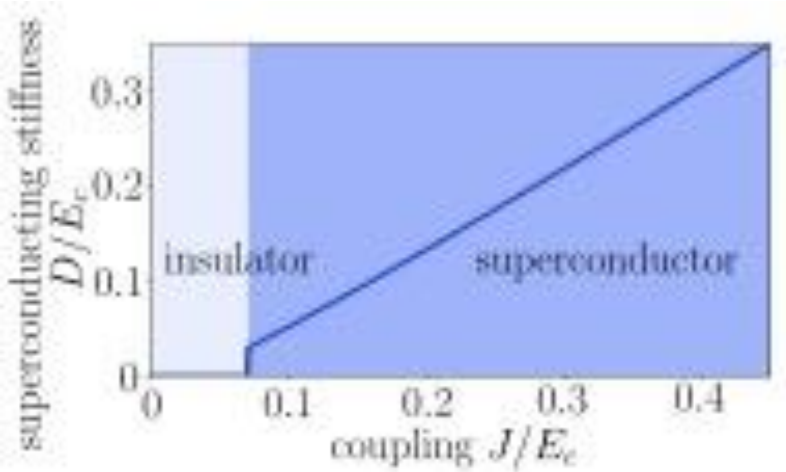
Superconductor-insulator quantum phase transitions

Variational approximation based on Free-energy ⇒

$$F \leq \langle H - H_v \rangle_v - \ln Z_v \qquad Z_v = \mathrm{Tr}[\exp(-\beta H_v)]$$

Self-consistent harmonic approximation (SCHA)

$$J \sum_{<ij>} \cos(\theta_i - \theta_j) \rightarrow \frac{1}{2} D \sum_{<ij>} (\theta_i - \theta_j)^2 \Rightarrow H_v$$



Phase stiffness

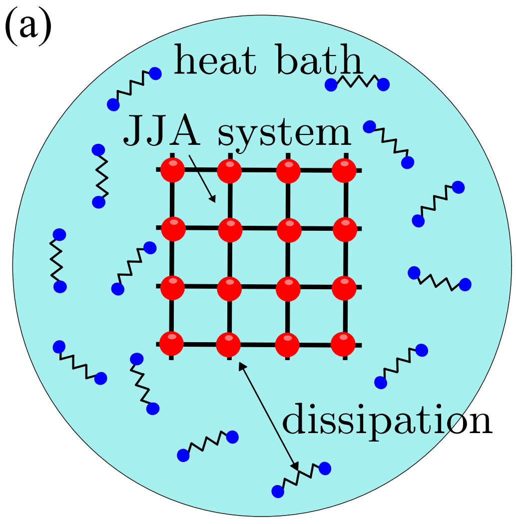
$$D = J \exp\left(-\frac{1}{2}\left\langle (\theta_i - \theta_j)^2 \right\rangle_{var}\right)$$

Typically first order transition

Dissipative phase transition in JJA

Chakravarty et al. PRL (1986), PRB (1988)

Ohmic heat bath



$$\mathcal{H} = \mathcal{H}_{JJA} + \mathcal{H}_{bath} + \mathcal{H}_c$$

$$\mathcal{H} = \sum_i \frac{1}{2} E_c n_i^2 + J \sum_{<ij>} (1 - \cos(\theta_i - \theta_j))$$

$$\mathcal{H}_{bath} = \sum_{<ij>} \frac{1}{2} m_l \left(\dot{x}_{l,ij}^2 + \omega_l^2 x_{l,ij}^2\right)$$

$$\Delta\theta_{ij} = \theta_i - \theta_j$$

$$\mathcal{H}_c = \sum_{<ij>} \Delta\theta_{ij} \sum_l f_{l,ij} x_{l,ij}$$

Integrate out the bath, assume Ohmic spectral density $(\sim \omega)$ of the bath

Imaginary-time action

$$S_{eff} = \int_0^\beta d\tau \left[\sum_i \frac{\dot{\theta}_i^2}{2E_c} + J \sum_{<ij>} \left(1 - \cos\Delta\theta_{ij}\right)\right] + \frac{\beta}{4\pi} \sum_{n,<ij>} \alpha|\omega_n| \left|\Delta\theta_{ij}(\omega_n)\right|^2$$

$$\alpha = \frac{h}{4e^2 R}$$

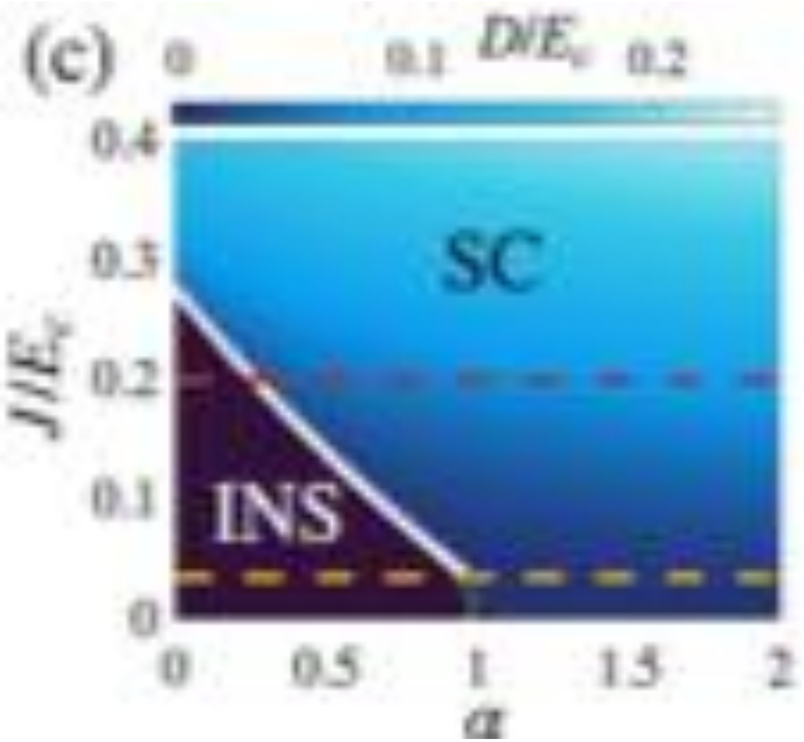
R, normal-state resistance

Self-consistent harmonic approximation ⇒

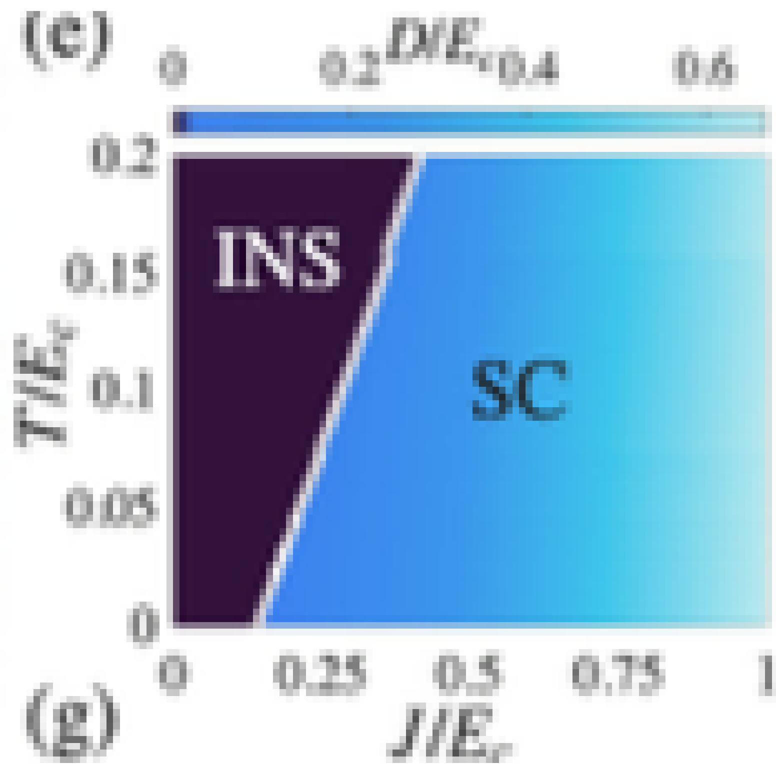
Chakravarty et al. PRL (1986), PRB (1988)

SCHA

$T = 0$

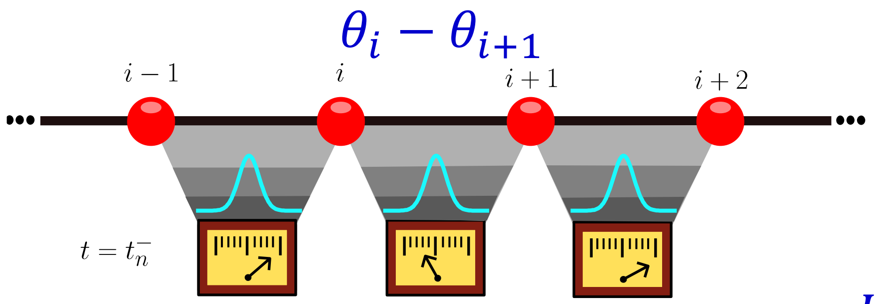


$T \neq 0$ (fixed α)



Second-order transition at $\alpha = \frac{1}{d}$ Independent of JJ coupling

Monitored Josephson junction array (JJA)



Detectors coupled to junctions $<ij>$

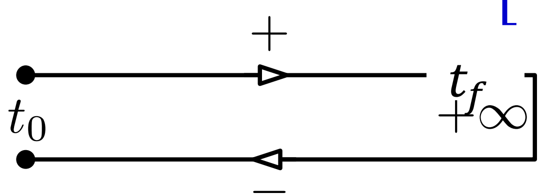
$$H(t) = H_{JJA} + \lambda \sum_{<ij>,n} \delta(t - n\tau)\Delta\theta_{ij} p_{ij,n}$$

$$\psi(\xi_{ij,n}) \sim \exp\left(-\frac{\xi_{ij,n}^2}{2\sigma}\right)$$

Readings $\{\xi_{ij,n}\}$

Schwinger-Keldysh action

$$S[\{\xi\},\theta] = \int_{t_0}^{t_f} dt \sum_{s=\pm} s \left[\sum_i \left\{ \frac{1}{2E_c} \dot{\theta}_{is}^2 + \left(\frac{\gamma}{E_c}\right) \dot{\theta}_{is} \sum_{j\in NN_i} \xi_{ij} + \frac{\imath s\hbar}{4\Delta} \sum_{j\in NN_i} \left(\Delta\theta_{ijs} - \xi_i\right)^2 \right\} - J \sum_{<ij>} (1 - \cos\Delta\theta_{ijs}) \right]$$



Generating function $\quad Z = \int \mathcal{D}\xi \mathcal{D}\theta \, e^{iS[\theta,\xi]}$

Observables averaged over measurement outcomes or quantum trajectories (Lindblad evolution)

$$\overline{\langle A \rangle} = \frac{1}{Z} \int \mathcal{D}\xi \mathcal{D}\theta A[\theta] e^{iS[\theta,\xi]}$$

$\Leftarrow$ SCHA

Self-consistent harmonic approximation (SCHA) for trajectory averaged steady state

Variational harmonic action. $S_v[\xi, \theta]$

Variational parameter D

$$-J \sum_{<ij>} \left(1 - \cos \Delta\theta_{ij,s}\right) \rightarrow \frac{1}{2} \sum_{<ij>} D \Delta\theta_{ij,s}^2$$

What is the variational principle?

No "free energy" for non-equilibrium state!

Variational principle for a quantum trajectory in a monitored dynamics – Maximize Born probability $Z[\xi] = e^{-F[\xi]}$

$$F[\xi] \leq F_v[\xi] = -\left(\frac{i}{\hbar}\right) \langle S - S_v \rangle_v - \ln Z_v[\xi]$$

Self-consistency condition → $D(t) = J \overline{\langle \cos \Delta\theta_{ij,s}(t) \rangle_v}$

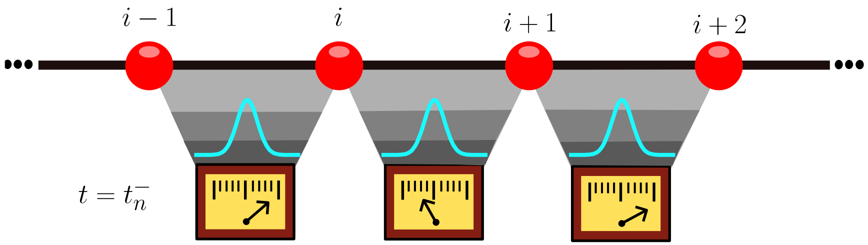
For steady state averaged over trajectories

$$D = J \exp\left(-\frac{\overline{\left\langle \Delta\theta_{ij,s}^2(t) \right\rangle_v}}{2}\right)$$

Superfluid stiffness $D \neq 0$ $\Rightarrow$ Superconductor

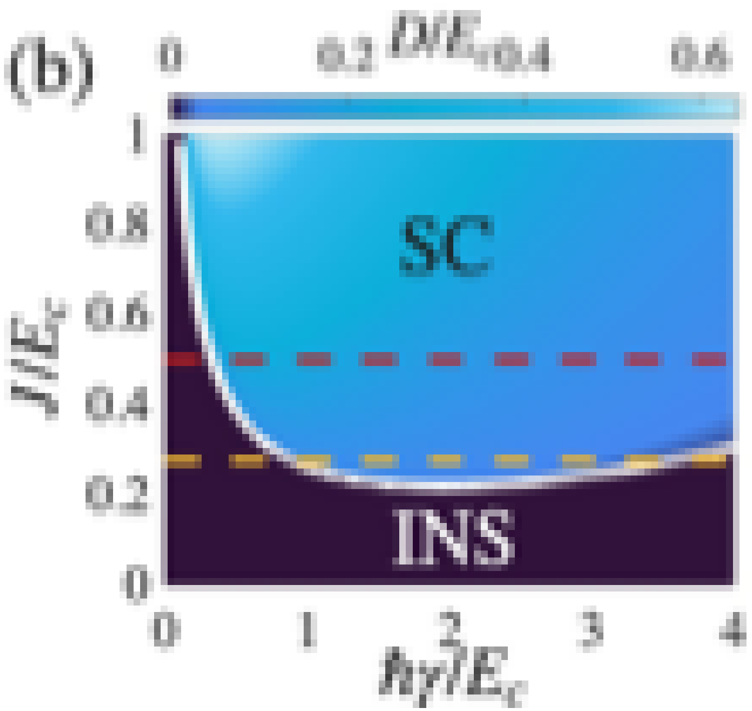
$D = 0$ $\Rightarrow$ Insulator

SC-Insulator transition in monitored Josephson junction array and phase diagram

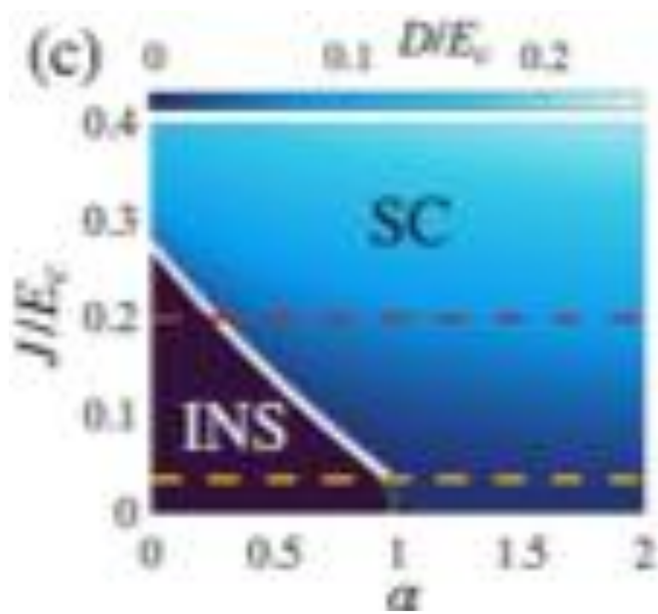


Three parameters

- Josephson coupling J/E_c
- Measurement strength $\hbar\Delta^{-1}/E_c$
- Feedback strength $\hbar\gamma/E_c$



Dissipative



Reentrant superconductor-insulator transitions

Is the measured JJA very different from dissipative JJA?

Trajectory averaged Green's function for non-equilibrium steady state $(t_f \to \infty)$ and effective temperature

$$G(q,\omega) = \begin{bmatrix} G^K(q,\omega) & G^R(q,\omega) \\ G^A(q,\omega) & 0 \end{bmatrix}$$

Causal structure

SCHA $\Rightarrow$ $D = J \exp\left[-\frac{1}{2}\overline{\left\langle \left(\theta_{i\pm}(t) - \theta_{i+1,\pm}(t)\right)^2 \right\rangle}\right]$

Measured and dissipative JJA can be directly compared within SCHA

	Measured		Dissipative	
Measurement strength Δ^{-1}	Feedback γ	$=$	$\frac{E_c \alpha}{2\pi}$	Dissipation
Effective temperature	$\frac{G^K(q,\omega)}{G^R(q,\omega) - G^A(q,\omega)} = \frac{2T_{eff}}{\hbar\omega} + \frac{\hbar\omega}{8T_{eff}}$	$\equiv$	$\coth\left(\frac{\hbar\omega}{2T}\right) \simeq \frac{2T}{\hbar\omega} + \frac{\hbar\omega}{6T} + \cdots$	

$\omega \to 0$ or $\hbar$ small

$$T_{eff} = \frac{\hbar^2 E_c}{4}\left(\frac{\Delta^{-1}}{\gamma}\right)$$

- In general, $T_{eff}(\omega)$
- No $T_{eff} \to 0$ limit

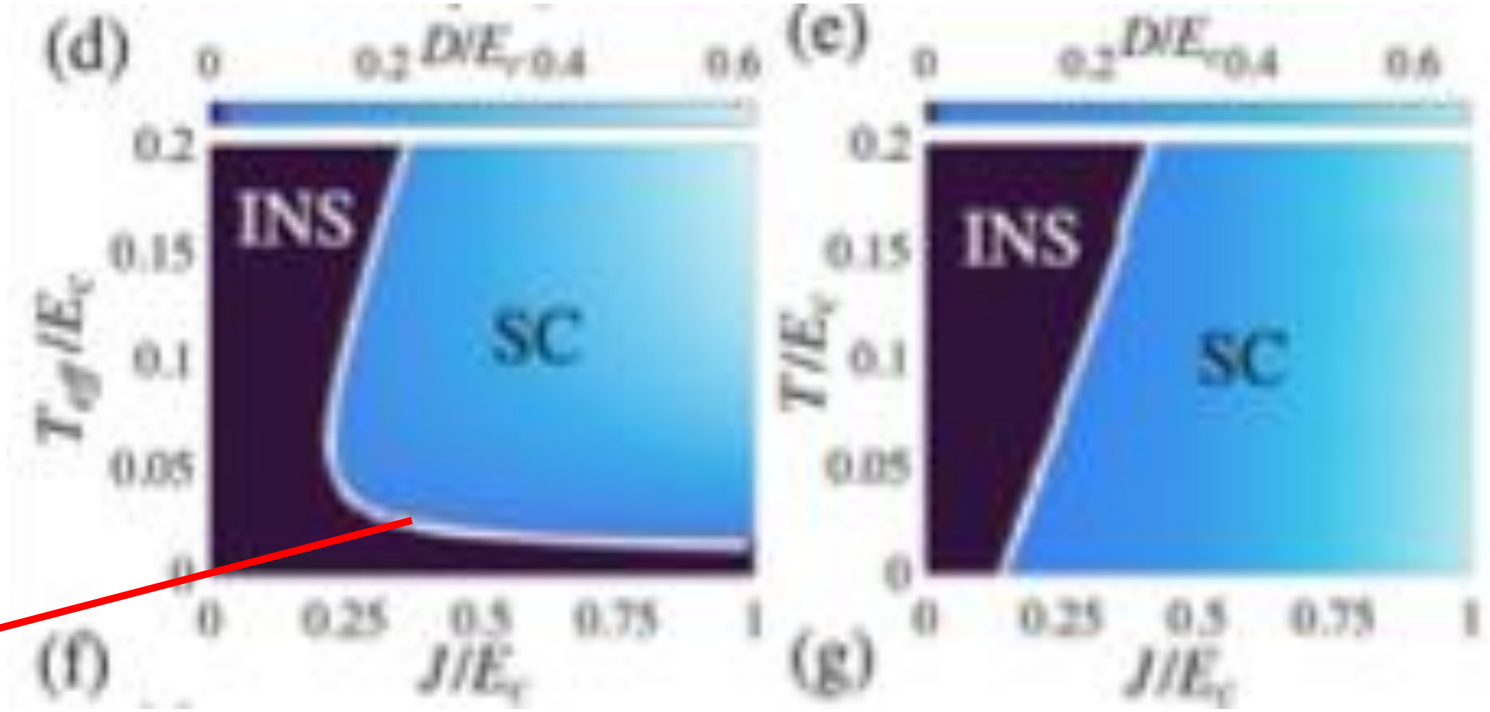
Measured Dissipative

Very different at low T_{eff}

Always an insulator

⇒

Reentrant transition



MIPT Insulator-SC transition

Seemingly low-temperature normal state (insulator)

→ *high-temperature* superconductor

⇒ Inverse superconductor-Insulator transition

Phase diagram valid beyond SCHA

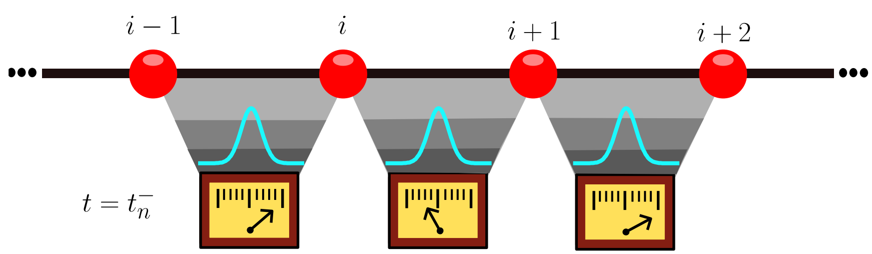
1. High temperature or semiclassical limit $\hbar \to 0$.
2. Strong measurement/ weak feedback limit $\Delta^{-1}/\gamma \gg 1$.
3. Strong feedback limit $\hbar\gamma \gg J, E_c$.
4. Weak coupling $J \ll E_c$ RG.

Conclusions (Part 2)

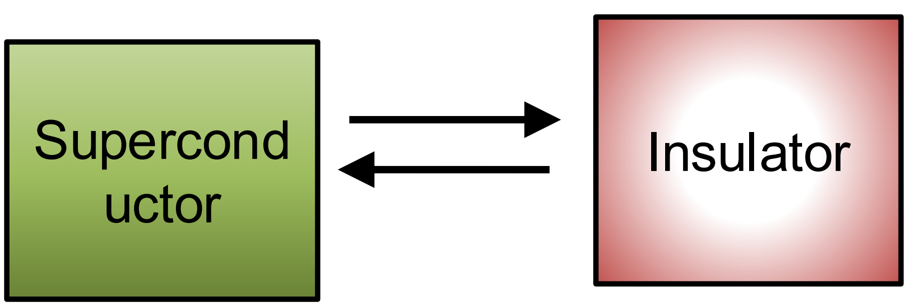
- Monitored quantum many-body systems – New paradigm for (non-unitary) quantum dynamics, non-equilibrium statistical mechanics, dynamical phase transitions and universality.

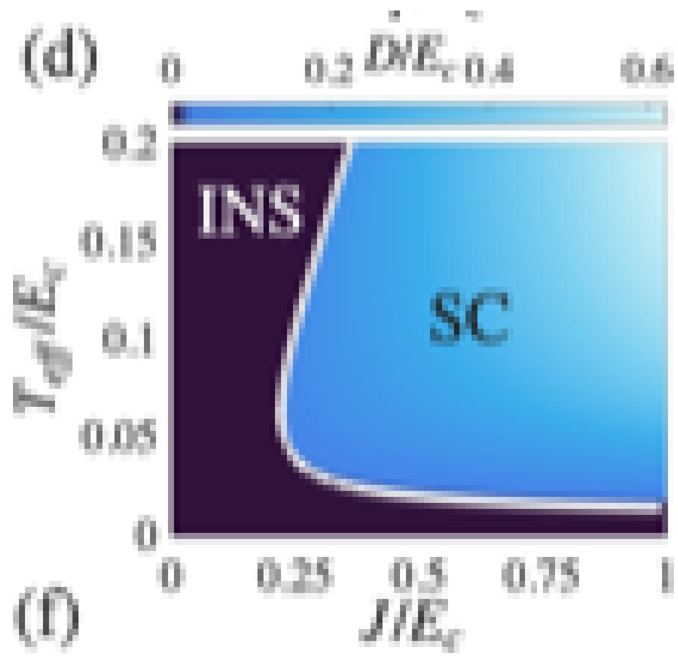
- Model of weak quantum measurements + feedback on JJA

$$\Delta\theta_{ij} = \theta_i - \theta_{i+1}$$



Measurements + feedback

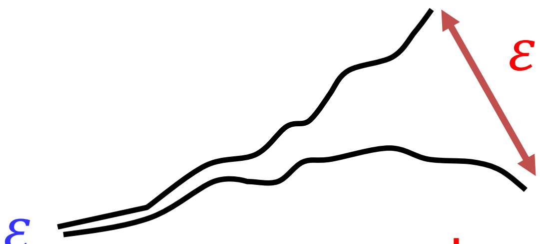




Classical Chaos

Single-particle chaos

Sensitivity to initial condition ⇒



$$\varepsilon e^{\lambda_L t} \lesssim \mathcal{O}(1)$$

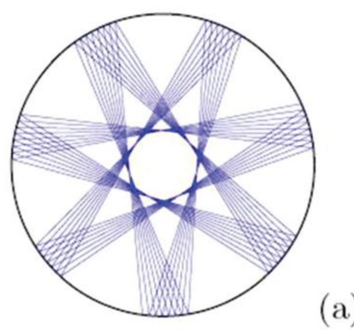
Lyapunov regime

$$\lambda_L^{-1} < t < t^* \sim \lambda_L^{-1} \log\left(\frac{1}{\varepsilon}\right)$$

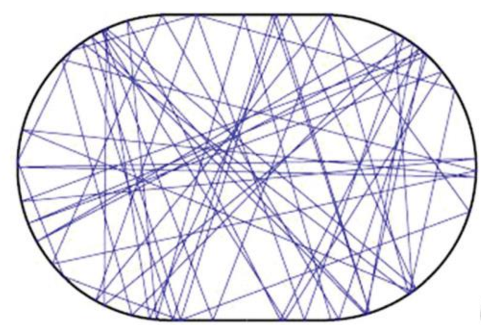
$$\frac{\delta x(t)}{\delta x(0)} \sim e^{\lambda_L t}$$

λ_L, Lyapunov exponent

Non chaotic billiard

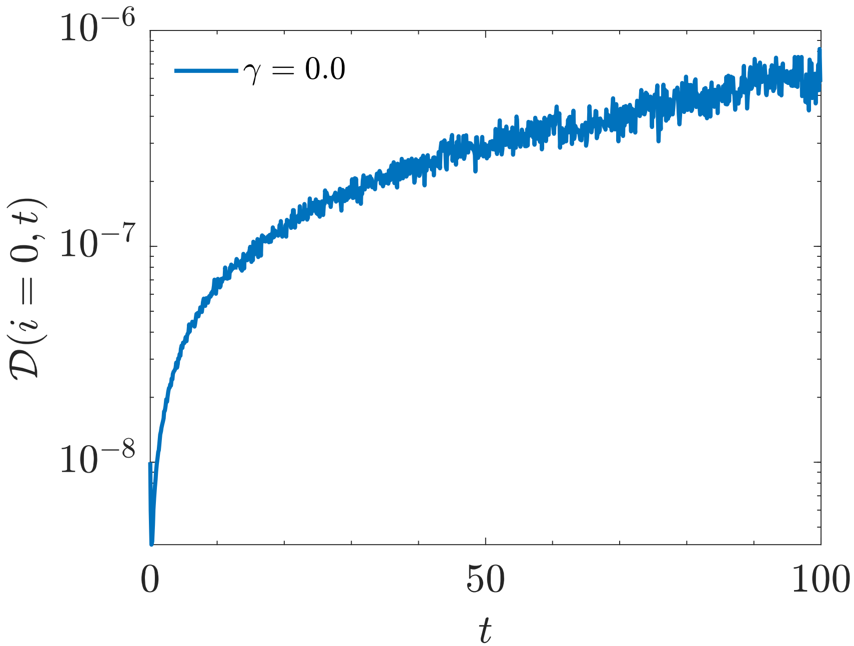


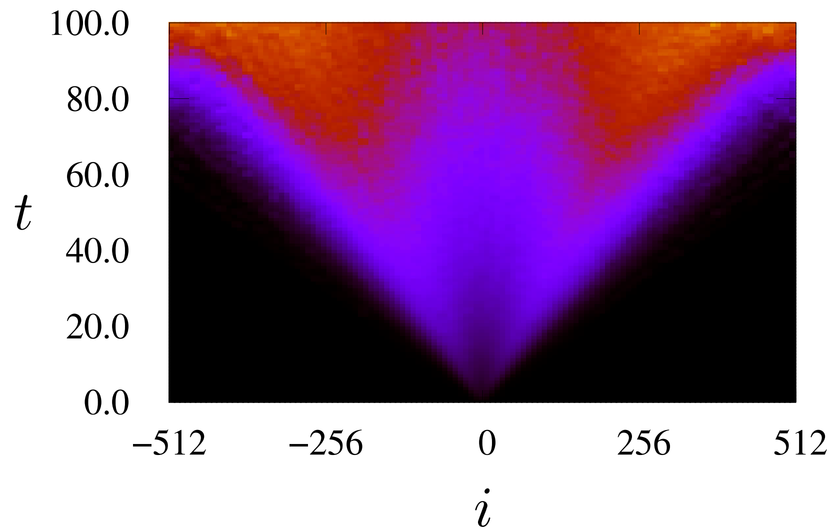
Chaotic billiard



Many-body chaos in integrable Toda chain

$a = 0.07, b = 15$

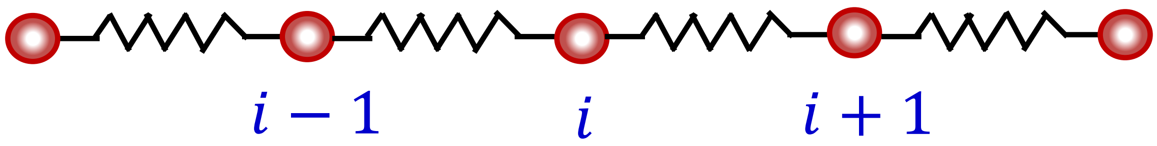


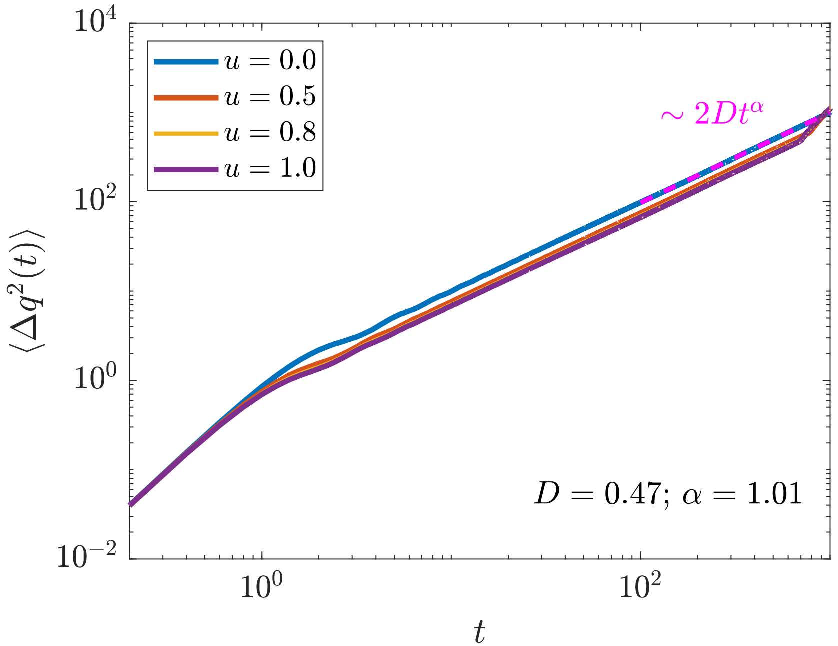


Light cone spread with butterfly velocity $v_B \neq 0$

- No exponential growth ($\lambda_L = 0$) in the integrable Toda chain.
- Non zero butterfly velocity

Is the transition visible in usual dynamical properties?





Diffusive for $\gamma = 0$

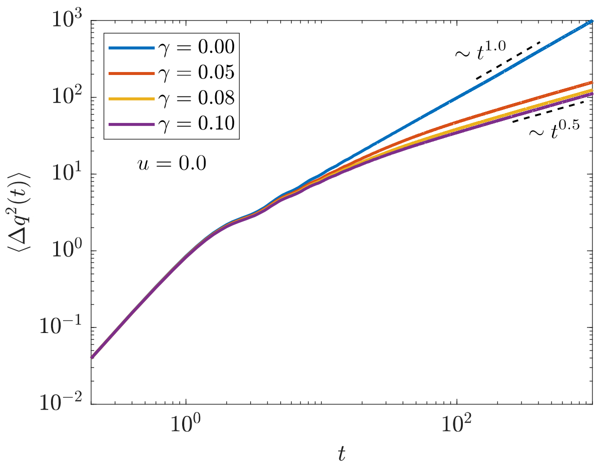
$u = 0$ (harmonic limit)

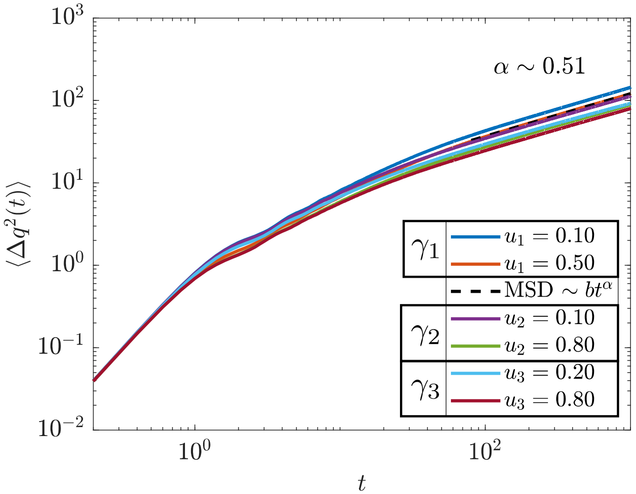
Diffusion constant

$$D = \frac{\mathrm{T}}{2}\sqrt{\frac{1}{mk}}$$

Florencio and Lee, Phys. Rev. A 31 (1985)

Unlike chaos, there is no transition in usual dynamical properties, e.g. diffusion



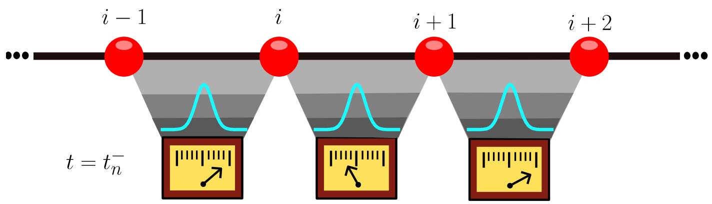


$\gamma \neq 0$, subdiffusion

Monomer subdiffusin in ploymers
e.g. Weber et al., Phys. Rev. E 82 (2010)

Similar model of weak measurements + feedback on JJA

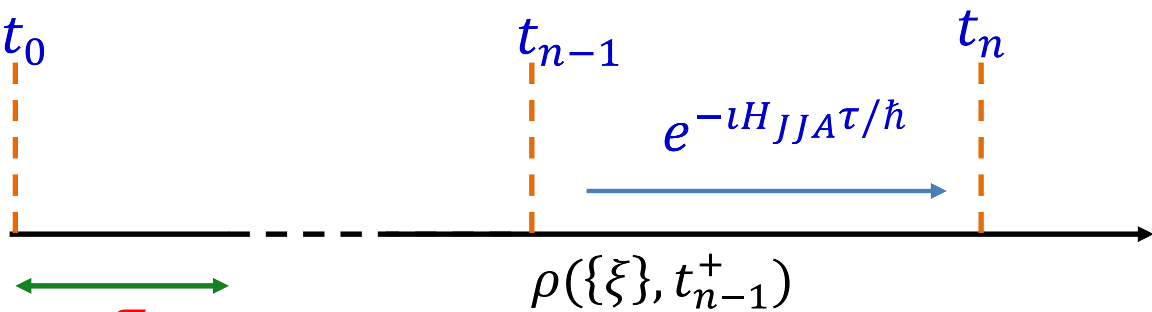
$$\Delta\theta_{ij} = \theta_i - \theta_{i+1}$$



Detectors coupled to phase difference junctions $< ij >$

$$H(t) = H_{JJA} + \sum_{<ij>,n} \delta(t - n\tau)\Delta\theta_{ij} p_{ij,n}$$

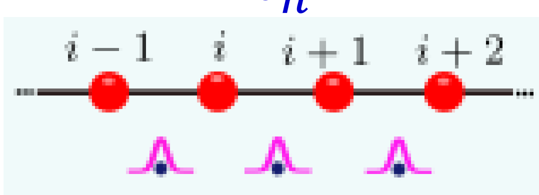
Detectors $[\hat{\xi}_{ij,n}, \hat{p}_{ij,n}] = i$



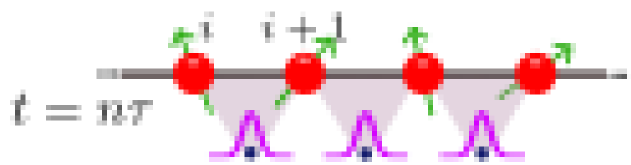
System under repeated weak measurements in intervals of τ

$t_n = n\tau$

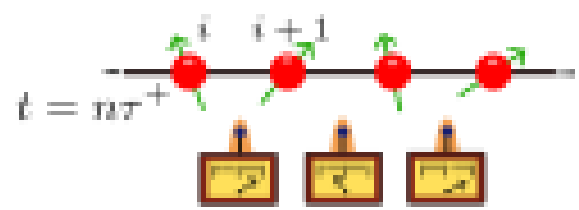
t_n^-



t_n



t_n^+



$$\psi(\xi_{ij,n}) \sim \exp\left(-\frac{\xi_{ij,n}^2}{2\sigma}\right)$$

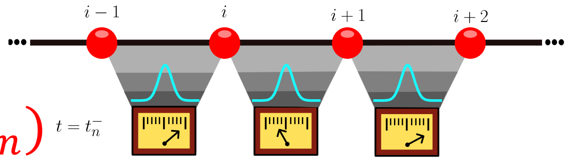
Apply

$$\sum_{<ij>,n} \delta(t - n\tau)\Delta\theta_{ij} p_{ij,n}$$

Readings $\{\xi_{ij,n}\}$

Projective measurements

Non-unitary time evolution of JJA density matrix



$$\rho(\{\xi\}_n, t_n^+) = M(\xi_n) e^{-\frac{\imath H_{JJA}\tau}{\hbar}} \rho(\{\xi\}_{n-1}, t_{n-1}^+) e^{\frac{\imath H_{JJA}\tau}{\hbar}} M^\dagger(\xi_n) \Big|_{t=t_n^-}$$

Measurement or Kraus operator, $M(\xi_n) \sim \exp\left(-\frac{(\xi_{ij,n} - \Delta\hat{\theta}_{ij})^2}{2\Delta}\tau\right)$

With only measurements system heats up indefinitely

Caves and Milburn, Phys. Rev. A (1987)

Measurement + feedback

$$M(\xi_n) \sim \prod_{\langle ij \rangle} \exp\left(\imath\gamma\tau\xi_{ij,n}(\hat{n}_i - \hat{n}_j)\right) \quad \exp\left(-\frac{(\xi_{ij,n} - \Delta\hat{\theta}_{ij})^2}{2\Delta}\tau\right)$$

"Feedback" γ, acts like dissipation

Limit of continuous weak measurement

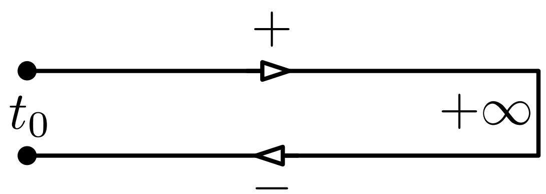
$\sigma \to \infty, \tau \to 0$ with Δ finite

$$\Delta = \sigma\tau$$

Measurement strength Δ^{-1}

Schwinger-Keldysh path integral for non-unitary dynamics

$$Z[\xi] = Tr[\rho(\{\xi(t)\})] = \int \mathcal{D}x \, e^{\frac{\imath S[\xi,\theta]}{\hbar}}$$



Alternative description of the monitored dynamics with continuous weak measurements and feedback

- Quantum state diffusion (QSD) or stochastic Schrodinger equation (SSE) for the normalized state of the system

$$|\psi(t+\delta t)\rangle = \left[1 - i\, H_{JJA}\delta t + \sum_{\langle ij\rangle} \zeta_{ij,t}\left(\hat{L}_{ij} - \langle \delta\hat{\theta}_{ij}\rangle\right)\right]|\psi(t)\rangle$$
$$-\frac{\delta t}{4\Delta}\sum_{\langle ij\rangle}\left[\left(\hat{L}_{ij}^{\dagger} - \langle \delta\hat{\theta}_{ij}\rangle\right)\hat{L}_{ij} - \left(\hat{L}_{ij} - \langle \delta\hat{\theta}_{ij}\rangle\right)\right]\langle \delta\hat{\theta}_{ij}\rangle\,|\psi(t)\rangle$$

$$\hat{L}_{ij} = \hat{\theta}_i - \hat{\theta}_j + i\gamma\Delta(\hat{n}_i - \hat{n}_j) \qquad \delta\hat{\theta}_{ij} = \hat{\theta}_i - \hat{\theta}_j\,,\delta\hat{n}_{ij} = \hat{n}_i - \hat{n}_j$$

$$\langle \zeta_{ij,t}\rangle = 0 \qquad \langle \zeta_{ij,t}\zeta_{kl,t'}\rangle = \frac{1}{2\Delta}\delta t\,\delta_{t,t'}\delta_{\langle ij\rangle,\langle kl\rangle}$$

- Dynamics of the trajectory averaged density matrix $\rho(t) = \overline{|\psi(t)\rangle\langle\psi(t)|}$

$$\frac{d\rho}{dt} = -i[\widetilde{H}_s, \rho] + \frac{1}{2\Delta}\sum_{\langle ij\rangle}\left(\hat{L}_{ij}\rho\hat{L}_{ij}^{\dagger} - \frac{1}{2}\{\hat{L}_{ij}^{\dagger}\hat{L}_{ij}, \rho\}\right)$$

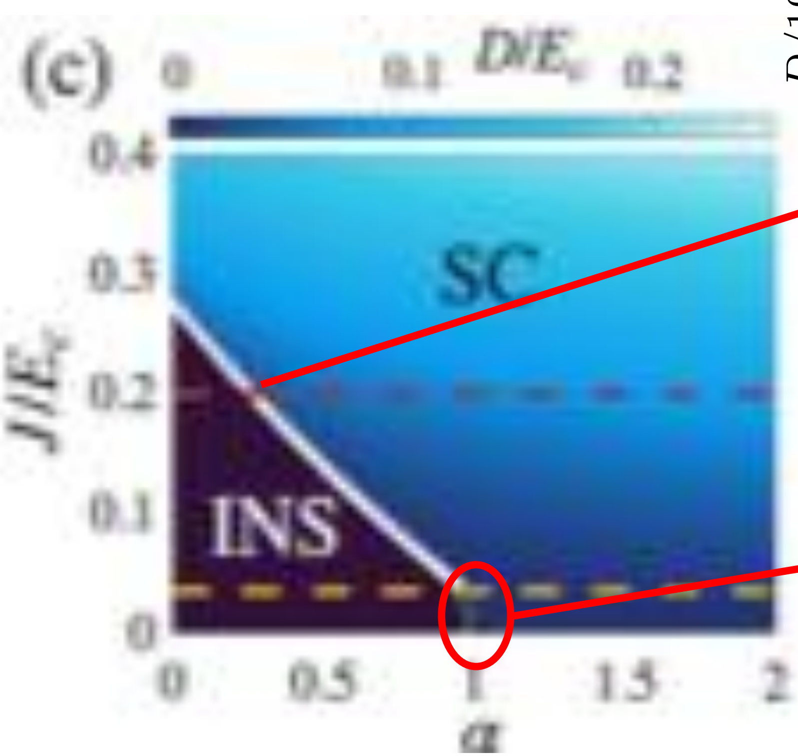
P. Das & SB, arXiv:2412.04556

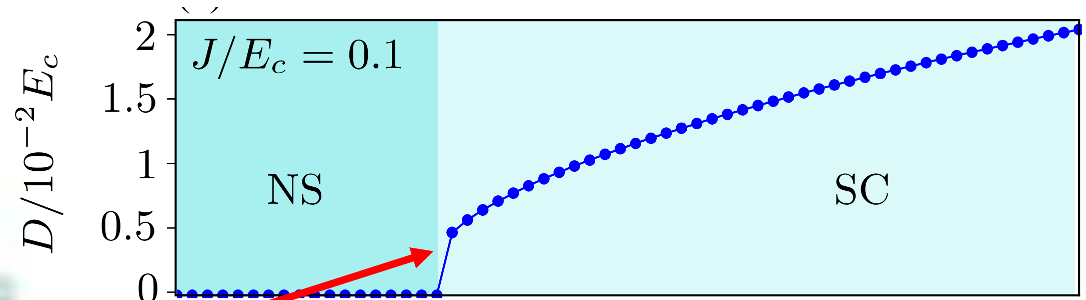
Self-consistent harmonic approximation ⇒

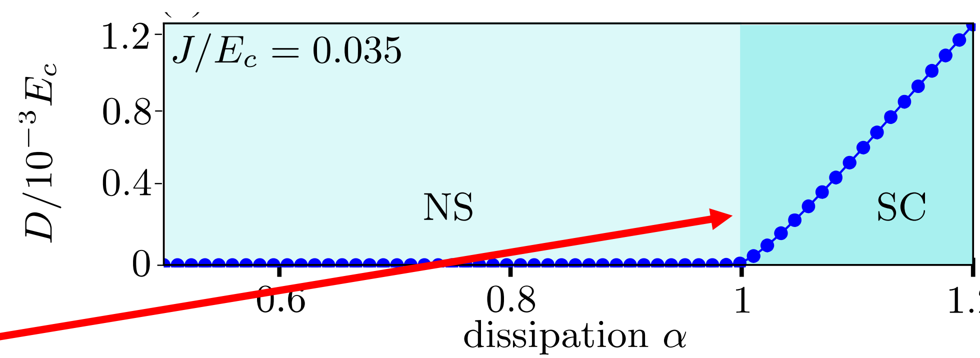
Chakravarty et al. PRL (1986), PRB (1988)

First-order transition

SCHA $T = 0$





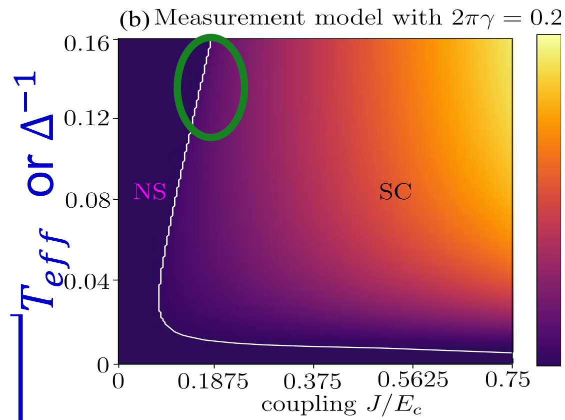


Second-order transition at

$\alpha = \frac{1}{d}$ Independent of JJ coupling

Beyond self-consistent harmonic approximation (SCHA)

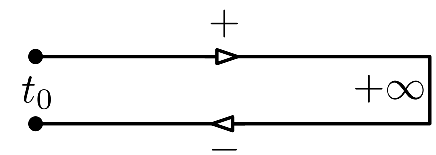
Semi-classical limit (high effective temperature)



SCHA $\Rightarrow$ $D = J\exp\left(-\frac{T_{eff}}{dD}\right)$

Same in measurement and dissipative model

$$S[\{\xi\},\theta] = \int_{t_0}^{t_f} dt \sum_{s=\pm} s \left[\sum_i \left\{ \frac{1}{2E_c}\ \dot{\theta}_{is}^2 + \left(\frac{\gamma}{E_c}\right) \dot{\theta}_{is} \sum_{j\in NN_i} \xi_{ij} \right\} + \frac{\imath s\hbar}{4\Delta} \sum_{ij\in NN_i} \left(\Delta\theta_{ijs} - \xi_i\right)^2 - J\sum_{<ij>} \left(1 - \cos\Delta\theta_{ijs}\right) \right]$$



Classical $(\theta_{ic} \equiv \theta_i)$ and quantum (x_{iq}) components $\qquad \theta_{i\pm} = \theta_i \pm \theta_{iq}$

Expand in θ_{iq} or $\hbar$ keeping $\mathcal{O}\left(\frac{1}{\sqrt{\hbar}}\right), \mathcal{O}(1)$ while scaling $\Delta \sim \hbar^2$

$\Rightarrow$ Stochastic Langevin equation

$$\langle \eta_{ij}(t)\eta_{kl}(t')\rangle = \frac{\hbar^2}{2\Delta}\delta_{ij}\delta_{kl}\delta(t-t')$$

$$\ddot{\theta}_i + \gamma \sum_{j\in NN_i} (\dot{\theta}_i - \dot{\theta}_j) = \frac{1}{m}\left[-J\sum_{j\in NN_i} \sin(\theta_i - \theta_j) + \sum_{j\in NN_i} \eta_{ij}\right]$$

$$m = \frac{\hbar^2}{E_c}$$

$$T_{eff} = \frac{\hbar^2 E_c}{4}\left(\frac{\Delta^{-1}}{\gamma}\right)$$

Usual thermal phase transitions,
e.g., BKT transition in 2D

Large feedback ("damping") limit $\frac{\hbar\gamma}{E_c} \gg 1$

Dynamics is effectively classical over a time scale $\gg m\gamma/J$

$\Rightarrow$ Fast mode $\theta_{js}^{>}(t) = \int_{J/m\gamma}^{\gamma} \frac{d\omega}{2\pi} e^{-i\omega t} \theta_{js}(\omega)$

Slow mode $\theta_{js}^{<}(t) = \int_{0}^{J/m\gamma} \frac{d\omega}{2\pi} e^{-i\omega t} \theta_{js}(\omega)$

Effective dynamics of slow modes

$$\ddot{\theta}_i^{<} + \gamma \sum_{j\in NN_i} (\dot{\theta}_i^{<} - \dot{\theta}_j^{<}) = \frac{1}{m}\left[-J_{eff} \sum_{j\in NN_i} \sin(\theta_i^{<} - \theta_j^{<}) + \sum_{j\in NN_i} \eta_{ij}\right]$$

$$J_{eff} = J\left(1 - \frac{\left\langle\left(\Delta\theta_{ij}^{c>}\right)^2\right\rangle}{2}\right) \approx J\left(1 - \frac{\gamma}{4\Delta^{-1}}\right)$$

Phase stiffness decreases with γ

$\rightarrow$ Insulator for large γ

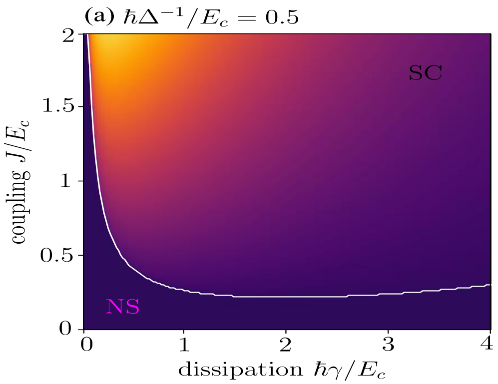
Dissipative case

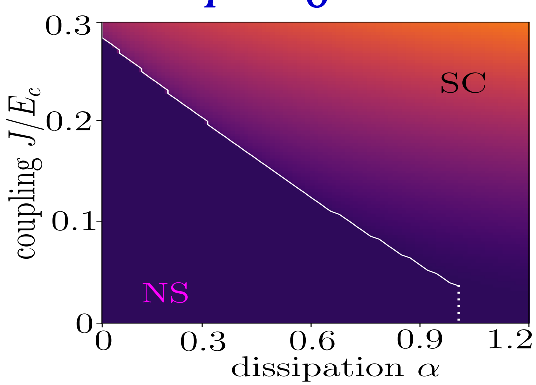
$$J_{eff} = J\left(1 - \frac{1}{2\alpha d}\ln\left(1 + \frac{\alpha}{2\pi}\right)\right) \quad \alpha \gg 1$$

$J_{eff} \rightarrow J$ for large dissipation

$\rightarrow$ Superconducting for large α

$T = 0$





Weak coupling limit $J \ll E_c$: Perturbative RG

Fast mode $\theta_{js}^{>}(t) = \int_{\omega_c/b}^{\omega_c} \frac{d\omega}{2\pi} e^{-i\omega t}\theta_{js}(\omega)$

Slow mode $\theta_{js}^{<}(t) = \int_{0}^{\omega_c/b} \frac{d\omega}{2\pi} e^{-i\omega t}\theta_{js}(\omega)$

$b = e^{\delta l}$

Treat the Josephson coupling J perturbatively

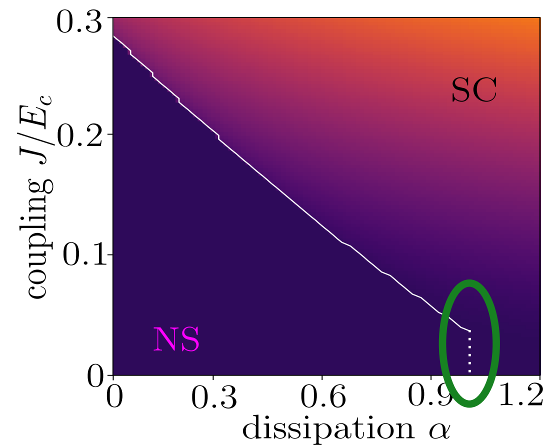
⇒ RG flow equation

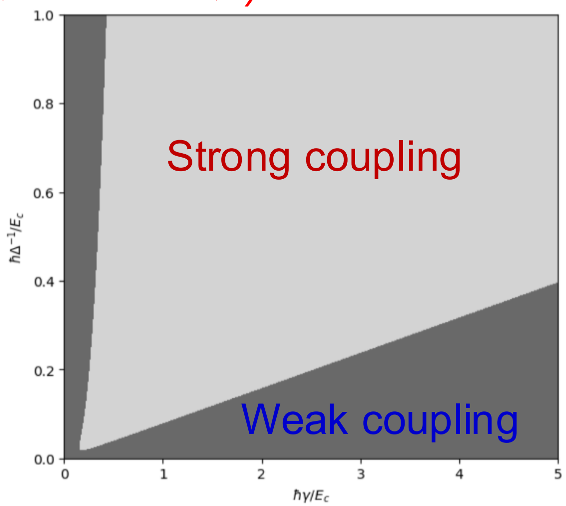
$$\frac{dJ}{dl} = J\left(1 - \frac{1}{4\pi\gamma d}\left(\frac{\Delta^{-1}}{\gamma^2} + \frac{\gamma^2}{\Delta^{-1}}\right)\right)$$

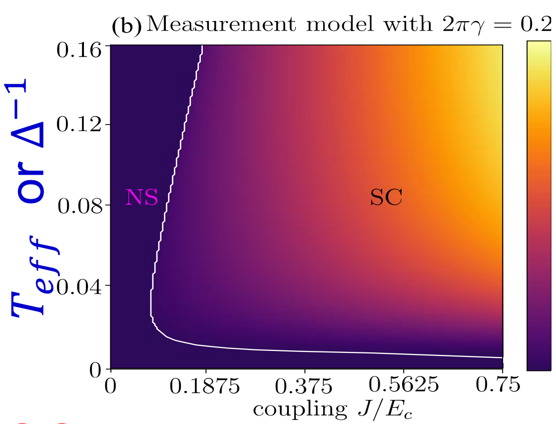
Dissipative case

$$\frac{dJ}{dl} = J\left(1 - \frac{1}{d\alpha}\right)$$

SCHA $T = 0$







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