

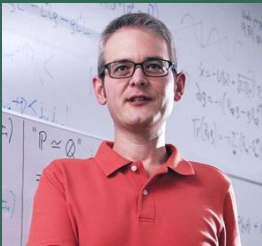
12th May 2026@YITP (32+8 min)
Frontiers in Nonequilibrium Physics 2026



Stochastic thermodynamics for non-Markov jump processes: the Fourier-embedding framework

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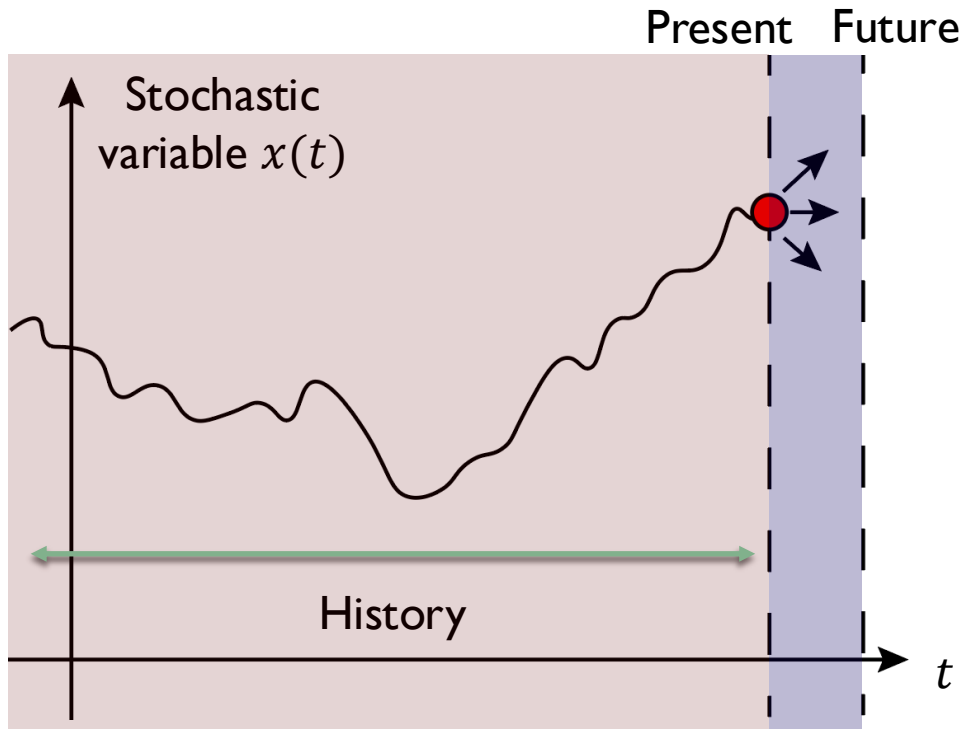
Collaborators



Andreas Dechant
Kyoto Univ.

- K. Kanazawa and A. Dechant, arXiv:2506.04726
- A. Dechant and K. Kanazawa, arXiv:2604.25095

Background: Markov vs. non-Markov processes



Let me review the standard solutions for Markov processes

- Markov = requiring only the current info $x(t)$ for 1-step future
- **Non**-Markov = all the history $\{x(s)\}_{s \leq t}$ is required for prediction
- NOTE: various tools are mainly prepared only for Markov processes

Solution: SDE \Leftrightarrow master equation

White Gaussian

White compound Poisson

Stochastic differential equation (SDE)

$$\frac{d\hat{v}}{dt} = -a(\hat{v}) + b(\hat{v}) \cdot \hat{\xi}^G + \hat{\xi}_{\lambda(y|\hat{v})}^{\text{CP}}$$

- 1-variable SDE = Gaussian + compound Poisson noise



Master equation (time-evolution of the probability density function (PDF))

$$\frac{\partial P(v, t)}{\partial t} = \left[\frac{\partial}{\partial v} a(v) + \frac{1}{2} \frac{\partial^2}{\partial v^2} b^2(v) \right] P(v, t) + \int_{-\infty}^{\infty} dy [\lambda(y|v - y) P(v - y, t) - \lambda(y|v) P(v, t)]$$

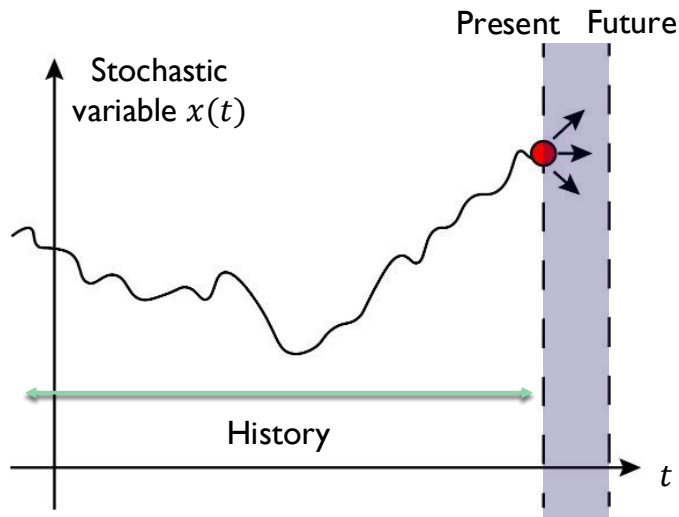
- Always linear regarding $P(v, t)$
- Within linear algebra in a wide sense



Standard solution: SDE \Leftrightarrow master equation
Master equation is solved within linear algebra

Tools are established for Markovian process
⇒ Stochastic thermo is established for Markovian systems

$$\frac{dv}{dt} = -U'(x) - \gamma v + \sqrt{2\gamma T} \xi^G \quad \longrightarrow \quad \frac{\partial P_t}{\partial t} = LP_t \quad \text{Master Eq. or Fokker-Planck Eq.}$$



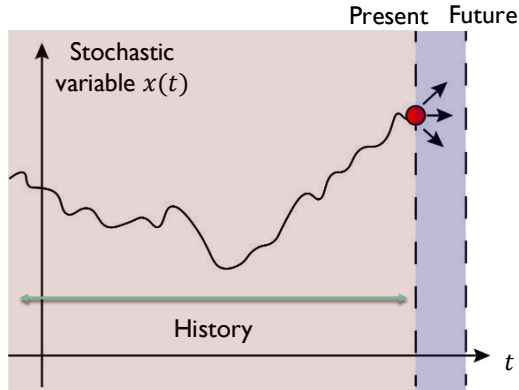
- Various tools available for Markovian processes
- Various thermodynamic inequalities
 - 2nd law
 - TUR
 - ...



Assuming Markovian, various thermodynamic bounds are revealed

Long-term motivation: stochastic thermodynamics for non-Markovian processes

$$\frac{dv}{dt} = [\text{history dependent noise}] \quad \Rightarrow \quad \frac{\partial P_t}{\partial t} = LP_t \quad \begin{array}{l} \text{Master Eq.??} \\ \text{Fokker-Planck Eq.??} \end{array}$$



Various bounds??:

- 2nd law
- TUR
- ...

■ Not fully formulated yet regarding non-Markovian systems...

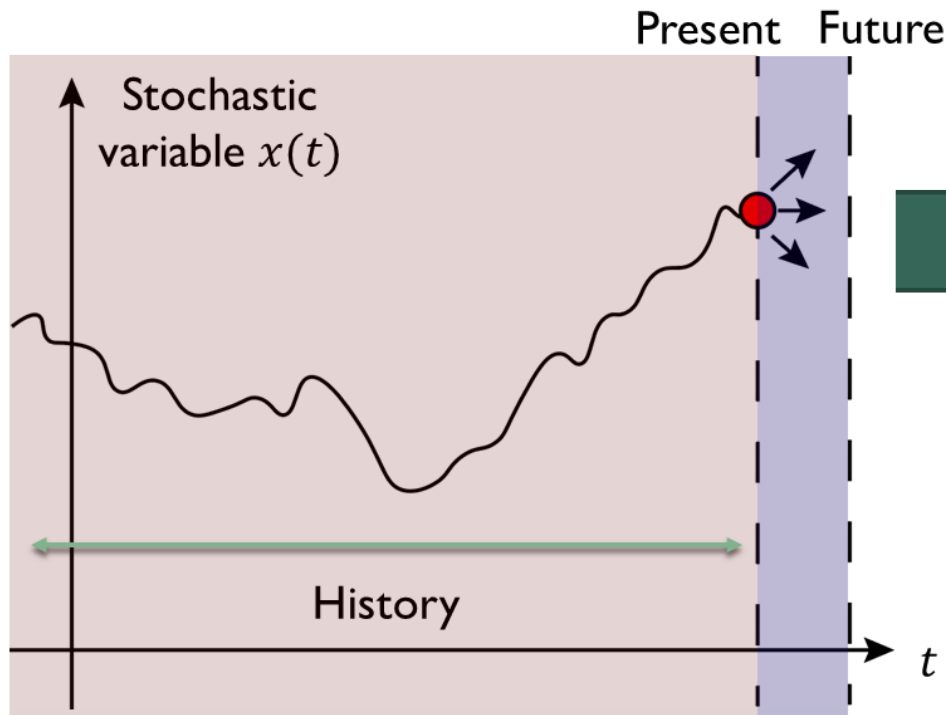
■ Elusive yet regarding the various thermodynamic bounds



Revelation of thermodynamic bounds for general non-Markovian systems

My research question:

Q: systematic solutions to non-Markovian systems?



Questions

Q: stochastic thermodynamics for non-Markov processes?

Q: master equations for non-Markov processes?

- Master Eq. available for *only specific class* of systems (e.g., renewal process)
- *Unavailable* for general non-Markovian dynamics...

Goal

Derivation of master Eqs. and stochastic thermos for *general* non-Markov dynamics by developing *Markovian embedding*

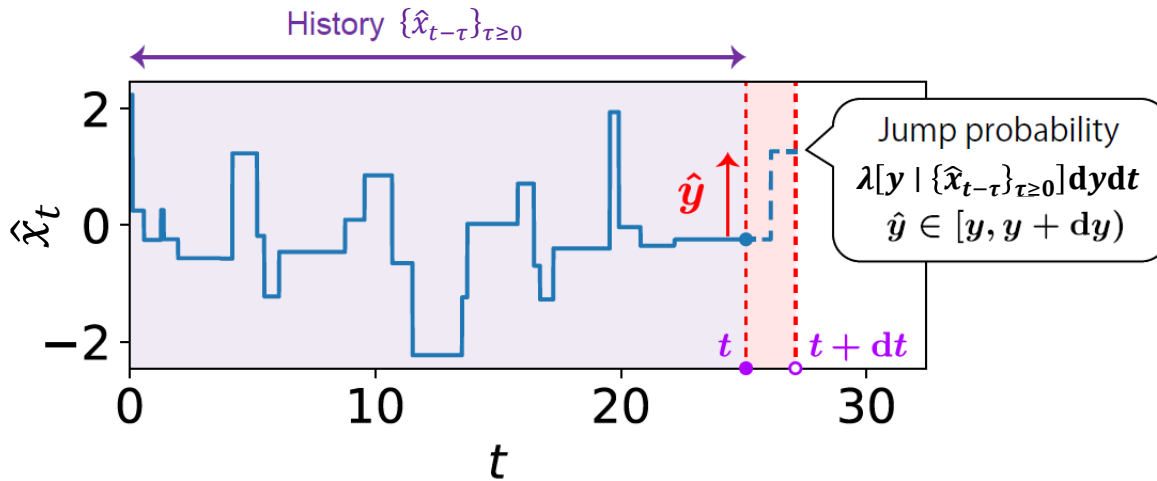


Main results: Fourier embedding and non-Markov stochastic thermodynamics

- [1] K. Kanazawa and A. Dechant, [arXiv:2506.04726](#)
- [2] A. Dechant and K. Kanazawa, [arXiv:2604.25095](#)

Model: non-Markov jump model for position \hat{x}_t

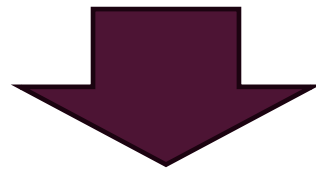
-- embedding based on the Fourier transform --



$$\frac{d\hat{x}_t}{dt} = \hat{\xi}^{\text{CP}}_{\lambda_a[y | \{x_{t-\tau}\}_{\tau \geq 0}]}$$

- Intensity: $\lambda_a[y | \{\hat{x}_{t-\tau}\}_{\tau \geq 0}]$
- Control parameter: a
- \hat{x}_t : even-parity variable

Fourier-transform
of historical velocity

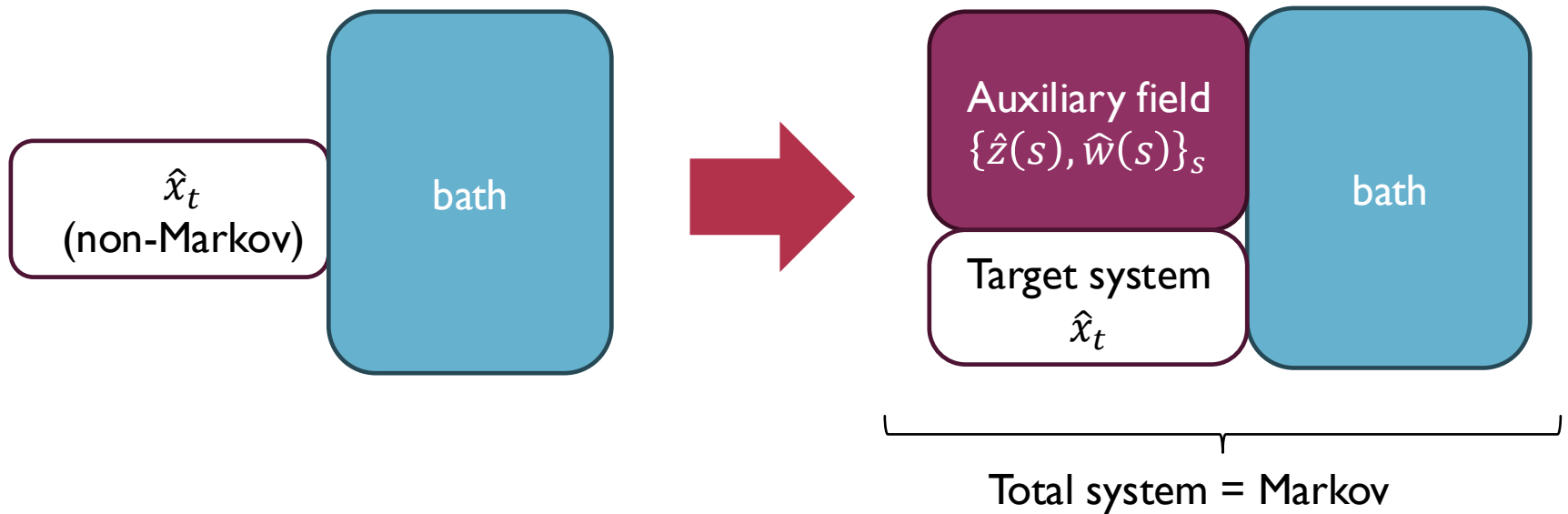


$$\hat{v}_t := \frac{d\hat{x}_t}{dt}$$

$$\hat{w}_t(s) := \int_0^\infty \hat{v}_{t-\tau} \cos(s\tau) d\tau, \quad \hat{z}_t(s) := \int_0^\infty \hat{v}_{t-\tau} \sin(s\tau) d\tau, \quad \hat{v}_{t-\tau} := \frac{d\hat{x}_{t-\tau}}{dt}$$

$$\hat{\Gamma}_t := (\hat{x}_t, \{\hat{z}_t(s)\}_{s>0}, \{\hat{w}_t(s)\}_{s>0})$$

Schematic of the embedding



- Total system is Markov after embedding
- Target system is non-Markov, after tracing out
- The embedding-variable set is not unique; there are many candidates, such as Laplace and Fourier embedding
 - ✓ Laplace embedding: K. Kanazawa and D. Sornette, Phys. Rev. Res. **6**, 023270 (2024).

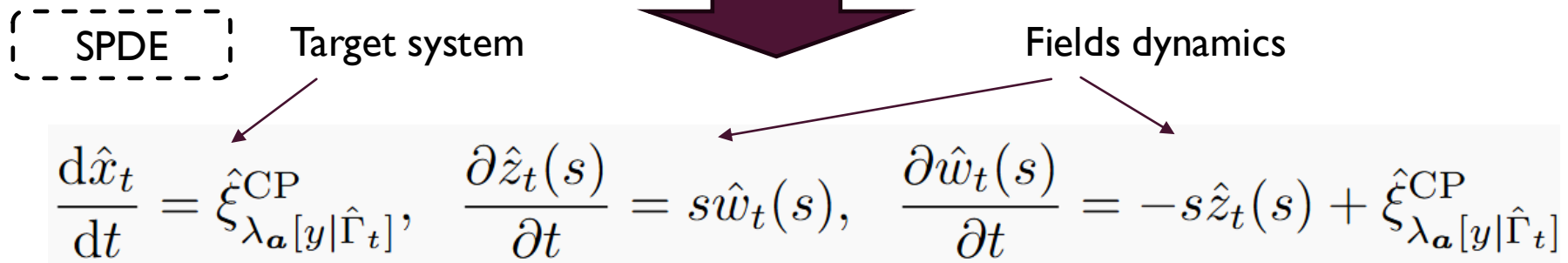
Fourier embedding deduces the Markov stochastic partial differential equation (SPDE) – *equivalent to Markov field theory* –

[Embedding]
$$\hat{w}_t(s) := \int_0^\infty \hat{v}_{t-\tau} \cos(s\tau) d\tau, \quad \hat{z}_t(s) := \int_0^\infty \hat{v}_{t-\tau} \sin(s\tau) d\tau, \quad \hat{v}_{t-\tau} := \frac{d\hat{x}_{t-\tau}}{dt}$$

$$\hat{\Gamma}_t := (\hat{x}_t, \{\hat{z}_t(s)\}_{s>0}, \{\hat{w}_t(s)\}_{s>0})$$

[SPDE]

Target system Fields dynamics



$$\frac{d\hat{x}_t}{dt} = \hat{\xi}_{\lambda_a[y|\hat{\Gamma}_t]}^{\text{CP}}, \quad \frac{\partial \hat{z}_t(s)}{\partial t} = s\hat{w}_t(s), \quad \frac{\partial \hat{w}_t(s)}{\partial t} = -s\hat{z}_t(s) + \hat{\xi}_{\lambda_a[y|\hat{\Gamma}_t]}^{\text{CP}}$$

- Auxiliary variables = harmonic oscillators + jump noise



The necessary and sufficient condition can be met for the time-reversal symmetry in the field master equation

Field master equation by the Fourier embedding

SPDE

$$\frac{d\hat{x}_t}{dt} = \hat{\xi}_{\lambda_{\mathbf{a}}[y|\hat{\Gamma}_t]}^{\text{CP}}, \quad \frac{\partial \hat{z}_t(s)}{\partial t} = s\hat{w}_t(s), \quad \frac{\partial \hat{w}_t(s)}{\partial t} = -s\hat{z}_t(s) + \hat{\xi}_{\lambda_{\mathbf{a}}[y|\hat{\Gamma}_t]}^{\text{CP}},$$

Master Eq.

$$\frac{\partial P_t[\Gamma]}{\partial t} = (\mathcal{L}_A + \mathcal{L}_J) P_t[\Gamma],$$

Advective Jump

$$\mathcal{L}_A P_t[\Gamma] := \int_0^\infty ds \left\{ sz(s) \frac{\delta P_t[\Gamma]}{\delta w(s)} - sw(s) \frac{\delta P_t[\Gamma]}{\delta z(s)} \right\},$$

$$\mathcal{L}_J P_t[\Gamma] := \int_{-\infty}^\infty dy \{ \lambda_{\mathbf{a}}[y | \Gamma - \Delta\Gamma_y] P_t[\Gamma - \Delta\Gamma_y] - \lambda_{\mathbf{a}}[y | \Gamma] P_t[\Gamma] \}$$

- Master equation for an arbitrary non-Markov jumps
- Advantage: *time-reversal symmetry* can be formulated

Necessary and sufficient condition of time-reversal symmetry I (regarding the total-system intensity)

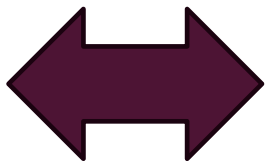
- Let us define the *total-system intensity*

$$\Lambda_{\mathbf{a}}[\Gamma | \Gamma'] := \lambda_{\mathbf{a}}[x - x' | \Gamma'] \delta[z - z'] \delta[w - w' - s(x - x')]$$

- The necessary and sufficient condition of the time-reversal symmetry:

$$\Lambda_{\mathbf{a}}[\Gamma | \Gamma'] P_{\text{can};\mathbf{a}}[\Gamma'] = \Lambda_{\mathbf{a}}[\Gamma'^* | \Gamma^*] P_{\text{can};\mathbf{a}}[\Gamma]$$

where * signifies the time reversal of the state variables.



$$\frac{1}{\beta} \ln \frac{\Lambda_{\mathbf{a}}[\Gamma^* | \Gamma'^*]}{\Lambda_{\mathbf{a}}[\Gamma' | \Gamma]} = E_{\text{tot}}[\Gamma', \mathbf{a}] - E_{\text{tot}}[\Gamma, \mathbf{a}]$$

Canonical dist. assumption $E_{\text{tot}}[\Gamma, \mathbf{a}] = E_{\text{tot}}[\Gamma^*, \mathbf{a}]$

$$P_{\text{can};\mathbf{a}}[\Gamma] = e^{\beta(F(\mathbf{a}) - E_{\text{tot}}[\Gamma, \mathbf{a}])}, \quad e^{-\beta F(\mathbf{a})} := \int d\Gamma e^{-\beta E_{\text{tot}}[\Gamma, \mathbf{a}]}$$

Assumption 1:

Necessary and sufficient condition of time-reversal symmetry 2 (regarding the *target-system intensity*)

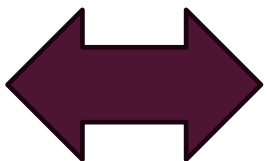
- Assume that the total intensity satisfies the time-reversal symmetry:

$$\Lambda_{\mathbf{a}} [\Gamma | \Gamma'] P_{\text{can};\mathbf{a}}[\Gamma'] = \Lambda_{\mathbf{a}} [\Gamma'^* | \Gamma^*] P_{\text{can};\mathbf{a}}[\Gamma]$$

- The jump intensity only for the target system (*the target-system intensity*) satisfies the following condition (which we call **Assumption 1**)

$$\frac{1}{\beta} \ln \frac{\lambda_{\mathbf{a}} [x - x' | \Gamma'^*]}{\lambda_{\mathbf{a}} [x' - x | \Gamma]} = E_{\text{tot}}[\Gamma', \mathbf{a}] - E_{\text{tot}}[\Gamma, \mathbf{a}]$$

- Remark: this condition is irrelevant to the selection of the embedding
- Time-reversal symmetry *only for the target system* is formulated, which is *independent of the Fourier embedding*



For the original variable $\hat{X}_t := (\{\hat{x}_{t-\tau}\}_{\tau \geq 0})$

$$\frac{1}{\beta} \ln \frac{\lambda_{\mathbf{a}} [x - x' | X'^*]}{\lambda_{\mathbf{a}} [x' - x | X]} = E_{\text{tot}}[X', \mathbf{a}] - E_{\text{tot}}[X, \mathbf{a}]$$

Assumption 2:

invariance of the total intensity under time reversal

- Let us define the total intensity: $\Lambda_{\text{tot};\mathbf{a}}[\Gamma] := \int \Lambda_{\mathbf{a}}[\Gamma' | \Gamma] d\Gamma'$
- Total intensity is assumed to be invariant by time reversal which we call **Assumption 2**

$$\Lambda_{\text{tot};\mathbf{a}}[\Gamma] = \Lambda_{\text{tot};\mathbf{a}}[\Gamma^*]$$

- Standard assumption even for Markovian stochastic thermodynamics
- Under Assumptions 1 and 2: the general solution is obtained for the stationary master equation (i.e., restriction of the canonical dist.)

$$(\mathcal{L}_A + \mathcal{L}_J)P_{\text{can}}[\Gamma] = 0$$

General solution of the steady-state field master equation (i.e., the restriction of the canonical dist.)

- The canonical dist. must satisfy the following master equation

$$(\mathcal{L}_A + \mathcal{L}_J)P_{\text{can};\mathbf{a}}[\Gamma] = 0$$

- While this is a functional-derivative equation, its general (complete) solution can be obtained by the method of characteristics

Necessary condition

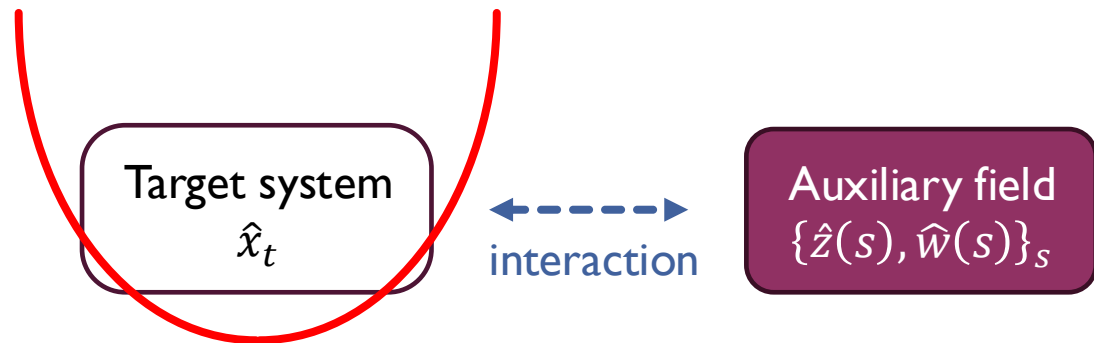
as the solution of the master equation

$$P_{\text{can};\mathbf{a}}[\Gamma] \propto e^{-E_{\text{tot}}[\Gamma, \mathbf{a}]}$$

$$E_{\text{tot}}[\Gamma, \mathbf{a}] = E_{\text{tot}}(x, \{\varepsilon_s\}_s, \mathbf{a}), \quad \varepsilon_s(z(s), w(s)) := \frac{1}{2} (z^2(s) + w^2(s))$$

Assumption 3: Controllable energy is associated only with the target system

Controlled by
the external parameter a



- Let us focus on a subclass where the total energy = controllable energy + non-controllable energy

$$E_{\text{tot}}[x, \{\varepsilon_s\}_s, \mathbf{a}] := \boxed{E_{\text{ctrl}}(x, \mathbf{a})} + E_{\text{nctrl}}[x, \{\varepsilon_s\}_s]$$

Controllable part depends only on x

- Sufficient condition that the second law does not depend on the selection of the embedding variable (i.e., the “gauge invariance” of the second law)

Simple example: quadratic non-controllable energy

$$P_{\text{can}; \mathbf{a}}[\Gamma] = \mathbb{1}_{S_C}[\Gamma] e^{\beta(F(\mathbf{a}) - E_{\text{tot}}[\Gamma, \mathbf{a}])}, \quad E_{\text{tot}}[\Gamma, \mathbf{a}] = E_{\text{ctrl}}(x, \mathbf{a}) + \frac{1}{2} \int_0^\infty ds M(s) [w^2(s) + z^2(s)]$$

$\mathbb{1}_{S_C}[\Gamma]$: the indicator function corresponding to the Fourier embedding
(Technical remark: conditional ergodicity, see Appendix)

- Example: Quadratic energy regarding the field variable
 - $E_{\text{ctrl}}(x, \mathbf{a})$: the controllable part of energy by the external parameter
 - $M(s)$: positive weight function, controlling the contribution of the wavenumber s
- We can construct specific models which has this stationary PDF of the field master equation (see the next slides)

1st law of stochastic thermodynamics (under Assumptions 1 and 2)

Stochastic heat and work

$$d\hat{Q} := \begin{cases} E_{\text{tot}}[\hat{\Gamma}', \mathbf{a}] - E_{\text{tot}}[\hat{\Gamma}, \mathbf{a}] = \frac{1}{\beta} \ln \frac{\Lambda_{\mathbf{a}}[\hat{\Gamma}^* | \hat{\Gamma}'^*]}{\Lambda_{\mathbf{a}}[\hat{\Gamma}' | \hat{\Gamma}]} & (\text{prob.} = \Lambda_{\mathbf{a}}[\hat{\Gamma}' | \hat{\Gamma}] d\Gamma' dt) \\ 0 & (\text{prob.} = 1 - \Lambda_{\mathbf{a}}[\hat{\Gamma}' | \hat{\Gamma}] d\Gamma' dt) \end{cases}$$

$$d\hat{W} := d\mathbf{a} \frac{\partial}{\partial \mathbf{a}} E_{\text{tot}}[\Gamma, \mathbf{a}]$$



1st law (energy conservation)

$$dE_{\text{tot}}[\hat{\Gamma}, \mathbf{a}] = d\hat{W} + d\hat{Q}$$

2nd law for the entropy production (under Assumptions 1 and 2)

Entropy production

$$\dot{\sigma}_{\text{sys}} := -\frac{d}{dt} \int P_t[\Gamma] \ln P_t[\Gamma] d\Gamma$$
$$\dot{\sigma}_{\text{bath}} := -\beta \left\langle \frac{d\hat{Q}}{dt} \right\rangle = \int \Lambda[\Gamma' | \Gamma] P_t[\Gamma] \ln \frac{\Lambda[\Gamma' | \Gamma]}{\Lambda[\Gamma^* | \Gamma'^*]} d\Gamma d\Gamma'$$



2nd law

$$\dot{\sigma}_{\text{tot}} := \dot{\sigma}_{\text{sys}} + \dot{\sigma}_{\text{bath}} = \int d\Gamma d\Gamma' \Lambda_{\mathbf{a}}[\Gamma' | \Gamma] P_t[\Gamma] \ln \frac{\Lambda_{\mathbf{a}}[\Gamma' | \Gamma] P_t[\Gamma]}{\Lambda_{\mathbf{a}}[\Gamma^* | \Gamma'^*] P_t[\Gamma']} \geq 0$$

“Gauge-invariance” of the 2nd law for the irreversible work (under Assumptions 1, 2, and 3)

- Free energy:

$$F(\mathbf{a}) := -\beta^{-1} \ln Z(\mathbf{a}), \quad Z(\mathbf{a}) := \int \exp(-\beta E_{\text{ctrl}}(x, \mathbf{a}) - \beta E_{\text{nctrl}}[x, \varepsilon]) d\Gamma.$$

- Free energy difference *does not depend* on the embedding variables:

$$\{\Delta F = \textit{irrelevant} \text{ to the embedding variables selection}\}$$

- Quasi-static work formula is closed only with $E_{\text{sys}}[\hat{x}, \mathbf{a}]$:

$$\langle \Delta \hat{W} \rangle := \int_{\mathbf{a}_{\text{ini}}}^{\mathbf{a}_{\text{fin}}} d\mathbf{a} \left\langle \frac{\partial}{\partial \mathbf{a}} E_{\text{sys}}(\hat{x}, \mathbf{a}) \right\rangle_{\text{can}} = F(\mathbf{a}_{\text{fin}}) - F(\mathbf{a}_{\text{ini}})$$

- “Gauge-invariant” 2nd law for the work: irrelevant to embedding variables

$$\langle \Delta \hat{W} \rangle \geq \Delta F$$

Quadratic non-controllable energy (QnCE) model

$$\lambda_{\mathbf{a}}[y | \Gamma] = \lambda_0 \rho_{\mathbf{a}}(y | x) \exp \left[-\frac{\beta}{2} \left\{ E_{\text{ctrl}}(x', \mathbf{a}) - E_{\text{ctrl}}(x, \mathbf{a}) + \int_0^{\infty} ds M(s) \left(\frac{y^2}{2} + yw(s) \right) \right\} \right],$$

$$\rho_{\mathbf{a}}(y | x) = \rho_{\mathbf{a}^*}(\epsilon y | \epsilon x), \quad \rho_{\mathbf{a}}(y | x) = \rho_{\mathbf{a}^*}(-\epsilon y | \epsilon x'), \quad y := x' - x.$$

- For the above intensity functional, we have the canonical PDF:

$$P_{\text{can}; \mathbf{a}}[\Gamma] = \mathbf{1}_{S_C}[\Gamma] e^{(F(\mathbf{a}) - E_{\text{tot}}[\Gamma, \mathbf{a}])}, \quad E_{\text{tot}}[\Gamma, \mathbf{a}] = E_{\text{sys}}(x, \mathbf{a}) + \int_0^{\infty} ds M(s) [w^2(s) + z^2(s)]$$

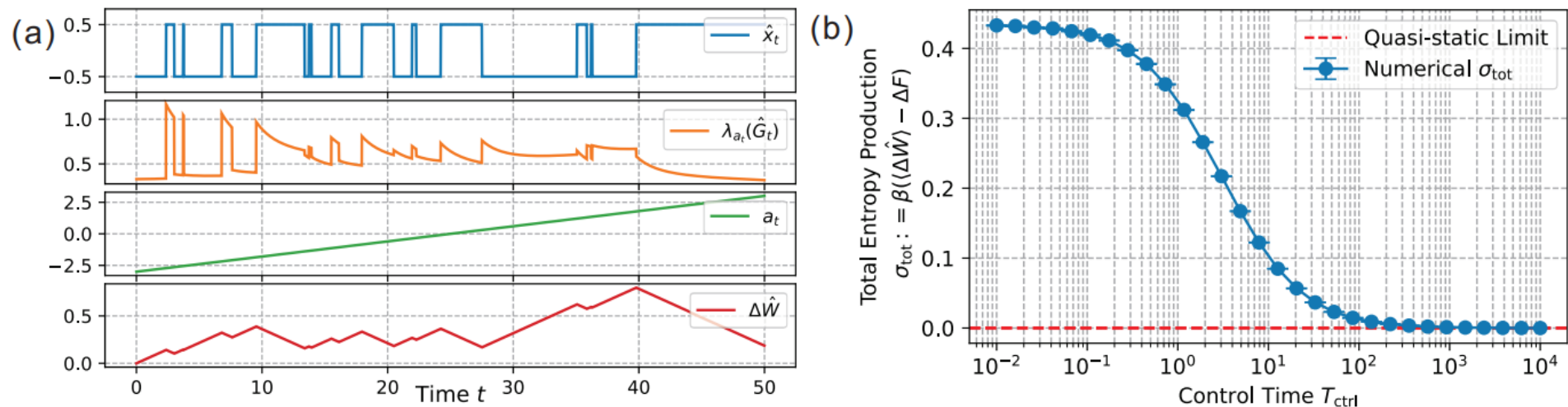
- Stochastic work and heat:

$$dQ := E_{\text{tot}}[\Gamma', \mathbf{a}] - E_{\text{tot}}[\Gamma, \mathbf{a}] = E_{\text{ctrl}}(x', \mathbf{a}) - E_{\text{ctrl}}(x, \mathbf{a}) + \int_0^{\infty} ds M(s) \left(\frac{y^2}{2} + yw(s) \right)$$

- Assumptions 1-3 are satisfied.
- Second law holds:

$$\sigma_{\text{tot}} = \beta (\langle \Delta W \rangle - \Delta F) \geq 0.$$

Example I: two-level system



$$E_{\text{ctrl}}(\hat{x}, a_t) = -a_t \hat{x}, \quad E_{\text{nctrl}}[\hat{z}, \hat{w}] = \frac{1}{2} \int_0^\infty M(s) [\hat{w}^2(s) + \hat{z}^2(s)] ds.$$

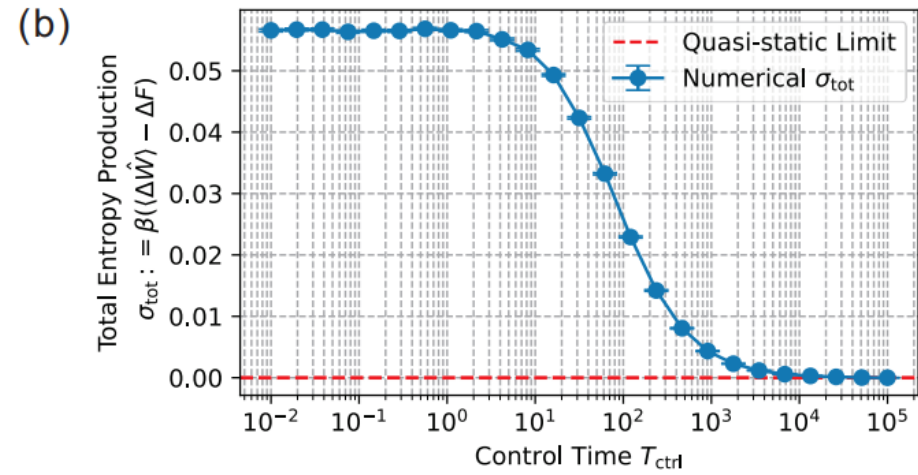
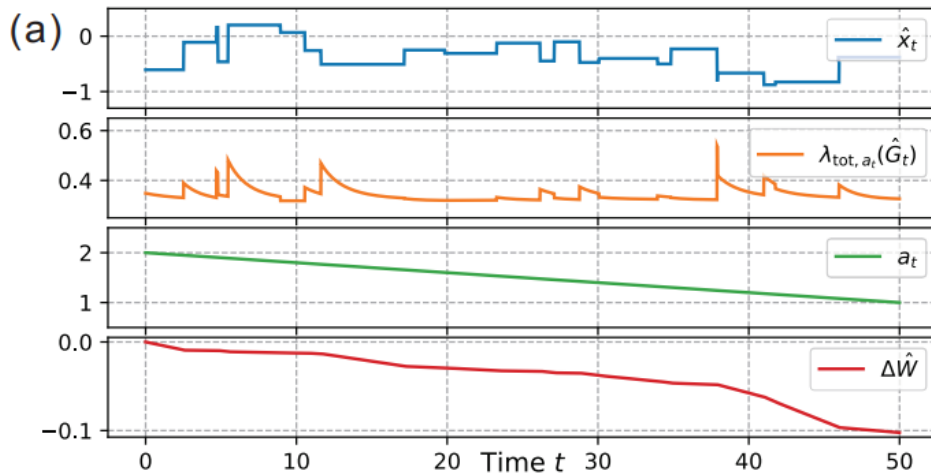
$$\lambda_a[y | \{x_{t-\tau}\}_{\tau \geq 0}] = \tilde{\lambda}_0 \exp \left[-\frac{y\beta}{2} \left\{ -a + \int_0^\infty \phi(\tau) \hat{v}_{t-\tau} d\tau \right\} \right], \quad \phi(\tau) := \int_0^\infty M(s) \cos(s\tau) ds.$$

Memory function



The 2nd law holds! $\sigma_{\text{tot}} = \beta (\langle\Delta W\rangle - \Delta F) \geq 0$

Example 2: non-Markov random walks



$$E_{\text{ctrl}}(\hat{x}, a) = \frac{a}{2} \hat{x}^2, \quad E_{\text{nctrl}}[\hat{z}, \hat{w}] = \frac{1}{2} \int_0^\infty M(s) [\hat{w}^2(s) + \hat{z}^2(s)] ds.$$

Harmonic potential

$$\lambda_a[y | \{x_{t-\tau}\}_{\tau \geq 0}] = \lambda_0 \rho(y) \exp \left[-\frac{\beta}{2} \left\{ E_{\text{ctrl}}(x+y, a) - E_{\text{ctrl}}(x, a) + \frac{y^2}{2} \phi(0) + y \int_0^\infty \phi(\tau) v_{t-\tau} d\tau \right\} \right]$$

Memory function



The 2nd law holds! $\sigma_{\text{tot}} = \beta (\langle \Delta W \rangle - \Delta F) \geq 0$



Discussion I: entropy production

Another approach: Laplace embedding

K. Kanazawa and D. Sornette, Phys. Rev. Lett. **125**, 138301 (2020)
K. Kanazawa and D. Sornette, Phys. Rev. Lett. **127**, 188301 (2021)
K. Kanazawa and D. Sornette, Phys. Rev. Res. **6**, 023270 (2024)

Laplace transform of the historical velocity: $\hat{Z}_t(s) := \int_0^\infty \hat{v}_{t-\tau} e^{-s\tau} d\tau.$



Markov SPDE:

$$\frac{d\hat{x}_t}{dt} = \hat{\xi}_{\lambda_a[y|\hat{G}_t]}^{\text{CP}}, \quad \frac{d\hat{Z}_t(s)}{dt} = -s\hat{Z}_t(s) + \hat{\xi}_{\lambda_a[y|\hat{G}_t]}^{\text{CP}}, \quad \hat{G}_t := (\hat{x}_t, \{\hat{Z}_t(s)\}_s)$$



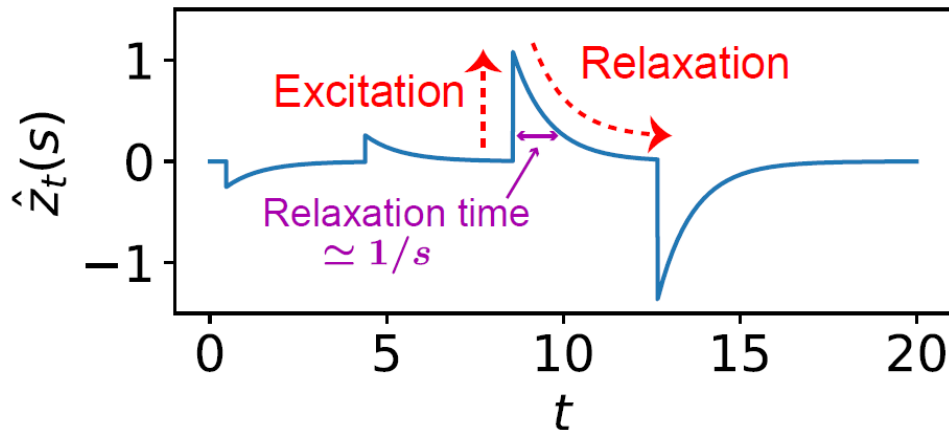
Field master equation:

$$\begin{aligned} \frac{\partial P_t[G]}{\partial t} &= \int ds \frac{\delta}{\delta Z(s)} (sZ(s)P_t[G]) + \int dy \{ \lambda[y | G - \Delta G_y] P_t[G - \Delta G_y] - \lambda[y | G] P_t[G] \} \\ &= \int ds \frac{\delta}{\delta Z(s)} (sZ(s)P_t[G]) + \int dG' \{ \tilde{\Lambda}[G | G'] P_t[G'] - \tilde{\Lambda}[G' | G] P_t[G] \} \end{aligned}$$

Q: Does the Laplace embedding work for stochastic thermodynamics?

A: No...

Problem of the Laplace-type embedding: time-reversal symmetry *cannot* be formulated in principle



$$\frac{d\hat{Z}_t(s)}{dt} = -s\hat{Z}_t(s) + \xi_{\lambda_a[y|\hat{G}_t]}^{\text{CP}}$$

Poisson jump

1st order SPDE

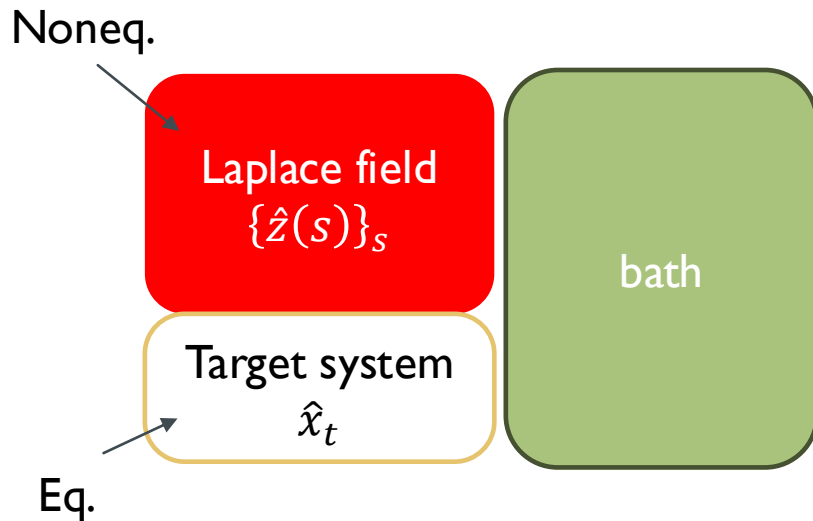
■ 1st-order differential Eq. + jump

- ✓ Trivially, irreversible relaxation occurs...
- ✓ In principle, no reverse paths are realized (exception = Gaussian limit)

■ Due to the 1st-order differential Eq., the time-reversal symmetry *never* exists

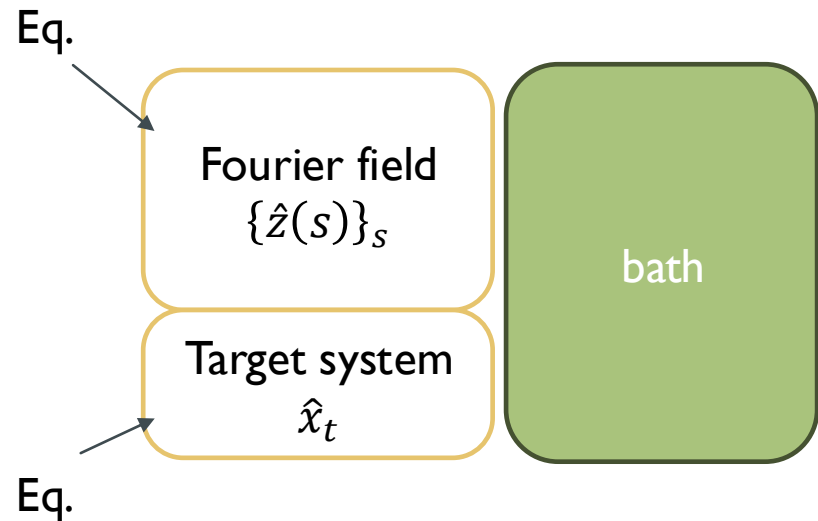
Discussion: what is the essential difference between the Laplace and Fourier embedding?

Laplace embedding



- Total intensity does not satisfy the time-reversal symmetry, even if the target system is in equilibrium
- Laplace field is always nonequilibrium

Fourier embedding



- Total intensity satisfies the time-reversal symmetry, if the target system is in equilibrium
- Fourier field can be equilibrium

Decomposition of the Laplace entropy based on the house-keeping entropy and excess entropy

Laplace embedding

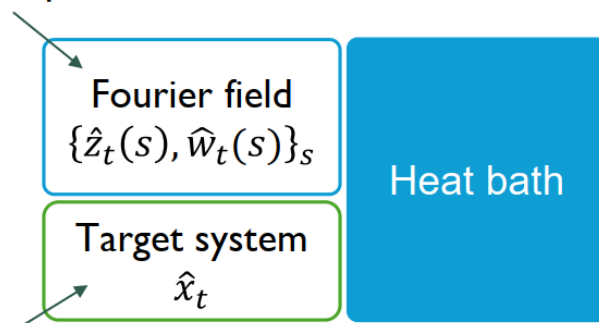
$$\hat{G}_t := (\hat{x}_t, \{\hat{Z}_t(s)\}_s) \quad \frac{d\hat{x}_t}{dt} = \hat{\xi}_{\lambda[y|\hat{G}_t]}^{\text{CP}}, \quad \frac{d\hat{Z}_t(s)}{dt} = -s\hat{Z}_t(s) + \hat{\xi}_{\lambda_{\alpha}[y|\hat{G}_t]}^{\text{CP}}$$



$$\left\{ \begin{array}{l} \dot{\Sigma}_{\text{hk}} \neq 0 \\ \dot{\Sigma}_{\text{ex}} := \int \lambda[y | \Gamma] P_t[\Gamma] \ln \frac{\lambda[y | \Gamma] P_t[\Gamma]}{\lambda[-y | \Gamma'^*] P_t[\Gamma']} dy d\Gamma = \dot{\sigma}_{\text{tot}} \end{array} \right. \quad \begin{array}{l} \dot{\Sigma}_{\text{tot}} = \dot{\Sigma}_{\text{hk}} + \dot{\Sigma}_{\text{ex}} \\ \text{Identical to Fourier's entropy!} \end{array}$$

- The total system by the Laplace embedding is always nonequilibrium $\dot{\Sigma}_{\text{tot}} \neq 0$, even if the target system is equilibrium (i.e., even if *Assumption 1* is satisfied).
- However, if we look at only the excess entropy (experimentally relevant), there is no difference between the Laplace and Fourier embedding
- Thermodynamic laws for the *target system* hold even for Laplace embedding, if *Assumptions 1-3* are satisfied, while the *total system* is nonequilibrium.

Equilibrium

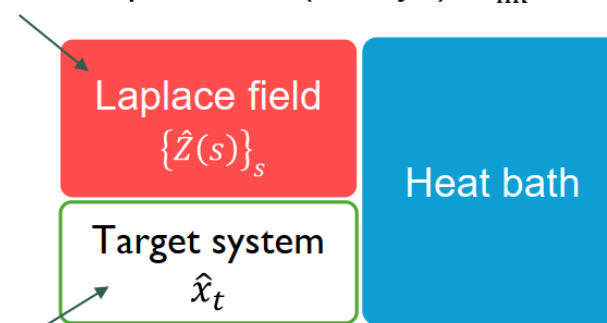


Equilibrium

Total entropy: σ_{tot}

(a) Fourier embedding

Non-equilibrium (always): $\Sigma_{\text{hk}} \neq 0$



Equilibrium: $\Sigma_{\text{ex}} = \sigma_{\text{tot}}$

Total entropy: $\Sigma_{\text{tot}} = \Sigma_{\text{hk}} + \Sigma_{\text{ex}} \neq \sigma_{\text{tot}}$

(b) Laplace embedding

See [A. Dechant and KK, arXiv:2604.25095] for the case of the generalized Langevin Eq.

$$\dot{\Sigma}_{\text{hk}} \neq 0$$

$$\dot{\Sigma}_{\text{tot}} = \dot{\Sigma}_{\text{hk}} + \dot{\Sigma}_{\text{ex}}$$

$$\dot{\Sigma}_{\text{ex}} := \int \lambda[y | \Gamma] P_t[\Gamma] \ln \frac{\lambda[y | \Gamma] P_t[\Gamma]}{\lambda[-y | \Gamma'^*] P_t[\Gamma']} dy d\Gamma \stackrel{\text{Identical to Fourier's entropy!}}{=} \dot{\sigma}_{\text{tot}}$$

- The total system by the Laplace embedding is always nonequilibrium $\dot{\Sigma}_{\text{tot}} \neq 0$, even if the target system is equilibrium (i.e., even if *Assumption 1* is satisfied).
- However, if we look at only the excess entropy (experimentally relevant), there is no difference between the Laplace and Fourier embedding
- Thermodynamic laws for the *target system* hold even for Laplace embedding, if *Assumptions 1-3* are satisfied, while the *total system* is nonequilibrium.



Summary

- KK and A. Dechant, arXiv:2506.04726
- A. Dechant and KK, arXiv:2604.25095

Summary:

stochastic thermodynamics for non-Markov jump models
based on Fourier's embedding + field master equations

$$\{\hat{x}_{t-\tau}\}_{\tau \geq 0}$$

$$\frac{d\hat{v}_t}{dt} = \hat{\xi}_{\lambda(y|\{\hat{v}_\tau\}_{\tau \leq t})}^{\text{CP}}$$



$$\hat{\Gamma} := (\hat{x}_t, \{\hat{z}_t(s)\}_{s>0}, \{\hat{w}_t(s)\}_{s>0})$$

$$\frac{d\hat{x}_t}{dt} = \hat{\xi}_{\lambda_{\alpha}[y|\hat{\Gamma}_t]}^{\text{CP}}, \quad \frac{\partial \hat{z}_t(s)}{\partial t} = \hat{w}_t(s)$$

$$\frac{\partial \hat{w}_t(s)}{\partial t} = -s^2 \hat{z}_t(s) + s \hat{\xi}_{\lambda_{\alpha}[y|\hat{\Gamma}_t]}^{\text{CP}}$$

■ Fourier embedding

- ✓ Time-reversal symmetry for the total system can be formulated

■ Stochastic thermodynamics formulated for general non-Markov jumps models

- ✓ “Gauge-invariance” of the second law:
embedding selection is *irrelevant* to the thermodynamic laws for irreversible work

■ Related works: Andreas Dechant (next talk) and Seita Shimoe (poster)