Hierarchy of Entropy Production and Thermodynamic Trade-off Relations in Non-Markovian Systems

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Based on: K. Funo, T. V. Vu, K. Saito, arXiv:2604.25245

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ERATO Sagawa Information-to-Energy Interconversion Project

QUANTUM ENERGY

Outline of this talk

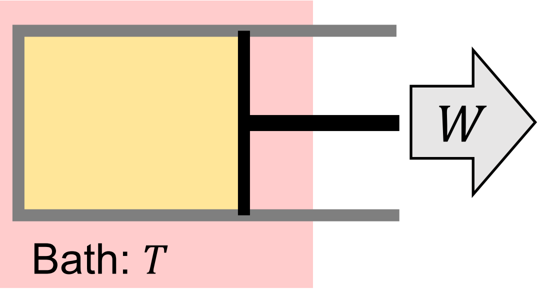
- Review: stochastic thermodynamics and non-Markovian effects
- Main result: Hierarchy of entropy production under Markovian embedding
- Application: Non-Markovian thermodynamic trade-off relations
- Generalization of the hierarchy to the quantum regime
- Conclusion

Thermodynamics in small systems

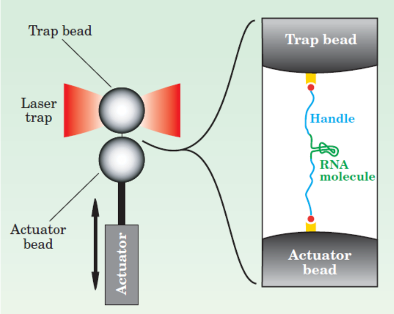
Classical thermodynamics

Steam engine

Wikipedia



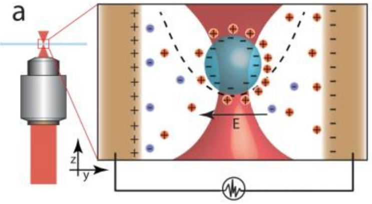
Stochastic thermodynamics



RNA molecules

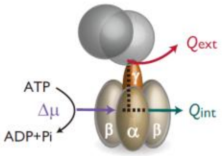
C. Bustamante *et al.*, Physics Today (2015)

Colloidal particles



S. Krishnamurthy, *et al.*, *Nat. Commun.* (2023)

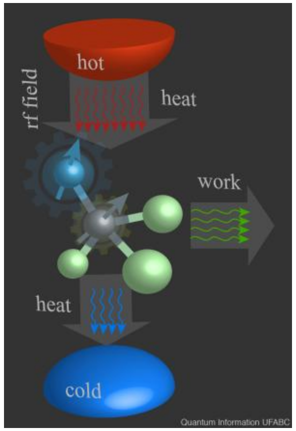
F1-ATPase



S. Toyabe *et al.*, PRL (2010)

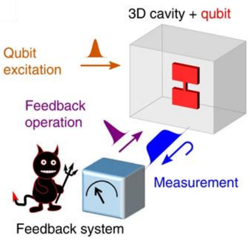
Quantum thermodynamics

NMR systems

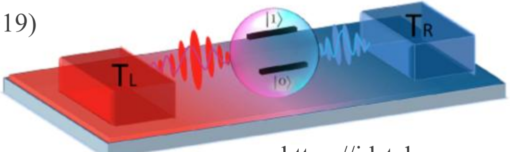


J. P. S. Peterson *et al.* (2019)

Superconducting qubits



Y. Masuyama, *et al.*, *Nat. Commun.* (2018)



https://idstch.com

~m

Thermal fluctuation

~μm

Quantum effects

μm~nm

Macroscopic

Microscopic

Equilibrium theory (established in 19th century)

Extended to **microscopic, out-of-equilibrium,** and **fluctuating** systems (late 1990s ~)

2nd law, fluctuation theorem, thermodynamic uncertainty relations, thermodynamic speed limits, …

Non-Markovian dynamics and memory effects

■ Non-Markovian dynamics

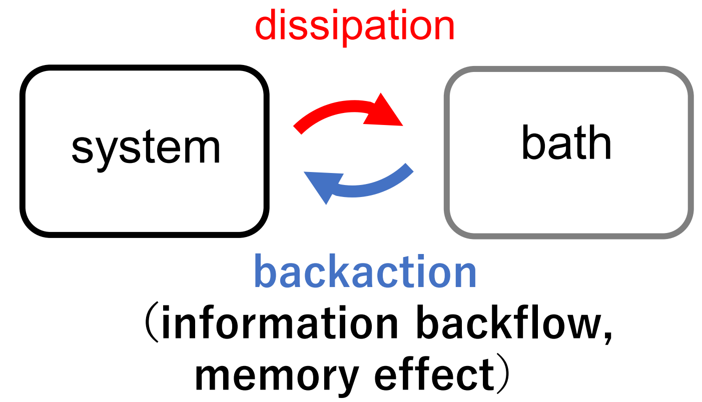
- bath time-scale is comparable or slower than that of the system
- the bath can temporarily store and return energy/excitation to the system

  “backflow of information” from the bath
- entropy production Σ: quantifies irreversibility and is always non-negative

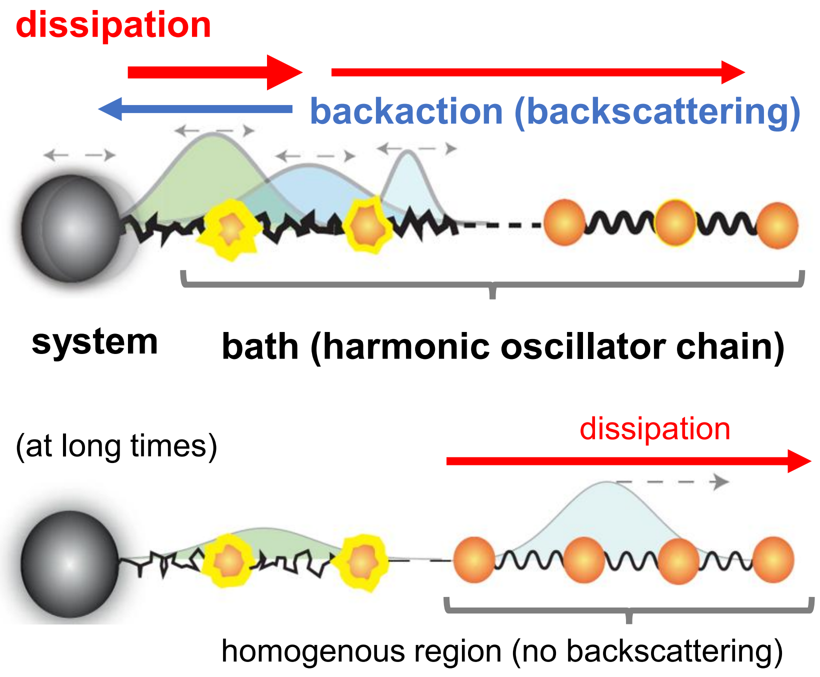
$$\Sigma = \underset{\text{system entropy change}}{\underline{\Delta S}} - \underset{\text{bath entropy change}}{\underline{\beta Q}} \geq 0$$

however, memory effects can render its ***rate transiently negative***

Strasberg & Esposito, PRE 99, 012120 (2019)



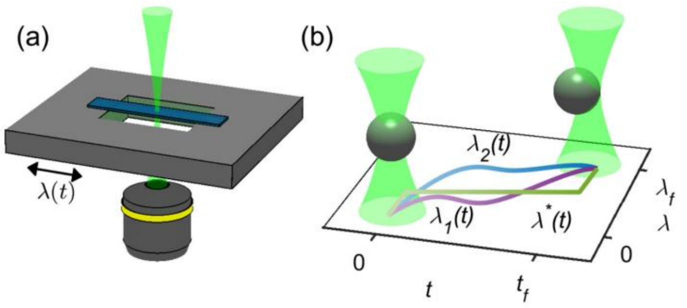
(information backflow, memory effect)



Chin, et al., Semicond. Semimet. (2011)

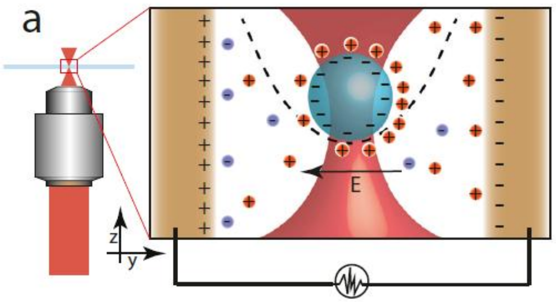
Non-Markovian experiments

☐ Optimal control



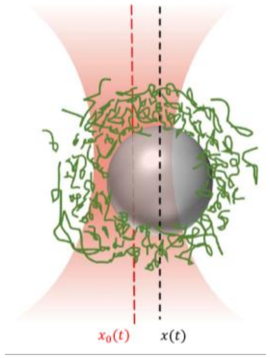
Loos, *et al*., PRX **14**, 021032 (2024)

☐ Heat engine with non-Markovian engineered noise



Kirshnamurthy, *et al*., *Nat. commun.* **14**, 6842 (2023)

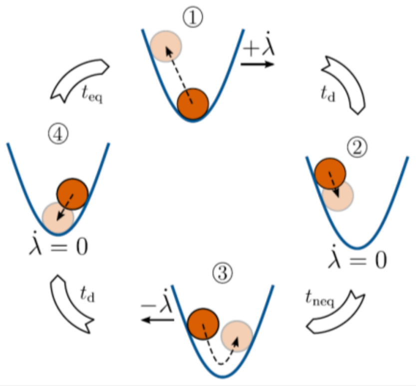
☐ Crooks fluctuation theorem under viscoelastic bath

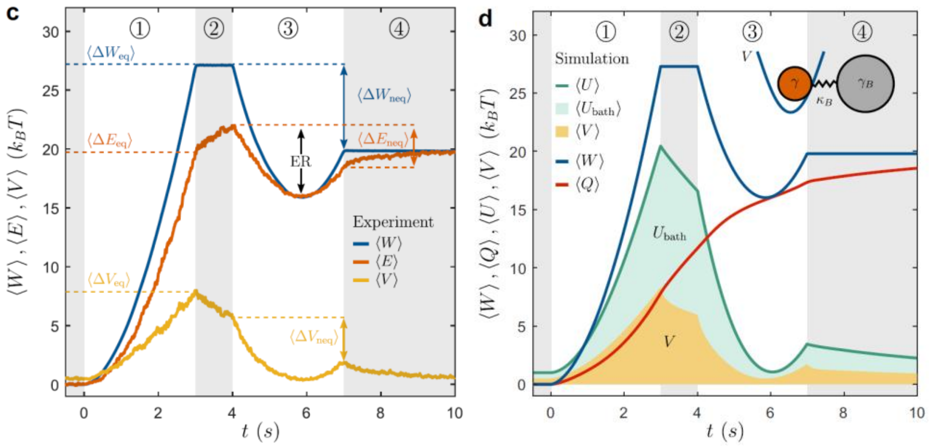


Das, et al., *NJP* **25,** 093051 (2023)

☐ Energy recuperation in non-Markovian baths

✓ **recovering** 30% of the **energy** injected into the surrounding medium **as useful work**





Ginot & Bechinger, *Nat. commun.* **16**, 10114 (2025)

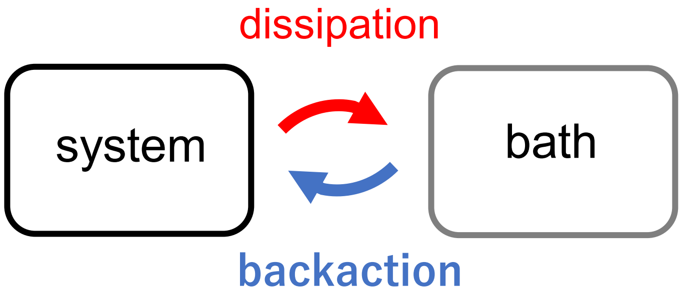
Motivation of this study

■ Non-Markovian dynamics

Review: de Vega, et al., Rev. Mod. Phys. **89**, 015001 (2017).

- colloidal particles/molecules in viscous fluids
- solid-state quantum devices
- complex molecular systems
- charge carrier dynamics in condensed matter systems

quantum



How memory effect modifies irreversibility (entropy production)?

Characterization of memory effect using stochastic thermodynamics?

we will mainly consider in the classical regime

Non-Markovian thermodynamics: related studies

■ Classical

• fluctuation theorem, Jarzynski equality for generalized Langevin dynamics	Ohkuma & Ohta, Stat. Mech.:Theory Exp. P10010 (2007), Speck & Seifert, Stat. Mech.:Theory Exp. L09002 (2007).
• stochastic thermodynamics for non-Markovian jump processes	Kanazawa & Dechant, arXiv:2506.04726, Dechant & Kanazawa, arXiv:2604.25095.
• strong-coupling thermodynamics	Seifert, PRL 116, 020601 (2016). Jarzynski, PRX 7, 011008(2017). Talkner & Hänggi, Rev. Mod. Phys. 92, 041002 (2020).

■ Quantum

• system-bath approach	Campisi, Hänggi, Talkner, Rev. Mod. Phys. 83, 771 (2011) Rivas, PRL 124, 160601 (2020), Vu, Honma, Saito, arXiv:2508.21567
• path-integral approach	**KF**, Quan, PRL 121, 040602 (2018) **KF**, Quan, PRE 98, 012113 (2018)
• reaction-coordinate approach	Strasberg, et al., NJP 18, 073007 (2016) Review: Nazir and Schaller (2018) [arXiv:1805.08307]
• pseudo-mode approach	Menczel, **KF**, et al., PRR 6, 033237 (2024)
• mesoscopic-leads approach	Brenes, et al., PRX 10, 031040 (2020)
• hierarchy of equation (HEOM) approach	Review: Kato and Tanimura, (2018) [arXiv:1804.02157]

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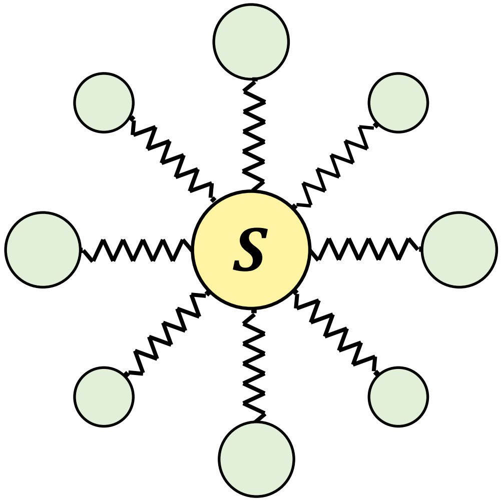
Setup: Generalized Langevin equation (GLE)

Gaussian bath model

$$H_S^{\lambda_t} + \underline{H_{int}} + \underline{H_B}$$

linear-coupling harmonic oscillators

$$H_S^{\lambda_t} = \frac{P^2}{2M} + V_S^{\lambda_t}$$



Generalized Langevin Equation (GLE)

$$\frac{dX}{dt} = \frac{P}{M}$$

$$\frac{dP}{dt} = -\partial_X V_S^{\lambda_t} - \int_0^t ds\, \underline{K(t-s)} \frac{P(s)}{M} + \underline{\eta_t}$$

memory kernel colored Gaussian noise

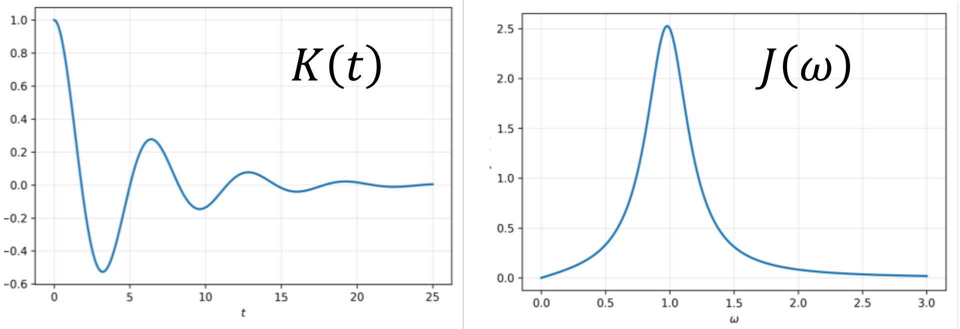
fluctuation-dissipation relation: $\mathbb{E}[\eta_t \eta_s] = K(t-s)/\beta$

Bath spectral density $J(\omega)$

- completely characterizes the property of the (Gaussian) bath
- connected to memory kernel via

$$K(t) := \frac{2}{\pi}\int_0^\infty d\omega \frac{J(\omega)}{\omega} \cos \omega\, t$$

e.g. underdamped Brownian motion spectral density



Entropy production in non-Markovian systems

■ Entropy production for strongly-coupled systems

Seifert, PRL (2016),
Jarzynski, PRX (2017),
Talkner, Hänggi, RMP (2020).

- Hamiltonian of mean force

$$H_S^* := H_S^{\lambda_t} - \beta^{-1} \ln \left[\int d\boldsymbol{z_B}\, e^{-\beta(H_{\mathrm{int}}+H_B)} / Z_B \right]$$

mean-force Gibbs state

$$\pi_S^* \propto \int d\boldsymbol{z_B}\, e^{-\beta(H_S+H_{\mathrm{int}}+H_B)} \propto \exp(-\beta H_S^*)$$

- Strong-coupling definitions

☐ internal energy

$$\mathcal{E}_S := H_S^* + \beta \partial_\beta H_S^*$$

☐ system entropy change

$$\Delta S_S^* := \underline{\Delta S_S} + \beta^2 \Delta \langle \partial_\beta H_S^* \rangle$$

Shannon entropy difference

☐ heat flux

$$\dot{Q}_{\mathrm{sys}} := \frac{d}{dt} \langle \mathcal{E}_S \rangle - \dot{W}$$

■ Harmonic oscillator bath model (with counter-term)

$$H_{\mathrm{int}} + H_B = \frac{1}{2} \sum_k \left(\frac{p_k^2}{M_k} + M_k \omega_k^2 \underline{\left(x_k - \frac{C_k X}{M_k \nu_k^2} \right)}^2 \right) \longrightarrow H_S^* = H_S^{\lambda_t}$$

interaction→shift of bath position

- non-Markovian entropy production (depends ***only on reduced system dynamics)***

$$\boldsymbol{\Sigma := \Delta S_S - \beta Q_{sys} \geq 0} \qquad \dot{Q}_{\mathrm{sys}} := \frac{d}{dt} \left\langle H_S^{\lambda_t} \right\rangle - \dot{W}$$

Markovian embedding: concept

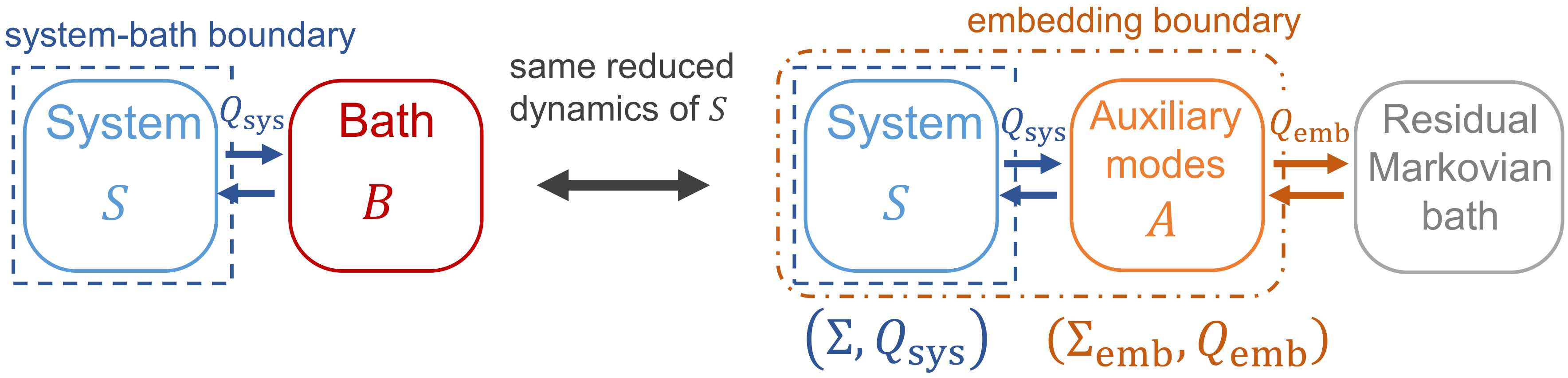
■ Gaussian bath model

- same $J(\omega)$ → equivalent reduced dynamics (GLE)
- represent the non-Markovian bath as:

Auxiliary modes A + **Residual Markovian bath**

(encoding memory) (irreversible dissipation)

☐ Original non-Markovian description ☐ Markovian embedded description



system-bath boundary

System S

Q_{sys}

Bath B

same reduced dynamics of S

embedding boundary

System S

Q_{sys}

Auxiliary modes A

Q_{emb}

Residual Markovian bath

(Σ, Q_{sys}) $(\Sigma_{\text{emb}}, Q_{\text{emb}})$

relation?

$$\Sigma = \Delta S_S - \beta Q_{\text{sys}} \longleftrightarrow \Sigma_{\text{emb}} = \Delta S_{SA} - \beta Q_{\text{emb}}$$

Non-Markovian entropy production

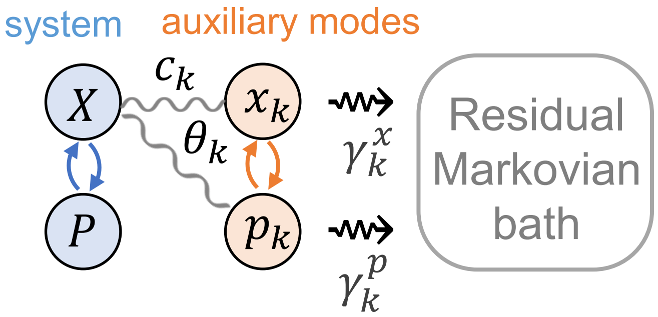
Markovian entropy production naturally defined for SA

Dalton, et al., (2025); Strasberg et al., (2016); Nazir and Schaller (2018); Kanazawa (2025);

Fokker-Planck equation for embedded system

■ Fokker-Planck equation for embedded Markovian system

$$\partial_t f_t^{SA} = \{H_{SA}^{tot}, f_t^{SA}\} + \mathcal{D} f_t^{SA}$$



c_k: coupling strength
θ_k: coupling angle

- steady-state: $\pi_{\lambda_t}^{SA} \propto \exp(-\beta H_{SA}^{tot})$

→ Markovian entropy production for SA:

$$\dot{\boldsymbol{\Sigma}}_{\mathbf{emb}} \coloneqq \dot{\boldsymbol{S}}_{\boldsymbol{SA}} - \boldsymbol{\beta} \dot{\boldsymbol{Q}}_{\mathbf{emb}} \geq \mathbf{0} \qquad \dot{Q}_{\mathrm{emb}} = \frac{d}{dt}\langle H_{SA}^{tot}\rangle - \dot{W}$$

- initial condition: $f_0^{SA} = f_0^S \underline{\pi^{A|S}}$ conditional thermal state $\pi^{A|S} \propto \exp(-\beta(H_{\mathrm{int}} + H_A))$

☐ reproduces GLE with $J(\omega)$ given by

c.f. convergence guarantees for discrete-mode approximations, Trivedi, et al., PRL (2021)

$$J(\omega) = \omega \sum_k \frac{c_k^2}{2 m_k \omega_k^2}\left[\frac{\Gamma_k + \delta_k \cos 2\,\theta_k (1 - \omega/\Omega_k)}{\Gamma_k^2 + (\Omega_k - \omega)^2} + \frac{\Gamma_k + \delta_k \cos 2\,\theta_k (1 + \omega/\Omega_k)}{\Gamma_k^2 + (\Omega_k + \omega)^2}\right]$$

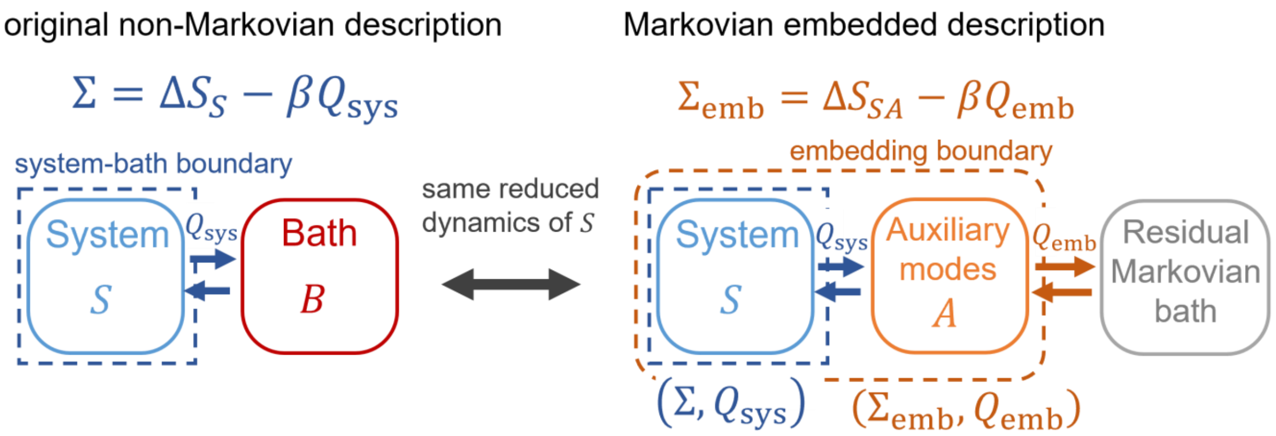
$$\Omega_k \coloneqq \sqrt{\omega_k^2 - \delta_k^2}$$

$$\gamma_k^p \coloneqq \Gamma_k + \delta_k \geq 0$$

$$\gamma_k^x \coloneqq \Gamma_k - \delta_k \geq 0$$

- Includes: Drude-Lorentz, Underdamped Brownian oscillator, Super-Ohmic spectral densities

Main result: Hierarchy of entropy production



original non-Markovian description

$\Sigma = \Delta S_S - \beta Q_{\mathrm{sys}}$

Markovian embedded description

$\Sigma_{\mathrm{emb}} = \Delta S_{SA} - \beta Q_{\mathrm{emb}}$

■ Hierarchy

$$\Sigma \geq \Sigma_{\mathrm{emb}}$$

■ Decomposition

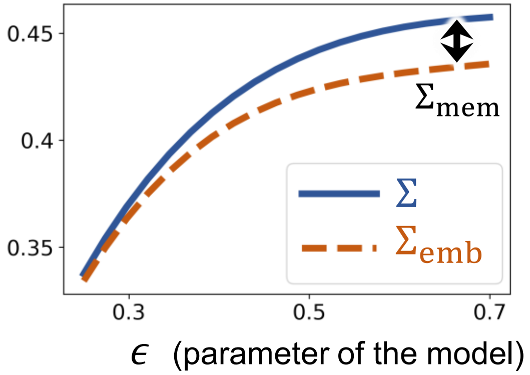
$$\Sigma = \Sigma_{\mathrm{emb}} + \Sigma_{\mathrm{mem}}$$

☐ memory contribution

$$\Sigma_{\mathrm{mem}} := D(f_\tau^{SA} || f_\tau^S \pi^{A|S}) \geq 0$$

- uses initial condition $f_0^{SA} = f_0^S \pi^{A|S}$

  conditional thermal state $\pi^{A|S} \propto \exp(-\beta(H_{\mathrm{int}} + H_A))$



■ Physical interpretation

Σ_{emb}: dissipation into the residual Markovian bath (memory-less part)

Σ_{mem}: correlation between S and A and deviation from $\pi^{A|S}$ (memory contribution)

$\Longrightarrow$ transiently negative *rate* $\dot{\Sigma}$ when $\dot{\Sigma}_{\mathrm{mem}}$ (memory contribution) is negative and dominates the memory-less part $\dot{\Sigma}_{\mathrm{emb}} \geq 0$

Funo, V. Vu, Saito, arXiv:2604.25245

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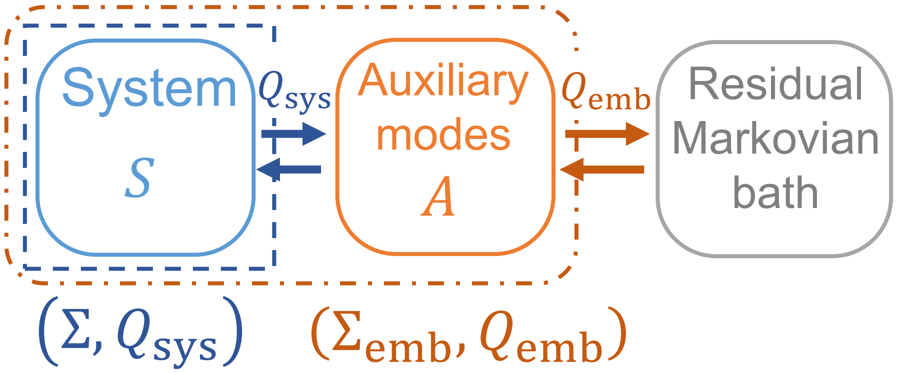
Applications: thermodynamic trade-off relations

■ Hierarchy



$$\Sigma \geq \Sigma_{\rm emb}$$

- utilize stochastic thermodynamic techniques developed for Markovian systems on SA
- bound based on system observables and Σ → *independent of the choice of embedding*

☐ underdamped regime

- entropic bound

$$\left(\int_0^\tau dt\,|\mathcal{J}_\mathcal{O}(t)|\right)^2 \leq \Theta\Sigma$$

- power-efficiency trade-off relation

$$\mathcal{P} \leq \beta_C \overline{\Theta}\eta(\eta_{Car} - \eta)$$

☐ overdamped regime

- thermodynamic speed limit

$$\frac{\mathcal{W}(f_0^\mathcal{S}, f_\tau^\mathcal{S})^2}{\tau} \leq \frac{\mu}{\beta}\Sigma$$

- thermodynamic uncertainty relation

$$\frac{2\left[\langle\mathcal{J}_\tau^\mathcal{S}\rangle + \Delta\langle\mathcal{J}_\tau^\mathcal{S}\rangle\right]^2}{\mathrm{Var}\left[\mathcal{J}_\tau^\mathcal{S}\right]} \leq \Sigma$$

Markovian: Shiraishi-Saito-Tasaki PRL (2016), Dechant-Sasa PRE (2018), Aurell, *et al.*, J. Stat. Phys. (2012), Koyuk-Seifert, PRL (2020)

Non-Markovian entropic bound

■ non-Markovian bound on heat current

$$\left(\int_0^\tau dt\,\underbrace{\left|\dot{Q}_{sys}\right|}_{\text{heat current}}\right)^2 \le \Theta\underbrace{\Sigma}_{\text{non-Markovian EP}}$$

$$\Theta := \frac{\mathcal{S}}{\beta}\int_0^\tau dt \int dXdP\,(P/M)^2 f_t^S(X,P)$$

$$\mathcal{S} := \sum_k \frac{c_k^2}{m_k\omega_k^2}\left(\frac{\cos^2\theta_k}{\gamma_k^x}+\frac{\sin^2\theta_k}{\gamma_k^p}\right)$$

- current is constrained by the entropy production
- prefactor Θ is controlled by
  - system-side observalble: $\int_0^\tau dt\langle(P/M)^2\rangle$
  - bath-side spectral property: $\mathcal{S}/\beta$
- does not depend on the choice of embedding

■ entropic bound using bath-induced current

- can be generalized for bath induced current

$$\mathcal{J}_{\mathcal{O}}(t) := -\int dXdP\,\mathcal{O}_t(X,P)\partial_P(F_t^{NM}f_t^S)$$

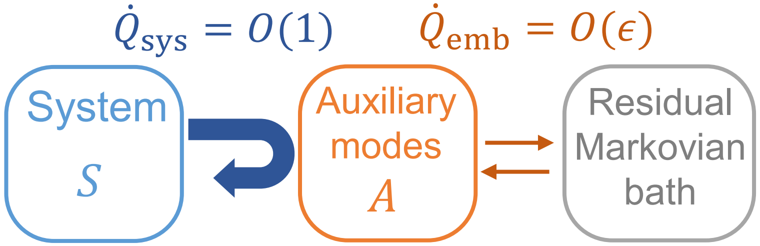
effective system dynamics

$$\partial_t f_t^S = \{H_S^{\lambda_t}, f_t^S\} - \partial_P\big(\underbrace{F_t^{\mathrm{NM}}}_{\text{non-Markovian bath contribution}} f_t^S\big)$$

$$F_t^{NM} := \int dxdp\partial_X H_{int} f_t^{A|S}(x,p|X,P)$$

Example: Memory-assisted heat exchange

- **Recovering energy injected into the bath?**

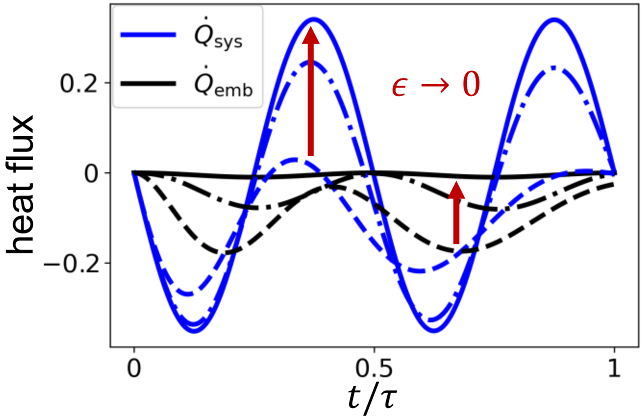


- **Explicit model**

- take $\gamma_k^x = \sin\theta_k = O(\epsilon), \gamma_k^p = O(\epsilon^{-1})$

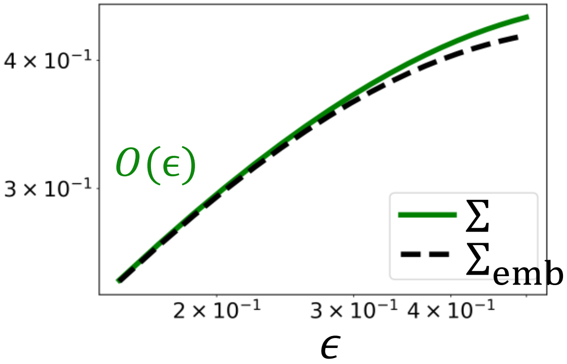
  $\longrightarrow$ **$\Theta = O(\epsilon^{-1})$: prefactor diverges**

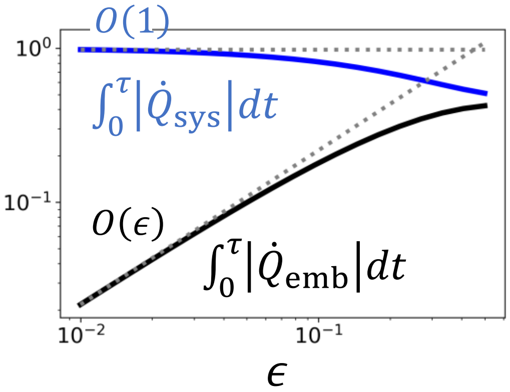
✓ oscillatory heat exchange between S and A with negligible dissipation to the residual Markovian bath



✓ Scaling

$$\underbrace{\left(\int_0^\tau dt\,\left|\dot{Q}_{\mathrm{sys}}(t)\right|\right)^2}_{O(1)} \le \underbrace{\Theta}_{O(\epsilon^{-1})}\,\underbrace{\Sigma}_{O(\epsilon)}$$



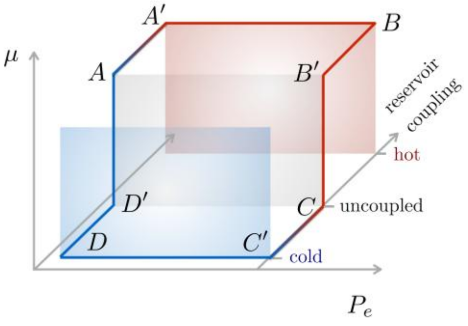


Power-efficiency trade-off relation

- **Heat engine setup:** multiple baths + time-dependent coupling g_t^a

Newman, Mintert, Nazir, PRE 95, 032139 (2017)

- incorporate the work cost of switching on/off the coupling



- **Power-efficiency trade-off relation**

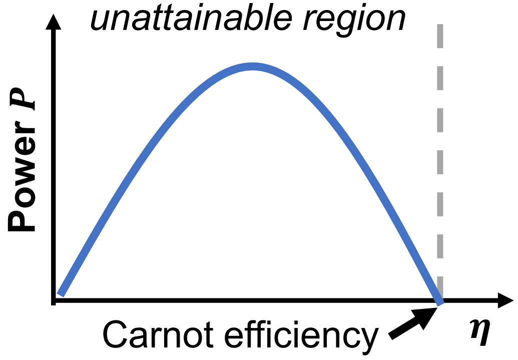
$$\underline{\mathcal{P}} \leq \beta_C \overline{\Theta}\, \underline{\eta}(\eta_{Car} - \underline{\eta})$$

power efficiency

Carnot efficiency
$\eta_{Car} = 1 - \beta_H/\beta_C$

$$\overline{\Theta} := \frac{1}{\tau}\sum_{a=\mathrm{H,C}} \frac{\mathcal{S}_a}{\beta_a}\int_0^\tau dt \int dXdP \left(\frac{g_t^a P}{M} + \dot{g_t^a} X\right)^2 f_t^S(X,P)$$

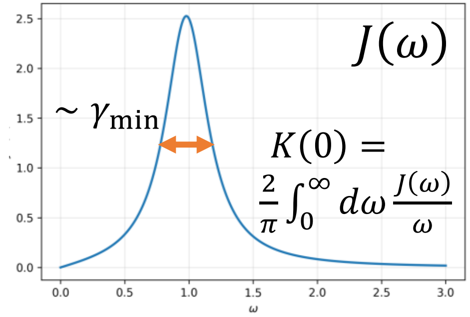
$\mathcal{S}_a$: spectral property of bath a



- finite $\overline{\Theta}$: Carnot efficiency at finite power is not attainable

$$\mathcal{S} \leq K(0)/\gamma_{\min} \qquad \gamma_{\min} = \min_k\{\gamma_k^x, \gamma_k^p\}$$

finite $K(0)/\gamma_{\min}$: $J(\omega)$ does not have sharp peaks and decays sufficiently fast



- Ohmic spectral density + constant coupling

→ recovers Shiraishi-Saito-Tasaki PRL 2016 for Markovian systems

Overdamped regime

■ Overdamped generalized Langevin equation

$$\frac{1}{\mu}\frac{dX}{dt} = \frac{\xi_t}{\mu} - \partial_X V_S^{\lambda_t}(X) - \int_0^t ds K^{od}(t-s)\dot{X}(s) + \eta_t^{od}$$

Markovian damping&noise

non-Markovian damping&noise

overdamped Drude-Lorentz spectrum

$$J^{od}(\omega) \coloneqq \omega \sum_k \frac{\mu_k c_k^2}{\omega^2 + \mu_k^2 \kappa_k^2}$$

$$K^{od}(t) \coloneqq \sum_k \frac{c_k^2}{\kappa_k} e^{-\mu_k \kappa_k |t|}$$

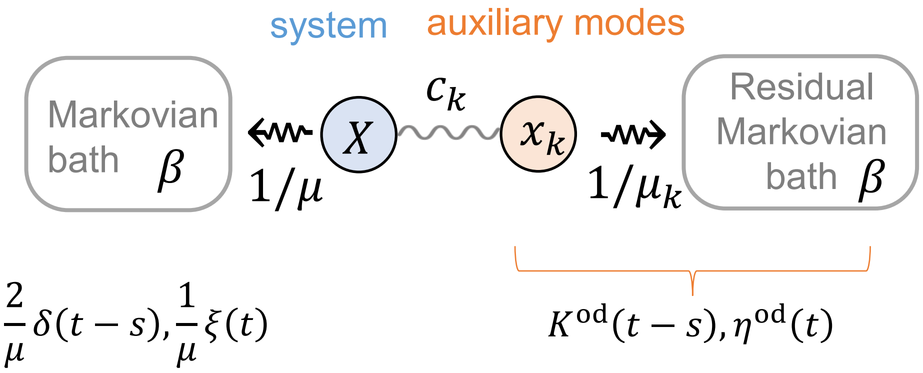
- underdamped GLE → small-mass limit

with the following choice: $K(t-s) = K^{\mathrm{od}}(t-s) + \frac{2}{\mu}\delta(t-s),\ \eta(t) = \eta^{\mathrm{od}}(t) + \frac{1}{\mu}\xi(t)$

■ Markovian embedding: bipartite overdamped systems

$$\frac{1}{\mu}\frac{dX}{dt} = -\partial_X V_S^{\lambda_t} + \sum_k c_k\left(x_k - \frac{c_k}{\kappa_k}X\right) + \frac{1}{\mu}\xi$$

$$\frac{1}{\mu_k}\frac{dx_k}{dt} = -\kappa_k\left(x_k - \frac{c_k}{\kappa_k}X\right) + \frac{1}{\mu_k}\xi_k$$

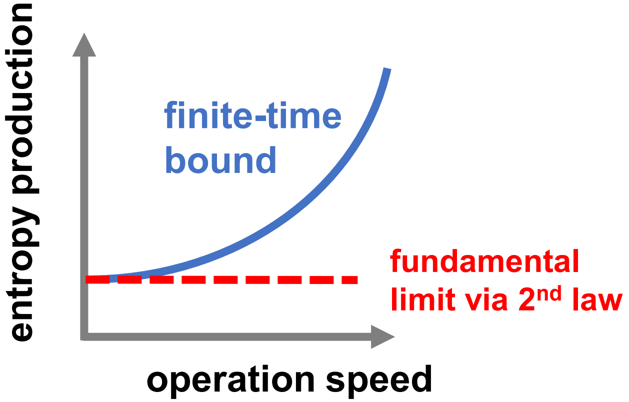


Overdamped applications

■ Thermodynamic speed limit

$$\underbrace{\frac{\mathcal{W}(f_0^{\mathcal{S}},f_\tau^{\mathcal{S}})^2}{\tau}}_{\text{speed of changing the probability distribution}} \leq \underbrace{\frac{\mu}{\beta}\Sigma}_{\text{thermodynamic cost (non-Markovian EP)}}$$

$\mathcal{W}(f,g)$: L^2-Wasserstein distance



■ Transient TUR with external driving

$$\underbrace{\frac{2[\langle\mathcal{J}_\tau^{\mathcal{S}}\rangle+\Delta\langle\mathcal{J}_\tau^{\mathcal{S}}\rangle]^2}{\mathrm{Var}[\mathcal{J}_\tau^{\mathcal{S}}]}}_{\text{precision of the current}} \leq \underbrace{\Sigma}_{\text{thermodynamic cost}}$$

$$\Delta \coloneqq \tau\partial_\tau - v\partial_v$$

time-dependent driving λ_{vt} with explicit speed parameter v

- $\mathcal{J}_\tau^{\mathcal{S}}$: arbitrary current constructed from the system trajectory

  e.g. $\mathcal{J}_\tau^{\mathcal{S}} = \frac{1}{\tau}\int_0^\tau dt\, f[X(t),\lambda_{vt}] \circ \dot{X}(t)$
- direct application of hierarchy to Koyuk-Seifert (2020)

formal expressions of the bounds remain the same as in the Markovian case → memory effect?

Extended hierarchy of entropy production

■ Continuity equation of the system

$$\partial_t f_t^S = -\partial_X J_t^S, \qquad J_t^S(X) := \mu\big[\underline{v_t(X)} + \underline{F_t^{\mathrm{NM}}(X)}\big] f_t^S(X)$$

conventional Markovian mean local velocity

memory force (effect of non-Markovian bath)

$$v_t(X) := -\partial_X V_S^{\lambda_t}(X) - \frac{1}{\beta}\partial_X \ln f_t^S(X) \qquad F_t^{\mathrm{NM}}(X) := \int d\boldsymbol{x}(\partial_X H_{\mathrm{int}}) f_t^{A|S}(\boldsymbol{x}|X)$$

■ Extended hierarchy of entropy production (overdamped case)

$$\underline{\widetilde{\Sigma}_{\mathrm{tp}}} \le \Sigma_{\mathrm{emb}} \le \Sigma = \Sigma_{\mathrm{M}} + \Sigma_{\mathrm{NM}} \le \Sigma_{\mathrm{M}}, \qquad \Sigma_{\mathrm{NM}} \le 0$$

transport cost: $\widetilde{\Sigma}_{\mathrm{tp}} := \frac{\beta}{\mu}\int_0^\tau dt \int dX (J_t^S)^2/f_t^S$

☐ *apparent* Markovian entropy production

$$\Sigma_{\mathrm{M}} := \mu\beta \int_0^\tau dt\, \langle v_t^2 \rangle \ge 0$$

- conventional Markovian expression

☐ ***negative*** non-Markovian correction term

$$\Sigma_{\mathrm{NM}} := \mu\beta \int_0^\tau dt\, \langle v_t F_t^{\mathrm{NM}} \rangle \le 0$$

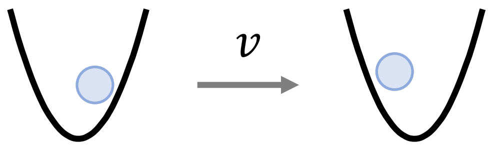
- negative time-integrated cross-correlation between v_t and F_t^{NM}

in the Markovian limit ($K^{\mathrm{od}} \to 0$), all quantities coincide: $\widetilde{\Sigma}_{\mathrm{tp}} = \Sigma_{\mathrm{emb}} = \Sigma = \Sigma_{\mathrm{M}}$ and $\Sigma_{\mathrm{NM}} = 0$

Example: TUR for dragged harmonic oscillator

dragged harmonic oscillator example

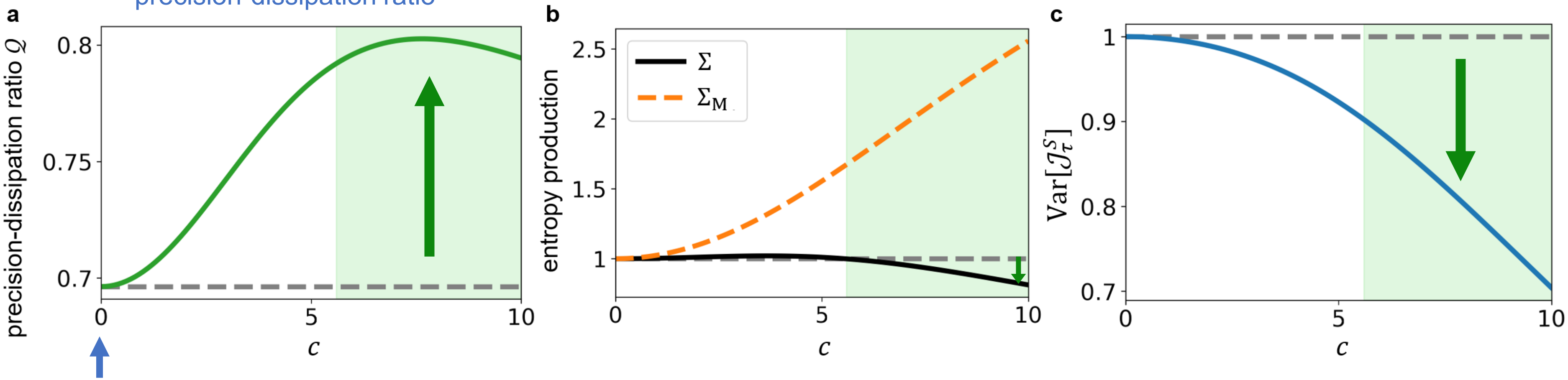
$$V_{\lambda_{vt}}(X)=\frac{\kappa}{2}(X-vt)^2 \qquad \mathcal{J}_\tau^{\mathcal{S}}=[X(\tau)-X(0)]/\tau$$



■ Effect of memory on the transient TUR

$$\mathcal{Q}:=\frac{2\left[\langle\mathcal{J}_\tau^{\mathcal{S}}\rangle+\Delta\langle\mathcal{J}_\tau^{\mathcal{S}}\rangle\right]^2}{\Sigma\mathrm{Var}\left[\mathcal{J}_\tau^{\mathcal{S}}\right]}\le 1 \qquad \Sigma\le\Sigma_{\mathrm{M}}$$

precision-dissipation ratio



Markovian limit

c: coupling to the non-Markovian bath

✓ bath memory effect can be utilized to improve the precision-dissipation ratio

F_t^{NM}
- *negative* time-integrated cross-correlation with ν_t → suppression of Σ
- *temporal correlation* of $X(t)$ → suppression of $\mathrm{Var}[\mathcal{J}_\tau^{\mathcal{S}}]$

Outline of this talk

- Review: stochastic thermodynamics and non-Markovian effects
- Main result: Hierarchy of entropy production under Markovian embedding
- Application: Non-Markovian thermodynamic trade-off relations
- Generalization of the hierarchy to the quantum regime
- Conclusion

Hierarchy in the quantum regime & general bath

Setup: GKLS master equation

$$\partial_t \rho_t^{SA} = -\frac{i}{\hbar}[H_{SA}^{\mathrm{tot}}, \rho_t^{SA}] + \mathcal{D}_t[\rho_t^{SA}]$$

$$H_{SA}^{\mathrm{tot}} = H_S^{\lambda_t} + H_{\mathrm{int}}^{g_t} + H_A$$

steady-state: $\pi_{\lambda_t, g_t}^{SA} \propto \exp(-\beta H_{SA}^{\mathrm{tot}})$

- Hamiltonian of mean force

$$H_S^* := -\beta^{-1} \ln\left(\mathrm{Tr}_A\left[e^{-\beta H_{\mathrm{tot}}^{SA}}/Z_A\right]\right)$$

→ strong-coupling definitions following Seifert, PRL (2016)

$$\mathcal{E}_S := H_S^* + \beta \partial_\beta H_S^*$$

$$\Delta S_S^* := \Delta S_S + \beta^2 \Delta\langle \partial_\beta H_S^* \rangle$$

$$\dot{Q}_{\mathrm{sys}} := \frac{d}{dt}\langle \mathcal{E}_S \rangle - \dot{W}$$

Hierarchy in the quantum regime & general bath

$$\Sigma = \Sigma_{\mathrm{emb}} + \Sigma_{\mathrm{mem}} \geq \Sigma_{\mathrm{emb}}$$

$$\Sigma_{\mathrm{mem}} := D(\rho_\tau^{SA} || \pi_{\lambda_\tau, g_\tau}^{SA}) - D(\rho_\tau^S || \pi_{\lambda_\tau, g_\tau}^S) \geq 0$$

See also Lacerda, et al., PRE (2024)
based on mesoscopic leads approach

- by assuming the initial condition as
  a. product state $\rho_0^{SA} = \rho_0^S \otimes \pi^A$ with the interaction switched off at $t = 0$: $H_{\mathrm{int}}^{g_0} = 0$
  b. thermal state $\rho_0^{SA} = \pi_{\lambda_0, g_0}^{SA}$

  more generally, ρ_0^{SA} that satisfies $D(\rho_0^{SA} || \pi_{\lambda_0, g_0}^{SA}) - D(\rho_0^S || \pi_{\lambda_0, g_0}^S) = 0$ ← $\rho_0^{SA} = \mathcal{R}_{\pi_0}^{\mathrm{Tr}_A}(\rho_0^S)$

  Petz recovery map

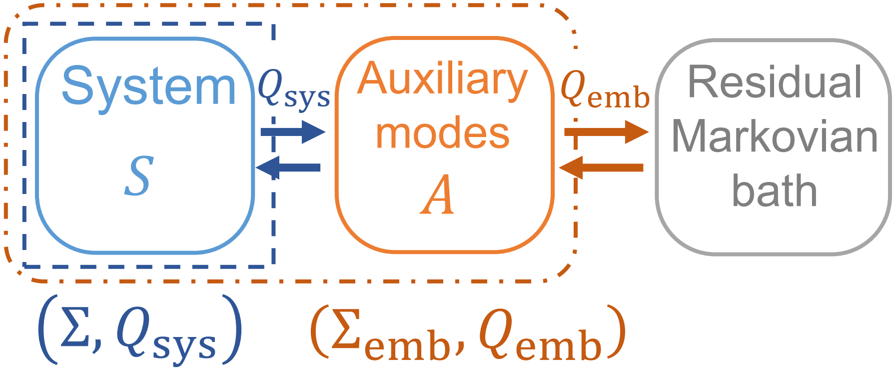
- classical case (general bath)

$\Sigma_{\mathrm{mem}} = D(f_\tau^{SA} || f_\tau^S \pi_{g_\tau}^{A|S})$, initial condition: $f_0^{SA} = f_0^S \pi_{g_0}^{A|S}$ (conditional thermal state)

Conclusion

✓ Hierarchy of entropy production under Markovian embedding

$$\Sigma = \Sigma_{\mathrm{emb}} + \Sigma_{\mathrm{mem}} \geq \Sigma_{\mathrm{emb}}$$



✓ non-Markovian thermodynamic trade-off relations

- underdamped regime: entropic bound, power-efficiency trade-off
- overdamped regime: thermodynamic speed limit, TUR

☐ further investigation of the memory effect

- thermodynamic meaning of non-Markovian correction term

$$\Sigma = \Sigma_{\mathrm{M}} + \Sigma_{\mathrm{NM}} \leq \Sigma_{\mathrm{M}}, \quad \Sigma_{\mathrm{NM}} = \mu\beta \int_0^\tau dt \left\langle v_t F_t^{\mathrm{NM}} \right\rangle \leq 0$$ (overdamped regime)

- hierarchy in the quantum regime: role of system-bath coherence, entanglement