Los espejos y la paternidad son abominables, porque multiplican y divulgan el universo.

Mirrors and fatherhood are abominable, because they multiply and spread the universe.

鏡と父は、その宇宙を繁殖させ、拡散させるがゆえに忌まわしいものである。

Jorge Luis Borges, "Tlön, Uqbar, Orbis Tertius", 1940

(in Ficciones; Eng. trans. Andrew Hurley; Jap. trans. Kazushi Shinoda)

Raphael Lefevere and Hal Tasaki, arXiv:2602.07988

Hierarchical Lorentz Mirror Model

Normal Transport and a Universal 2/3 Mean-Variance Law

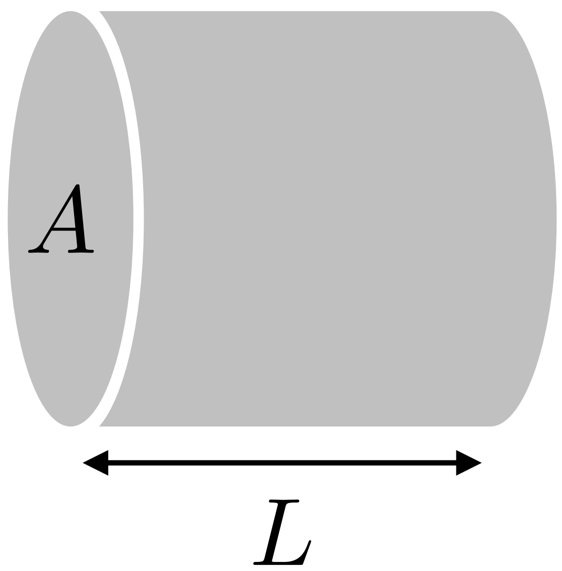
Hal Tasaki

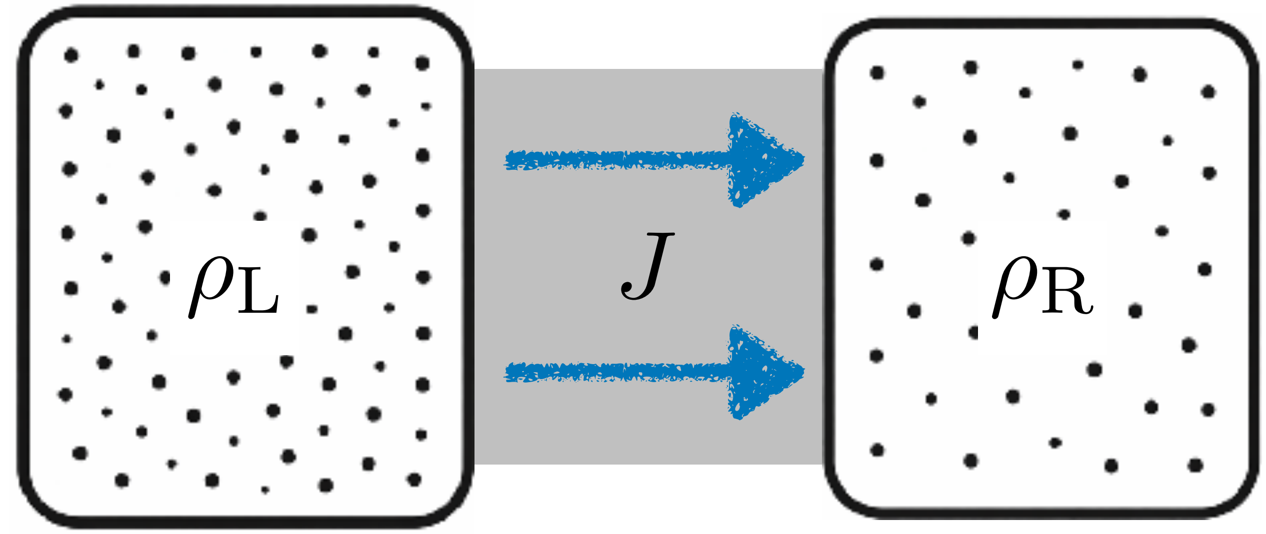
Frontiers in nonequilibrium physics, May 11,2026 @ YITP

Raphael Lefevere and Hal Tasaki, arXiv:2602.07988

normal transport

$$J = \mathcal{C}\,(\rho_{\mathrm{L}} - \rho_{\mathrm{R}})$$





conductance

$$\mathcal{C} \propto \frac{A}{L}$$

cross-section

length

follows from diffusive (Ohm's, Fick's, or Fourier's) laws

do we get normal transport from fully deterministic (and non-chaotic) dynamics in a quenched random environment?

Lorentz gas, especially its discrete version, the Lorentz mirror model

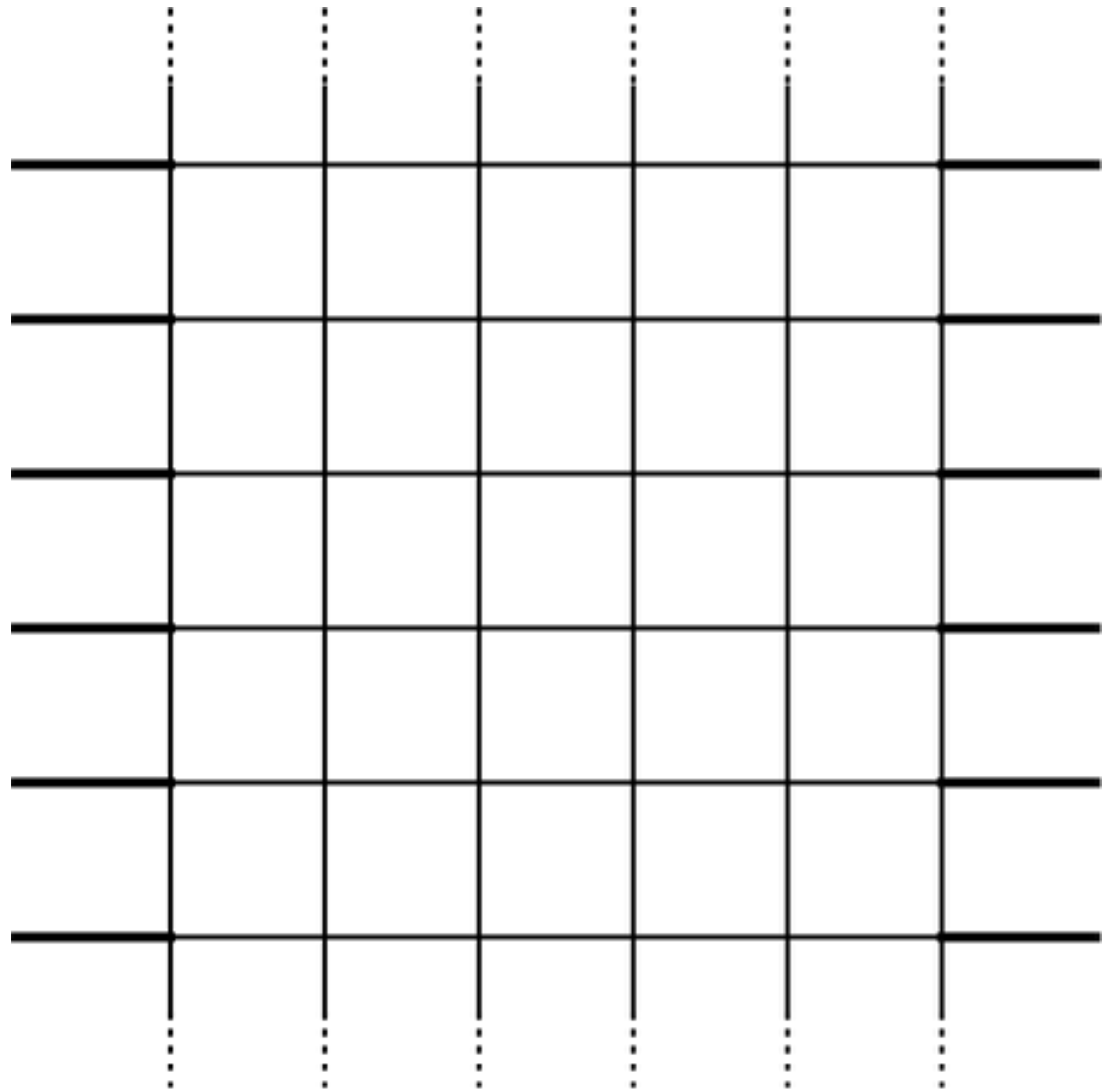
Lorentz 1905, Ruijgrok, Cohen 1988

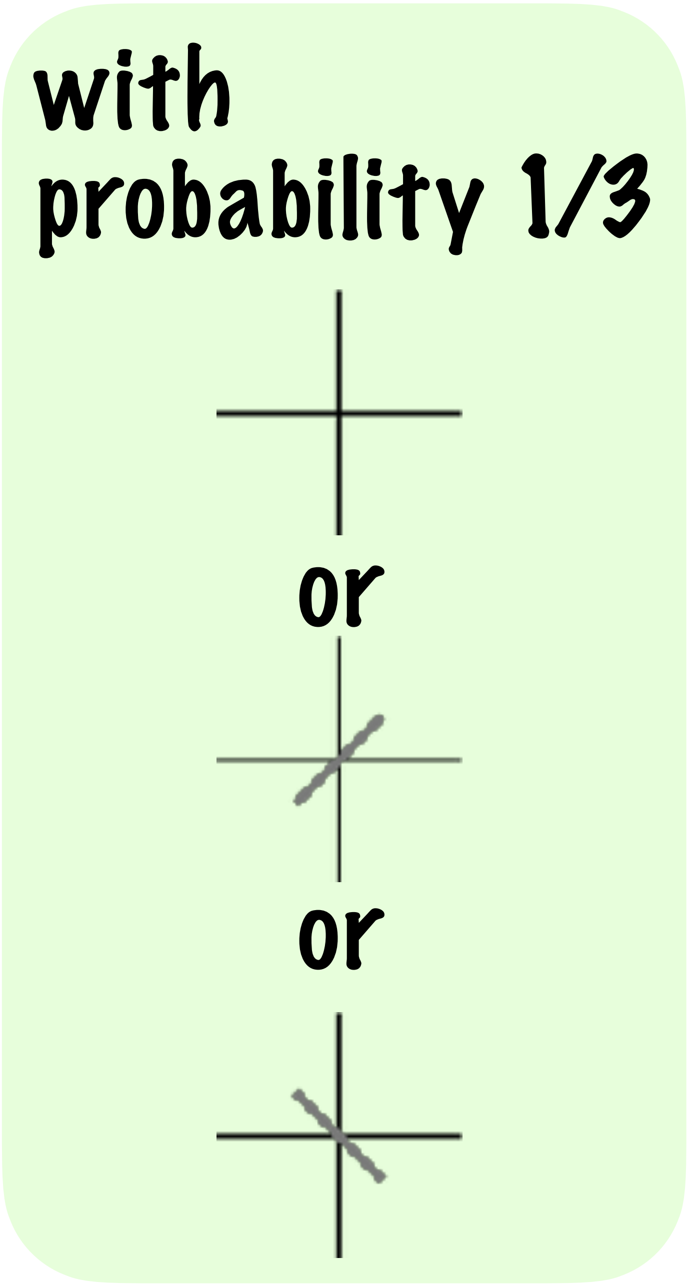
Lorentz mirror model

Hierarchical model

numerics for the $d = 3$ model

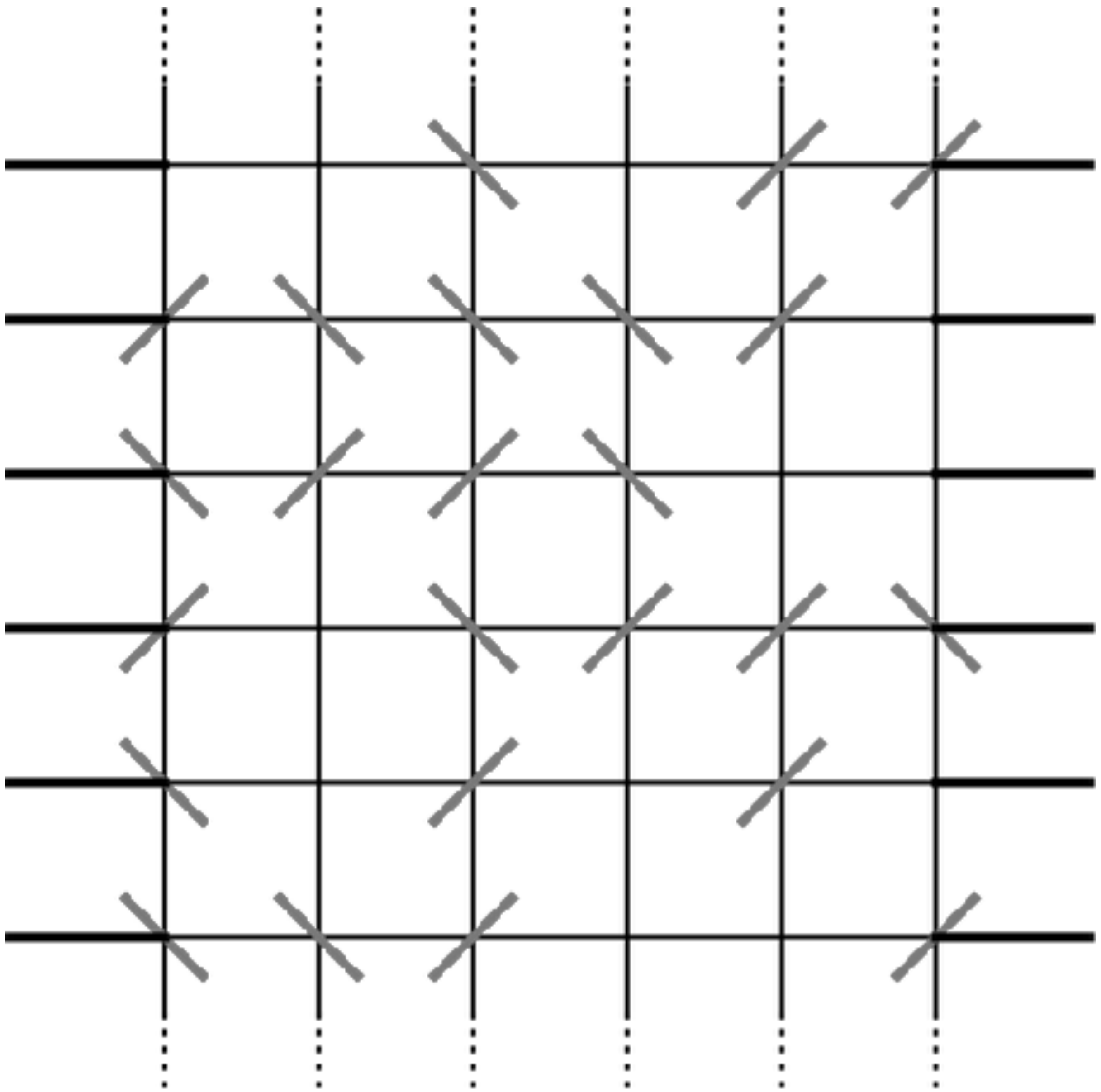
Lorentz mirror model (d=2)



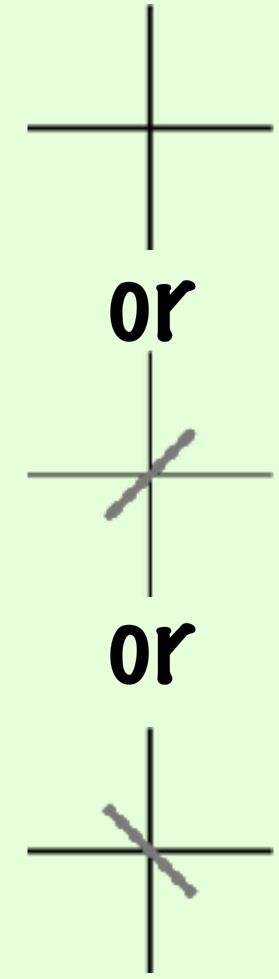


$L \times L$ square lattice
vertical: periodic b.c, horizontal: open b.c.
L external edges on the left and the right
place mirrors randomly

Lorentz mirror model (d=2)

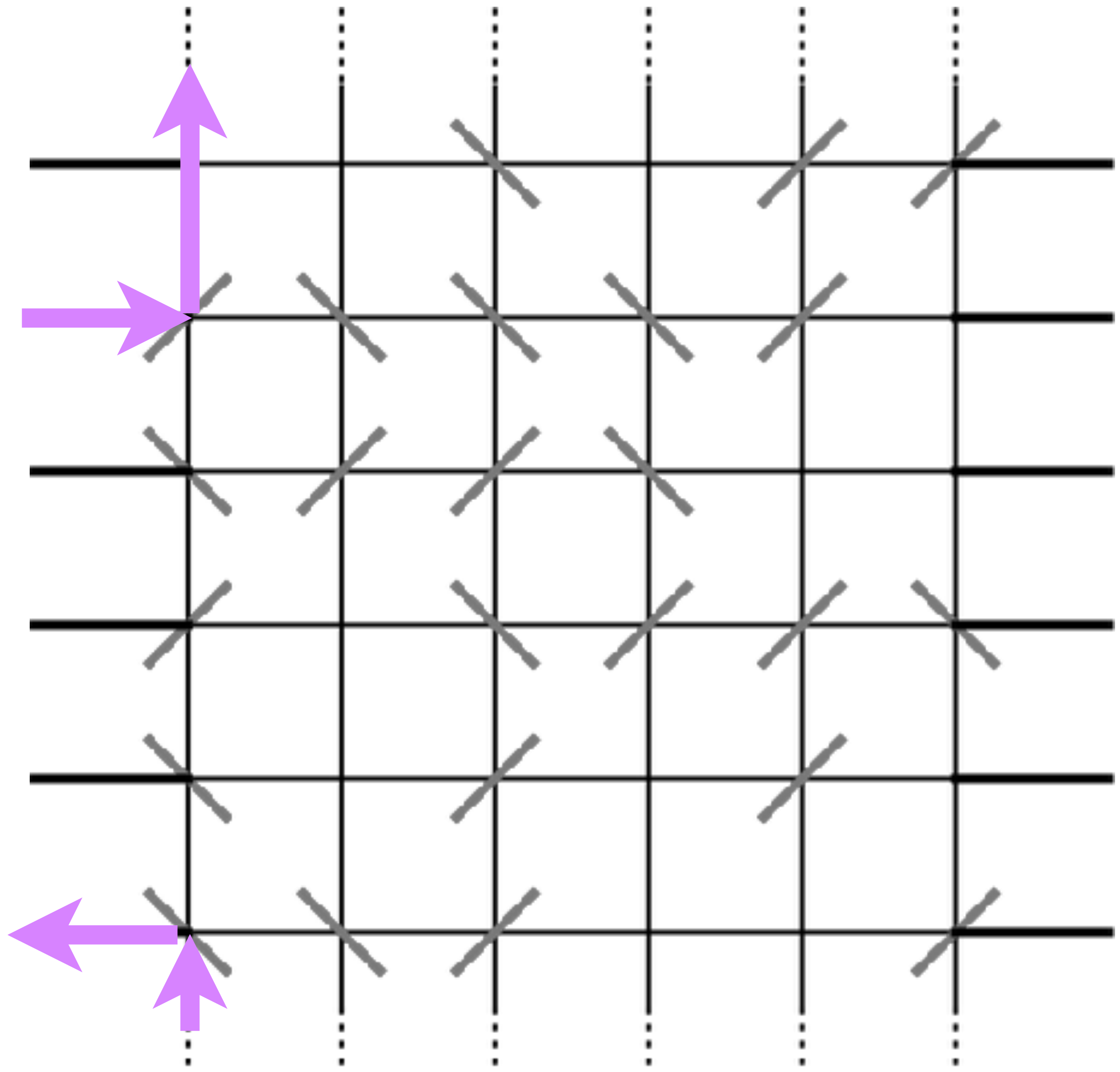


with probability 1/3



$L \times L$ square lattice
vertical: periodic b.c, horizontal: open b.c.
L external edges on the left and the right
place mirrors randomly

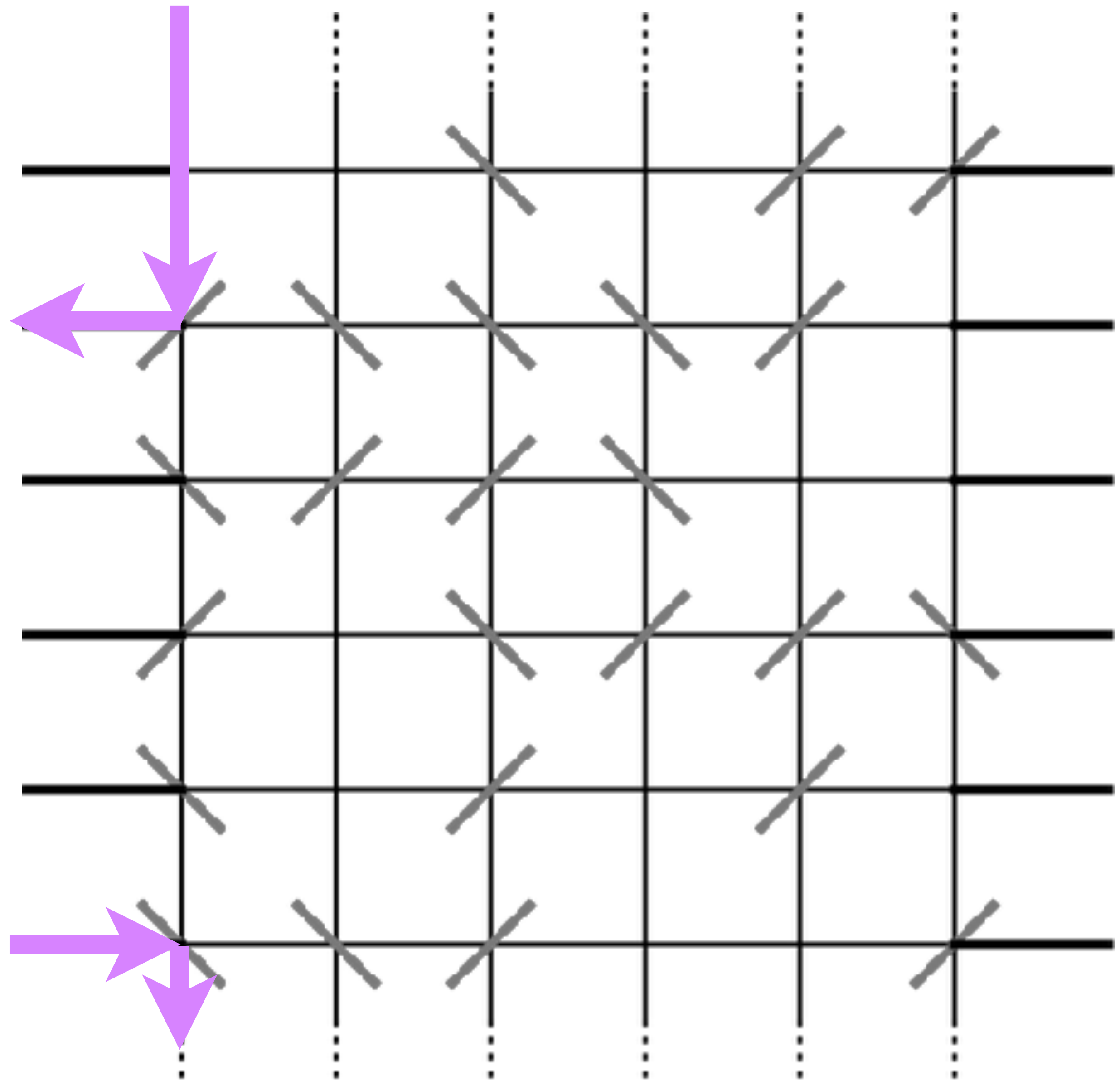
Lorentz mirror model (d=2)



a particle (or light ray) is reflected by mirrors

illustration by John Tenniel

Lorentz mirror model (d=2)

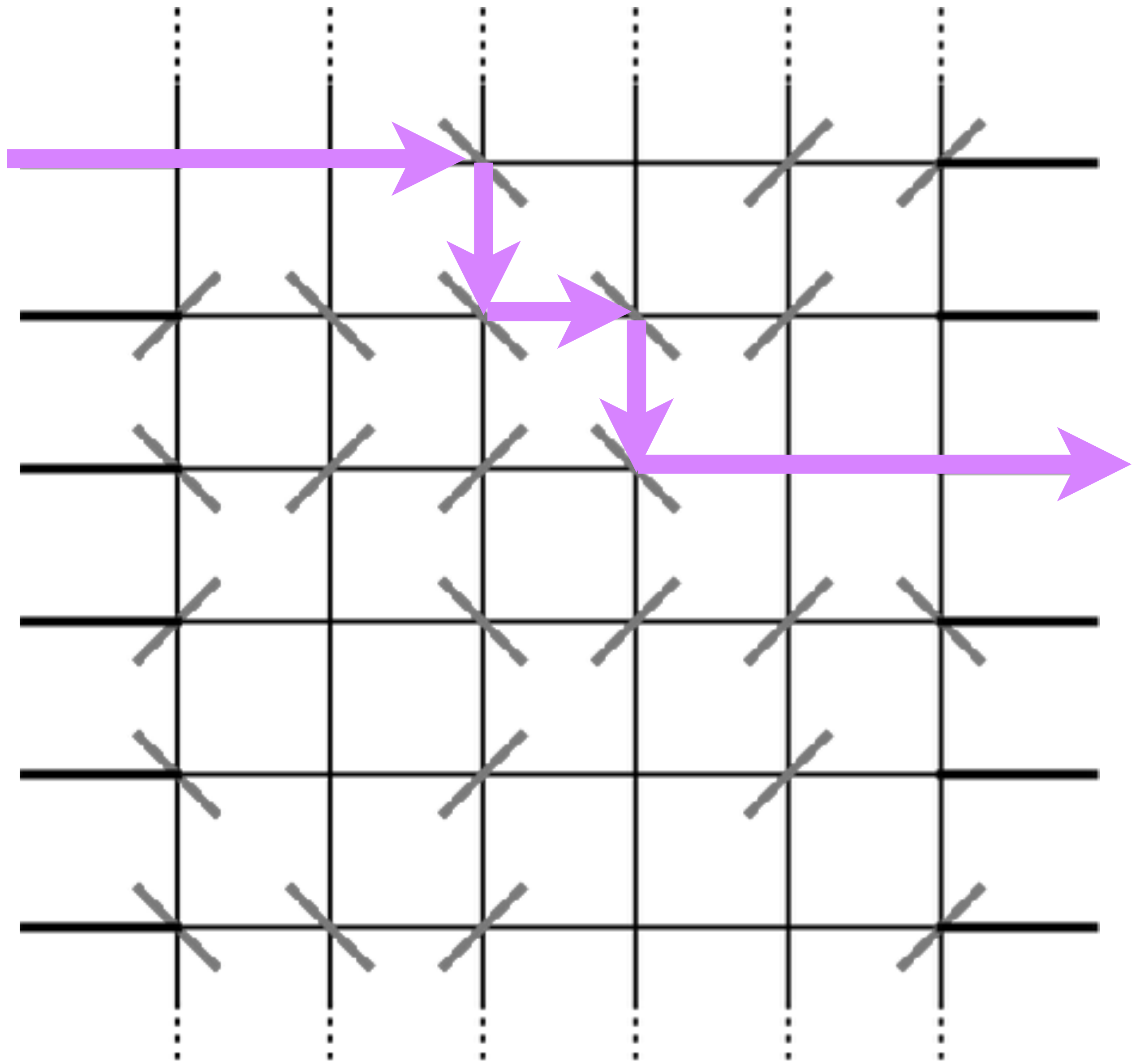


a particle (or light ray) is reflected by mirrors

trajectory is reversible

illustration by John Tenniel

Lorentz mirror model (d=2)



a particle (or light ray) is reflected by mirrors

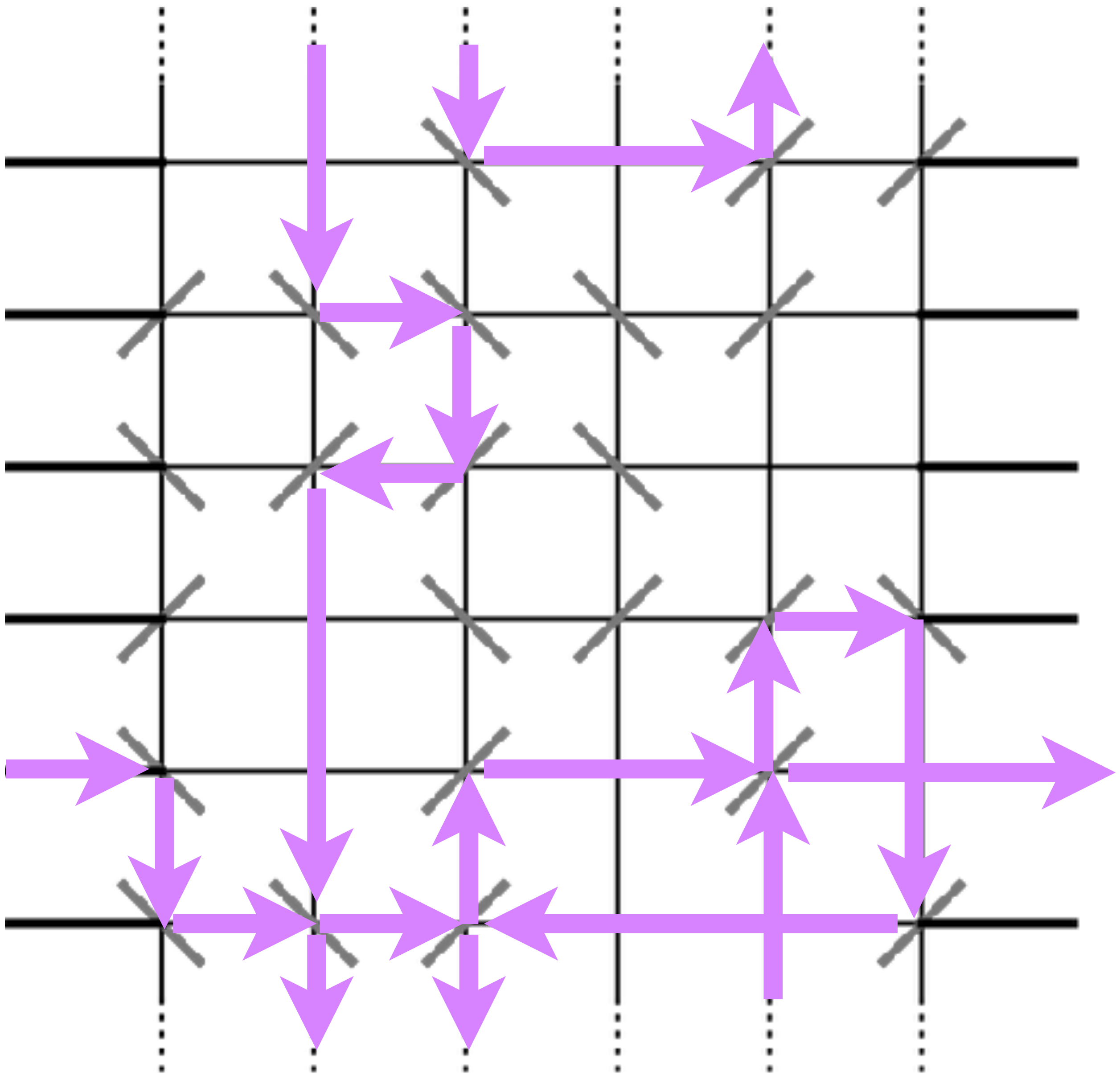
trajectory is reversible

looks like a random walk

non-backtracking random walk

illustration by John Tenniel

Lorentz mirror model (d=2)



a particle (or light ray) is reflected by mirrors

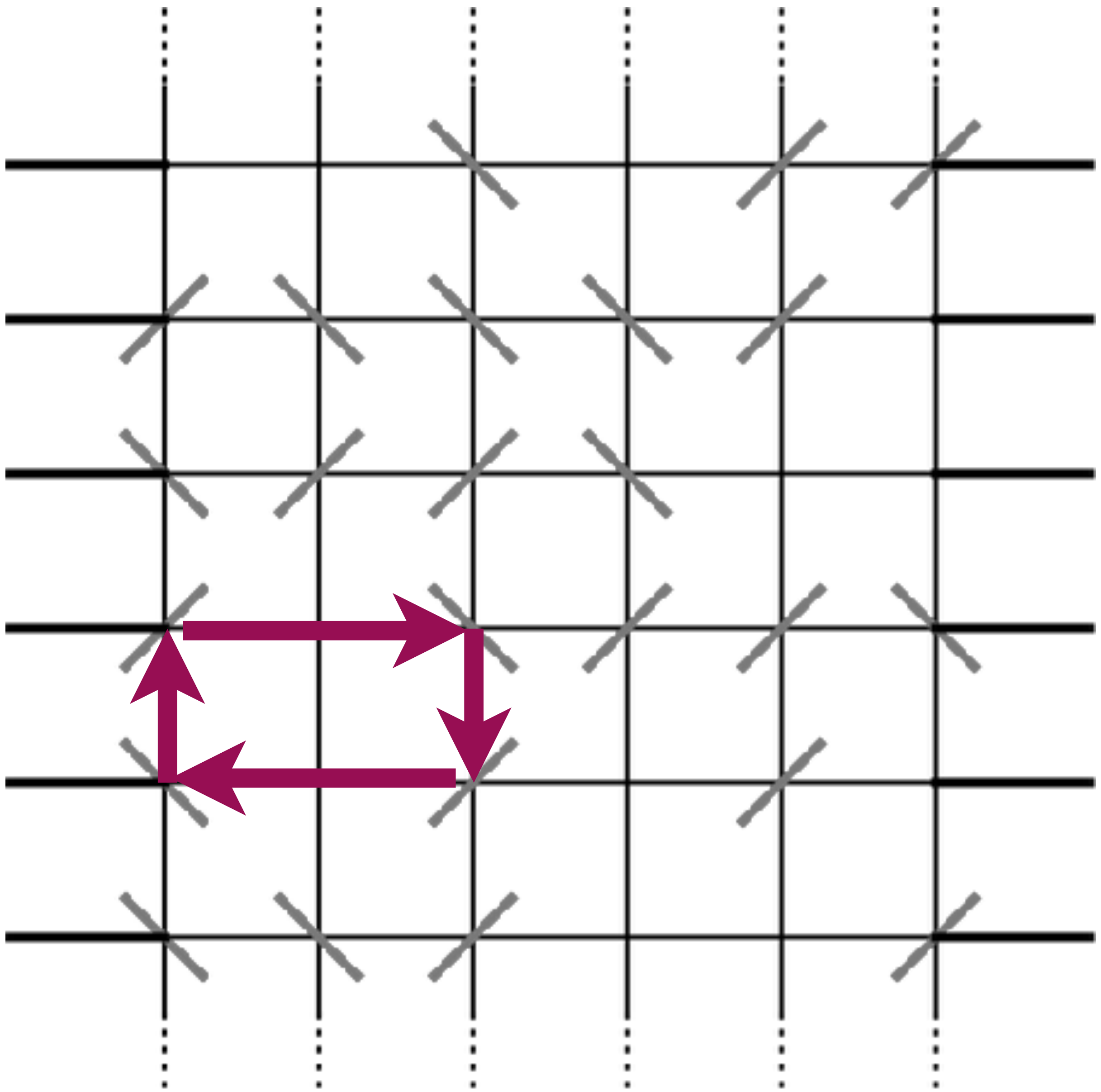
trajectory is reversible

looks like a random walk

but has strong memory!

illustration by John Tenniel

Lorentz mirror model (d=2)



a particle (or light ray) is reflected by mirrors

trajectory is reversible

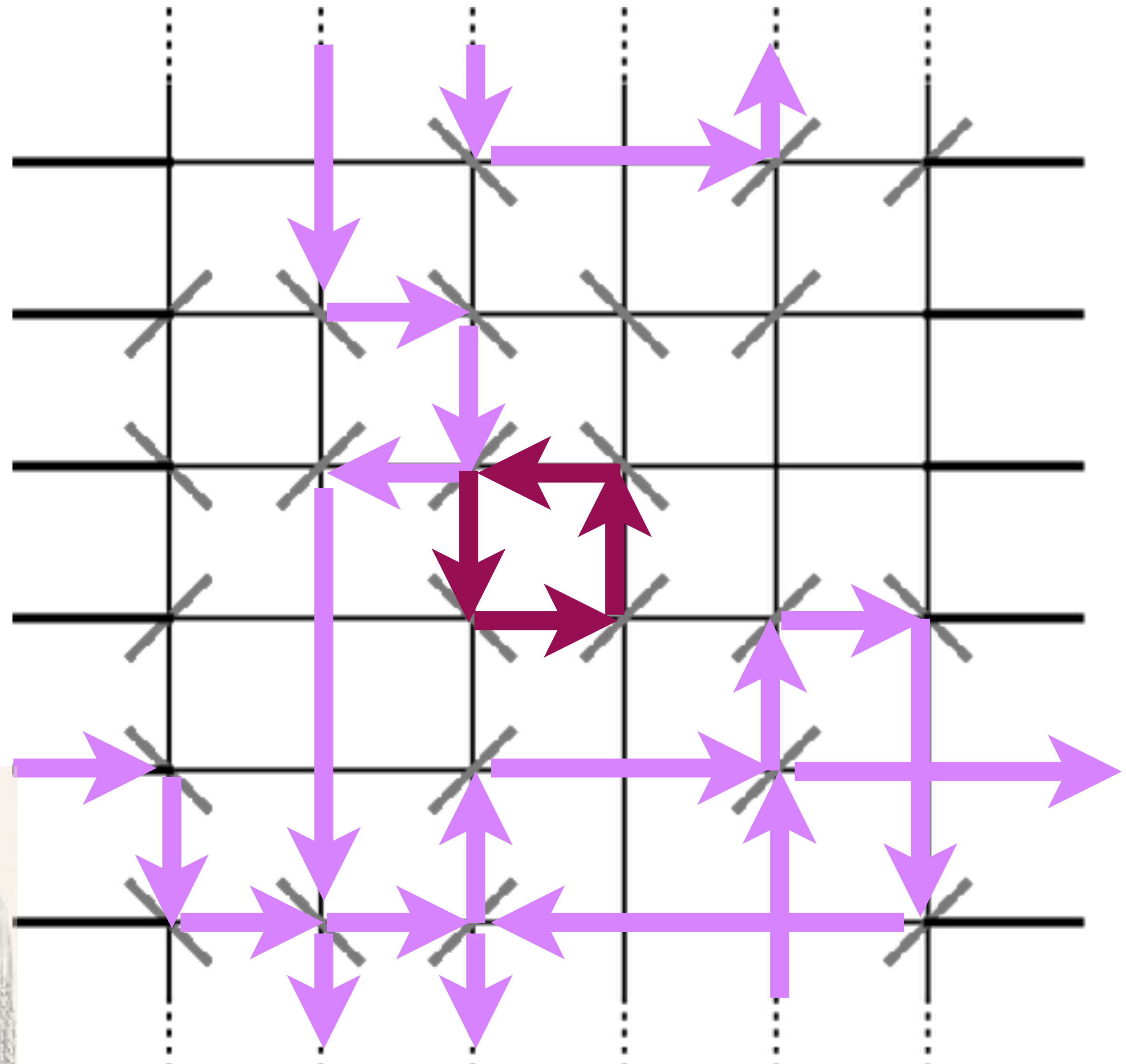
looks like a random walk

but has strong memory!

many loops!

illustration by John Tenniel

Lorentz mirror model (d=2)

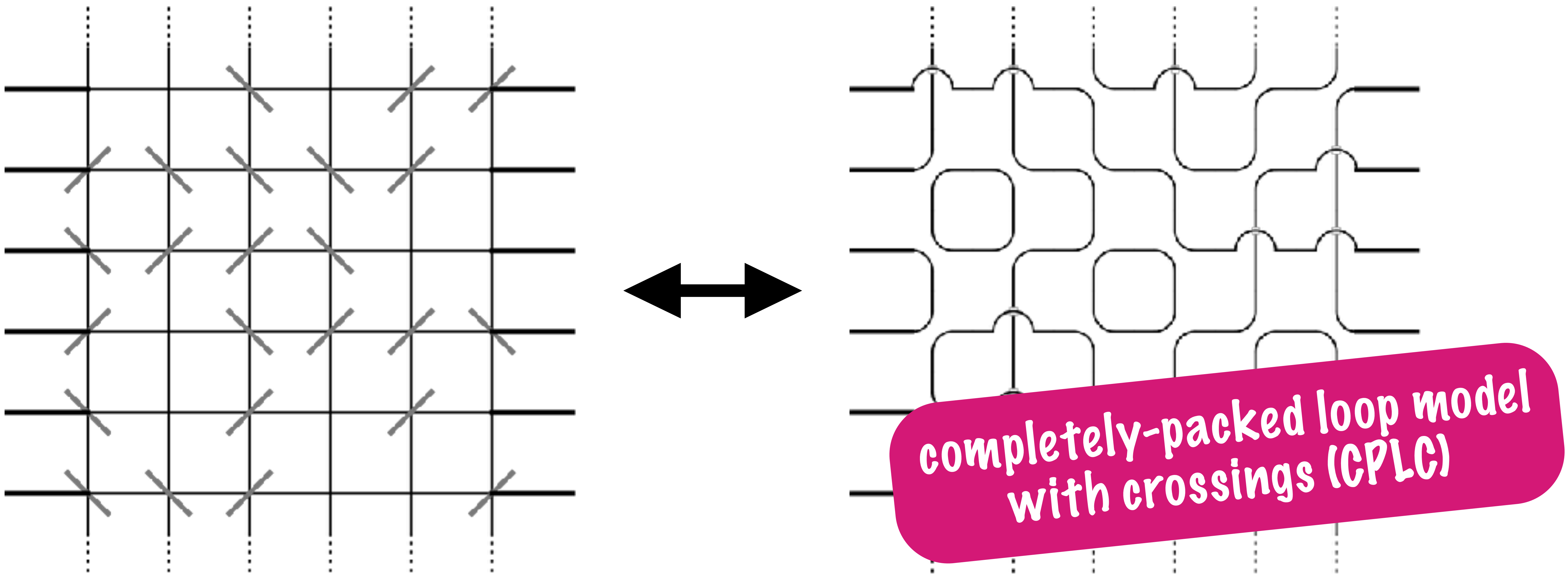


although there are many loops inside, a particle injected through an external edge must exit through another external edge (either on left or right)

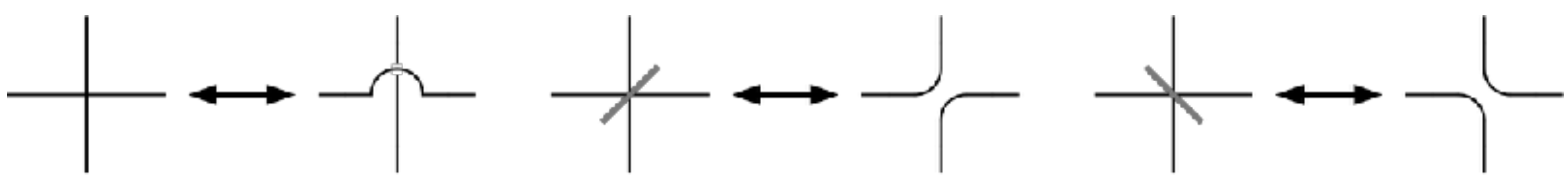
no traps!!

illustration by John Tenniel

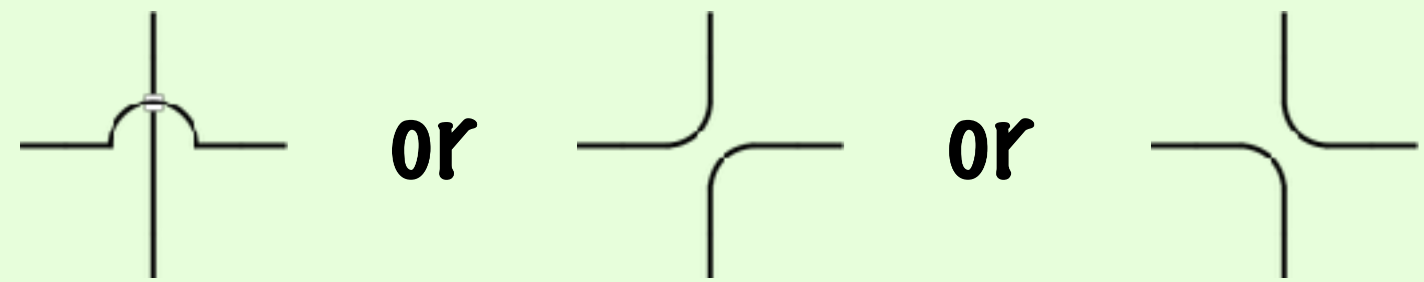
from mirrors to local pairings



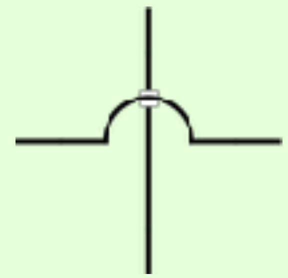
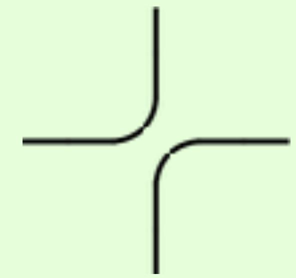
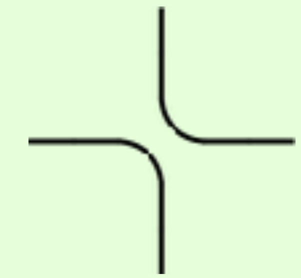
mirrors (and their absence) are equivalent to local pairings

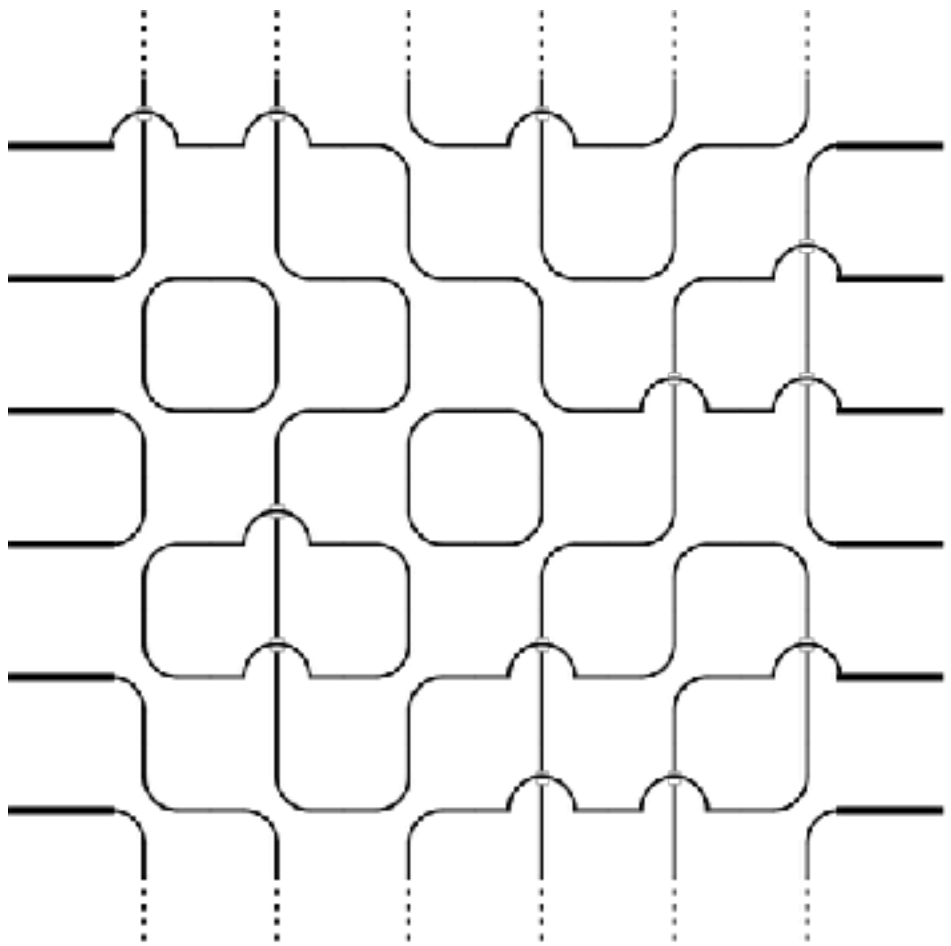


with probability 1/3



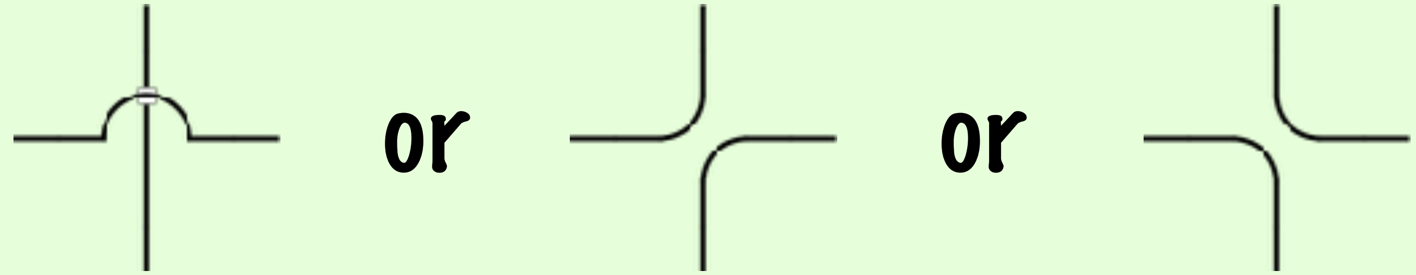
matching of external edges, and crossings

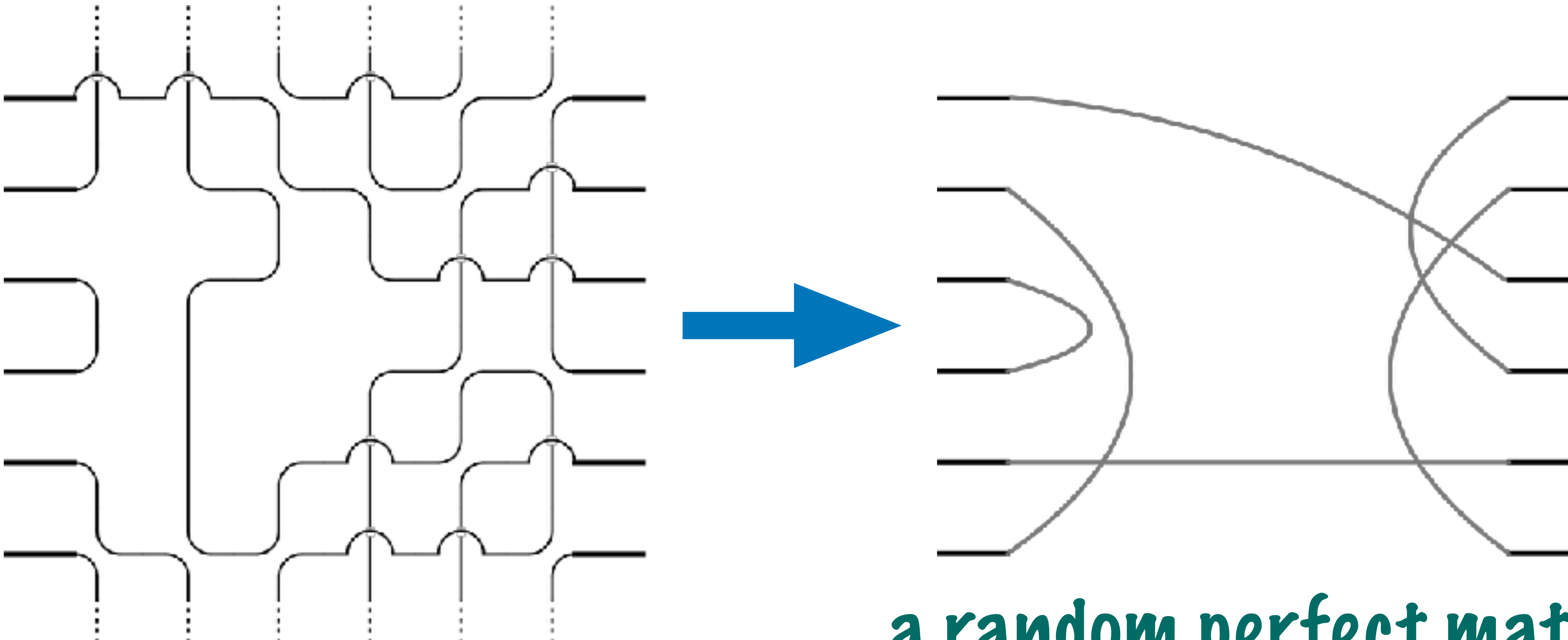
with probability 1/3  or  or 



matching of external edges, and crossings

with probability 1/3





remove loops

a random perfect matching (pairing) of external edges

left-right pair = crossing

conductance

$\mathscr{C}$ the number of crossings

Lorentz mirror model in $d \geq 2$

d dimensional $L \times \cdots \times L$ hyper cubic lattice
horizontal: open b.c., other directions: periodic b.c

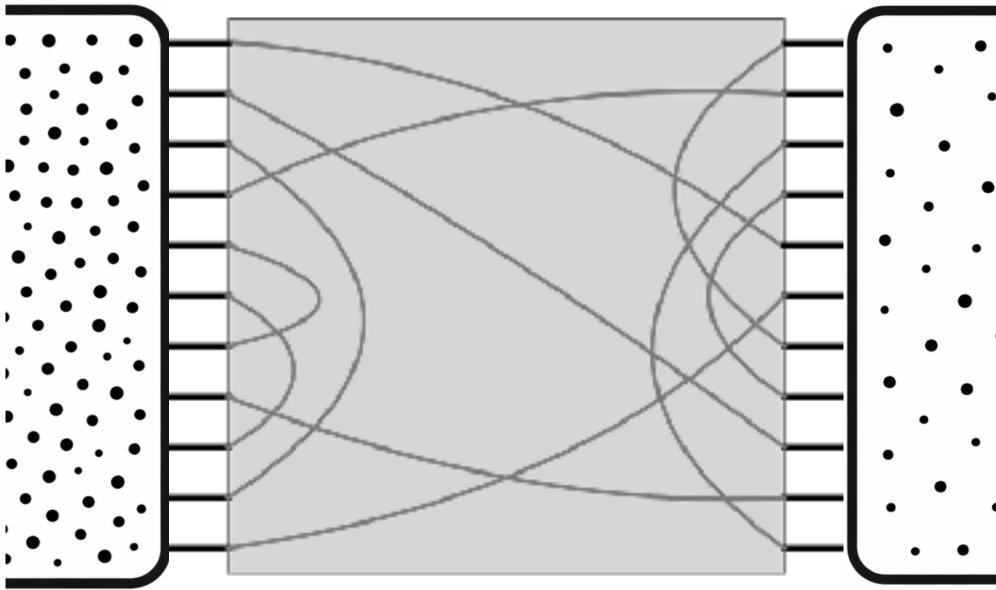
L^{d-1} external edges on the left and the right

at each vertex, independently choose one of the $(2d-1)!!$ pairings of the $2d$ incident edges with equal probability

local pairings induce random matching of the $2L^{d-1}$ external edges

conductance $\mathscr{C}$: the number of crossings (matched pairs of left and right external edges)

when left and right boundaries are exposed to particle baths with different densities, the stationary current is proportional to $\mathscr{C}$



Lefevere 2015 , Chiffaudel, Lefevere 2016

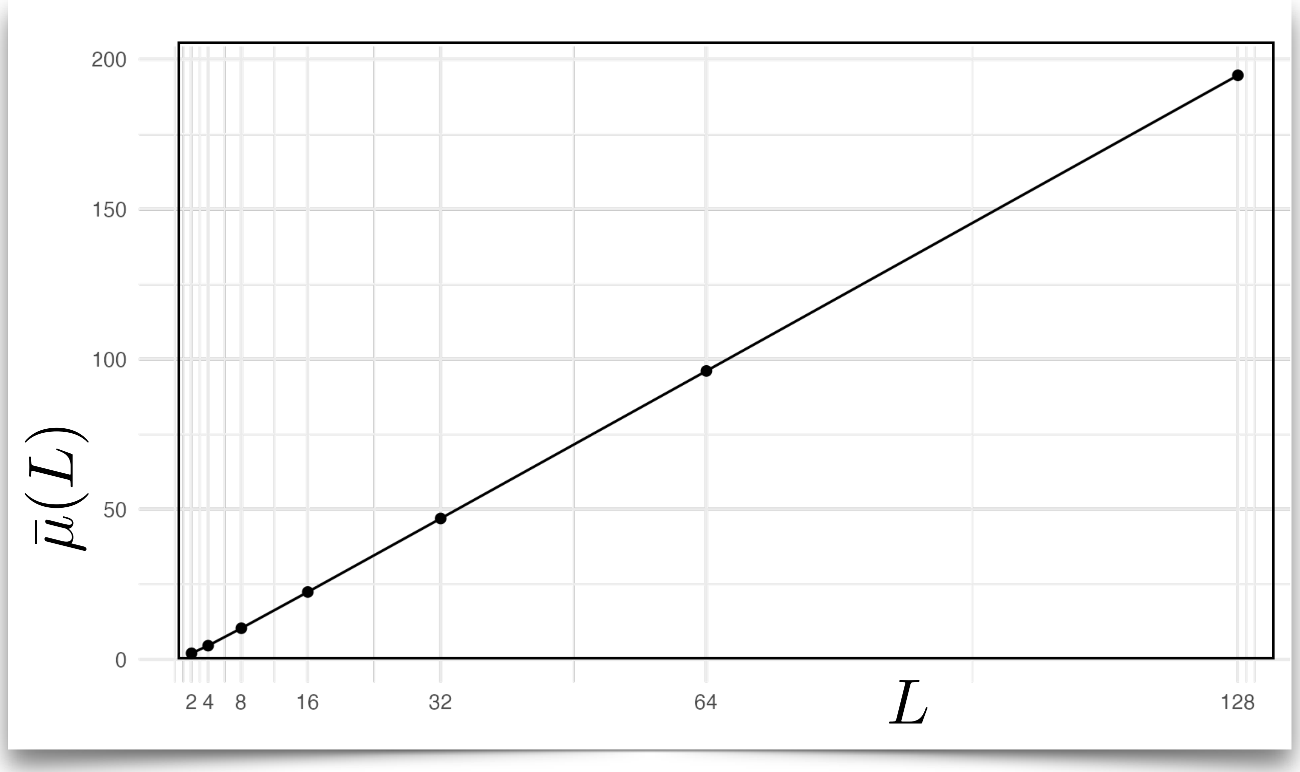
normal transport in the Lorentz mirror model

mean conductance $\bar{\mu}(L) = \langle \mathscr{C} \rangle$

normal transport $$\bar{\mu}(L) \propto \frac{A}{L} = \frac{L^{d-1}}{L} = L^{d-2}$$

numerical and theoretical support for normal transport in $d = 3$

Chiffaudel, Lefevere 2016, Lefevere 2025



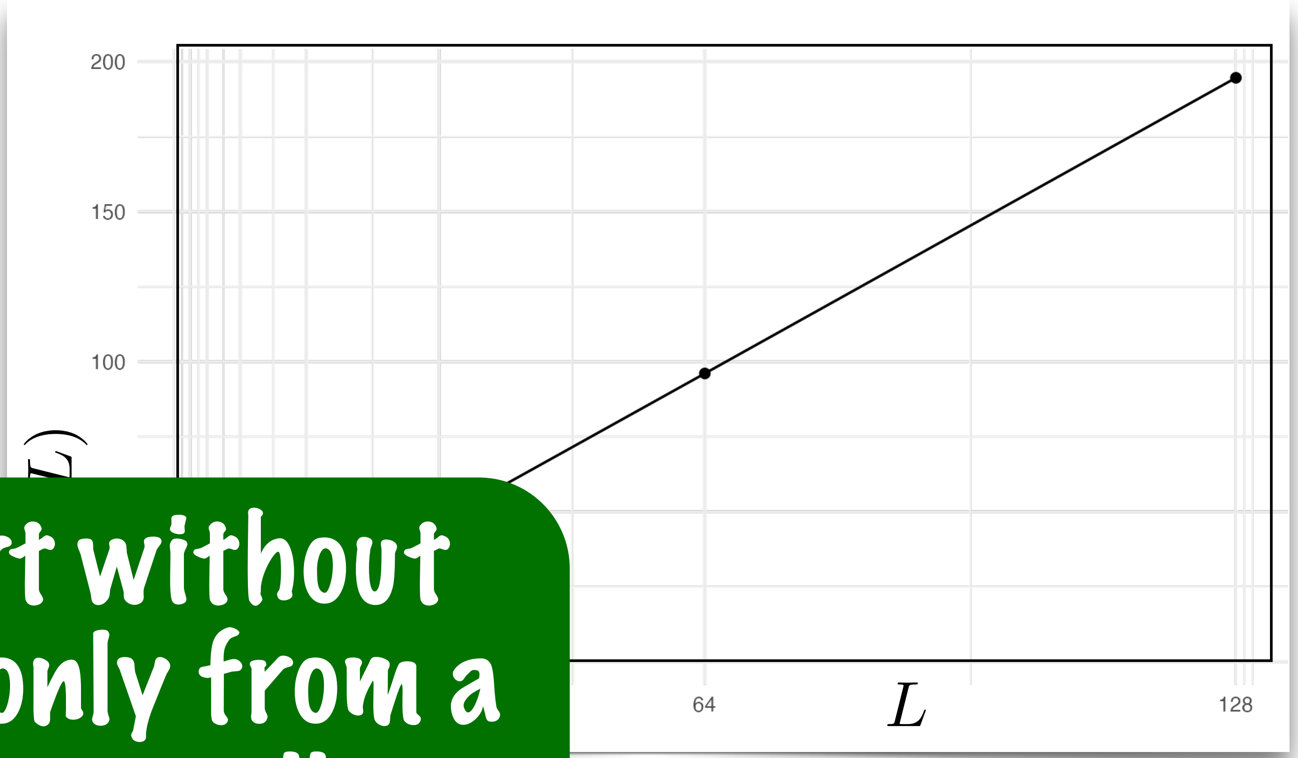
normal transport in the Lorentz mirror model

mean conductance $\bar{\mu}(L) = \langle \mathscr{C} \rangle$

normal transport $$\bar{\mu}(L) \propto \frac{A}{L} = \frac{L^{d-1}}{L} = L^{d-2}$$

numerical and theoretical support for normal transport in $d = 3$

Chiffaudel, Lefevere 2016, Lefevere 2025



normal (diffusive) transport without stochastic forcing or chaos, only from a quenched random environment!!

the structure of the trajectories is far from diffusive!

rigorous proof seems formidable...

tractable, yet nontrivial, hierarchical version of the model

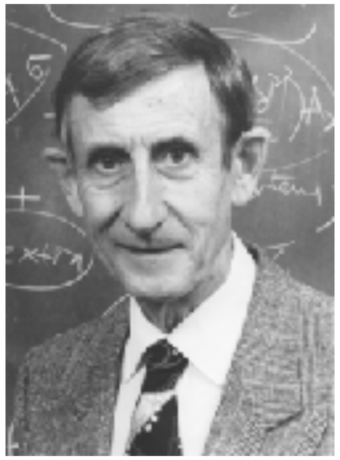
correct
dimensional scaling

mean-field-like
long-range interaction

Lorentz mirror model

Hierarchical model

Dyson 1969, ...



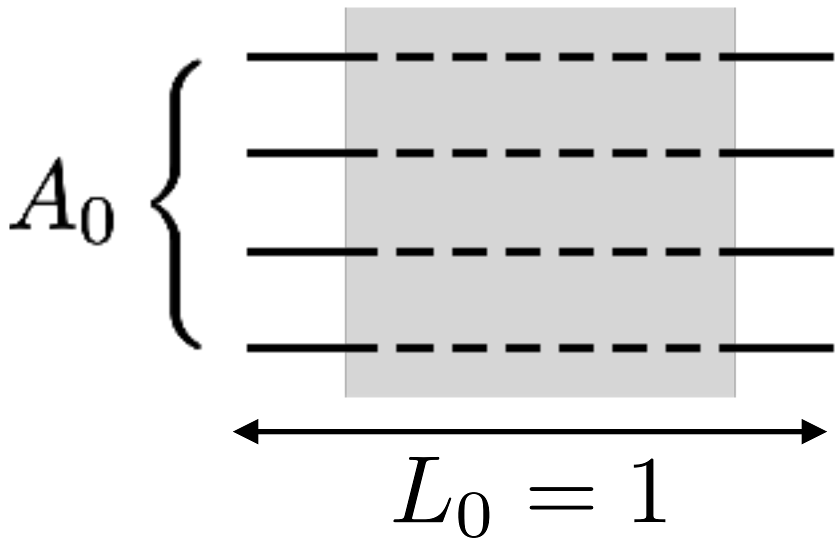
numerics for the $d = 3$ model

basic setting

dimension $d = 1,2,3,\ldots$ A_0 a positive even integer

generation $n = 0,1,2,\ldots$

generation-0 block



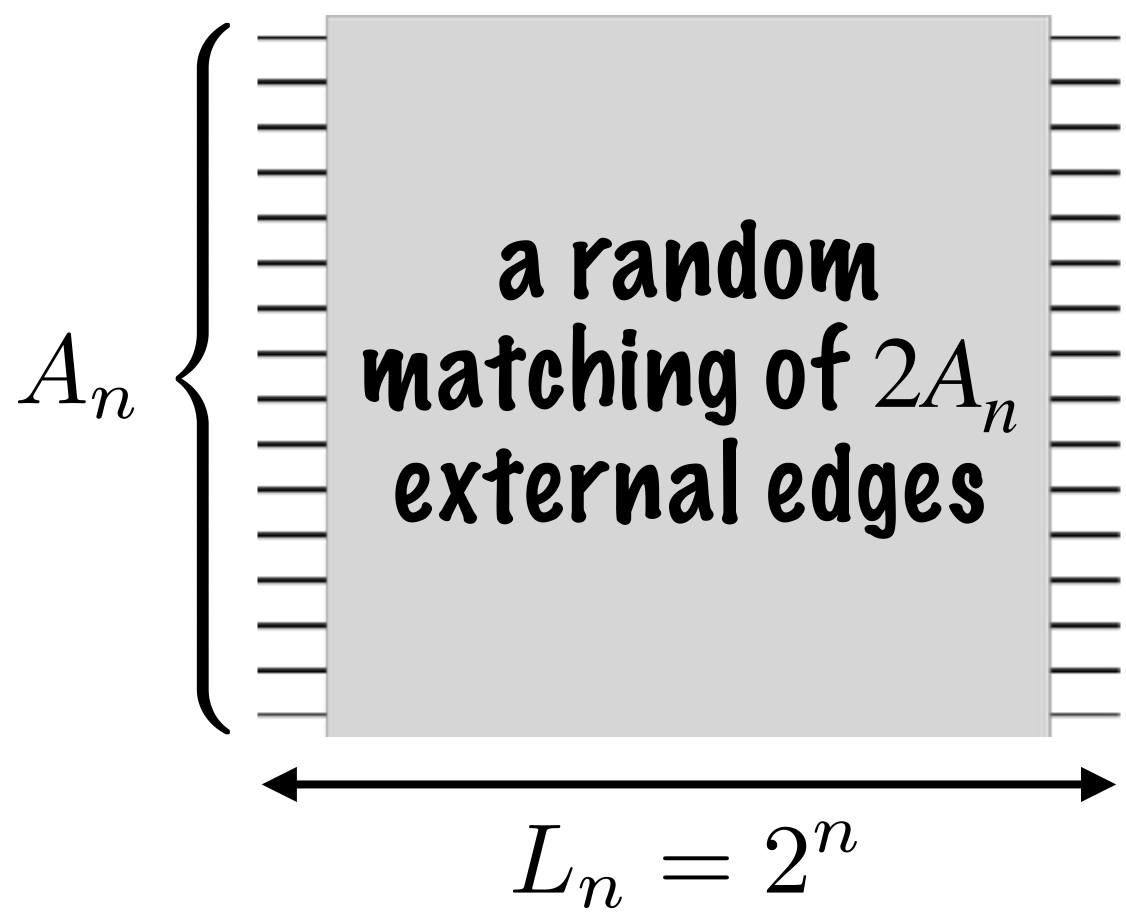
A_0 external edges on the left and right

exactly A_0 crossings

generation-n block

A_n external edges on the left and right

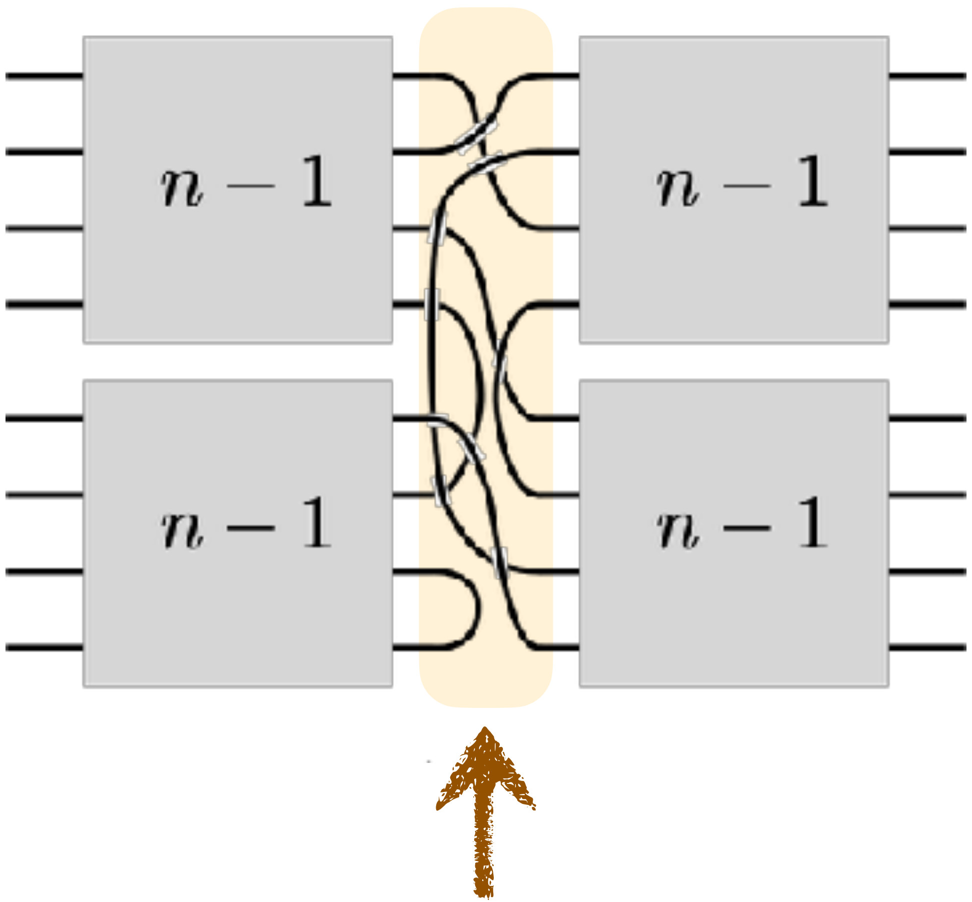
$$A_n = 2^{(d-1)n} A_0$$



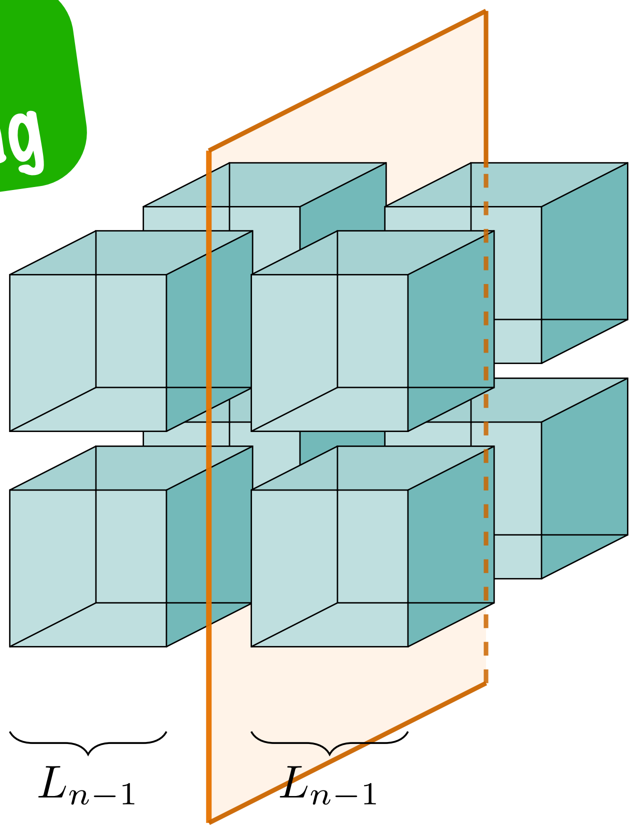
hierarchical construction

2^d independent copies of generation-$(n-1)$ blocks

2^{d-1} copies on the left and right



correct dimensional scaling

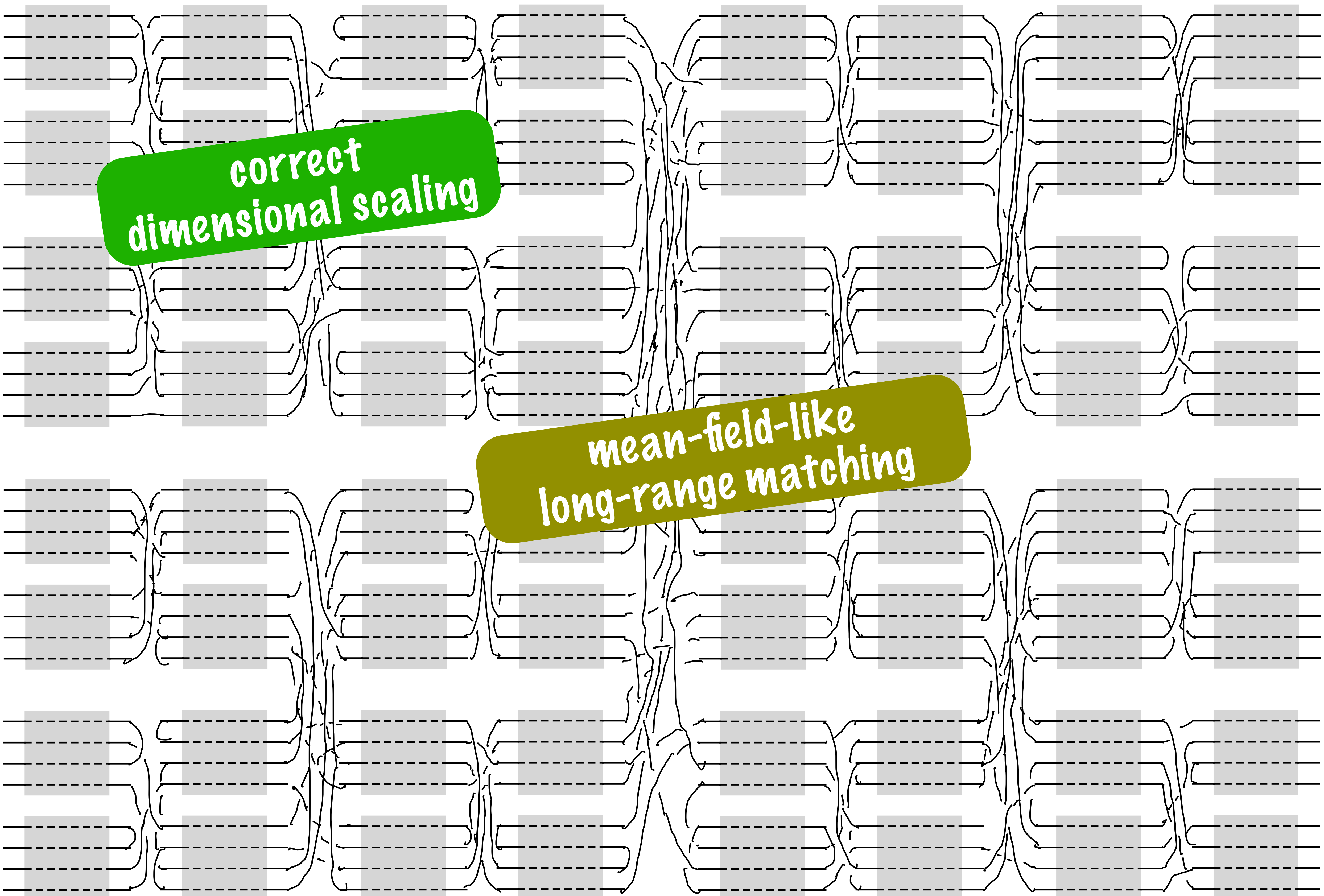


a random matching of $2^d A_{n-1} = 2A_n$ external edges at the interface (we choose one of the $(2A_n - 1)!!$ perfect matchings with equal probability)

mean-field-like long-range matching

hierarchical construction

a generation-3 block ($d = 2, A_0 = 4$)



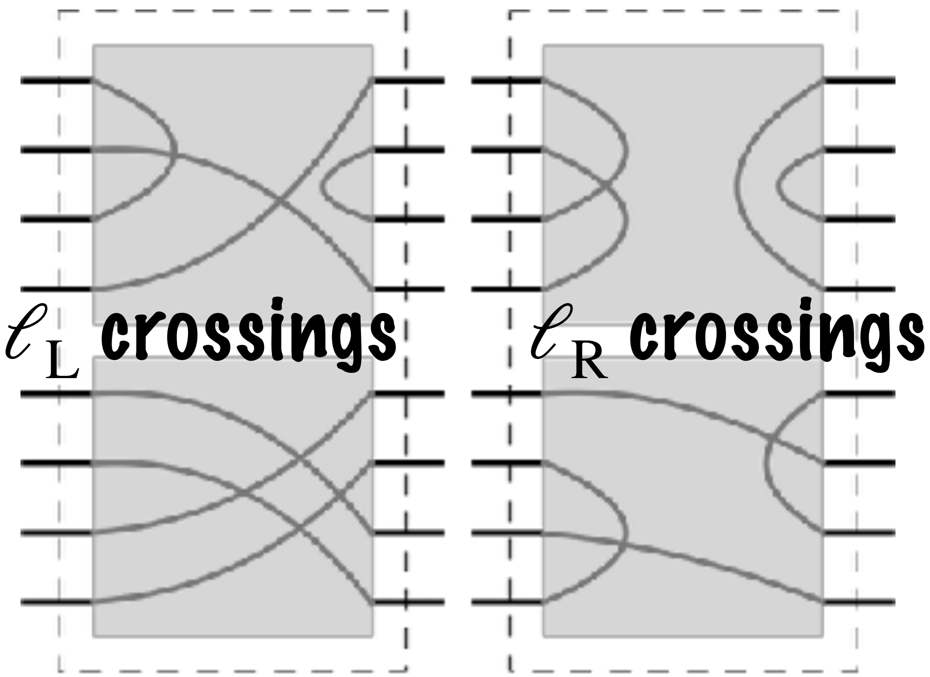
exact recursion relation

$P_n(\ell)$ probability that there are exactly ℓ crossings in generation-n block ($n = 0,1,\ldots$)

$$P_0(\ell) = \delta_{\ell,A_0}$$

$$P_n(\ell) = \sum_{\ell_1,\ldots,\ell_{2^d}} K(\ell\,|\,\ell_\mathrm{L},\ell_\mathrm{R}) \prod_{j=1}^{2^d} P_{n-1}(\ell_j)$$

over nonnegative even $\ell_1,\ldots,\ell_{2^d}$



$$\ell_\mathrm{L} = \sum_{j=1}^{2^{d-1}} \ell_j$$

$$\ell_\mathrm{R} = \sum_{j=2^{d-1}+1}^{2^d} \ell_j$$

$$K(\ell\,|\,\ell_\mathrm{L},\ell_\mathrm{R}) = \frac{(\ell_\mathrm{L}-\ell-1)!!(\ell_\mathrm{R}-\ell-1)!!}{(\ell_\mathrm{L}+\ell_\mathrm{R}-1)!!}\binom{\ell_\mathrm{L}}{\ell}\binom{\ell_\mathrm{R}}{\ell}\ell!$$

if ℓ is an even integer s.t. $0 \le \ell \le \min\{\ell_\mathrm{R},\ell_\mathrm{R}\}$

$K(\ell\,|\,\ell_\mathrm{L},\ell_\mathrm{R}) = 0$ otherwise

behavior of the mean conductance

$P_n(\ell)$ probability that there are exactly ℓ crossings in generation-n block ($n = 0,1,\ldots$)

mean conductance $\mu_n = \sum_{\ell=0,2,\ldots} \ell\, P_n(\ell)$

conditional mean

for $\ell_{\mathrm{L}}, \ell_{\mathrm{R}} \gg 1$

$$\tilde{\mu}(\ell_{\mathrm{L}}, \ell_{\mathrm{R}}) = \sum_{\ell} \ell\, K(\ell\,|\,\ell_{\mathrm{L}}, \ell_{\mathrm{R}}) = \frac{\ell_{\mathrm{L}}\,\ell_{\mathrm{R}}}{\ell_{\mathrm{L}} + \ell_{\mathrm{R}} - 1} \simeq \left(\frac{1}{\ell_{\mathrm{L}}} + \frac{1}{\ell_{\mathrm{R}}}\right)^{-1}$$

since the main contribution in the recursion should come from $\ell_{\mathrm{L}} \simeq \ell_{\mathrm{R}} \simeq 2^{d-1}\mu_{n-1}$, we expect

$$\mu_n \simeq 2^{d-2}\mu_{n-1}$$

normal transport!!

$$\mu_n \simeq 2^{(d-2)n}\mu_0 = 2^{(d-2)n}A_0 = \frac{A_n}{L_n}$$

$$A_n = 2^{(d-1)n}A_0$$
$$L_n = 2^n$$

$$P_n(\ell) = \sum_{\ell_1,\ldots,\ell_{2^d}} K(\ell\,|\,\ell_{\mathrm{L}}, \ell_{\mathrm{R}}) \prod_{j=1}^{2^d} P_{n-1}(\ell_j)$$

$$\ell_{\mathrm{L}} = \sum_{j=1}^{2^{d-1}} \ell_j \qquad \ell_{\mathrm{R}} = \sum_{j=2^{d-1}+1}^{2^d} \ell_j$$

main theorem: normal transport

theorem: let $d \geq 3$. for sufficiently large A_0 there exist positive constants C_d, C'_d such that

$$C_d \frac{A_n}{L_n} - 1 \leq \mu_n \leq C'_d \frac{A_n}{L_n}$$

for all n. if $A_0 \gg 1$, one has $C_d \simeq C'_d \simeq 1$ and hence $\mu_n \simeq A_n/L_n$ for all n.

for $d = 3$, $A_0 \geq 2$ suffices (no conditions)
$C_3 = 1 - 5/(4A_0)$, $C'_3 = 1 + 1/(3A_0)$

rigorous derivation of normal transport solely from a fully deterministic and non-chaotic motion in a quenched random environment!

in an artificial hierarchical setting...

proof of the upper bound

mean conductance $\mu_n = \sum_\ell \ell\, P_n(\ell)$

recursion $P_n(\ell) = \sum_{\ell_1,\ldots,\ell_{2^d}} K(\ell \,|\, \ell_\mathrm{L}, \ell_\mathrm{R}) \prod_{j=1}^{2^d} P_{n-1}(\ell_j)$

$$\mu_n = \sum_{\ell_1,\ldots,\ell_{2^d}} \boxed{\sum_\ell \ell\, K(\ell \,|\, \ell_\mathrm{L}, \ell_\mathrm{R})} \prod_{j=1}^{2^d} P_{n-1}(\ell_j)$$

$\ell_\mathrm{L} = \sum_{j=1}^{2^{d-1}} \ell_j$

$\ell_\mathrm{R} = \sum_{j=2^{d-1}+1}^{2^d} \ell_j$

$$\tilde{\mu}(\ell_\mathrm{L}, \ell_\mathrm{R}) = \frac{\ell_\mathrm{L}\,\ell_\mathrm{R}}{\ell_\mathrm{L} + \ell_\mathrm{R} - 1} \le \frac{\ell_\mathrm{L} + \ell_\mathrm{R}}{4} + \frac{1}{3}$$

$$\mu_n \le \sum_{\ell_1,\ldots,\ell_{2^d}} \Big(\frac{1}{4}\sum_{j=1}^{2^d} \ell_j + \frac{1}{3}\Big) \prod_{j=1}^{2^d} P_{n-1}(\ell_j) = 2^{d-2}\mu_{n-1} + \frac{1}{3}$$

for $d \neq 2$ $\mu_n + C''_d \le 2^{d-2}(\mu_{n-1} + C''_d)$

$\le 2^{(d-2)n}(\mu_0 + C''_d)$ **with** $C''_d = \{3(2^{d-2} - 1)\}^{-1}$

for $d = 2$ $\mu_n \le \dfrac{n}{3} + \mu_0$

suggests weakly anomalous transport with a log correction

proof of the lower bound

mean of the inverse $\eta_n = \sum_\ell (\ell+1)^{-1} P_n(\ell)$

$$u_n = \sum_{\ell_1,\ldots,\ell_{2^{d-1}}} \left(\sum_{j=1}^{2^{d-1}} \ell_j + 1\right)^{-1} \prod_{j=1}^{2^{d-1}} P_n(\ell_j)$$

Jensen's inequality implies $\mu_n \geq \eta_n^{-1} - 1$

recursion $P_n(\ell) = \sum_{\ell_1,\ldots,\ell_{2^d}} K(\ell \,|\, \ell_\mathrm{L}, \ell_\mathrm{R}) \prod_{j=1}^{2^d} P_{n-1}(\ell_j)$

$$\eta_n = \sum_{\ell_1,\ldots,\ell_{2^d}} \boxed{\sum_\ell (\ell+1)^{-1} K(\ell \,|\, \ell_\mathrm{L}, \ell_\mathrm{R})} \prod_{j=1}^{2^d} P_{n-1}(\ell_j)$$

$$= \tilde{\eta}(\ell_\mathrm{L}, \ell_\mathrm{R})$$

$$\tilde{\eta}(\ell_\mathrm{L}, \ell_\mathrm{R}) = \frac{\ell_\mathrm{L} + \ell_\mathrm{R} + 1}{(\ell_\mathrm{L}+1)(\ell_\mathrm{R}+1)} = \frac{1}{\ell_\mathrm{L}+1} + \frac{1}{\ell_\mathrm{R}+1} - \frac{1}{(\ell_\mathrm{L}+1)(\ell_\mathrm{R}+1)}$$

recursive identity $\eta_n = 2u_{n-1} - (u_{n-1})^2$

for $d = 1$, where $u_n = \eta_n$, $\eta_n \leq 2\eta_{n-1} \leq 2^n \eta_0 = 2^n/(A_0+1)$

$$\mu_n \geq (A_0+1)/L_n - 1$$

suggested by ChatGPT 5.2 Pro!!

for $d \geq 3$, we use further clever concavity argument

Gaussian approximation

$P_n(\ell)$ probability that there are exactly ℓ crossings in generation-n block ($n = 0,1,\ldots$)

mean conductance $\mu_n = \sum_{\ell=0,2,\ldots} \ell \, P_n(\ell)$

sample-to-sample variance $v_n = \sum_{\ell=0,2,\ldots} (\ell - \mu_n)^2 P_n(\ell)$

ansatz $P_n(\ell) \propto \exp\Big[-\dfrac{(\ell-\mu_n)^2}{2v_n}\Big]$

approximation

$$K(\ell \,|\, \ell_{\mathrm{L}}, \ell_{\mathrm{R}}) = \frac{(\ell_{\mathrm{L}}-\ell-1)!!(\ell_{\mathrm{R}}-\ell-1)!!}{(\ell_{\mathrm{L}}+\ell_{\mathrm{R}}-1)!!}\binom{\ell_{\mathrm{L}}}{\ell}\binom{\ell_{\mathrm{R}}}{\ell}\ell!$$

$$\propto \exp\Big[-\frac{\{\ell-\tilde{\mu}(\ell_{\mathrm{L}},\ell_{\mathrm{R}})\}^2}{2\,\tilde{v}(\ell_{\mathrm{L}},\ell_{\mathrm{R}})}\Big]$$

$$\tilde{\mu}(\ell_{\mathrm{L}},\ell_{\mathrm{R}}) = \frac{\ell_{\mathrm{L}}\,\ell_{\mathrm{R}}}{\ell_{\mathrm{L}}+\ell_{\mathrm{R}}-1} \qquad \tilde{v}(\ell_{\mathrm{L}},\ell_{\mathrm{R}}) = \frac{2\ell_{\mathrm{L}}\ell_{\mathrm{R}}(\ell_{\mathrm{L}}-1)(\ell_{\mathrm{R}}-1)}{(\ell_{\mathrm{L}}+\ell_{\mathrm{R}}-3)(\ell_{\mathrm{L}}+\ell_{\mathrm{R}}-1)^2}$$

recursion relations for μ_n and v_n

ansatz $P_n(\ell) \propto \exp\Big[-\dfrac{(\ell-\mu_n)^2}{2v_n}\Big]$

approximation $K(\ell\,|\,\ell_{\rm L},\ell_{\rm R}) \propto \exp\Big[-\dfrac{\{\ell-\tilde{\mu}(\ell_{\rm L},\ell_{\rm R})\}^2}{2\,\tilde{v}(\ell_{\rm L},\ell_{\rm R})}\Big]$

$$\tilde{\mu}(\ell_{\rm L},\ell_{\rm R}) = \frac{\ell_{\rm L}\,\ell_{\rm R}}{\ell_{\rm L}+\ell_{\rm R}-1} \qquad \tilde{v}(\ell_{\rm L},\ell_{\rm R}) = \frac{2\ell_{\rm L}\ell_{\rm R}(\ell_{\rm L}-1)(\ell_{\rm R}-1)}{(\ell_{\rm L}+\ell_{\rm R}-3)(\ell_{\rm L}+\ell_{\rm R}-1)^2}$$

recursion $P_n(\ell) = \sum_{\ell_1,\ldots,\ell_{2^d}} K(\ell\,|\,\ell_{\rm L},\ell_{\rm R}) \prod_{j=1}^{2^d} P_{n-1}(\ell_j)$

$$\ell_{\rm L} = \sum_{j=1}^{2^{d-1}} \ell_j \qquad \ell_{\rm R} = \sum_{j=2^{d-1}+1}^{2^d} \ell_j$$

for $d \geq 3$, the leading terms are

$$\mu_n \simeq 2^{d-2}\mu_{n-1}$$

$$v_n \simeq 2^{d-3}\mu_{n-1} + 2^{d-4}v_{n-1}$$

→ $\mu_n \simeq (\text{const})2^{(d-2)n} \propto \dfrac{A_n}{L_n}$

normal transport!!

universal "2/3 law"

for $d \geq 3$, the leading terms are

fluctuation from random matching

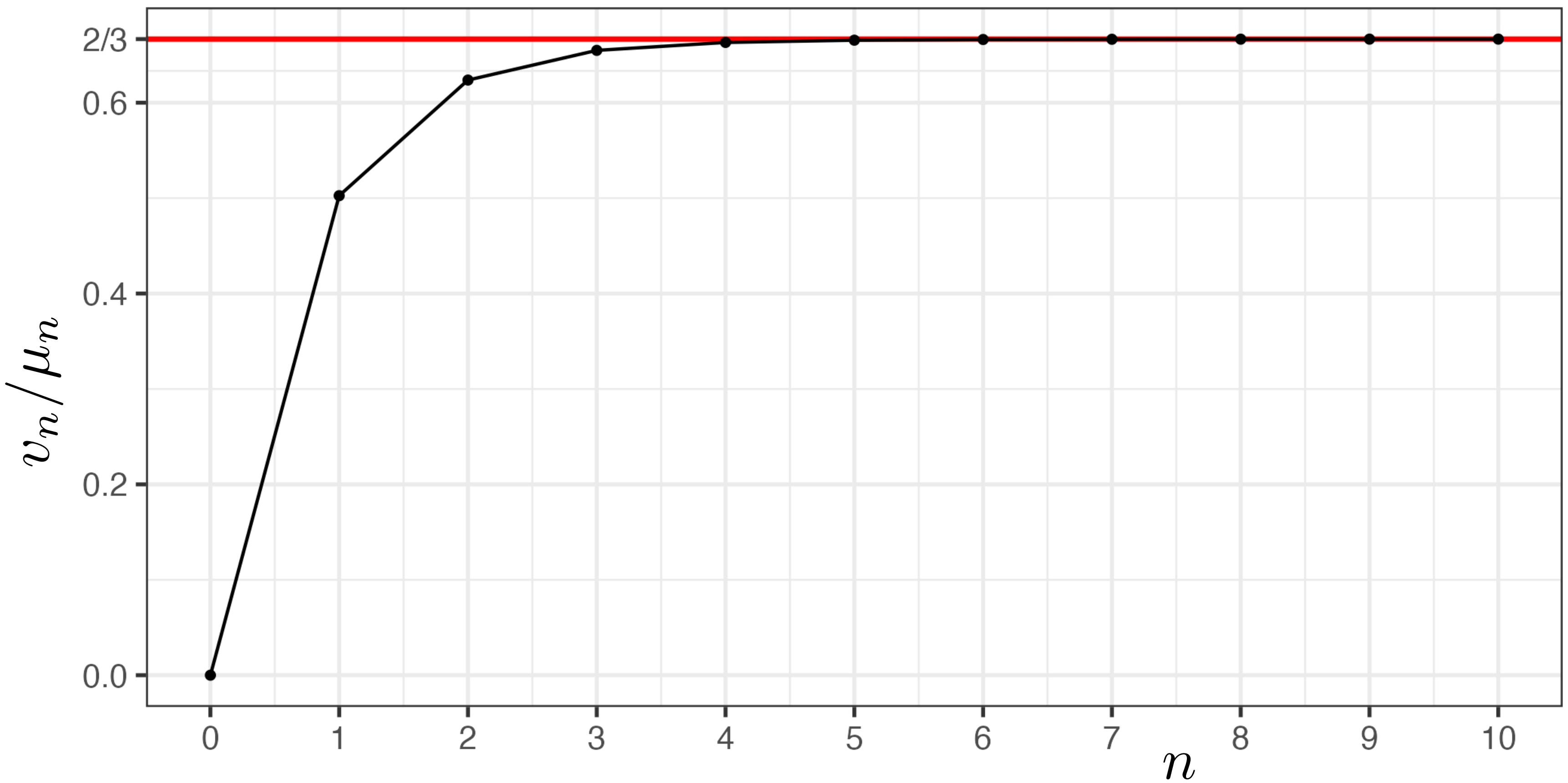
fluctuation from generation $n-1$

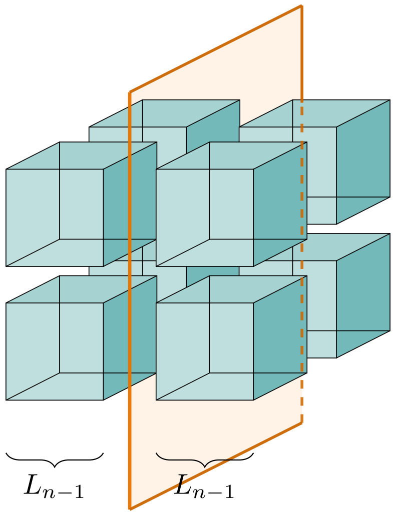
$$\mu_n \simeq 2^{d-2}\mu_{n-1} \qquad v_n \simeq 2^{d-3}\mu_{n-1} + 2^{d-4}v_{n-1}$$

$$\frac{v_n}{\mu_n} \simeq \frac{1}{2} + \frac{1}{4}\frac{v_{n-1}}{\mu_{n-1}}$$

d-independent recursion relation

$\dfrac{v_n}{\mu_n} \to \dfrac{2}{3}$ rapidly as $n \uparrow \infty$ in any dimension $d \geq 3$





numerical results for $d = 3$ with $A_0 = 2$

Lorentz mirror model

Hierarchical model

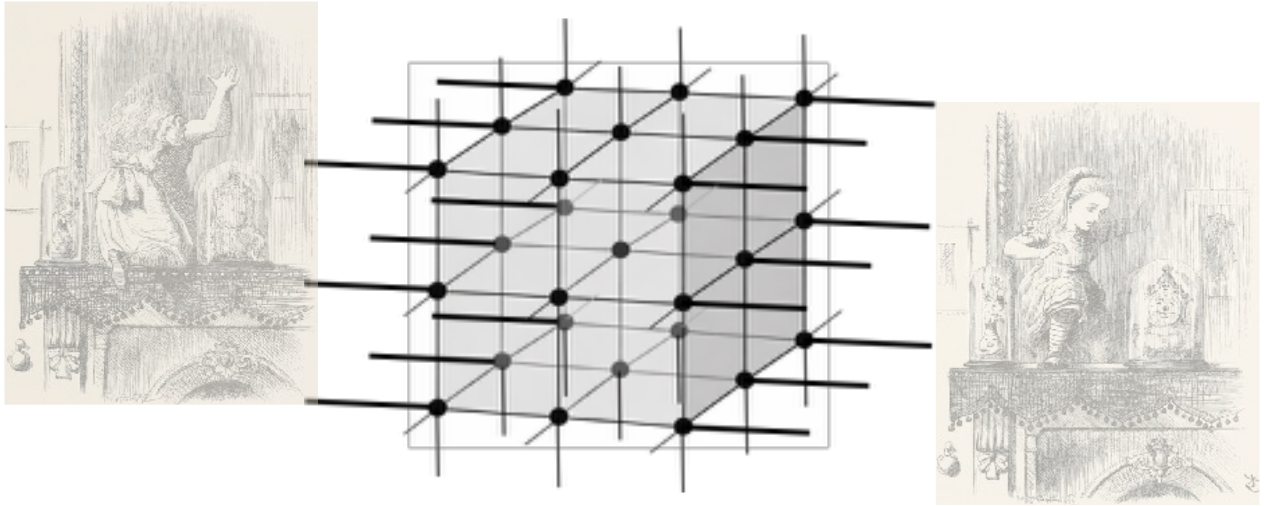
numerics for the $d = 3$ model

Monte Carlo simulation of the original Lorentz mirror model on the cubic lattice

the $L \times L \times L$ cubic lattice

$L = 2^n$ with $n = 1{,}2{,}\ldots{,}7$

choose one of the $5!! = 15$ pairings at each vertex

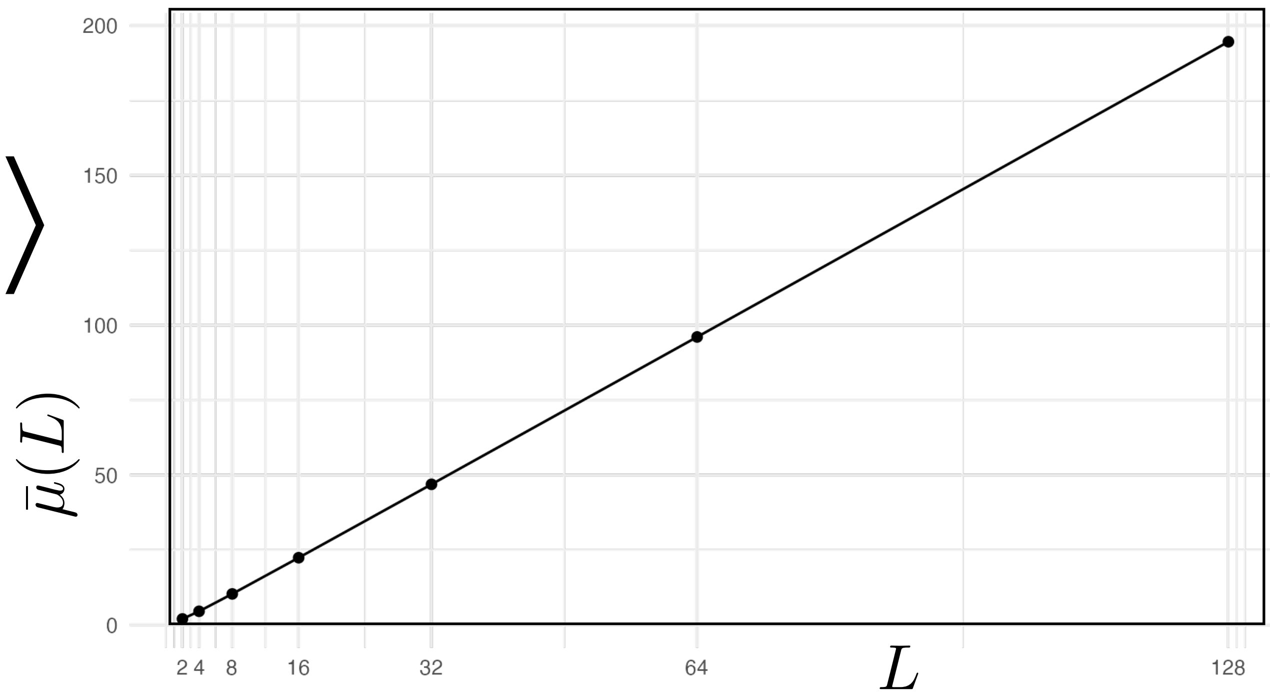


generate 6.4×10^5 independent environments

for each realization of the environment, faithfully count the number of crossings $\mathscr{C}$, i.e., the conductance

mean conductance $\bar{\mu}(L) = \langle \mathscr{C} \rangle$

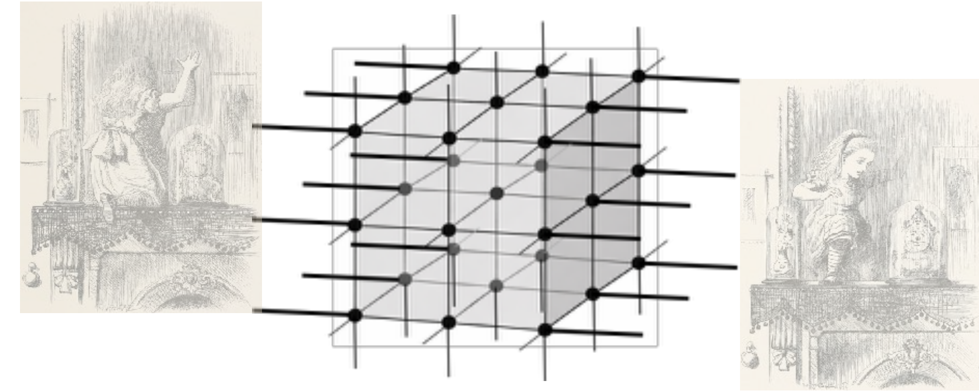
variance $\bar{v}(L) = \left\langle \left(\mathscr{C} - \bar{\mu}(L) \right)^2 \right\rangle$



we recover normal transport

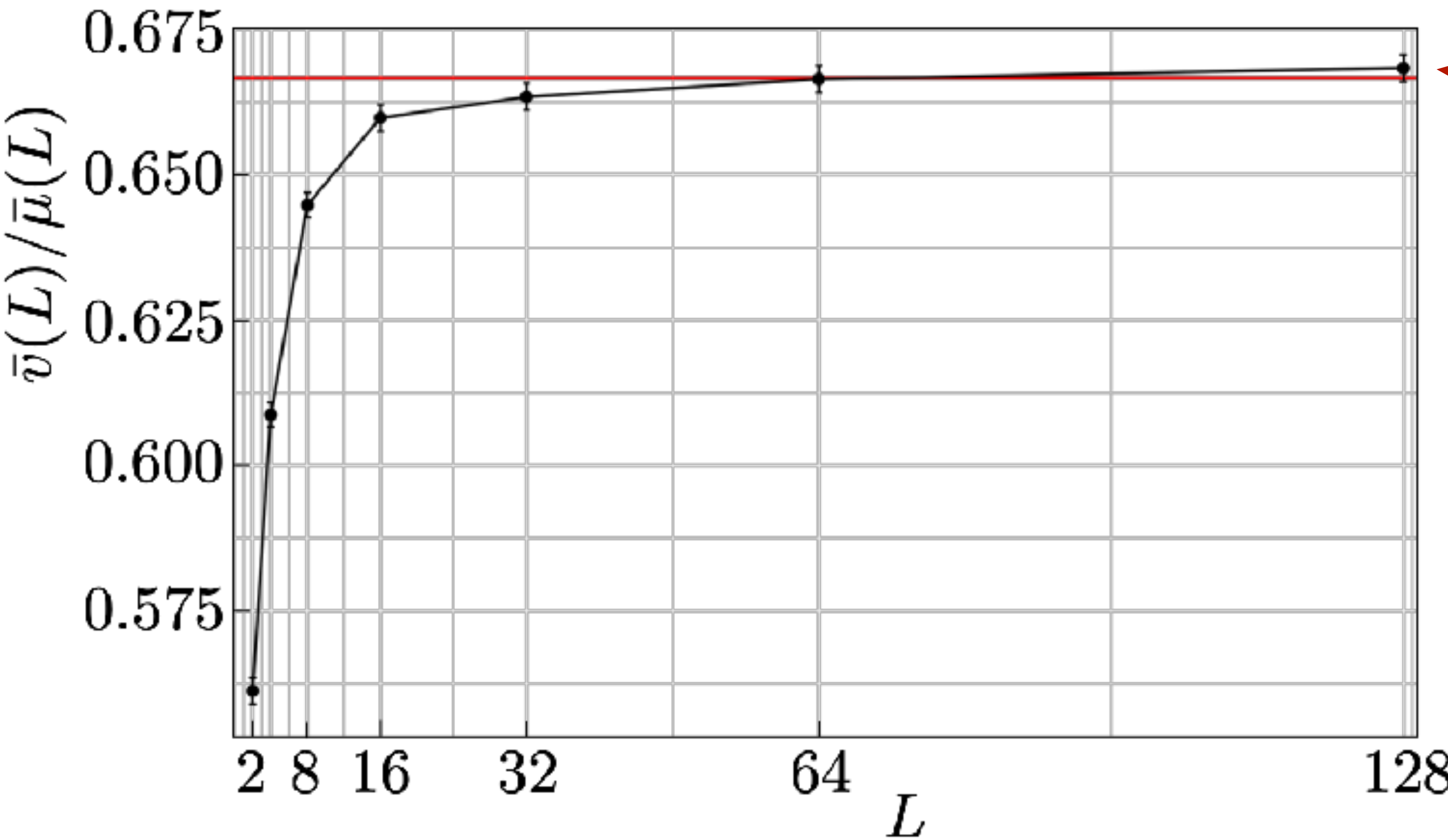
Chiffaudel, Lefevere 2016

variance-to-mean ratio



mean conductance $\bar{\mu}(L) = \langle \mathscr{C} \rangle$

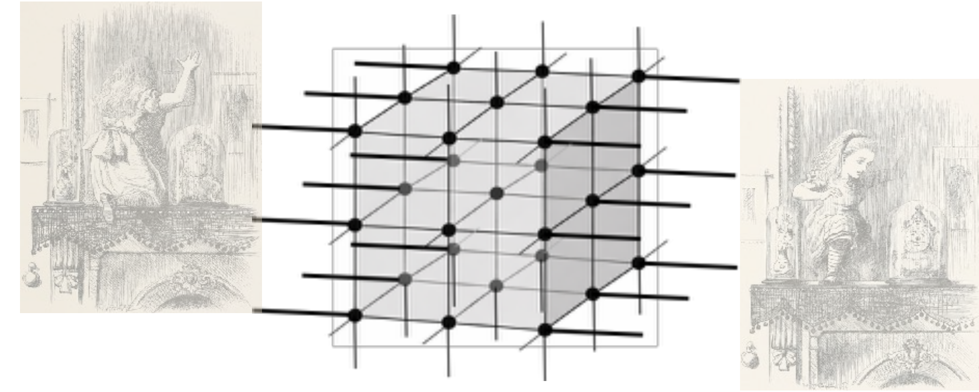
variance $\bar{v}(L) = \left\langle \left(\mathscr{C} - \bar{\mu}(L) \right)^2 \right\rangle$



$\frac{2}{3}$!!

Monte Carlo with 6.4×10^5 samples
error bars indicate 95% confidence intervals

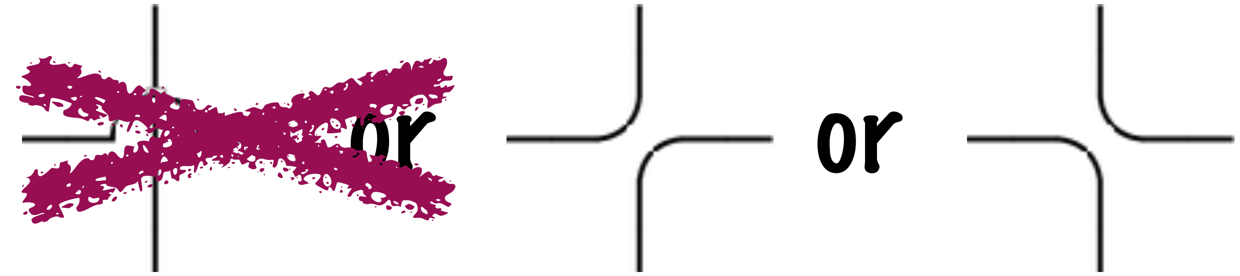
variance-to-mean ratio

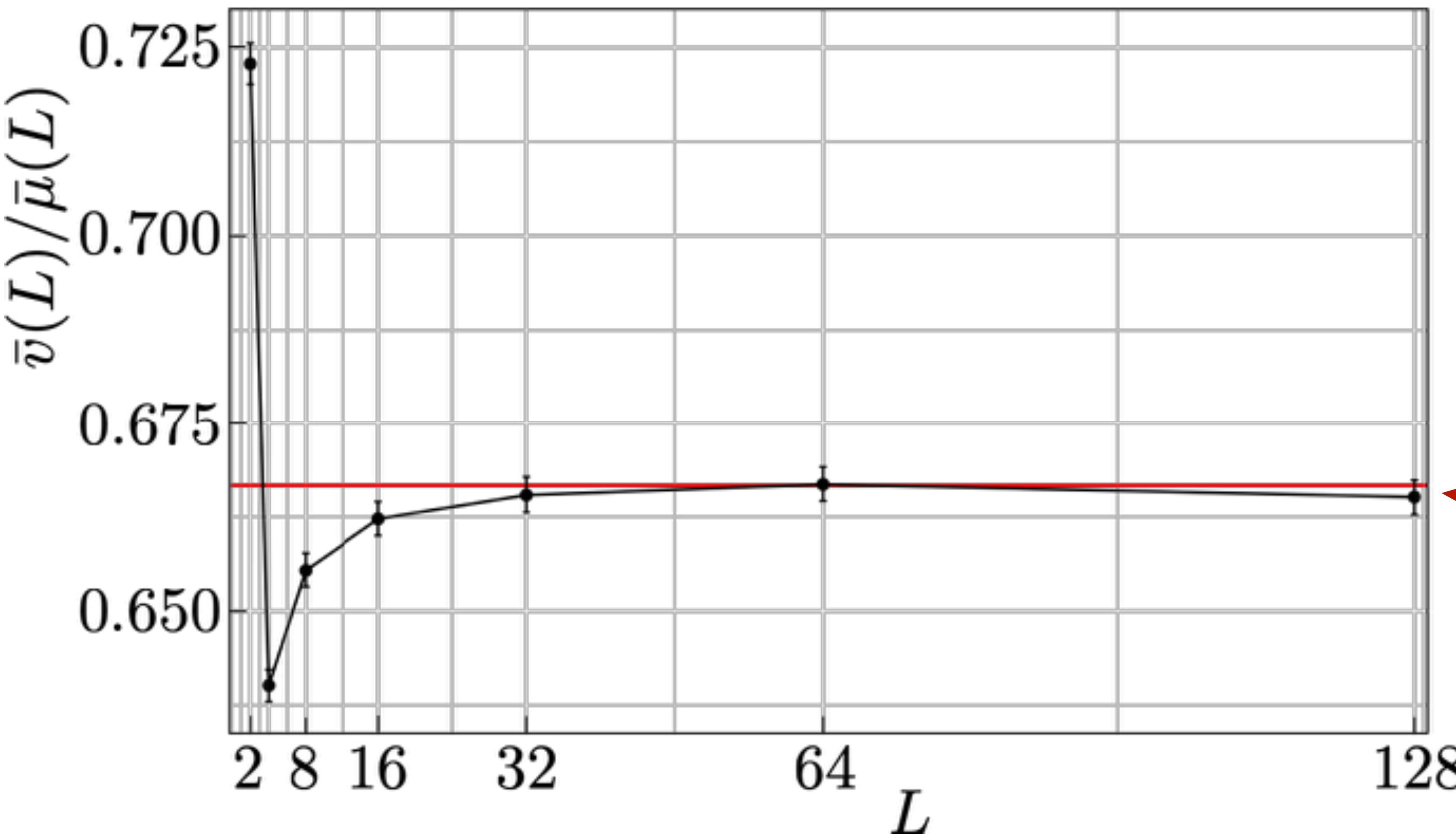


"orthogonal rule"

particles never go straight

choose one of the 8 orthogonal pairings at each vertex





still $\frac{2}{3}$!!!!

Monte Carlo with 6.4×10^5 samples
error bars indicate 95% confidence intervals

the marginal dimension $d = 2$

normal transport $\bar{\mu}(L) \propto \frac{A}{L} = \frac{L^{d-1}}{L} = L^{d-2}$

constant...

the marginal dimension $d = 2$

hierarchical model (Gaussian approximation +numerics)

2/3 law!

$$\mu_n \simeq \frac{n}{12} = \frac{\log L_n}{12\ \log 2} \qquad \frac{v_n}{\mu_n} \to \frac{2}{3} \qquad \text{as } n \uparrow \infty$$

weakly anomalous transport with a logarithmic correction

prediction: the original Lorentz mirror model in $d = 2$ also exhibits a logarithmic growth and the $2/3$ law

the marginal dimension $d = 2$

hierarchical model (Gaussian approximation +numerics)

$$\mu_n \simeq \frac{n}{12} = \frac{\log L_n}{12 \log 2} \qquad \frac{v_n}{\mu_n} \to \frac{2}{3} \quad \text{as } n \uparrow \infty$$

2/3 law!

weakly anomalous transport with a logarithmic correction

retrodiction: the original Lorentz mirror model in $d = 2$ also exhibits a logarithmic growth and the 2/3 law

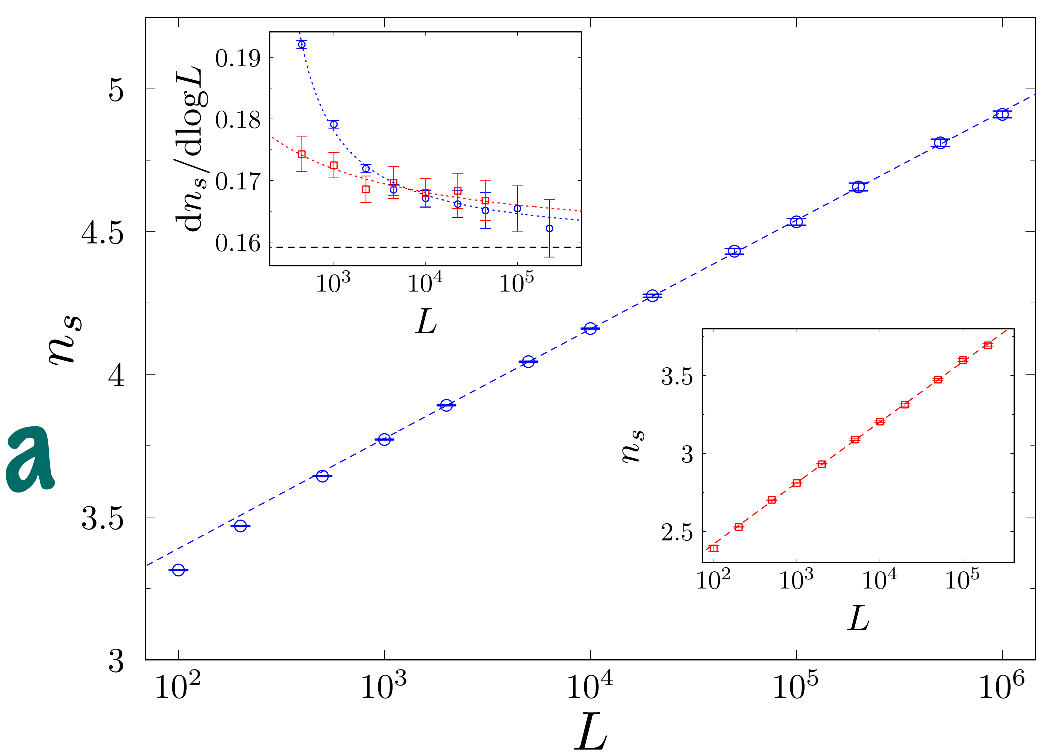
PHYSICAL REVIEW B **87**, 184204 (2013)

Loop models with crossings

Adam Nahum,[1] P. Serna,[2] A. M. Somoza,[2] and M. Ortuño[2]
[1]*Theoretical Physics, Oxford University, 1 Keble Road, Oxford OX1 3NP, United Kingdom*
[2]*Departamento de Física–CIOyN, Universidad de Murcia, Murcia 30.071, Spain*

(Received 10 March 2013; published 28 May 2013)

derivation from an effective nonlinear sigma model and numerical confirmation!!



the universal "2/3 law"

$$\frac{\text{(sample-to-sample variance of conductance)}}{\text{(mean conductance)}} \to \frac{2}{3}$$

observed theoretically (but not proved!) in the hierarchical Lorentz mirror model in $d \geq 2$, and numerically in the original Lorentz mirror model in $d = 2, 3$

emerges IF the effective description in terms of a nonlinear sigma model is valid

conjecture: v/μ (as $\mu \uparrow \infty$) is a universal amplitude ratio, and the value $2/3$ signifies a universality class of systems (including ours) exhibiting normal transport induced by random matching of conserved current

what models belong to this universality class?

any models in continuum like the Lorentz gas?

summary

- the Lorentz mirror model (both original and hierarchical) provides a clean setting to study transport generated solely by a quenched random environment
  - in the hierarchical version, we proved normal transport in $d \geq 3$ (normal transport in $d \geq 3$ in the original model has been expected)
    - in the marginal dimension $d = 2$, the hierarchical model reproduces known results
      - we found the "$2/3$ law" for the variance-to-mean ratio of conductance, and propose that it is a signature of a universality class of normal transport induced by random current matching

$\nu/\mu = 1$ if the trajectories were independent!

illustration by John Tenniel